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AN APPLICATION OF SHIFT OPERATORS TO ORDERED SYMMETRIC SPACES

by N.B. ANDERSEN and J.M. UNTERGERGER

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0. Introduction.

Let $\mathcal{M}$ be an ordered symmetric space, let $\mathfrak{a}^-$ be the negative Weyl chamber of a Cartan subspace $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ for $\mathcal{M}$ and let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the complex dual of $\mathfrak{a}$. Let Φ_λ denote the Harish-Chandra series on the Riemannian dual $\mathcal{M}^d$ of $\mathcal{M}$. The Ólafsson expansion formula for the spherical functions φ_λ on $\mathcal{M}$, see [Óla2, Theorem 5.7], states that

$$\varphi_\lambda(\exp t) = \sum_{w \in W_0} c(w\lambda)\Phi_{w\lambda}(\exp t), \quad t \in \mathfrak{a}^-,$$

where $c(\lambda)$ is the c-function for $\mathcal{M}$ and W_0 is some Weyl group.

Let $\mathcal{M}_{m,n}$, $m, n \in \mathbb{N}$ be a symmetric space of Cayley type, where m denotes the multiplicity of the short roots and n the rank of the symmetric space ($= \dim \mathfrak{a}$). Let also $\mathcal{M}_{0,n}$ denote the product of n copies of the rank 1 space $SO_o(1,2)/SO_o(1,1)$. We note that $\mathcal{M}_{0,n}$ is *not* a symmetric space of Cayley type. Using the Ólafsson expansion formula and the theory of shift operators introduced by E.M. Opdam (acting on the Harish-Chandra series), we relate the spherical functions on $\mathcal{M}_{m,n}$, via some differential operator given as a composition of shift operators, to the spherical functions on $\mathcal{M}_{k,n}$, where $k \in \{0,1\}$ satisfies $k \equiv m \bmod 2$. A similar result for

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the spherical functions on $\mathcal{M}^d_{m,n}$ is well-known, using the Harish-Chandra expansion formula.

The spherical Laplace transform $\mathcal{L}$ on $\mathcal{M}$ is defined in terms of integrating against the spherical functions. Let $m \in 2\mathbb{N}$. Using the above, the proof of the Paley-Wiener theorem for the spherical Laplace transform on $\mathcal{M}_{m,n}$ reduces to the rank 1 case, studied by G. Ólafsson and the first author in [AÓ]. The special case $\mathcal{M}_{2,n} = SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}^*_+$ was furthermore considered in [AU].

Finally we consider BC_n type root systems not necessarily corresponding to some ordered symmetric space. We define hypergeometric functions of the second type for these root systems and we define the corresponding hypergeometric Laplace transform. As above we can prove a Paley-Wiener theorem when the multiplicities of the short and intermediate roots are even and the multiplicity of the long roots is odd.

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1. Ordered symmetric spaces of Cayley type, spherical functions and the spherical Laplace transform.

Let $\mathcal{M}_{m,n} = G_{m,n}/H_{m,n}$ denote (ordered) symmetric spaces of Cayley type with root multiplicities (long roots, short roots) $= (1, m)$, $m \in \mathbb{N}$, and rank n, $n \in \mathbb{N}$, see [Óla1, Definition 5.7] for a precise definition of symmetric spaces of Cayley type. The irreducible symmetric spaces of Cayley type are classified by the following table (up to diffeomorphisms):

$\mathcal{M}_{m,n} = G_{m,n}/H_{m,n}$	m	rank
$Sp(n,\mathbb{R})/SL(n,\mathbb{R}) \times \mathbb{R}^*_+$	1	n
$SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}^*_+$	2	n
$SO^*(4n)/SU^*(2n) \times \mathbb{R}^*_+$	4	n
$SO(2,m+2)/SO(1,m+1) \times \mathbb{R}^*_+$, $m \geqslant 1$	m	2
$E_{7(-25)}/E_{6(-26)} \times \mathbb{R}^*_+$	8	3.

Below we present the properties of the ordered symmetric spaces $\mathcal{M}_{m,n}$ that we need in the following, we refer to [Far], [FHÓ] and [Óla2] for more details.

Let $\mathfrak{g} = \mathfrak{h} \oplus_\sigma \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{k} \oplus_\theta \mathfrak{p}$ be the decompositions of the Lie algebra $\mathfrak{g}$ of $G_{m,n}$ into the $\{\pm 1\}$-eigenspaces respectively of the involution

σ of $G_{m,n}$ fixing $H_{m,n} = G^{\sigma}_{m,n}$ and of a Cartan involution θ commuting with σ. Let $K_{m,n} = G^{\theta}_{m,n}$ be the maximal compact subgroup of $G_{m,n}$ fixed by θ with Lie algebra $\mathfrak{k}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$, then $\mathfrak{a}$ is a Cartan subspace for both $\mathcal{M}_{m,n}$ and the Riemannian dual $\mathcal{M}^d_{m,n} \cong G_{m,n}/K_{m,n}$.

Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ denote the associated root system of type C_n. Then $\Delta = \left\{ \frac{\pm\gamma_j \pm \gamma_i}{2} \right\} \cup \{\pm\gamma_i\}$ for some basis $(\gamma_1, \dots, \gamma_n)$ of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$, with multiplicities $m_\alpha = m$ for the short roots $\alpha = \frac{\pm\gamma_j \pm \gamma_i}{2}$ and $m_\alpha = 1$ for the long roots $\alpha = \pm\gamma_i$. Let $\Delta^+ = \left\{ \frac{\gamma_j \pm \gamma_i}{2}, i < j \right\} \cup \{\gamma_i\}$ be a set of positive roots. Let furthermore Δ_0 denote the root system $\Delta_0 = \left\{ \frac{\gamma_j - \gamma_i}{2} \right\}$ with positive roots $\Delta_0^+ = \left\{ \frac{\gamma_j - \gamma_i}{2}, i < j \right\}$. Let $W \cong \mathfrak{S}_\mathfrak{n} \times \{\pm 1\}^\mathfrak{n}$ and $W_0 \cong \mathfrak{S}_n$ (the permutation group of n elements) denote the Weyl groups of the root systems Δ and Δ_0 respectively.

We identify the complex dual $\mathfrak{a}^*_\mathbb{C}$ and $\mathbb{C}^n$ by the map

$$\mathbb{C}^n \ni \lambda = (\lambda_1, \dots, \lambda_n) \mapsto -\sum_j \lambda_j \gamma_j.$$

Let $\langle \cdot, \cdot \rangle$ denote the usual scalar product on $\mathbb{C}^n$ and identify $\mathfrak{a}$ with $\mathbb{R}^n$ such that the negative Weyl chamber $\mathfrak{a}^-$ is given by

$$\mathfrak{a}^- = \{t \in \mathbb{R}^n |\ 0 < t_1 < t_2 < \dots < t_{n-1} < t_n\}.$$

We write $x \geqslant r$ ($x > r$) if $x_j \geqslant r$ ($x_j > r$) for all j ($x \in \mathbb{R}^n$, $r \in \mathbb{R}$). Let $R > r > 0$ and define $S_{r,R} := \{t \in \mathbb{R}^n \mid t \geqslant r\} \cap \{t \in \mathbb{R}^n |\, |t| \leqslant R\}$. Let finally $S^o := \{t |\, t > 0\}$.

Let $c^d_{m,n}(\cdot)$ and $c_{m,n}(\cdot)$ denote the c-functions of respectively the Riemannian symmetric space $\mathcal{M}^d_{m,n}$ and the ordered symmetric space $\mathcal{M}_{m,n}$. Then

$$\begin{aligned} c^d_{m,n}(\lambda) &= \prod_{\alpha \in \Delta^+} \frac{\Gamma(\frac{1}{2}\langle \lambda, \check{\alpha} \rangle)}{\Gamma(\frac{1}{2}\langle \lambda, \check{\alpha} \rangle + \frac{m_\alpha}{2})} \\ &= \prod_j \frac{\Gamma(-\lambda_j)}{\Gamma(-\lambda_j + \frac{1}{2})} \prod_{i<j} \frac{\Gamma(-\lambda_j - \lambda_i)}{\Gamma(-\lambda_j - \lambda_i + \frac{m}{2})} \frac{\Gamma(-\lambda_j + \lambda_i)}{\Gamma(-\lambda_j + \lambda_i + \frac{m}{2})}, \end{aligned}$$

where $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$, see [Hel, Chapter 4, Theorem 6.14], and $c_{m,n}(\lambda) = c^0_{m,n}(\lambda) c^\Omega_{m,n}(\lambda)$, where

$$c^0_{m,n}(\lambda) = \prod_{\alpha \in \Delta_0^+} \frac{\Gamma(\frac{1}{2}\langle \lambda, \check{\alpha} \rangle)}{\Gamma(\frac{1}{2}\langle \lambda, \check{\alpha} \rangle + \frac{m_\alpha}{2})} = \prod_{i<j} \frac{\Gamma(-\lambda_j + \lambda_i)}{\Gamma(-\lambda_j + \lambda_i + \frac{m}{2})},$$

and

$$c^{\Omega}_{m,n}(\lambda) = \prod_{\alpha\in\Delta^+\setminus\Delta_0^+} \frac{\Gamma(-\frac{1}{2}\langle\lambda,\check{\alpha}\rangle - \frac{m_\alpha}{2}+1)}{\Gamma(-\frac{1}{2}\langle\lambda,\check{\alpha}\rangle+1)}$$

$$= \prod_j \frac{\Gamma(\lambda_j+\frac{1}{2})}{\Gamma(\lambda_j+1)} \prod_{i<j} \frac{\Gamma(\lambda_i+\lambda_j-\frac{m}{2}+1)}{\Gamma(\lambda_i+\lambda_j+1)},$$

see [FHÓ, §6] and [Far, §5].

The spherical functions on $\mathcal{M}_{m,n}$ and $\mathcal{M}^d_{m,n}$ will be denoted by $\varphi_{m,n}(\lambda,t)$ (defined for λ in some dense subset of $\mathbb{C}^n$ and $t\in S^o$) and $\psi^d_{m,n}(\lambda,t)$ ($\lambda\in\mathbb{C}^n$, $t\in\mathbb{R}^n$) respectively, and the Harish-Chandra series on $\mathcal{M}^d_{m,n}$ will be denoted by $\Phi_{m,n}(\lambda,t)$ ($\operatorname{Re}\lambda\in\overline{\mathfrak{a}^-}$, $t\in\mathfrak{a}^-$), see [FHÓ, §5], [Óla2, §4] and [Hel, Chapter 4] for the precise definitions and properties. We note that we here consider the spherical functions as W_0- (resp. $W-$) invariant functions on S^o (resp. on $\mathfrak{a}=\mathbb{R}^n$), instead of as (invariant) functions on (some subset) of $\mathcal{M}_{m,n}$ or $\mathcal{M}^d_{m,n}$.

