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DEGENERATION OF SCHUBERT VARIETIES OF SL_n/B TO TORIC VARIETIES

by R. DEHY and R. W.T. YU

Introduction.

In this paper, we complete our programme stated in [3] to prove the existence of degenerations of certain Schubert varieties of SL_n into toric varieties, thus generalizing the results of Gonciulea and Lakshmibai [5].

The essential idea is that we use the polytopes defined in [3] to construct a distributive lattice, and extend the method used by Gonciulea and Lakshmibai [5] for minuscule G/P to Schubert varieties in SL_n. Although they also prove the existence of degenerations for SL_n/P (and also Kempf varieties) in the same paper, their approach is different from the one for a minuscule G/P.

Since all the ingredients used here are based on standard monomials, we expect that it can be adapted in the other types. However, the difficult part is to construct a suitable distributive lattice and we shall make it more precise below.

Let $G = SL_{n+1}$, B be a Borel subgroup and W be the Weyl group of G which is the symmetric group of $n+1$ letters. Let α_i, $i = 1, \cdots, n$, be the corresponding set of simple roots so that $\langle \alpha_i, \alpha_j^\vee \rangle = a_{i,j}$ where $(a_{i,j})_{i,j}$ is the Cartan matrix, s_i the corresponding simple reflections in W and let ω_i be the corresponding fundamental weights. Denote also by $\ell(-)$ and $\preceq$ the length function and the Bruhat order on W.

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Recall that for $w \in W$, the Demazure module $E_w(\lambda)$ is the $\mathfrak{b}$-module $U(\mathfrak{b})v_{w\lambda}$, where $\mathfrak{b}$ is the Lie algebra of B, $U(\mathfrak{b})$ its enveloping algebra and $v_{w\lambda}$ a vector of extremal weight $w\lambda$ of the irreducible representation $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^{n} k_i\omega_i$, $k_i \geqslant 0$. Under certain conditions on w, in [3], we constructed n polytopes $\Delta_1, \dots, \Delta_n$, where n is the rank of G, such that the number of lattice points in the Minkowski sum $\sum_{i=1}^{n} k_i\Delta_i = \{\sum_{i=1}^{n}\sum_{j=1}^{k_i} x_{ij} \mid x_{ij} \in \Delta_i\}$ is equal to the dimension of $E_w(\lambda)$. The polytopes $\Delta_1, \dots, \Delta_n$ define a toric variety X equipped with n line bundles $\mathcal{L}_i, i = 1, \dots, n$ (see [11]). The aim of this paper is to degenerate the Schubert variety $S(w) = \overline{BwB}/B$ equipped with line bundles $\mathcal{L}_{\omega_i} = \overline{BwB} \times_B \mathbb{C}_{\omega_i}$ into X equipped with $\mathcal{L}_i$.

We consider the homogeneous coordinate ring of a multicone over S_w. This multicone is the $B\tau B$-orbit of $\bigoplus_{i=1}^{n} \mathbb{C}v_{\omega_i}$ in $\bigoplus_{i=1}^{n} V(\omega_i)$, and its coordinate ring is $R = \bigoplus_{\lambda \text{ dominant}} H^0(S_w, \mathcal{L}_\lambda)$, where $\mathcal{L}_\lambda = \bigotimes_{i=1}^{n} \mathcal{L}_{\omega_i}^{\otimes k_i}$ with $\lambda = \sum_i k_i\omega_i$. In [8], it has been shown that the map

$$\bigoplus_{k_1,\dots,k_n \geqslant 0} \bigotimes_{i=1}^{n} \operatorname{Sym}^{k_i} H^0(S_w, \mathcal{L}_{\omega_i}) \to R$$

is surjective and its kernel I is a multigraded ideal generated by elements of degree $(k_1, \dots, k_n)$ with $\sum_i k_i = 2$. On the other hand, we encounter an analogous situation considering the toric variety X defined by the polytopes $\Delta_1, \dots, \Delta_n$. Let $\mathcal{B}_{k_1,\dots,k_n}$ be the vector space over $\mathbb{C}$ generated by x^α, α a lattice point in $\sum_{i=1}^{n} k_i\Delta_i$. Then $S = \bigoplus_{k_1,\dots,k_n \geqslant 0} \mathcal{B}_{k_1,\dots,k_n}$ is the homogeneous coordinate ring of a multicone over the toric variety X, and $\mathcal{B}_{k_1,\dots,k_n} = H^0(X, \bigotimes_{i=1}^{n} \mathcal{L}_i^{\otimes k_i})$. Moreover, since the polytopes Δ_i can be triangulized by simplices of minimal volume, that is of volume $1/(\dim \Delta_i)!$, the map $\bigoplus_{k_1,\dots,k_n \geqslant 0} \bigotimes_{i=1}^{n} \operatorname{Sym}^{k_i} H^0(X, \mathcal{L}_i) \to S$ is surjective and its kernel J is a multigraded ideal generated by elements of degree $(k_1, \dots, k_n)$ with $\sum_i k_i = 2$; in other words $S = \mathbb{C}[x^{\alpha_{i,j}}]/J$ where $H := \{\alpha_{i,j}\}$ is the set of all lattice points in polytopes Δ_i, $i = 1, \dots, n$.

