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A NONLINEARIZABLE ACTION OF S_3 on $\mathbb{C}^4$

by G. FREUDENBURG and L. MOSER-JAUSLIN

1. Introduction.

An important problem in the theory of algebraic group actions is to understand how a finite group G can act algebraically on complex affine space $\mathbb{C}^n$. When G acts linearly, our understanding is fairly extensive; much less is known when G acts by general polynomial automorphisms. Consider the following question:

Given a finite group G, does there exist a non-linearizable action of G on $\mathbb{C}^n$?

By a *non-linearizable* G-action, we mean an algebraic action of G on $\mathbb{C}^n$ which is not conjugate to a linear action under any polynomial automorphism of $\mathbb{C}^n$.

For $n \leqslant 2$, the answer is no (see, for example, [Kr1]), and for $n = 3$ the question is open for all finite groups. For $n \geqslant 4$, the answer is yes. The first such non-linearizable actions of finite groups were given by Masuda and Petrie [MP]. Other examples were given in [MMP1], where the smallest example is an action of D_{10}, the dihedral group of 20 elements, acting on $\mathbb{C}^4$. Later, Mederer gave examples with D_5 and D_6 acting on $\mathbb{C}^4$ [Med]. In the present article we give an explicit example for $G = S_3$, the permutation group of order 6. It is the smallest non-abelian group.

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If one considers the analytic category, it has been shown by Derksen and Kutschebauch that for any finite group G, there are analytic non-linearizable actions of G on affine space [DK]. Their argument uses an idea of Asunuma, who studied the non-linearizability question over other fields. He showed in particular that there are non-linearizable actions of $\mathbb{R}^*$ on real affine space [A]. The actions given by Derksen and Kutschebauch are not algebraic.

One of the difficulties in finding non-linearizable actions is to construct candidates. For algebraic actions on complex space, the only method that has led to such candidates at this point is that of finding non-trivial equivariant vector bundles over representation spaces.

DEFINITION. — *Given a reductive group G, and a G-variety V, an* equivariant G-vector bundle with base V *is a vector bundle $\pi : E \to V$, where E is a G-variety, π is equivariant, and G acts linearly on the fibers. That is, for all $g \in G$ and all $v \in V$, the morphism induced by g of $\pi^{-1}(v)$ to $\pi^{-1}(gv)$ is linear. For example, if V has a fixed point, then the fiber is a G-module. An equivariant G-vector bundle over V is called* trivial *if it is isomorphic, as a G-vector bundle, to one of the form $V \times W \to V$ where W is a G-module.*

Bass and Haboush [BH2], Kraft [Kr2] and Masuda and Petrie [MP] all gave relations between finding non-linearizable actions on affine space and finding non-trivial equivariant vector bundles over representation spaces. Note that if V is a G-module, then, if one disregards the action, the total space is isomorphic to affine space by the result of Quillen and Suslin [Sus] and [Q]. The idea is then to consider the action of G, or possibly a slightly larger group, on the total space of a non-trivial equivariant vector bundle. It is not true in general that if the G-action on the total space of a G-vector bundle over a G-module is linearizable then the bundle is trivial. However, under certain additional hypotheses, this result is true. For example, one of the results of Masuda and Petrie states that if $E \to V$ is a G-vector bundle over a G-module V, and if G contains a subgroup H whose fixed point set in E is exactly the zero section, then the bundle is trivial if and only if the action is linearizable [MP].

The present example, and some of the other examples of non-linearizable actions of finite groups cited above, are obtained as a restriction of a non-linearizable action of $O(2,\mathbb{C}) = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^4$ given by Schwarz [Sch]. Thus, the present example gives a new proof that the Schwarz action is non-linearizable. In his article, Schwarz gave the first non-linearizable actions of reductive groups on affine space. In [KS], the case of actions with a one-dimensional quotient was studied.

The importance of this new example is twofold. First of all, it is known that any equivariant vector bundle over a representation of a finite *abelian* group is trivial (see [MMP2]), and therefore the action on the total space is linearizable. Thus, S_3 is the smallest group for which the method of equivariant vector bundles can be used to construct non-linearizable actions. Secondly, the proof is elementary and more transparent than in the other cases. One reason for this is that the action has a line of fixed points. Indeed, this is the first example of a non-linearizable action of any reductive group on $\mathbb{C}^4$ having a line of fixed points.

