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Hsiang-Ping HUANG

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COMMUTATORS ASSOCIATED TO A SUBFACTOR AND ITS RELATIVE COMMUTANTS

by Hsiang-Ping HUANG

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1. Introduction.

A central problem in subfactor theory is the classification of inclusions of II_1 factors, $N \subseteq M$. An important invariant for such an inclusion is the lattice of higher relative commutants, $\{M_i' \cap M_j\}_{i,j}$, known as the standard invariant, contained in the Jones tower $N \subseteq M \subseteq M_1 \cdots$. There are several approaches to studying the standard invariant, namely paragroups [3], λ-lattices [8], and planar algebras [6]. In the geometric framework of planar algebras, the existence of rotation operators is apparent. This is in contrast to the paragroup or λ-lattice setting where the existence of rotation operators is by no means obvious. However, for the moment, the planar algebra framework is restricted to the case then $N \subseteq M$ is extremal,

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a condition that is reflected in the spherical invariance of the corresponding planar algebra. For nonextremal inclusions $N \subseteq M$, it is not yet known if there are corresponding pictorial descriptions. Our paper provides a backbone to construct such "planar algebras". We prove the higher relative commutants of a subfactor inclusion are isomorphic to the cyclic tensor products. These cyclic tensor products admit natural rotation operators and other "planar" actions. In the future, we hope to use these actions to extend the definition of planar algebras, via the λ-lattice description, to the nonextremal setting.

The main result of this work is a structural result that decomposes a factor M into elements in $N' \cap M$ and commutators with a specific form:

MAIN THEOREM. — *Given a pair of II_1 factors $N \subseteq M$, $[M : N] < \infty$. There exists $m \in \mathbb{N}$ such that $\forall z \in M$ with $E_{N' \cap M}(z) = 0$, z is of the form*

$$z = \sum_{i=1}^{m} [a_i, b_i] \text{ where } a_i \in N,\ b_i \in M,$$

$$\|a_i\| \leqslant 2, \quad \|b_i\| \leqslant \frac{9}{2}\|z\|.$$

The number m depends solely on the Jones index; and thus, is independent of z. Here $E_{N' \cap M}$ denotes the trace-preserving conditional expectation.

This result can be viewed (by taking $N = M$) as an extension of a result appearing in [4], that in a II_1 factor, any element whose trace is zero can be written as a finite sum of commutators.

The proof of the main theorem follows the arguments in [4]: constructing a series of commutators with the support of the commutators being mutually orthogonal. In order for the series to converge, we need to uniformly bound the norms of the commutators. In [4], the uniform bound of norms heavily relies on the existence of spectral projections and the like, which is no longer possible in our case. We use S. Popa's relative Dixmier property [9] (which requires the Jones index to be finite) to get the norm estimate. Hence the series converges strongly.

In Section 2 we present the relative Dixmier property by S. Popa [10] which is the main technical part in this paper. In Section 3, with the aid of the relative Dixmier property, we prove the main theorem. In Section 4, we define the cyclic tensor products. We use the main theorem to prove the isomorphism between the graded vector space $\{N' \cap M_n\}$ and the cyclic tensor products. This isomorphism allows us to see clearly the existence of rotation operators on $\{N' \cap M_n\}$.

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2. Preliminaries.

The following is the relative Dixmier property by S. Popa for any inclusion of finite von Neumann algebras with finite Pimsner-Popa index.

THEOREM 1 [9], [10]. — *For any $c \geqslant 1$, any $\epsilon > 0$, there exists $m = m(c, \epsilon) \in \mathbb{N}$, such that given any inclusion $N \subseteq M$ of finite von Neumann algebras with a normal faithful conditional expectation $E_N : M \to N$ satisfying the Pimsner-Popa index $\operatorname{Ind} E_N \leqslant c$ and with a conditional expectation $E_{N' \cap M} : M \to N' \cap M$ satisfying $E_{N' \cap M}(ab) = E_{N' \cap M}(ba)$, where $a \in N, b \in M$, we have that, $\forall x \in M$ with $E_{N' \cap M}(x) = 0$, there are unitaries $u_1, \cdots, u_m$ in N such that $\| \sum_{k=1}^{m} \frac{1}{m} u_k x u_k^* \| \leqslant \epsilon \|x\|$.*

In [10] S. Popa proved the number m can be explicitly given.