The basic idea is that one can put a structure of a distributive lattice on the set H, of lattice points of $\Delta_1, \dots, \Delta_n$. This distributive lattice, denoted H equipped with operations $\vee$, $\wedge$, is such that for $\alpha, \beta \in H$ we have $\alpha + \beta = \alpha \vee \beta + \alpha \wedge \beta$. Hence that the algebra $\mathbb{C}[H]/I(H)$, where $I(H)$ is the homogeneous ideal generated by $x_\alpha x_\beta = x_{\alpha\vee\beta}x_{\alpha\wedge\beta}$, is the ring S. Therefore using Theorem 2.5 proved in [5], one obtains a flat deformation of R to $\mathbb{C}[H]/I(H)$ which is the homogeneous coordinate ring of a multicone over the toric variety X.

The paper is organized as follows. In Section 1, we recall results from

[3]. The theorem on degeneration of [5] is stated in Section 2. Sections 3,4 and 5 are devoted to showing that the conditions of the theorem are satisfied. Finally in Section 6, we discuss briefly which Schubert varieties fall into our context.

We shall use the above notations throughout this paper.

1. Distributive lattice on $\mathcal{W}^w$.

For a fundamental weight ω_i, $i = 1, \ldots, n$, let W_{ω_i} be the subgroup of the Weyl group W, stabilizing ω_i, that is $W_{\omega_i} = \{\tau \in W \mid \tau(\omega_i) = \omega_i\}$. Denote the quotient W/W_{ω_i} by W_i. The set W_i can, on the one hand, be identified with the subset of W consisting of elements τ such that $\tau \preceq \tau s_{\alpha_j}$ for $j \neq i$, *i.e.* the set of minimal representatives and, on the other hand, with the set of i-tuples $(r_1, \ldots, r_i)$ such that $0 \leqslant r_1 < \cdots < r_i \leqslant n$. The connection between these two identifications is that $(r_1, \ldots, r_i)$ corresponds to $s(r_1, 1)s(r_2, 2) \cdots s(r_i, i)$ where $s(a, b) = s_a s_{a-1} \cdots s_b$. The induced Bruhat order on W_i, which we shall also denote by $\preceq$ can be expressed under the above identifications by $\underline{a} = (a_1, \ldots, a_i) \preceq \underline{b} = (b_1, \ldots, b_i)$ if and only if $a_k \leqslant b_k$, $1 \leqslant k \leqslant i$. Furthermore, W_i becomes a distributive lattice (for generalities on distributive lattices, see [6] or Section 2 of [5]) under $\preceq$ where

$$\begin{aligned} &\underline{a} \vee \underline{b} = (\max\{a_1, b_1\}, \ldots, \max\{a_i, b_i\}) \quad \text{and} \\ &\underline{a} \wedge \underline{b} = (\min\{a_1, b_1\}, \ldots, \min\{a_i, b_i\}). \end{aligned} \tag{1.1}$$

Recall (see for example [3]) that any $w \in W$ has a unique factorization in the form $s(a_1, b_1)s(a_2, b_2) \cdots s(a_k, b_k)$ with $1 \leqslant a_1 < a_2 < \cdots < a_k \leqslant n$. We shall be interested in the w's satisfying $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k$.

For an element $w \in W$, let $W_i^w = \{\tau \in W_i \mid \tau \preceq \overline{w}\}$, where $\overline{w}$ is the representative of w in W_i. Denote by $\mathcal{W}^w := \coprod_{i=1}^n W_i^w$. Let us recall the following partial order from Section 8 of [3].

DEFINITION 1.1. — *Let $i \leqslant j$ and $w = s(a_1, b_1) \cdots s(a_k, b_k)$, with $1 \leqslant a_1 < \cdots < a_k \leqslant n$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k$. For $\phi = (r_1, \ldots, r_i) \in W_i^w$, we define*

$$\check{\phi} := (0, 1, \ldots, j - i - 1, \check{r}_{j-i+1}, \ldots, \check{r}_j) \in W_j^w$$

where $\check{r}_k = \max\{k-1, r_{k-j+i}\}$, $j-i+1 \leqslant k \leqslant j$ *and for* $\tau = (t_1, \ldots, t_j) \in W_j^w$, *let*

$$\tilde{\tau} := (t_{j-i+1}, \ldots, t_j) \in W_i^w.$$

We say that $\phi \preceq_w \tau$ *if* $\check{\phi} \preceq \tau$, *or equivalently if* $\phi \preceq \tilde{\tau}$, *and we define* $\tau \vee \phi := \tau \vee \check{\phi} \in W_j^w$ *and* $\tau \wedge \phi := \tilde{\tau} \wedge \phi \in W_i^w$ *(see Equation (1.1)).*

A simple consequence of the definition is the following lemma.

LEMMA 1.2. — *Let w be as in Definition 1.1. Then together with the above operations, $\mathcal{W}^w$ is a distributive lattice.*

An essential property of this partial order is the following theorem proved in [3].

THEOREM 1.3. — *We have $\phi \preceq_w \tau$ in $\mathcal{W}^w$ if and only if there exist liftings ϕ', τ' in W of ϕ, τ such that $\phi' \preceq \tau' \preceq w$.*

As we shall see in the next sections, this is used extensively in the proof.