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2. A non-linearizable action.

Let $S_3 =< \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma = \sigma^2\tau >$. It has 6 elements, and it is the smallest non-abelian group. Consider the action of S_3 on $E := \mathbb{C}^4$ given by

$$\sigma(a, b, x, y) = (\omega a, \omega^{-1} b, x, y)$$

where $\omega = e^{2\pi i/3}$, and

$$\tau(a, b, x, y) = (b, a, -b^3 x + (1 + ab + (ab)^2)y, (1 - ab)x + a^3 y).$$

One verifies easily that it is an action of S_3. Note that the action of the subgroup generated by σ is linear. It is also known that the action of the subgroup generated by τ is conjugate to a linear action (see [MJ] and [MMP2]), but we will show that the action of S_3 is not linearizable.

This action comes from Schwarz's first example of a non-linearizable $O(2, \mathbb{C})$-action on $\mathbb{C}^4$ [Sch]. We restrict the action to S_3. Note that the fixed point set of Schwarz's original action was just the origin, but when restricted to S_3, there is a line of fixed points.

The projection $\pi : E \to V$ given by $(a, b, x, y) \mapsto (a, b)$ is equivariant where V is the irreducible two-dimensional representation of S_3. This map defines the structure of an S_3-vector bundle over V. Indeed, π is equivariant, and S_3 acts linearly on the fibers. Thus S_3 preserves a natural grading on the coordinate ring $R := \mathbb{C}[E] = \mathbb{C}[a, b, x, y]$ of the total space. The grading is given by the degree in x and y. If $f \in R$, we can decompose f as $f = \sum_{i=0}^{d} f_i$ where f_i is homogeneous of degree i in x and y. Also, f is S_3-invariant if and only if f_i is invariant for each i.

LEMMA 1. — *Let $f \in R$ be an S_3-invariant function. Then there exists $\alpha \in \mathbb{C}$ such that $f_1(\alpha, \alpha, x, y) = 0$ for all x and y.*

Proof. — Since f is invariant, f_1 is also invariant. Let $f_1 = p(a, b)x + q(a, b)y$. First consider invariance under the action of σ. Since x and y are invariant under the action by σ, the polynomials $p(a, b)$ and $q(a, b)$ are in $\mathbb{C}[V]^\sigma = \mathbb{C}[a^3, b^3, ab]$. In particular, $p(a, a)$ and $q(a, a)$ are in $\mathbb{C}[a^2, a^3]$.

Now invariance under the action of τ means that

$$f_1 = p(b, a)(-b^3 x + (1 + ab + a^2 b^2)y) + q(b, a)((1 - ab)x + a^3 y).$$

In particular, when $a = b$, one finds that $p(a, a)(a^3 + 1) = q(a, a)(1 - a^2)$. Dividing both sides of the equation by $(1 + a)$, this yields

$$p(a, a)(a^2 - a + 1) = q(a, a)(1 - a). \tag{1}$$

Now we put the two parts together to show that $p(a, a)$ and $q(a, a)$ must have a common zero. The equation (1) implies that $1 - a$ divides $p(a, a)$. However, since $p(a, a) \in \mathbb{C}[a^2, a^3]$, it must have a root $\alpha \neq 1$. Then using again equation (1) we see that $q(\alpha, \alpha) = 0$, and the lemma is proven. □

THEOREM 1. — *The action of S_3 on E is algebraically non-linearizable.*

Proof. — Let $L \subset V$ be the line defined by $a = b$, and consider its inverse image $\pi^{-1}(L) \cong \mathbb{C}^3$. Note that $(a - b)$ is a semi-invariant of τ, and thus τ acts on $\pi^{-1}(L)$. This action is linearizable. Indeed consider the isomorphism $\varphi : \pi^{-1}(L) \to \mathbb{C}^3$ given by $(a, a, x, y) \mapsto (a, s, t)$ where

$$s = (1 - a)x + (1 - a + a^2)y \quad \text{and} \quad t = (1 + a)x - (1 + a + a^2)y. \tag{2}$$

The map φ is an isomorphism since $\begin{pmatrix} 1 - a & 1 - a + a^2 \\ 1 + a & -(1 + a + a^2) \end{pmatrix} = -2$. A short calculation shows that φ is equivariant with respect to the action of τ where $\tau(a) = a$, $\tau(s) = s$ and $\tau(t) = -t$. Note that s and t can also be considered as elements of R, but they are only semi-invariants of τ modulo $(a-b)$.