The properties of conditional expectations (see for example [11]) are vital and assumed without further reference in the paper. Let $N \subseteq M$ be a pair of finite von Neumann algebra with a faithful normal trace τ on M. Let E_N (resp. $E_{N' \cap M}$) be the trace preserving conditional expectation of M onto N (resp $N' \cap M$). Define the II-norm on M with respect to τ by

$$\|x\|_2 = \tau(x^* x)^{\frac{1}{2}}, \quad \forall x \in M.$$

We observe that $E_{N' \cap M}(ab) = E_{N' \cap M}(ba)$ and $E_{N' \cap M}([a, b]) = 0$ where $a \in N, b \in M$. And it is a useful fact [2] that for $a \in N, E_{N' \cap M}(a) = \tau(a)$ where τ denotes the trace of M.

Proof. — For $a \in N$, $b \in N' \cap M$,

$$\tau(E_{N' \cap M}(a) b) = \tau(ab) = \tau(a E_N(b)) = \tau(a)\tau(b).$$

Note that $E_N(b) \in N' \cap N = \mathbb{C}$. Again E_N is the trace-preserving conditional expectation [2]. $\square$

3. Main theorem.

In this section we use the relative Dixmier property in Section 2 to prove the following theorem.

MAIN THEOREM. — *Given a pair of II_1 factors $N \subseteq M$, $[M : N] < \infty$. There exists $m \in \mathbb{N}$ such that $\forall z \in M$ with $E_{N' \cap M}(z) = 0$, z is of the form*

$$z = \sum_{i=1}^{m} [a_i, b_i] \text{ where } a_i \in N,\ b_i \in M.$$

$$\|a_i\| \leqslant 2, \quad \|b_i\| \leqslant \frac{9}{2}\|z\|.$$

The number m depends solely on the Jones index; and thus, is independent of z. Here $E_{N' \cap M}$ denotes the trace-preserving conditional expectation.

3.1. Reduction to the left upper block.

This subsection is about reducing to a special case so that we can apply the techniques in [4]. First we can reduce the main theorem to the case where z is supported by a projection which is sufficiently small in the following sense:

LEMMA 1. — *For all $z \in M$ with $E_{N' \cap M}(z) = 0$, there exist mutually orthogonal projections p, q in N and elements z_1, z_{-1} in M, such that*

$$\begin{array}{llll} z_1 = p z_1 p, & E_{N' \cap M}(z_1) = 0, & \tau(p) = \dfrac{1}{2}, & \|z_1\| \leqslant 3\|z\| \\ z_{-1} = q z_{-1} q, & E_{N' \cap M}(z_{-1}) = 0, & \tau(q) = \dfrac{1}{2}, & \|z_{-1}\| \leqslant 3\|z\| \end{array}$$

and that $z - (z_1 + z_{-1})$ is the sum of three commutators of the form $[x, y]$ with $x \in N$, $y \in M$, $\|x\| \leqslant 1$, and $\|y\| \leqslant 2\|z\|$.

Proof. — Let p, q be two mutually orthogonal projections in N with $\tau(p) = \tau(q) = \frac{1}{2}$. Setting

$$z_1 = p(z - 2E_{N' \cap M}(pz))p, \quad z_{-1} = q(z - 2E_{N' \cap M}(qz))q$$

we have

$$z-(z_1+z_{-1}) = pzq + qzp + 2pE_{N'\cap M}(pz) + 2qE_{N'\cap M}(qz).$$

Let w be a partial isometry in N with q as the initial projection and p as the final projection. Then

$$\begin{aligned} pzq = [p, zq],\quad qzp = [q, zp] \\ 2pE_{N'\cap M}(pz) + 2qE_{N'\cap M}(qz) &= 2pE_{N'\cap M}(pz) - 2qE_{N'\cap M}(pz) \\ &= 2ww^*E_{N'\cap M}(pz) - 2E_{N'\cap M}(pz)w^*w \\ &= [w, 2E_{N'\cap M}(pz)w^*]. \end{aligned}$$