Remark 1.4. — In [3], we constructed for each fundamental weight ω_i, a polytope Δ_i^w such that the number of lattice points in the Minkowski sum $\sum_{i=1}^n k_i \Delta_i^w$ is equal to $\dim E_w(\sum_{i=1}^n k_i\omega_i)$. The set of vertices of the polytope Δ_i is indexed by the set W_i^w and these are the only lattice points of Δ_i. Moreover considering $\phi, \tau \in \mathcal{W}^w$ as vertices, we have $\phi+\tau = \phi\vee\tau+\phi\wedge\tau$. The polytopes Δ_i have also the important property that they can be triangulized by simplices of minimal volume so that a lattice point of $\sum_{i=1}^n k_i\Delta_i$ can be written as the sum of k_1 lattice points of Δ_1 and k_2 lattice points of Δ_2 and so on. This property gives information on the generators of the toric ideal defined by the Δ_i.

We shall end this section by proving certain facts concerning $\tau \vee \phi$ and $\tau \wedge \phi$ which will be needed throughout the paper. These are generalizations of certain results obtained in [5]. Let us suppose that w is as in Definition 1.1.

LEMMA 1.5. — *Let $j \geqslant i$ and $\phi \in W_i^w$, $\tau \in W_j^w$ be two non-comparable elements in $\mathcal{W}^w$. Let $\sigma = \tau \vee \phi$ and $\kappa = \tau \wedge \phi$. Then $\tau(\omega_j) + \phi(\omega_i) = \sigma(\omega_j) + \kappa(\omega_i)$.*

Proof. — This is just a direct consequence of the fact that $\phi + \tau =$

$\phi \vee \tau + \phi \wedge \tau$ in the polytope described in Remark 1.4, see [3].

It is also a straightforward computation by using the fact that if $\tau = (t_1, \ldots, t_j)$, then

$$\tau(\omega_j) = \omega_j - \sum_{k=1}^{j} (\alpha_k + \cdots + \alpha_{t_k}). \tag{1.2}$$

□

LEMMA 1.6. — *Let $j \geqslant i$ and $\phi \in W_i^w$, $\tau \in W_j^w$ with $\sigma = \tau \vee \phi$ and $\kappa = \tau \wedge \phi$. Then, we have the following:*

(1) *if $s_{i_1} \cdots s_{i_k} \tau = \sigma$ and $\ell(\sigma) = \ell(\tau) + k$, then $s_{i_1} \cdots s_{i_k} \breve{\kappa} = \breve{\phi}$ with $\ell(\breve{\kappa}) + k = \ell(\breve{\phi})$; or equivalently $s_{i_1} \cdots s_{i_k} \kappa = \phi$;*

(2) *if $s_{j_1} \cdots s_{j_l} \breve{\phi} = \sigma$ with $\ell(\sigma) = \ell(\breve{\phi}) + l$, then $s_{j_1} \cdots s_{j_l} \breve{\kappa} = \tau$ with $\ell(\tau) = \ell(\breve{\kappa}) + l$;*

(3) *the sets $\{\alpha_{i_p}\}$ and $\{\alpha_{j_q}\}$ have empty intersection and s_{i_p}, s_{j_q} commute.*

Proof. — Note that as a consequence of Definition 1.1, we have $\sigma = \tau \vee \breve{\phi}$ and $\breve{\kappa} = \tau \wedge \breve{\phi}$. Using Lemmas 7.17 and 7.18 of [5], we conclude that there exist $\alpha_{i_1}, \ldots, \alpha_{i_k}$ and $\alpha_{j_1}, \ldots, \alpha_{j_l}$ all simple enjoying the properties stated above. □

2. Theorem on degeneration.

Let us recall some basic facts on standard monomials.

Let $\phi \in W_i$ and $\phi = s_{i_r} \cdots s_{i_1}$ be a reduced expression for ϕ. Then the vector $Q_\phi := X_{-\alpha_{i_r}} \cdots X_{-\alpha_{i_1}} v_{\omega_i}$ is an extremal weight vector in $V(\omega_i)$ of weight $\phi(\omega_i)$. It is shown in [10] that Q_ϕ is independent of the choice of reduced expression of ϕ. Further, we have the following lemmas from [10]:

LEMMA 2.1. — *The set $\{Q_\tau \mid \tau \in W_i, \tau \preceq w\}$ is a $\mathbb{Z}$-basis for $E_{\mathbb{Z},w}(\omega_i)$.*

Let $\{P_\tau \mid \tau \in W_i\}$ be the $\mathbb{Z}$-basis of $V_{\mathbb{Z}}^*(\omega_i)$ dual to $\{Q_\tau \mid \tau \in W_i\}$. Then the set $\{P_\tau \mid \tau \in W_i, \tau \preceq w\}$ is a $\mathbb{Z}$-basis for $H^0(S_{\mathbb{Z}}(w), \mathcal{L}_{\mathbb{Z},\omega_i}) = E_{\mathbb{Z},w}^*(\omega_i)$.

LEMMA 2.2. — *Let $\sigma \succ \kappa \in W_i$ and $\sigma = s_{i_r} \cdots s_{i_1} \kappa$ and $\ell(\sigma) - \ell(\kappa) = r$. Then we have $P_\kappa = (-1)^r X_{-\alpha_{i_1}} \cdots X_{-\alpha_{i_r}} P_\sigma$.*

For a field k, let us denote the canonical image of P_w in $H^0(G/P_i, \mathcal{L}_{\omega_i})$ by $p_w, w \in W_i$.