The c-functions determine the asymptotic behaviour of the spherical functions. The Ólafsson and Harish-Chandra expansion formulae, expressing the spherical functions as sums over one of the Weyl groups in terms of the c-functions and the Harish-Chandra series, are given by

$$\varphi_{m,n}(\lambda,t) = \sum_{w\in W_0} c_{m,n}(w\lambda)\Phi_{m,n}(w\lambda,t)$$

$$= c^{\Omega}_{m,n}(\lambda)\sum_{w\in W_0} c^0_{m,n}(w\lambda)\Phi_{m,n}(w\lambda,t),$$

see [Óla2, Theorem 5.7], and

$$\psi^d_{m,n}(\lambda,t) = \sum_{w\in W} c^d_{m,n}(w\lambda)\Phi_{m,n}(w\lambda,t),$$

for $t\in\mathfrak{a}^-$. We note that $\psi^d_{m,n}(\lambda,t) = \sum_{w\in W_0\setminus W}\frac{c^d_{m,n}(w\lambda)}{c_{m,n}(w\lambda)}\varphi_{m,n}(w\lambda,t)$, $t\in\mathfrak{a}^-$.

We furthermore define the normalized spherical functions $\varphi^o_{m,n}(\lambda,\cdot)$ on $\mathcal{M}_{m,n}$ by $\varphi^o_{m,n}(\lambda,\cdot) := \varphi_{m,n}(\lambda,\cdot)/c^{\Omega}_{m,n}(\lambda)$.

LEMMA 1. — *Let $m\in\mathbb{N}$. The map $(\lambda,t)\mapsto\varphi^o_{m,n}(\lambda,t)$ extends to an analytic function for $\{\lambda\in\mathbb{C}^n \mid \operatorname{Re}\lambda > -1/4\}$ and $t\in S^o$.*

Proof. — Using the Ólafsson expansion formula, the proof follows from the same arguments used in e.g. [HS, Part 1, Chapter 4] to prove holomorphy for the hypergeometric function. □

Let $\delta_{m,n}(t) = \prod_{\alpha\in\Delta^+} \sinh^{m_\alpha} \langle -\alpha, t\rangle$ denote the Jacobian associated to the radial coordinates on $\mathfrak{a}^-$. The normalized spherical Laplace transform $\mathcal{L}^o_{m,n}$ on $\mathcal{M}_{m,n}$ and the spherical Fourier transform $\mathcal{F}_{m,n}$ on $\mathcal{M}^d_{m,n}$ are defined by (integrating against the spherical functions)

$$\mathcal{L}^o_{m,n}(f)(\lambda) = \int_{\mathfrak{a}^-} f(t)\varphi^o_{m,n}(\lambda,t)\delta_{m,n}(t)dt,$$

for $f \in C^\infty_c(S^o)^{W_0}$ (left-W_0-invariant functions in $C^\infty_c(S^o)$), whenever the integral converges, see [FHÓ, §5], and

$$\mathcal{F}_{m,n}(f)(\lambda) = \int_{\mathfrak{a}^-} f(t)\psi^d_{m,n}(\lambda,t)\delta_{m,n}(t)dt,$$

for $f \in C^\infty_c(\mathbb{R}^n)^W$, for $\lambda \in \mathbb{C}^n$, see [Hel, Chapter 4].

Given $f \in C^\infty_c(S^o)^{W_0}$, we denote by $f^d \in C^\infty_c(\mathbb{R}^n)^W$ the W-invariant function such that f^d coincides with f on S^o. Then

$$\mathcal{F}_{m,n}(f^d)(\lambda) = \sum_{w\in W_0\backslash W} c^1_{m,n}(\lambda)\mathcal{L}^o_{m,n}f(w\lambda),$$

almost everywhere (and the right hand side extends to an analytic function), where $c^1_{m,n}(\lambda) := c^d_{m,n}(\lambda)/c^0_{m,n}(\lambda)$. Using this, the inversion formula for the spherical Fourier transform yields the inversion formula for the normalized spherical Laplace transform

$$f(t) = \frac{|W|}{|W_0|}\int_{i\mathbb{R}^n} \mathcal{L}^o_{m,n}(f)(\lambda)\psi^d_{m,n}(\lambda,t)\frac{d\lambda}{c^d_{m,n}(-\lambda)c^0_{m,n}(\lambda)}, \tag{1}$$

for $t \in \mathfrak{a}^-$, see also [Óla2, Theorem 6.3].

Let P_λ and Q_λ denote Legendre functions of the first and second kind. We can view $P_{\lambda-\frac12}(\cosh t)$ and $Q_{\lambda-\frac12}(\cosh t)$ as spherical functions on respectively the Riemannian symmetric space $SO_o(1,2)/SO(2)$ and on the ordered symmetric space $SO_o(1,2)/SO_o(1,1)$, both of rank 1, see [FHÓ] and [AÓ]. We recall the following well-known growth estimates:

$$\left|P_{\lambda-\frac12}(\cosh t)\right| \leqslant ce^{(|\operatorname{Re}\lambda|-\frac12)|t|},$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, for some constant c; and, for any $r > 0$:

$$\left|\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac12)}\right|\left|Q_{\lambda-\frac12}(\cosh t)\right| \leqslant c_r e^{-(\operatorname{Re}\lambda+\frac12)t},$$

for $\operatorname{Re}\lambda \geqslant 0$ and $t \geqslant r > 0$, where c_r is a constant only depending on r.

