A nonlinearizable action of $S_3$ on $\mathbb{C}^4$


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A NONLINEARIZABLE ACTION OF $S_3$ on $\mathbb{C}^4$

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1. Introduction.

An important problem in the theory of algebraic group actions is to understand how a finite group $G$ can act algebraically on complex affine space $\mathbb{C}^n$. When $G$ acts linearly, our understanding is fairly extensive; much less is known when $G$ acts by general polynomial automorphisms. Consider the following question:

Given a finite group $G$, does there exist a non-linearizable action of $G$ on $\mathbb{C}^n$?

By a non-linearizable $G$-action, we mean an algebraic action of $G$ on $\mathbb{C}^n$ which is not conjugate to a linear action under any polynomial automorphism of $\mathbb{C}^n$.

For $n \leq 2$, the answer is no (see, for example, [Kr1]), and for $n = 3$ the question is open for all finite groups. For $n \geq 4$, the answer is yes. The first such non-linearizable actions of finite groups were given by Masuda and Petrie [MP]. Other examples were given in [MMP1], where the smallest example is an action of $D_{10}$, the dihedral group of 20 elements, acting on $\mathbb{C}^4$. Later, Mederer gave examples with $D_5$ and $D_6$ acting on $\mathbb{C}^4$ [Med]. In the present article we give an explicit example for $G = S_3$, the permutation group of order 6. It is the smallest non-abelian group.

Keywords: Nonlinearizable actions – Equivariant vector bundles – Invariants.
If one considers the analytic category, it has been shown by Derksen and Kutschebauch that for any finite group $G$, there are analytic non-linearizable actions of $G$ on affine space [DK]. Their argument uses an idea of Asunuma, who studied the non-linearizability question over other fields. He showed in particular that there are non-linearizable actions of $\mathbb{R}^*$ on real affine space [A]. The actions given by Derksen and Kutschebauch are not algebraic.

One of the difficulties in finding non-linearizable actions is to construct candidates. For algebraic actions on complex space, the only method that has led to such candidates at this point is that of finding non-trivial equivariant vector bundles over representation spaces.

**Definition.** Given a reductive group $G$, and a $G$-variety $V$, an equivariant $G$-vector bundle with base $V$ is a vector bundle $\pi : E \to V$, where $E$ is a $G$-variety, $\pi$ is equivariant, and $G$ acts linearly on the fibers. That is, for all $g \in G$ and all $v \in V$, the morphism induced by $g$ of $\pi^{-1}(v)$ to $\pi^{-1}(gv)$ is linear. For example, if $V$ has a fixed point, then the fiber is a $G$-module. An equivariant $G$-vector bundle over $V$ is called trivial if it is isomorphic, as a $G$-vector bundle, to one of the form $V \times W \to V$ where $W$ is a $G$-module.

Bass and Haboush [BH2], Kraft [Kr2] and Masuda and Petrie [MP] all gave relations between finding non-linearizable actions on affine space and finding non-trivial equivariant vector bundles over representation spaces. Note that if $V$ is a $G$-module, then, if one disregards the action, the total space is isomorphic to affine space by the result of Quillen and Suslin [Sus] and [Q]. The idea is then to consider the action of $G$, or possibly a slightly larger group, on the total space of a non-trivial equivariant vector bundle. It is not true in general that if the $G$-action on the total space of a $G$-vector bundle over a $G$-module is linearizable then the bundle is trivial. However, under certain additional hypotheses, this result is true. For example, one of the results of Masuda and Petrie states that if $E \to V$ is a $G$-vector bundle over a $G$-module, then the bundle is trivial. However, under certain additional hypotheses, this result is true. For example, one of the results of Masuda and Petrie states that if $E \to V$ is a $G$-vector bundle over a $G$-module $V$, and if $G$ contains a subgroup $H$ whose fixed point set in $E$ is exactly the zero section, then the bundle is trivial if and only if the action is linearizable [MP].

The present example, and some of the other examples of non-linearizable actions of finite groups cited above, are obtained as a restriction of a non-linearizable action of $O(2, \mathbb{C}) = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^4$ given by Schwarz [Sch]. Thus, the present example gives a new proof that the Schwarz action
is non-linearizable. In his article, Schwarz gave the first non-linearizable actions of reductive groups on affine space. In [KS], the case of actions with a one-dimensional quotient was studied.