DEFINITION 2.3 ([10]). — *A monomial $p_{\tau_{r,k_r}} \cdots p_{\tau_{r,1}} p_{\tau_{r-1,k_{r-1}}} \cdots p_{\tau_{1,1}}$, where $\tau_{i,j} \in W_i^w$, is called homogeneous of degree $(k_1, \dots, k_r)$ and of total degree $\sum_{j=1}^r k_j$.*

It is called standard on $S(w)$ if for each i, j, there exists $\tilde{\tau}_{i,j} \in W$, whose class in W_i is $\tau_{i,j}$, and $\tilde{\tau}_{1,1} \preceq \cdots \preceq \tilde{\tau}_{r,k_r} \preceq w$ in W. In other words $p_{\tau_{r,k_r}} \cdots p_{\tau_{1,1}}$ is standard on $S(w)$ if $\tau_{1,1} \preceq_w \cdots \preceq_w \tau_{r,k_r} \preceq_w w$.

THEOREM 2.4 ([10]).

(1) *Let $w \in W$. Then, denoting $\overline{w}$ the representative of w in W_i, for $\tau \in W_i$, $p_\tau |_{S(\overline{w})} \neq 0$ if and only if $\tau \preceq \overline{w}$. Furthermore, $\{p_\tau \mid \tau \in W_i^w\}$ is a k-basis for $H^0(S(\overline{w}), \mathcal{L}_{\omega_i})$.*

(2) *The standard monomials on $S(w)$ of degree $(k_1, \dots, k_n)$ form a basis of $H^0(S(w), \bigotimes_{i=1}^n \mathcal{L}_{\omega_i}^{\otimes k_i})$.*

Let H be a finite distributive lattice. Denote by $P = k[x_\alpha, \alpha \in H]$ and $I(H) \subset P$ the ideal generated by the binomials $\{x_\alpha x_\beta - x_{\alpha\vee\beta} x_{\alpha\wedge\beta} \mid \alpha, \beta \in H \text{ non-comparable}\}$.

Let $R = \bigoplus_{\lambda \text{ dominant}} H^0(S(w), \mathcal{L}_\lambda)$ be the homogeneous coordinate ring of a multicone over S_w. By the previous theorem, R has a basis indexed by standard monomials on $S(w)$. Thus we have the surjective map $\pi : P \to R$ sending $x_\alpha \mapsto p_\alpha$ where H is the set $\mathcal{W}^w$. Let $I = \ker \pi$ which is an ideal generated by relations in total degree 2 of the form

$$p_\tau p_\phi - \sum c_{\theta\psi} p_\theta p_\psi \tag{2.1}$$

where $p_\tau p_\phi$ is non standard and the $p_\theta p_\psi$'s are standard. These are called straightening relations ([1], [6], [10]).

THEOREM 2.5 ([5]). — *Assume that $\mathcal{W}^w$ is a distributive lattice such that the ideal I is generated by the straightening relations of the form*

$$p_\tau p_\phi - \sum c_{\theta\psi} p_\theta p_\psi \tag{2.2}$$

where τ, ϕ are non-comparable and $\theta \succeq \psi$. Further, suppose that we have

(1) *$c_{\tau\vee\phi,\tau\wedge\phi} = 1$, i.e. $p_{\tau\vee\phi} p_{\tau\wedge\phi}$ occurs on the right-hand side of Equation (2.2) with coefficient 1.*

(2) *$\tau, \phi \in]\psi, \theta[= \{\gamma \in \mathcal{W}^w \mid \psi \preceq \gamma \preceq \theta\}$ for every pair (θ, ψ) appearing on the right-hand side of Equation (2.2).*

(3) *There exist integers $n_1, \ldots, n_d \geqslant 1$ and an embedding of distributive lattices*

$$\iota : \mathcal{W}^w \hookrightarrow \bigcup_{d=1}^{n} \mathcal{C}(n_1, \ldots, n_d)$$

where $\mathcal{C}(n_1, \ldots, n_d)$ is the set of d-tuples $(i_1, \ldots, i_d)$ with $1 \leqslant i_j \leqslant n_j$, such that for every pair (θ, ψ) appearing on the right-hand side of Equation (2.2), $\iota(\tau) \mathbin{\dot\cup} \iota(\phi) = \iota(\theta) \mathbin{\dot\cup} \iota(\psi)$ where $\dot\cup$ denotes the disjoint union.

Then there exists a flat deformation whose special fiber is $P/I(\mathcal{W}^w)$ and whose general fiber is R.

By Lemma 1.2, if w is as in Definition 1.1, then $\mathcal{W}^w$ is a distributive lattice. In the next sections, we shall prove that all the conditions of the theorem are satisfied. Let us assume in the next sections that w is as in Definition 1.1.

3. Condition (2) of Theorem 2.5.

THEOREM 3.1 ([9], [10]). — *Let $i \leqslant j$, $\tau \in W_j^w$, $\phi \in W_i^w$ and $p_\tau p_\phi$ be a non standard monomial on $S(w)$. Let the corresponding straightening relation be given by*

$$p_\tau p_\phi = \sum_{l=1}^{N} c_l \; p_{\theta_l} p_{\psi_l}. \tag{3.1}$$

Then $\tau, \phi \prec_w \theta_l$, $\psi_l \prec_w \tau, \phi$ for all l such that $c_l \neq 0$.