Thus $z-(z_1+z_{-1})$ is the sum of three commutators with the special form. Now we proceed to check the properties of z_1, z_{-1}, x's and y's:

$$\begin{aligned} E_{N'\cap M}(z_1) &= E_{N'\cap M}(p(z - 2E_{N'\cap M}(pz))) \\ &= E_{N'\cap M}(pz) - 2E_{N'\cap M}(p)E_{N'\cap M}(pz) = 0, \end{aligned}$$

since $E_{N'\cap M}(p) = \tau(p) = \frac{1}{2}$.

$\|z_1\| \leqslant \|z\| + 2\|E_{N'\cap M}(pzp)\| \leqslant 3\|z\|$, because of the following: we can write $z = z^1 + iz^2$, where z^1, z^2 are self adjoint and have norms less than or equal to $\|z_1\|$:

$$\begin{aligned} E_{N'\cap M}(pz^1p) \leqslant \|z^1\|E_{N'\cap M}(p) = \frac{1}{2}\|z^1\| \leqslant \frac{1}{2}\|z\|. \\ E_{N'\cap M}(pz^2p) \leqslant \|z^2\|E_{N'\cap M}(p) = \frac{1}{2}\|z^2\| \leqslant \frac{1}{2}\|z\|. \end{aligned}$$

In the same manner we get $\|z_{-1}\| \leqslant 3\|z\|$.

For the x's and y's part, the only thing to check is

$$\|2E_{N'\cap M}(pz)w^*\| \leqslant 2\|z\|.$$

This follows from the above. □

From now on, we concentrate on showing z_1 is a finite sum of commutators with the specific form; while z_1, as in the lemma, is in M, satisfying $E_{N'\cap M}(z_1) = 0$, and $z_1 = pz_1p$. We take $p_1 = p$, a projection in N with $\tau(p_1) = \frac{1}{2}$. Likewise the case of z_{-1} follows.

3.2. Cutting into diagonal blocks.

This subsection is about setting up the environment we work on. Take p_1 as the first term of the sequence $(p_n)_{n\in\mathbb{N}}$ of mutually orthogonal projections in N, with $\tau(p_n) = 2^{-n}$ for all $n \in \mathbb{N}$. we get that $\sum_{n\in\mathbb{N}} p_n = 1$ (the sum converges strongly). For all $n \in \mathbb{N}$, we denote M_{p_n} (resp. $M_{p_n+p_{n+1}}$) as the reduced algebra of M by p_n (resp. $p_n + p_{n+1}$). Here the reduced algebra of M by a projection p_n in M is the von Neumann subalgebra $M_{p_n} = \{x \in M, x = p_n x p_n\}$ of M. Note the unit of the algebra M_{p_n}, which is p_n, is in general distinct from $1 \in M$. Similarly, we can define N_{p_n} and $N_{p_n+p_{n+1}}$.

Define $E_{p_nNp_n'\cap p_nMp_n} : p_nMp_n \to p_nNp_n' \cap p_nMp_n$ the conditional expectation by

$$E_{p_nNp_n'\cap p_nMp_n}(x) = \tau(p_n)^{-1} p_n E_{N'\cap M}(x) \text{ for } x \in p_nMp_n.$$

Note that $p_nNp_n' \cap p_nMp_n = p_nN' \cap Mp_n$ because $p_n \in (N' \cap M)' \cap M$ (see [7]). We have that

$$E_{p_nNp_n'\cap p_nMp_n}(a_nb_n) = E_{p_nNp_n'\cap p_nMp_n}(b_na_n),$$

where $a_n \in p_nNp_n, b_n \in p_nMp_n$.