The importance of this new example is twofold. First of all, it is known that any equivariant vector bundle over a representation of a finite abelian group is trivial (see [MMP2]), and therefore the action on the total space is linearizable. Thus, $S_3$ is the smallest group for which the method of equivariant vector bundles can be used to construct non-linearizable actions. Secondly, the proof is elementary and more transparent than in the other cases. One reason for this is that the action has a line of fixed points. Indeed, this is the first example of a non-linearizable action of any reductive group on $\mathbb{C}^4$ having a line of fixed points.

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Let $S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \tau \sigma = \sigma^2 \tau \rangle$. It has 6 elements, and it is the smallest non-abelian group. Consider the action of $S_3$ on $E := \mathbb{C}^4$ given by

$$\sigma(a, b, x, y) = (\omega a, \omega^{-1} b, x, y)$$

where $\omega = e^{2 \pi i /3}$, and

$$\tau(a, b, x, y) = (b, a, -b^3 x + (1 + ab + (ab)^2)y, (1 - ab)x + a^3 y).$$

One verifies easily that it is an action of $S_3$. Note that the action of the subgroup generated by $\sigma$ is linear. It is also known that the action of the subgroup generated by $\tau$ is conjugate to a linear action (see [MJ] and [MMP2]), but we will show that the action of $S_3$ is not linearizable.

This action comes from Schwarz's first example of a non-linearizable $O(2, \mathbb{C})$-action on $\mathbb{C}^4$ [Sch]. We restrict the action to $S_3$. Note that the fixed point set of Schwarz's original action was just the origin, but when restricted to $S_3$, there is a line of fixed points.

The projection $\pi : E \to V$ given by $(a, b, x, y) \mapsto (a, b)$ is equivariant where $V$ is the irreducible two-dimensional representation of $S_3$. This map
defines the structure of an $S_3$-vector bundle over $V$. Indeed, $\pi$ is equivariant, and $S_3$ acts linearly on the fibers. Thus $S_3$ preserves a natural grading on the coordinate ring $R := \mathbb{C}[E] = \mathbb{C}[a, b, x, y]$ of the total space. The grading is given by the degree in $x$ and $y$. If $f \in R$, we can decompose $f$ as $f = \sum_{i=0}^{d} f_i$ where $f_i$ is homogeneous of degree $i$ in $x$ and $y$. Also, $f$ is $S_3$-invariant if and only if $f_i$ is invariant for each $i$.

**Lemma 1.** Let $f \in R$ be an $S_3$-invariant function. Then there exists $\alpha \in \mathbb{C}$ such that $f_1(\alpha, \alpha, x, y) = 0$ for all $x$ and $y$.

**Proof.** Since $f$ is invariant, $f_1$ is also invariant. Let $f_1 = p(a, b)x + q(a, b)y$. First consider invariance under the action of $\sigma$. Since $x$ and $y$ are invariant under the action by $\sigma$, the polynomials $p(a, b)$ and $q(a, b)$ are in $\mathbb{C}[V]^\sigma = \mathbb{C}[a^3, b^3, ab]$. In particular, $p(a, a)$ and $q(a, a)$ are in $\mathbb{C}[a^2, a^3]$.

Now invariance under the action of $\tau$ means that $f_1 = p(b, a)(-b^3x + (1 + ab + a^2b^2)y) + q(b, a)((1 - ab)x + a^3y)$. In particular, when $a = b$, one finds that $p(a, a)(a^3 + 1) = q(a, a)(1 - a^2)$. Dividing both sides of the equation by $(1 + a)$, this yields

$$p(a, a)(a^2 - a + 1) = q(a, a)(1 - a).$$

(1)

Now we put the two parts together to show that $p(a, a)$ and $q(a, a)$ must have a common zero. The equation (1) implies that $1 - a$ divides $p(a, a)$. However, since $p(a, a) \in \mathbb{C}[a^2, a^3]$, it must have a root $\alpha \neq 1$. Then using again equation (1) we see that $q(\alpha, \alpha) = 0$, and the lemma is proven.