Proof. — The proof given here is just a generalization of the proof of Proposition 2.5 of [7]. Among the θ_i choose a minimal one, which we denote by θ. Let us reindex the θ_l so that $\theta = \theta_l$ for $1 \leqslant l \leqslant s$. Note that since θ is minimal we have $\theta_l \not\preceq \theta$ for $s < l \leqslant N$. Since $p_{\theta_l} p_{\psi_l}$ is standard, we can choose $\kappa_1^{(l)}, \kappa_2^{(l)} \in W$ such that $\kappa_2^{(l)} \preceq \kappa_1^{(l)} \preceq w$, the class of $\kappa_1^{(l)}$ in W_j is θ_l and the class of $\kappa_2^{(l)}$ in W_i is ψ_l. Let $Z_1 = \bigcup_{l=1}^{s} S(\kappa_1^{(l)})$ and restrict Equation (3.1) to Z_1. Then $p_{\theta_l} p_{\psi_l} \mid_{Z_1}$ is standard on Z_1 for $1 \leqslant l \leqslant s$ and $p_{\theta_l} p_{\psi_l} \mid_{Z_1} \equiv 0$ for $s < l \leqslant N$. By the linear independence of standard monomials, Equation (3.1) restricted to Z_1 is not zero. Hence $p_\tau p_\phi \mid_{Z_1} \neq 0$. This implies that $\tau, \phi \prec \kappa_1^{(l)}$. According to Theorem 1.3 (or Lemma 8.12 of [3]) we have $\tau, \phi \preceq_w \theta$; note that τ (or ϕ) cannot be equal to θ, because $p_\tau p_\phi$ is non standard. From this argument we deduce that $\tau, \phi \prec_w \theta_l$ for all l.

Let $\sigma = \tau \vee \phi \in W_j^w$ and $\kappa = \tau \wedge \phi \in W_i^w$. Now $\theta_l \in W_j^w$ and $\psi_l \in W_i^w$. By weight consideration, we have $\sigma(\omega_j)+\kappa(\omega_i) = \theta_l(\omega_j)+\psi_l(\omega_i)$. Furthermore $\tau, \phi \prec_w \theta_l$ implies that $\sigma \preceq_w \theta_l$, or equivalently $\sigma \preceq \theta_l$ since both belong to W_j. Therefore $\theta_l(\omega_j) \leqslant \sigma(\omega_j)$, which implies that $\kappa(\omega_i) \leqslant \psi_l(\omega_i)$. Therefore $\psi_l \preceq \kappa$. In other words $\psi_l \preceq_w \kappa \prec_w \tau, \phi$. □

COROLLARY 3.2. — *Let the notations be as in Lemma 1.5. Then in the straightening relation $p_\tau p_\phi = \sum c_{\theta\psi} p_\theta p_\psi$, either $\sigma \prec_w \theta$ or $\theta = \sigma$, $\psi = \kappa$.*

Proof. — From Theorem 3.1, we know that for any pair (θ, ψ) on the right-hand side, $\sigma \preceq_w \theta$ and $\psi \preceq_w \kappa$. Moreover if $\sigma = \theta$, then due to weight considerations, *i.e.* $\theta(\omega_j) + \psi(\omega_i) = \sigma(\omega_j) + \kappa(\omega_i)$, we see that $\kappa = \psi$. □

4. Condition (3) of Theorem 2.5.

Considering the set $\mathcal{W}^w := \coprod_{i=1}^n W_i^w$, we noted at the beginning of Section 1 that each set W_i^w can be identified with the subset of i-tuples $(a_1, \dots, a_i)$ where $0 \leqslant a_1 < \cdots < a_i \leqslant n$ and $(a_1, \dots, a_i)$ is smaller than the representative of w in W_i. Hence we have $\iota : \mathcal{W}^w \hookrightarrow \bigcup_{d=1}^n \mathcal{C}(n_1, \dots, n_d)$. For simplicity, we shall denote $\iota(\tau)$ also by τ. We want to prove the following lemma:

LEMMA 4.1. — *Let τ, ϕ be two non-comparable elements in $\mathcal{W}^w$. Then for any (θ, ψ) appearing on the right-hand side of the straightening relation (2.2), $\theta \,\dot\cup\, \psi = \tau \,\dot\cup\, \phi$.*

Proof. — Let $\tau = (t_1, \dots, t_j)$, $\phi = (r_1, \dots, r_i)$, $\theta = (a_1, \dots, a_j)$ and $\psi = (b_1, \dots, b_i)$. A necessary condition for $p_\theta p_\psi$ to appear on the right-hand side of Equation (2.2) is $\tau(\omega_j) + \phi(\omega_i) = \theta(\omega_j) + \psi(\omega_i)$. Here, we shall prove that this condition immediately implies the assertion, *i.e.* $\{t_1,\ldots,t_j\}\dot\cup\{r_1,\ldots,r_i\} = \{a_1,\ldots,a_j\}\dot\cup\{b_1,\ldots,b_i\}$. The proof is by induction on $i+j$.