Observe that $p_nNp_n \subseteq p_nMp_n$ is an inclusion of II_1 factors with the conditional expectation $E_{p_nNp_n} := E_N|_{p_nMp_n}$, inherited from the inclusion $N \subseteq M$,

$$\operatorname{Ind} E_{p_nNp_n}(p_nMp_n \to p_nNp_n) = [p_nMp_n : p_nNp_n] \leqslant [M : N].$$

Remark. — These conditional expectations are τ-preserving.

3.3. Transfer from the upper left to the lower right.

This subsection is about constructing the commutators via transferring as in [4]. The essence of our argument, inspired by [4], consists of constructing a sequence $(z_{p_n})_{n\in\mathbb{N}}$ where each z_n is in a proper reduced algebra M_{p_n} of M. Each z_n is supported by p_n, which is orthogonal to each other. $\{z_n\}_{n\in\mathbb{N}}$ is required to be bounded above in norm. Therefore, z_n goes to zero in the strong operator topology. One important feature is that the conditional expectation $E_{N_{p_n}'\cap M_{p_n}}$ of z_n is zero. We show that $z_n - z_{n+1}$ equals to a fixed number of commutators of the form $[x_n, y_n]$ where $x_n \in N_{p_n+p_{n+1}}$, $y_n \in M_{p_n+p_{n+1}}$. Thus x_n and y_n are supported by $p_n + p_{n+1}$. It is declared that x_n (resp. y_n) has a common upper bound in norm for all n. Similarly x_n, y_n tends to zero strongly as $n \to \infty$. We form two sets for x_n's (resp. y_n's): $X_{\text{odd}} = \{x_{2n-1}, n \in \mathbb{N}\}$, $X_{\text{even}} = \{x_{2n}, n \in \mathbb{N}\}$

(resp. $Y_{\text{odd}}, Y_{\text{even}}$). Every element of X_{odd} is supported by a mutually orthogonal projection. In a word these elements are independently working algebraically in their blocks. From the global view, a unique element $x_{\text{odd}} = \sum_{n=1}^{\infty} x_{2n-1}$ can represent the behaviors of all these elements. That is what we are seeking for. It also holds for $x_{\text{even}}, y_{\text{odd}}, y_{\text{even}}$. We use them to construct the commutators and show the latter do have the wanted properties. This completes the line of the proof.

Remark. — By the proceeding corollary in Section 2 and the fact that $[M_{p_n}, N_{p_n}] \leqslant [M, N]$, there exists $m = m([M : N], \frac{1}{2})$, $\forall n, \forall x_n \in M_{p_n}$, $E_{N_{p_n}{}' \cap M_{p_n}}(x_n) = 0$, there exist $u_{n,1}, \cdots, u_{n,m} \in \mathcal{U}(N_{p_n})$ such that $\| \sum_{k=1}^{m} \frac{1}{m} u_{n,k} x_n u_{n,k}^* \| \leqslant \frac{1}{2} \|x_n\|$. Please note that by the definition of $E_{N_{p_n}{}' \cap M_{p_n}}$, and viewing x_n as an element of $M_{p_n} \subset M$, the condition $E_{N' \cap M}(x_n) = 0$ ensures us that $E_{N_{p_n}{}' \cap M_{p_n}}(x_n) = 0$.

3.3.1. Cutting the norm by half. Let us recollect that $z_1 = p_1 z_1 p_1 \in M_{p_1} \subset M$, $E_{N' \cap M}(z_1) = 0$, and $\tau(p_1) = \frac{1}{2}$. Therefore $E_{N_{p_1}{}' \cap M_{p_1}}(z_1) = 0$. Accordingly

$$\exists u_{1,1}, \cdots, u_{1,m} \in \mathcal{U}(N_{p_1}) \subset N$$

such that

$$\text{if we define } z_1' = \frac{1}{m} \sum_{k=1}^{m} u_{1,k} z_1 u_{1,k}{}^*,$$

$$\|z_1'\| \leqslant \frac{1}{2} \|z_1\|.$$

It is observed that z_1' is supported in M_{p_1} and $E_{N' \cap M}(z_1') = 0$ and $z_1 - z_1'$ is the sum of m- commutators in the desired form,