**Theorem 1.** The action of $S_3$ on $E$ is algebraically non-linearizable.

**Proof.** Let $L \subset V$ be the line defined by $a = b$, and consider its inverse image $\pi^{-1}(L) \cong \mathbb{C}^3$. Note that $(a - b)$ is a semi-invariant of $\tau$, and thus $\tau$ acts on $\pi^{-1}(L)$. This action is linearizable. Indeed consider the isomorphism $\varphi : \pi^{-1}(L) \to \mathbb{C}^3$ given by $(a, a, x, y) \mapsto (s, t)$ where

$$s = (1 - a)x + (1 - a + a^2)y \quad \text{and} \quad t = (1 + a)x - (1 + a + a^2)y.$$

(2)

The map $\varphi$ is an isomorphism since $\begin{pmatrix} 1 - a & 1 - a + a^2 \\ 1 + a & -(1 + a + a^2) \end{pmatrix} = -2$. A short calculation shows that $\varphi$ is equivariant with respect to the action of $\tau$ where $\tau(a) = a$, $\tau(s) = s$ and $\tau(t) = -t$. Note that $s$ and $t$ can also be
considered as elements of $R$, but they are only semi-invariants of $\tau$ modulo $(a - b)$.

Now consider the fixed point sets $E^{S_3} \subset E^\tau \subset \pi^{-1}(L) \cong \mathbb{C}^3$. One sees easily that $E^{S_3}$ is a line isomorphic to $\mathbb{C}$ in $E^\tau$, which is isomorphic to $\mathbb{C}^2$. For a subset $Z \subset E$, denote by $I(Z)$ the vanishing ideal in $R$ of polynomials which are identically zero on $Z$. We have that

$$I(E^{S_3}) = (a, b, t) \quad \text{and} \quad I(E^\tau) = (a - b, t).$$

Define $\tilde{R} := \mathbb{C}[E^\tau] = R/(a - b, t) = \mathbb{C}[\bar{a}, \bar{s}]$, where for a polynomial $h \in R$, $\bar{h}$ denotes its class in $\tilde{R}$. The image $\tilde{I} \subset \tilde{R}$ of the ideal $I(E^{S_3})$ is an ideal generated by $\bar{a}$.

Now suppose that the $S_3$-action on $E$ were linearizable. Then it would be isomorphic to the action on the tangent space at the origin. Thus $E$ would be equivariantly isomorphic to $V \oplus \mathbb{C} \oplus \mathbb{C}_{\text{sign}}$, where $\mathbb{C}$ is the one-dimensional trivial representation and $\mathbb{C}_{\text{sign}}$ is the one-dimensional signature representation. This means that $R = \mathbb{C}[a, b, x, y] = \mathbb{C}[u, v, f, g]$ where $Cu \oplus Cv \cong V$ as $S_3$-representations, $f$ is invariant and $g$ is semi-invariant. Therefore, we would have that

$$I(E^{S_3}) = (u, v, g) \quad \text{and} \quad I(E^\tau) = (u - v, g).$$

This means that the ideal $\tilde{I} \subset \tilde{R}$ would be generated by $\bar{u}$. Hence, we have that $\bar{u}$ is a non-zero multiple of $\bar{a}$. Therefore,

$$\tilde{R} = \mathbb{C}[\bar{a}, \bar{s}] = \mathbb{C}[\bar{u}, \bar{f}] = \mathbb{C}[\bar{a}, \bar{f}].$$

Thus $\bar{f} = \gamma \bar{s} + r(\bar{a})$ where $\gamma$ is a non-zero constant and $r$ is a polynomial of one variable. Lifting back to $R$, we have

$$f = \gamma s + r(a) + m \cdot t + n \cdot (a - b)$$

for some $m, n \in R$. Then it follows that the linear part $f_1$ satisfies

$$f_1 = \gamma s + m_0 \cdot t + n_1 \cdot (a - b)$$

because $s$ and $t$ are homogeneous of degree 1, and $a - b$ and $r(a)$ are homogeneous of degree 0.

In particular, $m_0 \in \mathbb{C}[a, b]$, and therefore

$$f_1(a, a, x, y) = \gamma s + q(a) \cdot t$$

for $q = m_0(a, a) \in \mathbb{C}[a]$. It follows from Lemma 1 that there exists $\alpha \in \mathbb{C}$ with $f_1(\alpha, \alpha) = 0$. But this is impossible, since $s(\alpha)$ and $t(\alpha)$ are linearly independent functions in $\mathbb{C}[x, y]$ and $\gamma \neq 0$. 

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Thus the action is not linearizable, and the theorem is proven. □

Remark 1. — The first attempt to prove Theorem 1 was to show that $R$ has no invariant variable. A variable of $R$ is a polynomial $f$ such that $R = C[f, g_1, g_2, g_3]$ for some polynomials $g_i$, and the set $\{f, g_1, g_2, g_3\}$ is called a system of variables of $R$. What we showed was that there is no invariant variable contained in any system of variables corresponding to a linear action; it is still unclear whether the invariant ring of the above action contains any other variable $f$. If so, then the action restricted to the zero set of $f$ is an action of $S_3$ on $C^3$, and this action could be non-linearizable. Thus, the question of whether there is an invariant variable is of interest.