Now consider the fixed point sets $E^{S_3} \subset E^\tau \subset \pi^{-1}(L) \cong \mathbb{C}^3$. One sees easily that E^{S_3} is a line isomorphic to $\mathbb{C}$ in E^τ, which is isomorphic to $\mathbb{C}^2$. For a subset $Z \subset E$, denote by $I(Z)$ the vanishing ideal in R of polynomials which are identically zero on Z. We have that

$$I(E^{S_3}) = (a, b, t) \quad \text{and} \quad I(E^\tau) = (a-b, t).$$

Define $\bar{R} := \mathbb{C}[E^\tau] = R/(a-b,t) = \mathbb{C}[\bar{a}, \bar{s}]$, where for a polynomial $h \in R$, $\bar{h}$ denotes its class in $\bar{R}$. The image $\bar{I} \subset \bar{R}$ of the ideal $I(E^{S_3})$ is an ideal generated by $\bar{a}$.

Now suppose that the S_3-action on E were linearizable. Then it would be isomorphic to the action on the tangent space at the origin. Thus E would be equivariantly isomorphic to $V \oplus \mathbb{C} \oplus \mathbb{C}_{\text{sign}}$, where $\mathbb{C}$ is the one-dimensional trivial representation and $\mathbb{C}_{\text{sign}}$ is the one-dimensional signature representation. This means that $R = \mathbb{C}[a,b,x,y] = \mathbb{C}[u,v,f,g]$ where $\mathbb{C}u \oplus \mathbb{C}v \cong V$ as S_3-representations, f is invariant and g is semi-invariant. Therefore, we would have that

$$I(E^{S_3}) = (u, v, g) \quad \text{and} \quad I(E^\tau) = (u-v, g).$$

This means that the ideal $\bar{I} \subset \bar{R}$ would be generated by $\bar{u}$. Hence, we have that $\bar{u}$ is a non-zero multiple of $\bar{a}$. Therefore,

$$\bar{R} = \mathbb{C}[\bar{a}, \bar{s}] = \mathbb{C}[\bar{u}, \bar{f}] = \mathbb{C}[\bar{a}, \bar{f}].$$

Thus $\bar{f} = \gamma\bar{s} + r(\bar{a})$ where γ is a non-zero constant and r is a polynomial of one variable. Lifting back to R, we have

$$f = \gamma s + r(a) + m \cdot t + n \cdot (a-b)$$

for some $m, n \in R$. Then it follows that the linear part f_1 satisfies

$$f_1 = \gamma s + m_0 \cdot t + n_1 \cdot (a-b)$$

because s and t are homogeneous of degree 1, and $a-b$ and $r(a)$ are homogeneous of degree 0.

In particular, $m_0 \in \mathbb{C}[a,b]$, and therefore

$$f_1(a,a,x,y) = \gamma s + q(a) \cdot t$$

for $q = m_0(a,a) \in \mathbb{C}[a]$. It follows from *Lemma 1* that there exists $\alpha \in \mathbb{C}$ with $f_1(\alpha, \alpha) = 0$. But this is impossible, since $s(\alpha)$ and $t(\alpha)$ are linearly independent functions in $\mathbb{C}[x,y]$ and $\gamma \neq 0$.

Thus the action is not linearizable, and the theorem is proven. □

Remark 1. — The first attempt to prove *Theorem 1* was to show that R has no invariant variable. A *variable* of R is a polynomial f such that $R = \mathbb{C}[f, g_1, g_2, g_3]$ for some polynomials g_i, and the set $\{f, g_1, g_2, g_3\}$ is called a *system* of variables of R. What we showed was that there is no invariant variable contained in any system of variables corresponding to a linear action; it is still unclear whether the invariant ring of the above action contains any other variable f. If so, then the action restricted to the zero set of f is an action of S_3 on $\mathbb{C}^3$, and this action could be non-linearizable. Thus, the question of whether there is an invariant variable is of interest.