Let $\mathcal{M}_{0,n}$ and $\mathcal{M}^d_{0,n}$ be the products of n copies of $SO_o(1,2)/SO_o(1,1)$ respectively of n copies of $SO_o(1,2)/SO(2)$. The Harish-Chandra series on $\mathcal{M}^d_{0,n}$ is given by

$$\Phi_{0,n}(\lambda,t) = \pi^{-n/2}\prod_{i=1}^n \frac{\Gamma(\lambda_i+1)}{\Gamma(\lambda_i+\frac12)}\prod_{i=1}^n Q_{\lambda_i-\frac12}(\cosh t_i).$$

We use the Ólafsson expansion formula to define (normalized) pseudo-spherical functions $\varphi^o_{0,n}$ on $\mathcal{M}_{0,n}$ by

$$\varphi^o_{0,n}(\lambda,t) := \sum_{w\in W_0} c_{0,n}(w\lambda)\Phi_{0,n}(w\lambda,t)$$
$$= \prod_{i=1}^n \frac{\Gamma(\lambda_i+1)}{\Gamma(\lambda_i+\frac{1}{2})} \sum_{w\in W_0} \prod_{i=1}^n Q_{w\lambda_i-\frac{1}{2}}(\cosh t_i).$$

We easily get the following estimate: Let $r>0$. There exists a constant c_r such that

$$(2) \qquad |\varphi^o_{0,n}(\lambda,t)| \leqslant c_r e^{-\min_{w\in W_0}\langle w\operatorname{Re}\lambda,t\rangle} \leqslant c_r e^{-\langle \operatorname{Re}\lambda, rt_o\rangle},$$

for $\operatorname{Re}\lambda \geqslant 0$ and $t\geqslant r$, where $t_o=(1,\dots,1)$. We similarly define pseudo-spherical functions $\psi^d_{0,n}$ on $\mathcal{M}^d_{0,n}$ by $\psi^d_{0,n}(\lambda,t) := \sum_{w\in W} c^d_{0,n}(w\lambda)\Phi_{0,n}(w\lambda,t)$.

2. Shift operators.

In this section we briefly discuss the action of the elementary shift operators on the Harish-Chandra series and thus on the spherical functions. We follow the 3 survey papers [HS, Part 1], [Hec] and [Opd], which we refer to for more information and references.

For the remainder of this paper we fix the rank $n\in\mathbb{N}$ and use subscript $_m$ instead of $_{m,n}$. Let $m\in\mathbb{N}\cup\{0\}$ and let G^+_m and G^-_{m+1} denote the elementary (raising, respectively lowering) shift operators associated with the orbit of the short roots, cf. [HS, Part 1, Definition 3.2.1] and [Opd, Definition 5.9]. Both operators preserve the space $C^\infty(\mathbb{R}^n)^W$ and, for all $m\in\mathbb{N}\cup\{0\}$:

$$\int_{\mathbb{R}^n} \{G^+_m f_1\}(t) f_2(t)|\delta_{m+1}(t)|dt = \int_{\mathbb{R}^n} f_1(t)\{G^-_{m+1}f_2\}(t)|\delta_m(t)|dt,$$

for all $f_1,\, f_2 \in C^\infty_c(\mathbb{R}^n)^W$. They 'shift' the spherical functions in the following sense:

LEMMA 2. — *For all $m\in\mathbb{N}\cup\{0\}$ and λ generic, we have*

$$\varphi^o_{m+2}(\lambda,\cdot) = \frac{c^0_{m+2}(\lambda)c^d_{m+2}(-\lambda)}{c^0_m(\lambda)c^d_m(-\lambda)} G^+_m \varphi^o_m(\lambda,\cdot).$$

Proof. — By [Hec, Corollary 3.4.4] we have

$$G^+_m\Phi_m(\lambda,\cdot) = \frac{c^d_m(-\lambda)}{c^d_{m+2}(-\lambda)}\Phi_{m+2}(\lambda,\cdot),$$

which yields

$$G_m^+\varphi_m^o(\lambda,\cdot) = \sum_{w\in W_0} \frac{c_m^d(-w\lambda)c_m^0(w\lambda)}{c_{m+2}^d(-w\lambda)c_{m+2}^0(w\lambda)} c_{m+2}^0(w\lambda)\Phi_{m+2}(\lambda,\cdot),$$

whence the result, since the fraction $\frac{c_m^0(\lambda)c_m^d(-\lambda)}{c_{m+2}^0(\lambda)c_{m+2}^d(-\lambda)}$ is W_0-invariant. □

We notice that the above expression for the spherical function $\varphi_{m+2}^o(\lambda,t)$, $m \in \mathbb{N}\cup\{0\}$, is W-invariant in the t-variable, since $\varphi_0^o(\lambda,t)$ obviously is so.

LEMMA 3. — *For all $m \in \mathbb{N}\cup\{0\}$ and λ generic, we have*

$$\psi_{m+2}^d(\lambda,\cdot) = \frac{c_{m+2}^d(\lambda)c_{m+2}^d(-\lambda)}{c_m^d(\lambda)c_m^d(-\lambda)} G_m^+\psi_m^d(\lambda,\cdot).$$