3. Equivariant vector bundles.

It is clear that Theorem 1 implies that the equivariant $S_3$-vector bundle $E \to V$ is non-trivial. But actually, Lemma 1 implies a stronger result:

**Proposition 1.** — Let $X := S_3L = L \cup \sigma L \cup \sigma^2L \subset V$. Then $\pi^{-1}(X) \to X$ is a non-trivial $S_3$-vector bundle.

**Proof.** — Since $X$ is invariant, the restriction of $\pi$ to $\pi^{-1}(X) \to X$ is an equivariant $S_3$-vector bundle. If it were trivial, it would have to be isomorphic to $X \times (C \oplus C_{\text{sign}})$ because the fiber over 0 is the representation $C \oplus C_{\text{sign}}$. Thus it would have a subbundle isomorphic to $X \times C_{\text{sign}}$. This subbundle would be defined by an invariant function of $\pi^{-1}(X)$ which is linear on the fiber. In other words, it would be the restriction of a linear function $f_1 = p(a, b)x + q(a, b)y$ to $C(\pi^{-1}(X))$. But Lemma 1 implies that there is a value $\alpha \in C$ such that $f_1(\alpha, \alpha) = 0$. Thus the entire fiber of $(\alpha, \alpha) \in X$ would be in the zero set of $f_1$. This contradicts the assumption that $f_1$ defines a rank one subbundle. □

In fact, the idea of the proof of Theorem 1 is to show that non-triviality of the bundle over $X$ implies non-linearity of the action on $E$. None of the arguments cited in the introduction which compare the triviality of vector bundles and linearity of the action on the total space can be used precisely as stated for the example given here. However, we use an idea of the proof of a result of Masuda and Petrie (Lemma 5.1 of [MP]).
We show that if $\varphi$ were an equivariant isomorphism of $E$ to $V \times (\mathbb{C} \oplus \mathbb{C}_{\text{sign}})$, then we can assume that $S_3$ fixes $X$ in the zero section of the two bundles. If one knew that $\varphi$ preserved the bundle over $X$, one could then show that the normal derivative of $\varphi$ gives an isomorphism to the trivial $S_3$-vector bundle. However, one cannot be sure that $\varphi$ does in fact preserve the bundles over $X$, and thus the argument in [MP] must be modified.

Using methods developed by Mederer in [Med], one can study the set of all equivariant $S_3$-vector bundles over $V$ and over $X$ whose fiber over the fixed point is isomorphic to $\mathbb{C} \oplus \mathbb{C}_{\text{sign}}$. In fact, one can show that this is the only non-linearizable action of $S_3$ on $\mathbb{C}^4$ coming from a vector bundle over $V$ with the given zero fiber, which is non-trivializable when restricted to $X$.

**Remark 2.** — It has been shown that all analytic equivariant vector bundles over a representation space of a reductive group are trivial [HK]. In particular, this means that the given action is analytically linearizable. For the action of Section 2, this can be seen directly, and the difference between the analytic and algebraic categories becomes clear. To see this, note that Lemma 1 is not true in the analytic category, because one can choose $p(a, a) = (1 - a)e^a$ and $q(a, a) = (1 - a + a^2)e^a$. One can show that this extends to a system of analytic generators. This gives an analytic linearization of the action with $u = a$, $v = b$, and
\[
\begin{pmatrix}
  f \\
  g
\end{pmatrix} = P
\begin{pmatrix}
  x \\
  y
\end{pmatrix},
\]
where $P = (p_{ij})$ for
\[
\begin{align*}
p_{11}(a, b) &= \left(1 - \frac{b^3}{d^2}\right) \cosh(d) - d \left(1 - \frac{b^3}{d^4}\right) \sinh(d) \\
p_{12}(a, b) &= \left(1 - \frac{a^3}{d^2} + d^2\right) \cosh(d) - d \left(1 - \frac{a^3}{d^4} - a^2\right) \sinh(d) \\
p_{21}(a, b) &= \left(1 + \frac{b^3}{d^2}\right) \cosh(d) - d \left(1 + \frac{b^3}{d^4}\right) \sinh(d) \\
p_{22}(a, b) &= -\left(1 + \frac{a^3}{d^2} + d^2\right) \cosh(d) + d \left(1 + \frac{a^3}{d^4} - a^2\right) \sinh(d)
\end{align*}
\]
and $d = \sqrt{ab}$. Note that the coefficients in the matrix $P$ are all analytic on $\mathbb{C}^2$.