3. Equivariant vector bundles.

It is clear that *Theorem 1* implies that the equivariant S_3-vector bundle $E \to V$ is non-trivial. But actually, *Lemma 1* implies a stronger result:

PROPOSITION 1. — *Let $X := S_3 L = L \cup \sigma L \cup \sigma^2 L \subset V$. Then $\pi^{-1}(X) \to X$ is a non-trivial S_3-vector bundle.*

Proof. — Since X is invariant, the restriction of π to $\pi^{-1}(X) \to X$ is an equivariant S_3-vector bundle. If it were trivial, it would have to be isomorphic to $X \times (\mathbb{C} \oplus \mathbb{C}_{\text{sign}})$ because the fiber over 0 is the representation $\mathbb{C} \oplus \mathbb{C}_{\text{sign}}$. Thus it would have a subbundle isomorphic to $X \times \mathbb{C}_{\text{sign}}$. This subbundle would be defined by an invariant function of $\pi^{-1}(X)$ which is linear on the fiber. In other words, it would be the restriction of a linear function $f_1 = p(a,b)x + q(a,b)y$ to $\mathbb{C}(\pi^{-1}(X))$. But *Lemma 1* implies that there is a value $\alpha \in \mathbb{C}$ such that $f_1(\alpha, \alpha) = 0$. Thus the entire fiber of $(\alpha, \alpha) \in X$ would be in the zero set of f_1. This contradicts the assumption that f_1 defines a rank one subbundle. □

In fact, the idea of the proof of *Theorem 1* is to show that non-triviality of the bundle over X implies non-linearity of the action on E. None of the arguments cited in the introduction which compare the triviality of vector bundles and linearity of the action on the total space can be used precisely as stated for the example given here. However, we use an idea of the proof of a result of Masuda and Petrie (Lemma 5.1 of [MP]).

We show that if φ were an equivariant isomorphism of E to $V \times (\mathbb{C} \oplus \mathbb{C}_{\text{sign}})$, then we can assume that φ fixes X in the zero section of the two bundles. If one knew that φ preserved the bundle over X, one could then show that the normal derivative of φ gives an isomorphism to the trivial S_3-vector bundle. However, one cannot be sure that φ does in fact preserve the bundles over X, and thus the argument in [MP] must be modified.

Using methods developed by Mederer in [Med], one can study the set of all equivariant S_3-vector bundles over V and over X whose fiber over the fixed point is isomorphic to $\mathbb{C} \oplus \mathbb{C}_{\text{sign}}$. In fact, one can show that this is the only non-linearizable action of S_3 on $\mathbb{C}^4$ coming from a vector bundle over V with the given zero fiber, which is non-trivializable when restricted to X.

Remark 2. — It has been shown that all *analytic* equivariant vector bundles over a representation space of a reductive group are trivial [HK]. In particular, this means that the given action is analytically linearizable. For the action of Section 2, this can be seen directly, and the difference between the analytic and algebraic categories becomes clear. To see this, note that *Lemma 1* is not true in the analytic category, because one can choose $p(a,a) = (1-a)e^a$ and $q(a,a) = (1-a+a^2)e^a$. One can show that this extends to a system of analytic generators. This gives an analytic linearization of the action with $u = a$, $v = b$, and

$$\begin{pmatrix} f \\ g \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix},$$

where $P = (p_{ij})$ for

$$\begin{aligned}
p_{11}(a,b) &= \Big(1 - \frac{b^3}{d^2}\Big)\cosh(d) - d\Big(1 - \frac{b^3}{d^4}\Big)\sinh(d)\\
p_{12}(a,b) &= \Big(1 - \frac{a^3}{d^2} + d^2\Big)\cosh(d) - d\Big(1 - \frac{a^3}{d^4} - \frac{a^3}{d^2}\Big)\sinh(d)\\
p_{21}(a,b) &= \Big(1 + \frac{b^3}{d^2}\Big)\cosh(d) - d\Big(1 + \frac{b^3}{d^4}\Big)\sinh(d)\\
p_{22}(a,b) &= -\Big(1 + \frac{a^3}{d^2} + d^2\Big)\cosh(d) + d\Big(1 + \frac{a^3}{d^4} + \frac{a^3}{d^2}\Big)\sinh(d)
\end{aligned}$$

and $d = \sqrt{ab}$. Note that the coefficients in the matrix P are all analytic on $\mathbb{C}^2$.