Proof. — As before. □

Fix $m \in \mathbb{N}$ and define two differential operators $D_m^+ = G_{m-2}^+ \circ\cdots\circ G_{k+2}^+ \circ G_k^+$ and $D_m^- = G_{k+2}^- \circ\cdots\circ G_{m-2}^- \circ G_m^-$, where $k \in \{0,1\}$ is given by $k \equiv m \bmod 2$. We have the following relation between the normalized spherical Laplace transforms $\mathcal{L}_m^o$ and $\mathcal{L}_k^o$ associated with the root systems $(1,m)$ and $(1,k)$:

$$\begin{aligned}
\mathcal{L}_m^o(f)(\lambda) &= \frac{1}{|W|}\int_{\mathbb{R}^n} f^d(t)\varphi_m^o(\lambda,t)|\delta_m(t)|dt \\
&= \frac{c_m^0(\lambda)c_m^d(-\lambda)}{c_k^0(\lambda)c_k^d(-\lambda)}\frac{1}{|W|}\int_{\mathbb{R}^n} f^d(t)\{D_m^+\varphi_k^o\}(\lambda,t)|\delta_m(t)|dt \\
&= \frac{c_m^0(\lambda)c_m^d(-\lambda)}{c_k^0(\lambda)c_k^d(-\lambda)}\frac{1}{|W|}\int_{\mathbb{R}^n} \{D_m^- f^d\}(t)\varphi_k^o(\lambda,t)|\delta_k(t)|dt \\
&= \frac{c_m^0(\lambda)c_m^d(-\lambda)}{c_k^0(\lambda)c_k^d(-\lambda)}\mathcal{L}_k^o(D_m^- f)(\lambda),
\end{aligned}\tag{3}$$

for $f \in C_c^\infty(S^o)^{W_0}$ (using a 'bump' function to compensate for the fact that $\varphi_k^0(\lambda,.)$ is not compactly supported). We can obtain a similar result for the Fourier transform.

3. The Paley-Wiener theorem.

We recall the definition of the Paley-Wiener space $\mathcal{H}^R(\mathbb{C}^n)$ for the spherical Fourier transform: it is the space of W-invariant holomorphic functions g on $\mathbb{C}^n$ satisfying the estimate

$$\sup_{\lambda\in\mathbb{C}^n} e^{-R|\operatorname{Re}\lambda|}(1+|\lambda|)^N|g(\lambda)| < \infty,$$

for all $N \in \mathbb{N}$. The Paley-Wiener theorem for the spherical Fourier transform states that $\mathcal{F}_m$ is a bijection of $C_R^\infty(\mathbb{R}^n)^W = \{f \in C_c^\infty(\mathbb{R}^n)^W \mid f(t) \equiv 0$ for $|t| > R\}$ onto $\mathcal{H}^R(\mathbb{C}^n)$ for all $R > 0$, see e.g. [Hel, Chapter 4, Theorem 7.1].

We define the Paley-Wiener spaces $PW_m^{r,R}(\mathbb{C}^n)$ and $PW_m(\mathbb{C}^n)$ for the (normalized) spherical Laplace transform as:

DEFINITION 4. — *Let $m \in \mathbb{N} \cup \{0\}$ and let $R > r > 0$. We define the Paley-Wiener space $PW_m^{r,R}(\mathbb{C}^n)$ as the space of W_0-invariant meromorphic functions g on $\mathbb{C}^n$, holomorphic on the extended tube $\operatorname{Re}\lambda > -1/4$, satisfying*

(i)

$$\sup_{\operatorname{Re}\lambda \geqslant 0} e^{\operatorname{Re}\langle\lambda, rt_o\rangle}(1+|\lambda|)^N |g(\lambda)| < \infty,$$

for all $N \in \mathbb{N}$, where $t_o = (1, \ldots, 1)$, and

(ii) *the c_m^1-weighted average $P_m^{\mathrm{av}} g(\lambda) = \sum_{w \in W_0 \backslash W} c_m^1(w\lambda) g(w\lambda)$ extends to a function in $\mathcal{H}^R(\mathbb{C}^n)$.*

Furthermore denote by $PW_m(\mathbb{C}^n)$ the union of the spaces $PW_m^{r,R}(\mathbb{C}^n)$ over all $R > r > 0$.

LEMMA 5. — *Let $m \in \mathbb{N}$ and let $k \in \{0,1\}$ be given by $k \equiv m \bmod 2$. Let $R > r > 0$ and assume that $g_k \in PW_k^{r,R}(\mathbb{C}^n)$. Assume that the function*

$$g_m(\lambda) = \frac{c_m^0(\lambda) c_m^d(-\lambda)}{c_k^0(\lambda) c_k^d(-\lambda)} g_k(\lambda),$$

is holomorphic for $\operatorname{Re}\lambda > -1/4$ and that $P_m^{\mathrm{av}} g_m$ is holomorphic on $\mathbb{C}^n$, then $g_m \in PW_m^{r,R}(\mathbb{C}^n)$.

Proof. — We note that

$$P_m^{\mathrm{av}} g_m(\lambda) = \frac{c_m^d(\lambda) c_m^d(-\lambda)}{c_k^d(\lambda) c_k^d(-\lambda)} P_k^{\mathrm{av}} g_k(\lambda).$$

Since $\frac{c_k^d(\cdot)}{c_m^d(\cdot)}$ is a polynomial and $P_k^{\mathrm{av}} g_k(\lambda) \in \mathcal{H}^R(\mathbb{C}^n)$, [Hel, Ch. III, Lemma 5.13] shows that $P_m^{\mathrm{av}} g_m \in \mathcal{H}^R(\mathbb{C}^n)$. Since also $\frac{c_k^0(\cdot)}{c_m^0(\cdot)}$ is a polynomial, we furthermore conclude from the proof of the aforementioned lemma that g_m satisfies the first growth inequality in the definition of $PW_m^{r,R}(\mathbb{C}^n)$. $\square$

THEOREM 6 (The Paley-Wiener Theorem). — *Let $m \in 2\mathbb{N} \cup \{0\}$. The normalized spherical Laplace transform $\mathcal{L}_m^o$ is a bijection of $C_c^\infty(S^o)^{W_0}$*

onto $PW_m(\mathbb{C}^n)$. More precisely it is a bijection of $C^\infty_{r,R}(S^o)^{W_0} := \{f \in C^\infty_c(S^o)^{W_0} \mid \operatorname{supp} f \subset S_{r,R}\}$ onto $PW^{r,R}_m(\mathbb{C}^n)$ for all $R > r > 0$.