The fact that $\tau(\omega_j)+\phi(\omega_i) = \theta(\omega_j)+\psi(\omega_i)$ implies, by using Equation (1.2) in the proof of Lemma 1.5,

$$(4.1)\qquad \sum_{k=1}^{j}(\alpha_k+\cdots+\alpha_{t_k}) + \sum_{l=1}^{i}(\alpha_l+\cdots+\alpha_{r_l}) = \sum_{k=1}^{j}(\alpha_k+\cdots+\alpha_{a_k}) + \sum_{l=1}^{i}(\alpha_l+\cdots+\alpha_{b_l})$$

Note that

$$\max\{t_1,\ldots,t_j,r_1,\ldots,r_i\} = \max\{t_j,r_i\}$$

and that

$$\max\{a_1,\ldots,a_j,b_1,\ldots,b_i\} = \max\{a_j,b_i\}.$$

Then due to the equality in Equation (4.1), we must have $\max\{t_j,r_i\} = \max\{a_j,b_i\}$. There are four cases to consider.

- Case (1) $t_j = a_j \geqslant b_i$. This implies that $\alpha_j+\cdots+\alpha_{t_j} = \alpha_j+\cdots+\alpha_{a_j}$. Hence denoting $\tau' = (t_1,\ldots,t_{j-1})$ and $\theta' = (a_1,\ldots,a_{j-1})$, Equation (4.1) implies that $\tau'(\omega_{j-1})+\phi(\omega_i) = \theta'(\omega_{j-1})+\psi(\omega_i)$. By induction we are done.

- Case (2) $t_j = b_i > a_j$. Let m be the smallest number such that $a_{j-m} > b_{i-m}$ (if such an m less than $i-1$ does not exist, let $m=i$). Note that $b_{i-m+1} \geqslant a_{j-m+1} > a_{j-m} > b_{i-m}$. Set

$$\begin{aligned}
&\tau' = (t_1,\ldots,t_{j-1}) \in W_{j-1},\\
&\theta' = (a_1,\ldots,a_{j-m},b_{i-m+1},b_{i-m+2},\ldots,b_{i-1}) \in W_{j-1},\\
&\psi' = (b_1,\ldots,b_{i-m},a_{j-m+1},a_{j-m+2},\ldots,a_j) \in W_i \ \text{ if } \ m\neq i \ \text{ and}\\
&\psi' = (a_{j-i+1},\ldots,a_j) \ \text{ if } \ m=i.
\end{aligned}$$

Since $i \leqslant j$, we have $i-k-1 \leqslant j-k-1 \leqslant a_{j-k}$ for $0\leqslant k < i$. Therefore $\psi' \in W_i$. Using the fact that for $0 \leqslant k < m$, we have $i-k-1 \leqslant j-k-1 \leqslant a_{j-k} \leqslant b_{i-k}$, then

$$\begin{aligned}
(4.2)\quad &(\alpha_{j-k}+\cdots+\alpha_{a_{j-k}}) + (\alpha_{i-k}+\cdots+\alpha_{b_{i-k}})\\
&= (\alpha_{j-k}+\cdots+\alpha_{a_{j-k}}+\alpha_{a_{j-k}+1}+\cdots+\alpha_{b_{i-k}}) + (\alpha_{i-k}+\cdots+\alpha_{a_{j-k}}).
\end{aligned}$$

From Equations (4.1) and (4.2), we can conclude that $\tau'(\omega_{j-1})+\phi(\omega_i) = \theta'(\omega_{j-1})+\psi'(\omega_i)$. The rest follows by induction.

- Case (3) $r_i = b_i \geqslant a_j$ is similar to case (1).

- Case (4) $r_i = a_j > b_i$ is similar to case (2). □

In fact, we have proved:

Lemma 4.2. — *Let $j \geqslant i$, $\tau, \theta \in W_j$, $\phi, \psi \in W_i$ be such that $\tau(\omega_j) + \phi(\omega_i) = \theta(\omega_j) + \psi(\omega_i)$. Then $\theta \mathbin{\dot{\cup}} \psi = \tau \mathbin{\dot{\cup}} \phi$.*

5. Condition (1) of Theorem 2.5.

Proposition 5.1. — *Let $\tau, \phi \in \mathcal{W}^w$ be two non-comparable elements. Then in the straightening relation (3.1), $p_{\tau\vee\phi}p_{\tau\wedge\phi}$ occurs with coefficient ± 1.*

Proof. — As before, denote $\sigma = \tau \vee \phi$, $\kappa = \tau \wedge \phi$. Note that $\tau, \phi \prec_w \sigma$ (that is there exist liftings $\tilde{\tau}, \tilde{\phi}, \tilde{\sigma}$ in W such that $\tilde{\tau}, \tilde{\phi} \preceq \tilde{\sigma} \preceq w$). Corollary 3.2 implies that the restriction of Equation (3.1) to the Schubert variety $S(\tilde{\sigma})$ is $p_\tau p_\phi = ap_\sigma p_\kappa$, where $a \neq 0$. Since standard monomial basis is characteristic free, this holds in any characteristics. Hence $a = \pm 1$. □

So now we have to prove that $a = 1$. Since the irreducible representation $V(\omega_i + \omega_j)$, appears as a direct sum in the decomposition of $V(\omega_j) \otimes V(\omega_i)$ into irreducible representations, we have an imbedding $V(\omega_i + \omega_j) \hookrightarrow V(\omega_j) \otimes V(\omega_i)$. Note that since the weight space of weight $\omega_i + \omega_j$ is one-dimensional, the element $v_{\omega_j} \otimes v_{\omega_i}$ belongs to $V(\omega_i + \omega_j)$. The imbedding above induces a projection $H^0(G/B, \mathcal{L}_{\omega_i}) \otimes H^0(G/B, \mathcal{L}_{\omega_j}) \to H^0(G/B, \mathcal{L}_{\omega_i} \otimes \mathcal{L}_{\omega_j})$. For simplicity we shall denote the image of $f \otimes g$ under this projection by fg. We shall construct a basis for $E_{\mathbb{Z},w}(\omega_i + \omega_j)$ which is a "rank two" version of the one given in [10].