$$z_1 - z_1' = \sum_{k=1}^{m} \Big[\frac{1}{m} u_{1,k}{}^*, u_{1,k} z_1 \Big].$$

where $\frac{1}{m} u_{1,k}{}^* \in N_{p_1} \subseteq N$, $\| \frac{1}{m} u_{1,k}{}^* \| \leqslant \frac{1}{m}$,

$$u_{1,k} z_1 \in M_{p_1} \subseteq M, \|u_{1,k} z_1\| \leqslant \|z_1\|.$$

3.3.2. Moving down to the lower right. Find p_1' and p_1'' two mutually orthogonal equivalent projections in N_{p_1}, with the sum equal to p_1. Get w_1' and w_1'', partial isometries in $N_{p_1+p_2}$ with

$$w_1'{}^* w_1' = p_1', \quad w_1''{}^* w_1'' = p_1'', \quad w_1' w_1'{}^* = p_2 = w_1'' w_1''{}^*$$

and define

$$z_2 = w_1' z_1' w_1'{}^* + w_1'' z_1' w_1''{}^*.$$

It is clear that z_2 is supported in M_{p_2} with

$$E_{N'\cap M}(z_2) = E_{N'\cap M}((p_1' + p_1'')z_1') = E_{N'\cap M}(p_1 z_1') = 0,$$

$$\|z_2\| \leqslant 2\|z_1'\| \leqslant \|z_1\|.$$

We want to show that $z_1' - z_2$ is a finite sum of commutators in the desired form in $M_{p_1+p_2}$.

3.3.3. Finding the specific commutators. Consider the partial isometry $w_1 = {w_1'}^* w_1'' \in N_{p_1} \subseteq N_{p_1+p_2}$ with the initial projection p_1' and the final projection p_1''. And define

$$x_1 = z_1' w_1 + {w_1}^* z_1' {w_1'}^* + z_1' {w_1''}^* \in N_{p_1+p_2}, \quad y_1 = {w_1}^* + w_1'' \in N_{p_1+p_2}.$$

It is easy to see that $\|x_1\| \leqslant 3\|z_1'\| \leqslant \frac{3}{2}\|z_1\|$, and $\|y_1\| \leqslant 2$.

LEMMA 2.

$$z_1' = z_2 + [-y_1, x_1] + [{w_1''}^*, w_1' z_1' {w_1''}^*] + [-w_1'', w_1'' z_1' {w_1'}^*],$$

$$w_1' z_1' {w_1''}^* \in N_{p_1+p_2}, \quad w_1'' z_1' {w_1'}^* \in N_{p_1+p_2}.$$

Note that the norm of either is less than the norm of z_1'.

Proof. — It is noted that the calculation can be simplified in the matricial form relative to p_1', p_1'', p_2 and the associated partial isometries: set

$$p_1' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p_1'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad w_1'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$z_1' = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a+d \end{pmatrix}.$$

$$x_1 = \begin{pmatrix} 0 & a & b \\ 0 & c & a+d \\ 0 & 0 & 0 \end{pmatrix}, \quad y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$w_1' z_1' w_1''^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}, \quad w_1'' z_1' w_1'^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}.$$

And

$$[-y_1, x_1] = \begin{pmatrix} a & b & 0 \\ c & a+d & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & c & a+d \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & -b \\ 0 & -c & -(a+d) \end{pmatrix}$$

$$[w_1''^*, w_1' z_1' w_1''^*] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

$$[-w_1'', w_1'' z_1' w_1'^*] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}.$$

□

3.3.4. Constructing the sequence. Then we obtain step by step the sequences $(z_n)_{n\in\mathbb{N}}, (u_{n,1}), \cdots, (u_{n,m})$,

$(z_n'), (p_n'), (p_n''), (w_n'), (w_n''), (w_n), (x_n), (y_n)$ in M satisfying

$z_n, z_n' \in M_{p_n}, u_{n,1}, \cdots, u_{n,m}$ unitaries $\in N_{p_n}$,

$E_{N'\cap M}(z_n) = E_{N'\cap M}(z_n') = 0$,

$z_n = z_n' + \sum_{k=1}^m \left[\frac{1}{m} u_{n,k}{}^*, u_{n,k} z_n\right]$,

$\|z_n'\| \leqslant \frac{1}{2}\|z_n\|$,

p_n', p_n'' projections $\in N_{p_n}$, w_n', w_n'', w_n partial isometries $\in N_{p_n+p_{n+1}}$,