Proof. — Let $0 < r < R$. From the estimate (2) it is easily seen that $\mathcal{L}^o_0$ maps $C^\infty_{r,R}(S^o)$ into $PW^{r,R}_0(\mathbb{C}^n)$ for all $R > r > 0$. Lemma 5 and (3) thus imply that $\mathcal{L}^o_m$ maps $C^\infty_{r,R}(S^o)$ into $PW^{r,R}_m(\mathbb{C}^n)$ for all $R > r > 0$.

It remains to show that $\mathcal{L}^o_m$ is onto. Consider the wave packet $\mathcal{I}_m g \in C^\infty(S^o)^{W_0}$ defined by the inversion formula (1):

$$\mathcal{I}_m g(t) = \frac{|W|}{|W_0|}\int_{i\mathbb{R}^n} g(\lambda)\psi^d_m(\lambda,t)\,\frac{d\lambda}{c^d_m(-\lambda)c^0_m(\lambda)},\quad t \in \mathfrak{a}_-.$$

Define the auxiliary function Ξ^d_0 by

$$\Xi^d_0(\lambda,t) := \sum_{w\in\{\pm1\}^n} c^d_0(w\lambda)\Phi_0(w\lambda,t) = \prod_j P_{\lambda_j-\frac{1}{2}}(\cosh t_j).$$

Hence $\psi^d_0(\lambda,\cdot) = \sum_{w\in W_0}\Xi^d_0(w\lambda,\cdot)$. We also define the ($W_0$-invariant) fraction

$$Q_m(\lambda) = \frac{c^d_m(\lambda)}{c^d_0(\lambda)c^0_m(\lambda)} = \prod_{i<j}\frac{\Gamma(-\lambda_j-\lambda_i)}{\Gamma(-\lambda_j-\lambda_i+\frac{m}{2})}$$
$$= \left[\prod_{i<j}(-\lambda_i-\lambda_j)\dots(-\lambda_i-\lambda_j+\frac{m}{2}-1)\right]^{-1}.$$

Let $g \in PW^{r,R}_m(\mathbb{C}^n)$. Assume that $t \not> r$, that is, $t_j \leqslant r$ for a certain $j \leqslant n$: then there exists a $\lambda_r > 0$ such that $\langle\lambda_r, t - rt_o\rangle = -\epsilon < 0$. Let also $\lambda_s = s(1,\dots,1)$ for some $s > m$, so that $Q_m(\lambda+\lambda_s)$ (and also $\frac{1}{c^d_m(-\lambda)}$ and $\frac{1}{c^0_m(\lambda)}$ since m is even) are well-defined for $\operatorname{Re}\lambda \geqslant 0$. Using Cauchy's theorem and W_0-invariance we get:

$$\begin{aligned}
\frac{1}{|W|}\mathcal{I}_m g(t) &= \frac{1}{|W_0|}\int_{i\mathbb{R}^n} g(\lambda+\lambda_s)\psi_m^d(\lambda+\lambda_s,t)\frac{d\lambda}{c_m^d(-\lambda-\lambda_s)c_m^0(\lambda+\lambda_s)}\\
&= \frac{1}{|W_0|}\int_{i\mathbb{R}^n} g(\lambda+\lambda_s)D_m^+\psi_0^d(\lambda+\lambda_s,t)\frac{Q_m(\lambda+\lambda_s)d\lambda}{c_0^d(-\lambda-\lambda_s)}\\
&= \frac{1}{|W_0|}D_m^+\int_{i\mathbb{R}^n} g(\lambda+\lambda_s)\psi_0^d(\lambda+\lambda_s,t)\frac{Q_m(\lambda+\lambda_s)d\lambda}{c_0^d(-\lambda-\lambda_s)}\\
&= \frac{1}{|W_0|}D_m^+\int_{i\mathbb{R}^n} g(\lambda+\lambda_s)\left(\sum_{w\in W_0}\Xi_0^d(w\lambda+\lambda_s,t)\right)\frac{Q_m(\lambda+\lambda_s)d\lambda}{c_0^d(-\lambda-\lambda_s)}\\
&= D_m^+\int_{i\mathbb{R}^n} g(\lambda+\lambda_s)\Xi_0^d(\lambda+\lambda_s,t)\frac{Q_m(\lambda+\lambda_s)d\lambda}{c_0^d(-\lambda-\lambda_s)}\\
&= D_m^+\int_{i\mathbb{R}^n} g(\lambda+\lambda_s+\mu\lambda_r)\Xi_0^d(\lambda+\lambda_s+\mu\lambda_r,t)\frac{Q_m(\lambda+\lambda_s+\mu\lambda_r)d\lambda}{c_0^d(-\lambda-\lambda_s-\mu\lambda_r)}\\
&\to 0 \quad \text{for} \quad \mu\to\infty,
\end{aligned}$$

by the growth estimates of the Legendre functions, and we conclude that $\mathcal{I}_m g$ is zero outside $\{t|t>r\}$. The first contour shift is permissible since

$$\left|\psi_m^d(\lambda,t)\right| \leqslant (1+|t|)^{|W|-1}e^{\langle \operatorname{Re}\lambda,t\rangle},$$

for all $\lambda,t\in\overline{\mathfrak{a}^-}$, see e.g. [Wal, Proposition 4.6.3]. We can interchange the differential operator D_m^+ and the integral by the Paley-Wiener growth estimates and Lebesgue's dominated convergence theorem, since D_m^+ is of the form

$$D_m^+ = \sum_{i_1=0,\dots,i_n=0}^{m} T_i(t)\frac{\partial^{i_1}}{\partial t_1^{i_1}}\cdots\frac{\partial^{i_n}}{\partial t_n^{i_n}},$$

where the functions $\{T_i\}$ are differentiable functions on $\mathfrak{a}^-$.