It is also known that any equivariant vector bundle over a representation space of a reductive group is stably trivial [BH1]. This means that there exists a representation space W such that the Whitney sum of Θ_W and the given bundle is trivial, where Θ_W denotes the trivial bundle $V \times W \to V$. In our case we have:

PROPOSITION 2. — *The Whitney sum $E \oplus \Theta_V$ is a trivial S_3-vector bundle. In particular, the S_3-action on $E \times V$ is trivial.*

This result was known by Schwarz [Sch], but we will give an explicit isomorphism.

Proof. — The isomorphism $\Phi : E \times V \cong V \times V \times \mathbb{C} \times \mathbb{C}_{\text{sign}}$ is given by

$$\begin{aligned}\Phi((a,b,x,y),(u,v)) = {} & ((a,b),(-b^2x + a(1+ab)y - u + bv,\\ & bx - a^2y - v + au),\\ & ((1-ab)x + (a^3+1)y + av + bu)/2,\\ & (-(1-ab)x + (1-a^3)y - av + bu)/2),\end{aligned}$$

and its inverse is

$$\begin{aligned}\Phi^{-1}((a,b),(u',v'),z,w) = {} & ((a,b,a^2u' + a(1+ab)v' + (1-a^3)z - (1+a^3)w,\\ & bu' + b^2v' + (1-ab)z + (1-ab)w,\\ & (-u' - bv' + a(z+w), -t - as + b(z-w))).\end{aligned}$$

This proves that the S_3-vector bundle $E \oplus \Theta_V$ is trivial. □

However, as we shall show in the next section, $E \times \mathbb{C}^n \times \mathbb{C}^m_{\text{sign}}$ is non-linearizable for all non-negative n and m.

4. Generalization to affine spaces of higher dimension.

THEOREM 2. — *For any pair of non-negative integers n and m, the S_3-action on $E \times \mathbb{C}^n \times \mathbb{C}^m_{\text{sign}}$ is non-linearizable.*

COROLLARY 1. — *For any integer $N \geqslant 4$ there exist non-linearizable actions of S_3 on $\mathbb{C}^N$.*

The fact that the statement of *Theorem 2* holds for $n = 0$ can actually be proven just as in Section 2, because the fixed point sets are identical. This is enough to prove *Corollary 1.* However, the theorem implies more than that. Since E is the total space of an equivariant vector bundle, we know that there exists a G-module W such that $E \times W$ is linearizable. A direct consequence of *Theorem 2* and *Proposition 2* is:

COROLLARY 2. — *If W is a G-module, then $E \times W$ is linearizable if and only if W contains V as a submodule.*

The proof of *Theorem 2* is similar to the proof of *Theorem 1*, but the notation is a bit more involved.

Proof of Theorem 2. — Let $z_1, \dots, z_n$ be the coordinates of $\mathbb{C}^n$ and $w_1, \dots, w_m$ be the coordinates of $\mathbb{C}^m_{\mathrm{sign}}$. Denote by E' the S_3-variety $E' := E \times \mathbb{C}^n \times \mathbb{C}^m_{\mathrm{sign}}$, and by $R' := \mathbb{C}[E'] = \mathbb{C}[a, b, x, y, z_1, \dots, z_n, w_1, \dots, w_m]$. The projection $\pi' : E' \to V$ given by $(a, b, x, y, z_1, \dots, z_n, w_1, \dots, w_m) \mapsto (a, b)$ defines the structure of an S_3-vector bundle. As before, S_3 preserves the grading defined by the degree in x, y, $z_1, \dots, z_n$ and $w_1, \dots, w_m$. Define s and t as in equation (2), this time considering them as elements of R'.

The vanishing ideals of the fixed point sets of τ and of S_3 are given by

$$J := I((E')^\tau) = (a - b, t, w_1, \dots, w_m), \quad I := I((E')^{S_3}) = J + (a).$$

Denote by $\bar{R}' := \mathbb{C}[(E')^\tau] = R/J = \mathbb{C}[\bar{a}, \bar{s}, \bar{z}_1, \dots, \bar{z}_n]$, and by $\bar{I}$ the image of I in $\bar{R}'$.