$p_n' + p_n'' = p_n$, $w_n'^* w_n' = p_n'$, $w_n''^* w_n'' = p_n''$, $w_n' w_n'^* = w_n'' w_n''^* = p_{n+1}$, $w_n = w_n'^* w_n''$,

$z_{n+1} = w_n' z_n' w_n'^* + w_n'' z_n' w_n''^*$,

$\underline{\underline{\|z_{n+1}\| \leqslant 2\|z_n'\| \leqslant \|z_n\|}}$

$E_{N'\cap M}(z_{n+1}) = 0$

$z_n' = z_{n+1} + [-y_n, x_n] + [w_n''^*, w_n' z_n' w_n''^*] + [-w_n'', w_n'' z_n' w_n'^*]$,

$x_n = z_n' w_n + w_n{}^* z_n' w_n'^* + z_n' w_n''^*$, $y_n = w_n{}^* + w_n''$,

$x_n \in M_{p_n+p_{n+1}}$, $y_n \in N_{p_n+p_{n+1}}$.

Note that the sequence $(\|z_n\|)_{n\in\mathbb{N}}$ is bounded and the construction implies that $\|x_n\| \leqslant \frac{3}{2}\|z_n\| \leqslant \frac{3}{2}\|z_1\| \leqslant \frac{9}{2}\|z\|$, $\|y_n\| \leqslant 2$ for all $n \in \mathbb{N}$.

The terms of the sequence $(x_{2k})_{k\in\mathbb{N}}$ are in the respective reduced algebras of M supported by the projections $(p_{2k}+p_{2k+1})_{k\in\mathbb{N}}$ which are mutually orthogonal; as they are uniformly bounded, the series $\sum_{k=1}^{\infty} x_{2k}$ converge for the strong operator topology to an element whose uniform norm is equal to $\sup_k \|x_{2k}\| \leqslant \frac{3}{2}\|z_1\| \leqslant \frac{9}{2}\|z\|$. Similarly, we can define:

$$u_{i_{\text{even}}} = \sum_{k=1}^{\infty} \frac{1}{m} u_{2k,i}{}^{*} \in N \quad ; \quad u_{i_{\text{odd}}} = \sum_{k=1}^{\infty} \frac{1}{m} u_{2k-1,i}{}^{*} \in N$$

$$v_{i_{\text{even}}} = \sum_{k=1}^{\infty} u_{2k,i} z_{2k}{}' \in M \quad ; \quad v_{i_{\text{odd}}} = \sum_{k=1}^{\infty} u_{2k-1,i} z_{2k-1}{}' \in M$$

$$x_{\text{even}} = \sum_{k=1}^{\infty} x_{2k} \in M \quad ; \quad x_{\text{odd}} = \sum_{k=1}^{\infty} x_{2k-1} \in M$$

$$y_{\text{even}} = \sum_{k=1}^{\infty} y_{2k} \in N \quad ; \quad y_{\text{odd}} = \sum_{k=1}^{\infty} y_{2k-1} \in N$$

$$r_{\text{even}} = \sum_{k=1}^{\infty} w_{2k}{}'' \in N \quad ; \quad r_{\text{odd}} = \sum_{k=1}^{\infty} w_{2k-1}{}'' \in N$$

$$s_{\text{even}} = \sum_{k=1}^{\infty} w_{2k}{}' z_{2k}{}' w_{2k}{}''^{*} \in M \quad ; \quad s_{\text{odd}} = \sum_{k=1}^{\infty} w_{2k-1}{}' z_{2k-1}{}' w_{2k-1}{}'^{*} \in M$$

$$t_{\text{even}} = \sum_{k=1}^{\infty} w_{2k}{}'' z_{2k}{}' w_{2k}{}'^{*} \in M \quad ; \quad t_{\text{odd}} = \sum_{k=1}^{\infty} w_{2k-1}{}'' z_{2k-1}{}' w_{2k-1}{}'^{*} \in M.$$