In the following let $i \leqslant j$ (that is no element of W_i^w can be bigger than an element of W_j^w) and recall from Lemma 2.1 that, for $\phi \in W_i$, we have denoted by Q_ϕ an extremal weight vector in $V_{\mathbb{Z}}(\omega_i)$ of weight $\phi(\omega_i)$.

Let $\Sigma(w) := \{(\tau, \sigma) \in W_j^w \times W_i^w \mid$ there exist liftings $\widetilde{\tau}, \widetilde{\sigma}$ in W such that $\widetilde{\sigma} \preceq \widetilde{\tau} \preceq w\}$.

Definition 5.2. — *Let w be as in Definition 1.1. Let $\kappa \in W_i^w$, $\sigma \in W_j^w$ be such that $(\sigma, \kappa) \in \Sigma(w)$ and let $\sigma = s_{i_r} \cdots s_{i_1} \breve{\kappa}$ where $r = \ell(\sigma) - \ell(\breve{\kappa})$. Define $E_{\breve{\kappa},\kappa} := Q_{\breve{\kappa}} \otimes Q_\kappa \in V_{\mathbb{Z}}(\omega_j) \otimes V_{\mathbb{Z}}(\omega_i)$ and define $E_{\sigma,\kappa} := X_{-\alpha_{i_r}} \cdots X_{-\alpha_{i_1}} E_{\breve{\kappa},\kappa}$.*

Note that $E_{\breve{\kappa},\kappa}$ is an extremal weight vector since $\breve{\kappa}$ is the image of κ (the minimal representative in W) in W_j. It is also clear that $E_{\sigma,\kappa}$ is a weight vector of weight $\kappa(\omega_i) + \sigma(\omega_j)$.

PROPOSITION 5.3. — *Let $w \in W$ be as in Definition 1.1. Then $E_{\sigma,\kappa}$ does not depend on the choice of reduced expression and the set $\{E_{\sigma,\kappa} \mid \kappa \in W_i^w, \sigma \in W_j^w, \kappa \preceq_w \sigma\}$ is a $\mathbb{Z}$-basis for the Demazure module $E_{\mathbb{Z},w}(\omega_i + \omega_j)$.*

Proof. — Let $\sigma = s_{i_r} \cdots s_{i_1}\breve{\kappa} = s_{j_r} \cdots s_{j_1}\breve{\kappa}$. Denote by $\phi = s_{j_{r-1}} \cdots s_{j_1}\breve{\kappa}$. Then we have $\sigma = s_{j_r}\phi$. Now if $i_r = j_r$, then we proceed by induction on the length of σ. Otherwise, let k be the largest integer such that $s_{i_{k-1}} \cdots s_{i_1}\breve{\kappa} \preceq \phi$. Then $s_{i_k} \cdots s_{i_1}\breve{\kappa} \npreceq \phi$ and we have $\phi \vee s_{i_k} \cdots s_{i_1}\breve{\kappa} = \sigma$, $\phi \wedge s_{i_k} \cdots s_{i_1}\breve{\kappa} = s_{i_{k-1}} \cdots s_{i_1}\breve{\kappa}$. By Lemma 1.6, we have that $j_r = i_k$ and s_{j_r} commute with s_{i_l} for $l \geqslant k$. Thus

$$X_{-\alpha_{i_r}} \cdots X_{-\alpha_{i_1}} E_{\breve{\kappa},\kappa} = X_{-\alpha_{i_k}} X_{-\alpha_{i_r}} \cdots X_{-\alpha_{i_{k+1}}} X_{-\alpha_{i_{k-1}}} \cdots X_{-\alpha_{i_1}} E_{\breve{\kappa},\kappa}.$$

By induction, $E_{\phi,\kappa} = X_{-\alpha_{j_{r-1}}} \cdots X_{-\alpha_{j_1}} E_{\breve{\kappa},\kappa}$. Therefore the right-hand side is $X_{-\alpha_{j_r}} E_{\phi,\kappa}$ and we have proved that the definition of $E_{\sigma,\kappa}$ does not depend on the choice of the reduced expression.

We are left to show that these elements form a basis for $E_{\mathbb{Z},w}(\omega_i + \omega_j)$.

We claim that $E_{\sigma,\kappa} \in E_{\mathbb{Z},w}(\omega_i + \omega_j)$. It is clear that $E_{\breve{\kappa},\kappa} \in E_{\mathbb{Z},\kappa}(\omega_i + \omega_j)$. Now, since w satisfies the condition of Definition 1.1, we have $w \succeq s_{i_r} \cdots s_{i_1}\kappa$ (if κ is $[a_1, 1]...[a_i, i]$, then $s_{i_r}...s_{i_1}\kappa$ is $[b_1, 1]...[b_{j-i}, j-i][b_{j-i+1}, 1]...[b_j, i]$ where $\sigma = (b_1, \cdots, b_j)$), thus

$$E_{\sigma,\kappa} \in X_{-\alpha_{i_r}} \cdots X_{-\alpha_{i_1}} E_{\mathbb{Z},\kappa}(\omega_i + \omega_j) \subset E_{\mathbb{Z},w}(\omega_i + \omega_j).$$

We have therefore our claim.