An easy calculation shows that (for $t\in\mathfrak{a}^-$):

$$\mathcal{I}_m g(t) = \int_{i\mathbb{R}^n} P_m^{\mathrm{av}}g(\lambda)\psi_m^d(\lambda,t)\left|c_m^d(\lambda)\right|^{-2}d\lambda,$$

which we recognize as the inverse Fourier transform of $P_m^{\mathrm{av}}g\in\mathcal{H}^R(\mathbb{C}^n)$, whence $\mathcal{I}_m g(t)=0$ for $|t|>R$ by the Paley-Wiener theorem for the spherical Fourier transform on $\mathcal{M}_m^d$. All this implies that $\mathcal{I}_m g\in C_{r,R}^\infty(S^o)$.

Since $P_m^{\mathrm{av}}\mathcal{L}_m^o f=\mathcal{F}_m f^d$ for all $f\in C_c^\infty(S^o)^{W_0}$, the above also yields

$$P_m^{\mathrm{av}}\mathcal{L}_m^o\mathcal{I}_m g = \mathcal{F}_m(\mathcal{I}_m g)^d = P_m^{\mathrm{av}}g,$$

for all $g\in PW_m(\mathbb{C}^n)$. Let $h=\mathcal{L}_m^o\mathcal{I}_m g-g$, $h^1(\lambda):=h(\lambda)/c_m^d(-\lambda)c_m^0(\lambda)$ and

$$\operatorname{av}h^1(\lambda) := \frac{1}{|W|}\sum_{w\in W}h^1(w\lambda) = \frac{|W_0|}{|W|}\sum_{w\in W_0\backslash W}h^1(w\lambda).$$

Then $P_m^{\mathrm{av}} h(\lambda) = |W| c_m^d(\lambda) c_m^d(-\lambda)\,\mathrm{av}\, h^1(\lambda) = 0$, whence also $\mathrm{av}\, h^1(\lambda) = 0$. The function h^1 also satisfies item (i) of Definition 4, in particular, $h^1(i\,\cdot\,) \in L^1(\mathbb{R}^n)$. Let

$$\gamma(s) = \int_{\mathbb{R}^n} h^1(i\lambda) e^{i\langle s,\lambda\rangle}\, d\lambda, \qquad s \in \mathbb{R}^n,$$

denote the Euclidean Fourier transform of $h^1(i\,\cdot\,)$. The condition (i) implies that h^1 is holomorphic in an open set containing $\{z \in \mathbb{C}^n |\ \mathrm{Re}\, z \geqslant 0\}$, and the standard argument with Cauchy's theorem gives that γ is supported on $\{t | t \geqslant r\}$. On the other hand, the average $\mathrm{av}\,\gamma$ of γ is the Fourier transform of $\mathrm{av}\, h^1(i\cdot)$, which vanishes, hence $\mathrm{av}\,\gamma$ vanishes as well and $\gamma = 0$ by the support condition. Since the Euclidean Fourier transform is injective on $L^1(\mathbb{R}^n)$, we conclude that h^1, and hence also $h = \mathcal{L}_m^o \mathcal{I}_m g - g$, vanishes. This implies that $\mathcal{L}_m^o \mathcal{I}_m g = g$ for all $g \in PW_m(\mathbb{C}^n)$ and hence that $\mathcal{L}_m^o$ maps $C_{r,R}^\infty(S^o)^{W_0}$ onto $PW_m^{r,R}(\mathbb{C}^n)$ for all $R > r > 0$. $\square$

Remarks. — It follows from the above that the proof of the full Paley-Wiener theorem for all ordered symmetric spaces of Cayley type basically has been reduced to the $\mathcal{M}_{1,n} = Sp(n,\mathbb{R})/SL(n,\mathbb{R}) \times \mathbb{R}_+^*$-case. Furthermore we can use the same approach, but using shift operators associated with both the orbit of the short roots and the orbit of the (unique) long root, to prove a Paley-Wiener theorem for $Sp(n,n)/Sp(n,\mathbb{C})$ (where the root multiplicities are 4 for the short roots and 3 for the long root), see also the next section.

Remark. — We proved the Paley-Wiener theorem for

$$\mathcal{M}_2 = SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$$

in [AU] using explicit expressions of the spherical functions in terms of Legendre functions.