It is also known that any equivariant vector bundle over a representation space of a reductive group is stably trivial [BH1]. This means that there exists a representation space $W$ such that the Whitney sum of $\Theta_W$ and the
given bundle is trivial, where $\Theta_W$ denotes the trivial bundle $V \times W \to V$. In our case we have:

**Proposition 2.** — The Whitney sum $E \oplus \Theta_V$ is a trivial $S_3$-vector bundle. In particular, the $S_3$-action on $E \times V$ is trivial.

This result was known by Schwarz [Sch], but we will give an explicit isomorphism.

**Proof.** — The isomorphism $\Phi : E \times V \cong V \times V \times \mathbb{C} \times \mathbb{C}_{\text{sign}}$ is given by

$$
\Phi((a,b,x,y),(u,v)) = ((a,b),(-b^2x + a(1 + ab)y - u + bv, bx - a^2y - v + au),
((1 - ab)x + (a^3 + 1)y + av + bu)/2, 
(- (1 - ab)x + (1 - a^3)y - av + bu)/2),
$$

and its inverse is

$$
\Phi^{-1}((a,b),(u',v'),z,w) = ((a,b,a^2u' + a(1 + ab)v' + (1 - a^3)z - (1 + a^3)w, 
bu' + b^2v' + (1 - ab)z + (1 - ab)w, 
-u' - bv' + a(z + w), -t - as + b(z - w))).
$$

This proves that the $S_3$-vector bundle $E \oplus \Theta_V$ is trivial. □

However, as we shall show in the next section, $E \times \mathbb{C}^n \times \mathbb{C}^m_{\text{sign}}$ is non-linearizable for all non-negative $n$ and $m$.

4. Generalization to affine spaces of higher dimension.

**Theorem 2.** — For any pair of non-negative integers $n$ and $m$, the $S_3$-action on $E \times \mathbb{C}^n \times \mathbb{C}^m_{\text{sign}}$ is non-linearizable.

**Corollary 1.** — For any integer $N \geq 4$ there exist non-linearizable actions of $S_3$ on $\mathbb{C}^N$.

The fact that the statement of Theorem 2 holds for $n = 0$ can actually be proven just as in Section 2, because the fixed point sets are identical. This is enough to prove Corollary 1. However, the theorem implies more than that. Since $E$ is the total space of an equivariant vector bundle, we know that there exists a $G$-module $W$ such that $E \times W$ is linearizable. A direct consequence of Theorem 2 and Proposition 2 is:
COROLLARY 2. — If $W$ is a $G$-module, then $E \times W$ is linearizable if and only if $W$ contains $V$ as a submodule.

The proof of Theorem 2 is similar to the proof of Theorem 1, but the notation is a bit more involved.

Proof of Theorem 2. — Let $z_1, \ldots, z_n$ be the coordinates of $\mathbb{C}^n$ and $w_1, \ldots, w_m$ be the coordinates of $\mathbb{C}^m_{\text{sign}}$. Denote by $E'$ the $S_3$-variety $E' := E \times \mathbb{C}^n \times \mathbb{C}^m_{\text{sign}}$, and by $R' := \mathbb{C}[E'] = \mathbb{C}[a, b, x, y, z_1, \ldots, z_n, w_1, \ldots, w_m]$. The projection $\pi' : E' \to V$ given by $(a, b, x, y, z_1, \ldots, z_n, w_1, \ldots, w_m) \mapsto (a, b)$ defines the structure of an $S_3$-vector bundle. As before, $S_3$ preserves the grading defined by the degree in $x, y, z_1, \ldots, z_n$ and $w_1, \ldots, w_m$. Define $s$ and $t$ as in equation (2), this time considering them as elements of $R'$.

The vanishing ideals of the fixed point sets of $\tau$ and of $S_3$ are given by

$$J := I((E')^\tau) = (a - b, t, w_1, \ldots, w_m), \quad I := I((E')^{S_3}) = J + (a).$$

Denote by $\tilde{R}' := \mathbb{C}[I(E')] = R/J = \mathbb{C}[\tilde{a}, \tilde{s}, \tilde{z}_1, \ldots, \tilde{z}_n]$, and by $\tilde{I}$ the image of $I$ in $\tilde{R}'$.