Then we have

$$\begin{aligned} z_1 = \sum_{n=1}^{\infty}(z_n - z_{n+1}) = & \sum_{i=1}^{m}([u_{i_{\text{even}}}, v_{i_{\text{even}}}] + [u_{i_{\text{odd}}}, v_{i_{\text{odd}}}]) \\ & + [-y_{\text{even}}, x_{\text{even}}] + [-y_{\text{odd}}, x_{\text{odd}}] \\ & + [r_{\text{even}}{}^{*}, s_{\text{even}}] + [r_{\text{odd}}{}^{*}, s_{\text{odd}}] \\ & + [-r_{\text{even}}, t_{\text{even}}] + [-r_{\text{odd}}, t_{\text{odd}}] \end{aligned}$$

with

$$\|u_{i_{\text{even}}}\|, \|u_{i_{\text{odd}}}\| \leqslant \frac{1}{m}, \qquad \|v_{i_{\text{even}}}\|, \|v_{i_{\text{odd}}}\| \leqslant \|z_1\| \leqslant 3\|z\|,$$

$$\|x_{\text{even}}\|, \|x_{\text{odd}}\| \leqslant \frac{3}{2}\|z_1\| \leqslant \frac{9}{2}\|z\|, \qquad \|y_{\text{even}}\|, \|y_{\text{odd}}\| \leqslant 2,$$

$$\|r_{\text{even}}\|, \|r_{\text{odd}}\| \leqslant 1,$$

$$\|s_{\text{even}}\|, \|s_{\text{odd}}\| \leqslant \frac{1}{2}\|z_1\| \leqslant \frac{3}{2}\|z\|,$$

$$\|t_{\text{even}}\|, \|t_{\text{odd}}\| \leqslant \frac{1}{2}\|z_1\| \leqslant \frac{3}{2}\|z\|.$$

We have therefore showed the following statement, combined with the lemma in Subsection 3.1: given a pair of II_1 factors $N \subseteq M$ with finite Jones index, and z is an element of M. The following conditions are equivalent:

(1) $E_{N'\cap M}(z) = 0$;

(2) z is a sum of $2m + 9$ commutators of the form $[a, b]$ where $a \in N, b \in M$. Here m, giving in [10] depends solely on the index.

4. Application.

In this section we prove the vector space isomorphism between the relative commutants and the cyclic tensor products. We also show the existence of the rotations operators on them.

THEOREM 2 [5]. — *Let $N \subseteq M$ be an inclusion of II_1 factors with finite Jones index. The tower of II_1 factors M_n is defined by $M_{-1} = N$, $M_0 = M$, $M_n = \langle M_{n-1}, e_n\rangle$ where e_n is the Jones projection of M_n onto M_{n-1}. We have $M_n \simeq M \otimes_N M \otimes_N \otimes_N \cdots \otimes_N M$ ($n+1$-terms) as a $M-M$ $(N-N)$ bimodule.*

Proof. — The proof can be found in [5] and is included here for the convenience of the reader.

CLAIM. — *The map $\pi : M\otimes_N M \to M_1$ defined by $\pi(x\otimes_N y) = xe_1y$ is a $M - M$ bimodule isomorphism.*

The 2-sided ideal $\sum_{k=1}^{n} x_k e_1 y_k$ is equal to M_1, since M_1 is algebraically simple. This proves the map is onto. We need to prove it is injective too. Because $[M : N] < \infty$, we have a finite orthonormal basis $\{m_k\}_k$ for $N \subseteq M$. Take z in $M \otimes_N M$. It is easily seen that $z = \sum_{k,l} m_k z_{k,l} \otimes_N m_l^*$, where $z_{k,l}$ is in N,

$$e_1 m_p^* \pi(z) m_q e_1 = \sum_{k,l} e_1 m_p^* m_k z_{k,l} e_1 m_l^* e_1 = f_p z_{p,q} f_q e_1 = 0, \ \forall p,\ q,$$

where $E_N(m_k^* m_k) = f_k \in \mathcal{P}(N)$ and $m_k = m_k f_k$,

$$\sum_{p,q} m_p z_{p,q} \otimes_N m_q^* = \sum m_p f_p z_{p,q} f_q \otimes_N m_q^* = 0.$$