4. The hypergeometric Laplace transform on BC_n type root systems.

Consider the BC_n type root system $\Delta = \left\{\frac{\pm\gamma_j \pm \gamma_i}{2}\right\} \cup \left\{\frac{\pm\gamma_i}{2}\right\} \cup \{\pm\gamma_i\}$. We choose a set of positive roots by $\Delta^+ = \left\{\frac{\gamma_j \pm \gamma_i}{2},\ i < j\right\} \cup \left\{\frac{\gamma_i}{2}\right\} \cup \{\gamma_i\}$. Let again Δ_0 denote the root system $\Delta_0 = \left\{\frac{\gamma_j - \gamma_i}{2}\right\}$ with positive roots $\Delta_0^+ = \left\{\frac{\gamma_j - \gamma_i}{2}, i < j\right\}$ and let $W \cong \mathfrak{S}_n \times \{\pm 1\}^n$ and $W_0 \cong \mathfrak{S}_n$ denote the

Weyl groups of the two root systems. Let $m = (m_1, m_2, m_3)$ denote the root multiplicities m_1 for the short roots $\frac{\pm\gamma_j \pm\gamma_i}{2}$, m_2 for the intermediate roots $\frac{\pm\gamma_i}{2}$ and m_3 for the long roots $\pm\gamma_i$.

We identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}^n$ and $\mathfrak{a}$ with $\mathbb{R}^n$ as before and we define the c-functions c_m^d, $c_m = c_m^0 c_m^\Omega$ and the Jacobian δ_m using the product formulae in §1 (e.g. $c_m^d(\lambda) = \prod_{\alpha\in\Delta^+} \frac{\Gamma(\frac{1}{2}\langle\lambda,\check{\alpha}\rangle)}{\Gamma(\frac{1}{2}\langle\lambda,\check{\alpha}\rangle + \frac{m_\alpha}{2})}$).

We define the Harish-Chandra series $\Phi_m(\lambda, t)$ ($\mathrm{Re}\,\lambda\in\overline{\mathfrak{a}^-}$, $t\in\mathfrak{a}^-$) and the hypergeometric functions (of the first type) $\psi_m^d(\lambda, t) = \sum_{w\in W} c_m^d(w\lambda)$ $\Phi_m(w\lambda, t)$ ($\lambda \in \mathbb{C}^n$, $t \in \mathbb{R}^n$) as in e.g. [HS, Part 1, Chapter 4]. We use the Ólafsson expansion formula to define hypergeometric functions of the second type $\varphi_m(\lambda, t)$ as follows:

$$\varphi_m(\lambda, t) = \sum_{w\in W_0} c_m(w\lambda)\Phi_m(w\lambda, t) = c_m^\Omega(\lambda) \sum_{w\in W_0} c_m^0(w\lambda)\Phi_m(w\lambda, t),$$

defined for λ in some dense subset of $\mathbb{C}^n$ and $t \in S^o$. We also define the normalized hypergeometric functions of the second type $\varphi_m^o(\lambda, \cdot)$ by $\varphi_m^o(\lambda, \cdot) := \varphi_m(\lambda, \cdot)/c_m^\Omega(\lambda)$. We note that Lemma 1 still holds for $m \in \mathbb{N}^3 \cup \{0\}$. We define the normalized hypergeometric Laplace transform $\mathcal{L}_m^o$ by integrating against $\varphi_m^o(\lambda, t)$:

$$\mathcal{L}_m^o(f)(\lambda) = \int_{\mathfrak{a}^-} f(t)\varphi_m^o(\lambda, t)\delta_m(t)dt,$$

for $f \in C_c^\infty(S^o)^{W_0}$ whenever the integral converges. The hypergeometric Fourier transform is similarly defined by integrating against $\psi_m^d(\lambda, t)$ (for $f \in C_c^\infty(\mathbb{R}^n)^W$).

As has been shown by Heckman and Opdam, see e.g. [Hec, §5] and [Opd, §8], the Paley-Wiener theorem for the spherical Fourier transform on a Riemannian symmetric space can be generalized to the hypergeometric Fourier transform associated to a root system, even for non-integer (non-negative) values of the multiplicity parameter m (for certain values of m this is of course the same transform).

We define the Paley-Wiener spaces $PW_m^{r,R}(\mathbb{C}^n)$ and $PW_m(\mathbb{C}^n)$ for the (normalized) hypergeometric Laplace transform as in Definition 4 (with $m \in \mathbb{N}^3 \cup \{0\}$).

THEOREM 7 (The Paley-Wiener Theorem). — *Let $m_1, m_2 \in 2\mathbb{N}$ and $m_3 \in 2\mathbb{N} - 1$. The normalized hypergeometric Laplace transform $\mathcal{L}_m^o$ is a bijection of $C_c^\infty(S^o)^{W_0}$ onto $PW_m(\mathbb{C}^n)$. More precisely it is a bijection of $C_{r,R}^\infty(S^o)^{W_0}$ onto $PW_m^{r,R}(\mathbb{C}^n)$ for all $R > r > 0$.*

Proof. — As before, using (compositions of) shift operators associated with the orbits of respectively the short roots, the intermediate roots and the long roots. Lemma 2 and Lemma 3 still hold with $m \in \mathbb{N}^3 \cup \{0\}$ (and the obvious changes in notation) and Lemma 5 also still holds with the modification that $k = (k_1, k_2, k_3) \in \{0,1\}^3$ satisfies $k_i \equiv m_i \bmod 2$ for $i \in \{1,2,3\}$. $\square$

Remarks. — A similar definition of the Laplace transform in the rank 1 case (for non-integer values of the multiplicity parameter) has been considered by M. Mizony in [Miz], in which he defines the Laplace-Jacobi transform by integration against the (normalized) Jacobi function of the second type. He also obtains a partial Paley-Wiener theorem, and an obvious question is of course if the above Paley-Wiener theorem can be generalized to non-integer values of the multiplicity parameter m. With modifications, the above definition of the hypergeometric functions of the second type and the associated hypergeometric Laplace transform also carries over to other root systems.

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