If the $S_3$-action on $E'$ were linearizable, then the ring $R'$ could be expressed as $R' = \mathbb{C}[u, v, f_0, \ldots, f_n, g_0, \ldots, g_m]$, where the $f_i$'s are invariants, the $g_i$'s are semi-invariants whose weight is the signature character of $S_3$, and $Cu \oplus Cv \cong V$ as $S_3$-representations. Thus we would have

$$J = (u - v, g_0, \ldots, g_m) \quad \text{and} \quad I = J + (u).$$

In particular, $\tilde{R}'$ would be $\mathbb{C}[\tilde{u}, \tilde{f}_0, \ldots, \tilde{f}_n]$, and $\tilde{u}$ would be a non-zero constant multiple of $\tilde{a}$. We would have that

$$\tilde{R}' = \mathbb{C}[\tilde{a}, \tilde{s}, \tilde{z}_1, \ldots, \tilde{z}_m] = \mathbb{C}[\tilde{a}, \tilde{f}_0, \ldots, \tilde{f}_n] = \mathbb{C}[\tilde{a}, \tilde{f}_0, \ldots, \tilde{f}_n].$$

This shows that the map $\psi : \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}$ defined by $\psi(\tilde{a}, \tilde{s}, \tilde{z}_1, \ldots, \tilde{z}_n) = (\tilde{a}, \tilde{f}_0, \ldots, \tilde{f}_n)$ is an automorphism. For $i = 1, \ldots, n$ denote the linear part of $f_i$ by

$$f_{i1} = p_i x + q_i y + \sum_{j=1}^n r_{ij} z_j + \sum_{k=1}^m \tilde{r}_{ik} w_k$$

with $p_i, q_i, r_{ij}$ and $\tilde{r}_{ik}$ polynomials in $\mathbb{C}[a, b]$. Using equation (2), we find that for $i = 1, \ldots, n$, the linear part of $\tilde{f}_i$ is given by

$$\tilde{f}_{i1} = p_i(a, a) \tilde{x} + q_i(a, a) \tilde{y} + \sum_{j=1}^n r_{ij}(a, a) \tilde{z}_j$$

$$= \frac{1}{2} (p_i(a, a)(1 + a + a^2) + q_i(a, a)(1 + a)) \tilde{s} + \sum_{j=1}^n r_{ij}(a, a) \tilde{z}_j.$$
By calculating the Jacobian of \( \psi \) on the line \( \{(a,0,\ldots,0) | a \in \mathbb{C}\} \), one finds that the determinant of the matrix
\[
A(a) := \begin{pmatrix}
\frac{1}{2}(p_0(a,a)(1+a+a^2)+q_0(a,a)(1+a)) & r_{01}(a,a) & \cdots & r_{0n}(a,a) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{2}(p_n(a,a)(1+a+a^2)+q_n(a,a)(1+a)) & r_{n1}(a,a) & \cdots & r_{nn}(a,a)
\end{pmatrix}
\]
is a non-zero constant \( \delta \).

As in the proof of Lemma 1, since \( f_{ij} \) is invariant for all \( i \), we find that \( p_i(a,a), q_i(a,a) \) and \( r_{ij}(a,a) \in \mathbb{C}[a^2,a^3] \), for all \( i, j \) and \( 1 \leq i \) and that
\[
p_i(a,a)(1-a+a^2) = q_i(a,a)(1-a)
\]
for all \( i \).

By multiplying the first column of \( A(a) \) by \( 1-a \) and applying equation (3), we find that \( (1-a)\delta \) is the determinant of
\[
B(a) = \begin{pmatrix}
p_0(a,a) & r_{01}(a,a) & \cdots & r_{0n}(a,a) \\
\vdots & \vdots & \ddots & \vdots \\
p_n(a,a) & r_{n1}(a,a) & \cdots & r_{nn}(a,a)
\end{pmatrix}
\]
But since all of the coefficients of \( B(a) \) are in \( \mathbb{C}[a^2,a^3] \), this is impossible. Thus the action of \( S_3 \) on \( E' \) is non-linearizable. This finishes the proof of the theorem.

**Remark 3.** — Note that we may replace the underlying field \( \mathbb{C} \) with any field \( k \) of characteristic zero containing 3 cube roots of unity, thus obtaining a non-linearizable \( S_3 \)-action on \( A_n^a \) for any \( n \geq 4 \).

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A NONLINEARIZABLE ACTION OF $S_3$ on $\mathbb{C}^d$