If the S_3-action on E' were linearizable, then the ring R' could be expressed as $R' = \mathbb{C}[u, v, f_0, \dots, f_n, g_0, \dots, g_m]$, where the f_i's are invariants, the g_i's are semi-invariants whose weight is the signature character of S_3, and $\mathbb{C}u \oplus \mathbb{C}v \cong V$ as S_3-representations. Thus we would have

$$J = (u - v, g_0, \dots, g_m) \quad \text{and} \quad I = J + (u).$$

In particular, $\bar{R}'$ would be $\mathbb{C}[\bar{u}, \bar{f}_0, \dots, \bar{f}_n]$, and $\bar{u}$ would be a non-zero constant multiple of $\bar{a}$. We would have that

$$\bar{R}' = \mathbb{C}[\bar{a}, \bar{s}, \bar{z}_1, \dots, \bar{z}_m] = \mathbb{C}[\bar{u}, \bar{f}_0, \dots, \bar{f}_n] = \mathbb{C}[\bar{a}, \bar{f}_0, \dots, \bar{f}_n].$$

This shows that the map $\psi : \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}$ defined by $\psi(\bar{a}, \bar{s}, \bar{z}_1, \dots, \bar{z}_n) = (\bar{a}, \bar{f}_0, \dots, \bar{f}_n)$ is an automorphism. For $i = 1, \dots, n$ denote the linear part of f_i by

$$f_{i1} = p_i x + q_i y + \sum_{j=1}^{n} r_{ij} z_j + \sum_{k=1}^{m} \tilde{r}_{ik} w_k$$

with p_i, q_i, r_{ij} and $\tilde{r}_{ik}$ polynomials in $\mathbb{C}[a, b]$. Using equation (2), we find that for $i = 1, \dots, n$, the linear part of $\bar{f}_i$ is given by

$$\begin{aligned}\bar{f}_{i1} &= p_i(a, a)\bar{x} + q_i(a, a)\bar{y} + \sum_{j=1}^{n} r_{ij}(a, a)\bar{z}_j \\ &= \frac{1}{2}(p_i(a, a)(1 + a + a^2) + q_i(a, a)(1 + a))\bar{s} + \sum_{j=1}^{n} r_{ij}(a, a)\bar{z}_j.\end{aligned}$$

By calculating the Jacobian of ψ on the line $\{(\bar{a},0,\ldots,0)|a\in\mathbb{C}\}$, one finds that the deteminant of the matrix

$$A(a):=\begin{pmatrix} \frac{1}{2}(p_0(a,a)(1+a+a^2)+q_0(a,a)(1+a)) & r_{01}(a,a) & \ldots & r_{0n}(a,a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}(p_n(a,a)(1+a+a^2)+q_n(a,a)(1+a)) & r_{n1}(a,a) & \ldots & r_{nn}(a,a) \end{pmatrix}$$

is a non-zero constant δ.

As in the proof of *Lemma 1*, since f_{i1} is invariant for all i, we find that $p_i(a,a)$, $q_i(a,a)$ and $r_{ij}(a,a)\in\mathbb{C}[a^2,a^3]$, for all i, j and k and that

$$p_i(a,a)(1-a+a^2)=q_i(a,a)(1-a) \tag{3}$$

for all i.

By multiplying the first column of $A(a)$ by $1-a$ and applying equation (3), we find that $(1-a)\delta$ is the determinant of

$$B(a)=\begin{pmatrix} p_0(a,a) & r_{01}(a,a) & \ldots & r_{0n}(a,a) \\ \vdots & \vdots & \ddots & \vdots \\ p_n(a,a) & r_{n1}(a,a) & \ldots & r_{nn}(a,a) \end{pmatrix}.$$

But since all of the coefficients of $B(a)$ are in $\mathbb{C}[a^2,a^3]$, this is impossible. Thus the action of S_3 on E' is non-linearizable. This finishes the proof of the theorem. □

Remark 3. — Note that we may replace the underlying field $\mathbb{C}$ with any field $\mathbf{k}$ of characteristic 0 containing 3 cube roots of unity, thus obtaining a non-linearizable S_3-action on $\mathbb{A}^n_{\mathbf{k}}$ for any $n\geqslant 4$.

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