Now by the definition of $E_{\sigma,\kappa}$, we have

$$E_{\sigma,\kappa} = Q_\sigma \otimes Q_\kappa + \sum_{(u,v)\in I} Q_u \otimes Q_v$$

where $I \subset W_j \times W_i$ and for each $(u, v) \in I$, we have $u \prec \sigma$ in W_j, $v \succ \kappa$ in W_i and $\sigma(\omega_j) + \kappa(\omega_i) = u(\omega_j) + v(\omega_i)$. It is now clear that the $E_{\sigma,\kappa}$'s are independent.

Further, one deduces from the expression for $E_{\sigma,\kappa}$ above that the $\mathbb{Z}$-submodule generated by the $E_{\sigma,\kappa}$'s is a direct summand of the tensor product $V_{\mathbb{Z}}(\omega_j) \otimes V_{\mathbb{Z}}(\omega_i)$. Finally, by standard monomial theory, the cardinal of $\Sigma(w)$ is the rank of $E_{\mathbb{Z},w}(\omega_i + \omega_j)$. So the result follows. □

We can now prove that $a = 1$.

COROLLARY 5.4. — *Let the notations be as in Lemma 1.5, then in the straightening relation* $p_\tau p_\phi = \sum_{l=1}^{N} c_l \, p_{\theta_l} p_{\psi_l}$, *the term* $p_\sigma p_\kappa$ *occurs on the right hand side with coefficient* 1.

Proof. — Recall from the proof of Proposition 5.3 that

$$E_{\sigma,\kappa} = Q_\sigma \otimes Q_\kappa + \sum_{(u,v)\in I} Q_u \otimes Q_v$$

where $I \subset W_j \times W_i$ and for each $(u, v) \in I$, we have $u \prec \sigma$ in W_j, $v \succ \kappa$ in W_i and $\sigma(\omega_j) + \kappa(\omega_i) = u(\omega_j) + v(\omega_i)$.

Let us apply $p_\tau p_\phi$ to $E_{\sigma,\kappa}$. Then from the explicite expression of $E_{\sigma,\kappa}$ above, this is either 0 or 1 depending if $Q_\tau \otimes Q_\phi$ appears in the right hand side or not.

On the other hand, if we replace $p_\tau p_\phi$ by the right hand side of the straightening relation, then it is clear from Theorem 3.1 that the same evaluation yields $a_{\sigma,\kappa}$ where $a_{\sigma,\kappa}$ is the coefficient of $p_\sigma p_\kappa$ in the straightening relation. But this is non zero from Proposition 5.1. So it must be 1. □

6. Consequence.

As an immediate consequence, we have:

THEOREM 6.1. — *Let* w *be as in Definition 1.1. Then there exists a flat deformation whose special fiber is a toric variety and whose general fiber is* $S(w)$.

Proof. — By Theorem 2.4, there exists a flat deformation whose general fiber is $S(w)$ and whose special fiber is a variety defined by a binomial ideal associated to a distributive lattice. This latter is toric as shown in [4]. □

Remark 6.2. — If we look closely at the proofs, then we realize that Theorem 2.4 can be replaced by the following.

Suppose that $\mathcal{W}^w$ admits a structure of distributive lattice such that

(1) the partial order corresponds to standardness, *cf.* Theorem 1.3;

(2) weights are preserved, *cf.* Lemma 1.5;

(3) Lemma 1.6 is satisfied.

Then there exists a flat deformation whose special fiber is a toric variety and whose general fiber is $S(w)$.

In particular, consider the bijection Θ of W defined by $s_i \mapsto s_{n+1-i}$ induced by the non trivial Dynkin diagram automorphism. This induces a bijection between W_i and W_{n+1-i} which preverses the induced Bruhat order. Now let w be as in Definition 1.1, then Θ induces a structure of distributive lattice on $\mathcal{W}^{\Theta(w)}$. It is easy to check that the same proof works. Thus we have,

THEOREM 6.3. — *Let w or $\Theta(w)$ be as in Definition 1.1. Then there exists a flat deformation whose special fiber is a toric variety and whose general fiber is $S(w)$.*

Remark 6.4. — As noticed in [3], we can extend our results to the following elements. Let $0 \leqslant k_1 < k_2 < \cdots < k_{r+1} \leqslant n+1$, and for $1 \leqslant i \leqslant r$, let S_i be the subgroup of W generated by the reflections $s_{k_i+1}, \cdots s_{k_{i+1}-1}$.

Now suppose that $w = w_1 \cdots w_r$ where $w_i \in S_i$. Then it is clear that w_i and w_j commute if $i \neq j$ and it follows easily that if each w_i satisfies the condition of Theorem 6.3, *i.e.* either w_i or $\Theta(w_i)$ is as in Definition 1.1, then the conclusion of the same theorem holds for w.

For example, the element $s_1s_2s_5s_4$ satisfies the above conditions.

Our results apply to all the elements of W in the case of SL_3 thus giving a more general proof to [2]. However, in the case of SL_4, there are precisely 4 elements for which the condition of the theorem is not satisfied. Namely, they are $s_1s_3s_2$, $s_2s_1s_3$, $s_2s_1s_3s_2$ and $s_1s_2s_3s_2s_1$. The main problem in these cases is that standardness is not transitive in all the obvious "orderings".

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