The rest is an induction process:

$$\begin{aligned} M_n &= M_{n-1} \otimes_{M_{n-2}} M_{n-1} \\ &= (M_{n-2} \otimes_{M_{n-3}} M_{n-2}) \otimes_{M_{n-2}} (M_{n-2} \otimes_{M_{n-3}} M_{n-2}) \\ &= M_{n-2} \otimes_{M_{n-3}} M_{n-2} \otimes_{M_{n-3}} M_{n-2} \\ &= \cdots \end{aligned}$$

□

COROLLARY [1]. — *Let $N \subseteq M$ be an inclusion of II_1 factors with finite Jones index. Then $M = (N' \cap M) \oplus [N, M]$ as a vector space. Here $[N, M]$ denotes the vector space spanned by the commutators of the form $[a, b]$ where $a \in N$, $b \in M_n$. And the $(n+1)$th cyclic tensor product V_{n+1}, defined to be the quotient space of $M \otimes_N M \otimes_N \cdots \otimes_N M$ ($n+1$-terms) by the subspace generated by the vectors of the form $[a, b_n]$ where $a \in N$, $b_n \in M_n$, is isomorphic to the relative commutant $N' \cap M_n$, where the tower of II_1 factors is defined by $M_0 = M$, $M_n = \langle M_{n-1}, e_n \rangle$.*

Proof. — The main theorem showed that $M = (N' \cap M) + [N, M]$. And it is obvious that $(N' \cap M) \cap [N, M] = 0$. This gives the first half of the corollary. Now $[M_n : N] = [M : N]^{n+1} < \infty$. Therefore we have $M_n = (N' \cap M_n) \oplus [N, M_n]$ as a vector space. By the above theorem, $M_n \simeq M \otimes_N M \otimes_N \cdots \otimes_N M$ ($n+1$ terms) as an $N-N$ bimodule. And the cyclic tensor product V_{n+1} is isomorphic to $M_n/[N, M_n] \simeq N' \cap M_n$ as a vector space. □

Under the above isomorphism, we can consider all the $N-N$ bimodule maps on $M \otimes_N \cdots \otimes_N M$ ($n+1$ terms). These maps preserve the N- central vectors V_{n+1} and hence have their counterparts on $N' \cap M_n$. However there is another operator, the rotation operator, which does not arise in this manner. We define this operator, $\tilde{\rho_n} : V_{n+1} \to V_{n+1}$ by

$$\tilde{\rho_n}([x_1, x_2, \cdots, x_{n+1}]) = [x_2, \cdots, x_{n+1}, x_1],$$

where $[x_1, x_2, \cdots, x_{n+1}]$ represents an equivalence class in $M \otimes_N \cdots \otimes_N M$.

This map is well defined because

$$[x_2, \cdots, x_{n+1}, x_1 n] \equiv [n x_2, \cdots x_{n+1}, x_1] \quad (\text{mod by } [N, M_n]);$$
$$[x_2, \cdots, x_{n+1} n, x_1] \equiv [x_2, \cdots, x_{n+1}, n x_1],$$

where $x_1, x_2, \cdots, x_{n+1} \in M$ and $n \in N$. There is a natural multiplication operators, $\otimes_N$, on the graded cyclic tensor product spaces. The multiplication operator on the relative commutants can be recovered via the above maps. Equipped with all the maps mentioned above, it should be possible to build planar algebras within the frames of cyclic tensor product spaces. The existence of rotation operators is an immediate advantage.

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Hsiang-Ping HUANG,
University of California
Department of Mathematics
Los Angeles, CA 90095 (USA).
hphuang@math.ucla.edu