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STRUCTURE OF THREE INTERVAL EXCHANGE TRANSFORMATIONS I: AN ARITHMETIC STUDY

by S. FERENCZI, C. HOLTON & L.Q. ZAMBONI

1. Introduction.

A fundamental problem in arithmetic concerns the extent to which an irrational number θ is approximated (in a suitable sense) by rational numbers p/q. Such questions are intimately related to the underlying algebraic nature of the parameter θ. The problem of minimizing the quantity $|q\theta - p|$ leads naturally to the *regular continued fraction expansion* of θ:

$$\theta = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \cfrac{1}{\ddots}}}}}$$

The expansion is obtained by iterating the *Gauss map* $S\colon (0,1) \to [0,1)$ given by

$$S(x) = \left\{\frac{1}{x}\right\}$$

where $0 \leq \{x\} < 1$ denotes the fractional part of x. The transformation S may be iterated indefinitely ($S^k(\theta) \neq 0$ for all $k \geq 1$), if and only if

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$\theta \in (0,1)$ is irrational. This leads to the continued fraction expression for θ given above. The positive integers (n_k) are called the *partial quotients.* By truncating the expansion of θ at the kth level one obtains a rational number p_k/q_k (called the k-th *convergent* of θ) which gives the *best rational approximation* of θ in the following sense:

$$\|q_k\theta\| = |q_k\theta - p_k| \quad \text{and} \quad \|q'\theta\| > \|q_k\theta\| \text{ for all } 0 < q' < q_k$$

where $\|\theta\|$ denotes the difference (taken positively) between θ and the nearest integer. This best approximation property of the convergents, coupled with the entirely arbitrary nature of the sequence of partial quotients is what distinguishes the classical continued fraction algorithm from all other known continued fraction type algorithms. The following theorem, due to Lagrange [27], constitutes a fundamental result in the theory of regular continued fractions:

THEOREM 1.1 (Lagrange, 1769). — *The partial quotients in the continued fraction expansion of an irrational number θ are ultimately periodic if and only if θ is algebraic of degree* 2.

The problem of simultaneously approximating an n-tuple of real numbers $(\theta_1, \theta_2, \ldots, \theta_n)$ by rational numbers $(p_1/q, p_2/q, \ldots, p_n/q)$ (with the same denominator) has been and continues to be an important area of investigation with a wide range of applications to different areas of mathematics. The question dates back to Hermite in [21] where he suggests finding a generalization of the continued fraction algorithm which reflects the algebraic nature of the parameter(s). As a response to this problem Jacobi developed a special case of what is now called the Jacobi-Perron algorithm (see [6], [32], [37]). Since then, a number of other multidimensional division algorithms have been studied including [2], [5], [8], [10], [18], [19], [24], [25], [26], [28], [29], [33], [43], [44] to name just a few. (See [9], [38], [39] for nice surveys on multidimensional continued fractions). It is known that for each n-tuple of irrationals $(\theta_1, \theta_2, \ldots, \theta_n)$ the system of inequalities

$$\left|\frac{p_i}{q} - \theta_i\right| < \frac{1}{q^{1+\frac{1}{n}}} \quad \text{for } i = 1, 2, \ldots, n$$

has infinitely many solutions [HaWr]. Moreover the exponent $1 + \frac{1}{n}$ is optimal in the sense that for any $\mu > 1 + \frac{1}{n}$ there exist $(\theta_1, \theta_2, \ldots, \theta_n)$ for which the system of inequalities $|p_i/q - \theta_i| < 1/q^{\mu}$ has only finitely many solutions. However, unlike the 1-dimensional case where the convergents of the regular continued fraction yield the best rational approximations in the sense mentioned above, in dimension greater than one most of the usual continued fraction algorithms stop short of producing optimal simultaneous approximations of $(\theta_1, \theta_2, \ldots, \theta_n)$. Moreover it is not known whether many of the higher dimensional division algorithms cited above (including Jacobi-Perron) satisfy a full Lagrange type theorem: The algorithm is ultimately periodic if and only if all the parameters lie in the same algebraic extension of $\mathbb{Q}$. Generally one can prove the "only if" part while the "if" part is only conjectural. There are a few exceptions: One is a geometric construction due to Korkina in [25] based on ideas originally due to Klein [24] and later modified by Arnold [2]. Another is a 2-dimensional multiplicative algorithm due to Hara-Mimachi and Ito in [19]; still another is a recent n-dimensional algorithm due to Garrity in [18], where the author considers purely periodic expansions of period length one. A fourth example concerns a recent paper of Boshernitzan and Carroll [8] in which they show that a certain family of vectorial division algorithms, when applied to quadratic vector spaces, yields sequences of remainders which are ultimately periodic. However, in this case the converse is false: periodicity does not imply that the parameters are quadratic. The division algorithm of Boshernitzan-Carroll is based on a renormalization process associated with interval exchange transformations.

In this paper we describe a new 2-dimensional division algorithm we call the *negative slope algorithm* which stems from the dynamics of a three interval exchange transformation on $[0,1]$. We consider a generalization of the Gauss map given by the following transformation $T\colon (0,1)\times(0,1) \to [0,1)\times[0,1)$ defined by

$$T(x,y) = \begin{cases} \left(\left\{\dfrac{y}{(x+y)-1}\right\}, \left\{\dfrac{x}{(x+y)-1}\right\}\right) & \text{if } (x+y) > 1, \\ \left(\left\{\dfrac{1-y}{1-(x+y)}\right\}, \left\{\dfrac{1-x}{1-(x+y)}\right\}\right) & \text{if } (x+y) < 1, \end{cases}$$

where again $0 \le \{x\} < 1$ denotes the fractional part of x. The map T may be iterated indefinitely (the algorithm does not stop) on a pair $z = (\alpha, \beta)$ if and only if z lies off a special set of rational lines with negative slopes (see Proposition 2.3). Thus if α and β are not both rational, then T may be iterated indefinitely on either (α,β) or $(\alpha, 1-\beta)$ or both. Iterating T on a point (α,β) leads to an expansion of the form (ϵ_k, n_k, m_k) where $\epsilon_k = \pm 1$ and n_k, m_k are positive integers. The quantity ϵ_k records which of the two defining rules for T is used at stage k of the iteration, while n_k and m_k record the integer parts in each coordinate. The sequence (ϵ_k, n_k, m_k) is analogous to the sequence of partial quotients in the regular continued fraction algorithm.

Geometrically this iteration produces a sequence of nested quadrilaterals Q_k in the plane converging to the point (α, β). At stage k of the iteration, the quadrilateral Q_k is partitioned according to a family of quadrilaterals whose main diagonal has slope -1, and whose boundary is made up of lines of negative rational slope. This partition into quadrilaterals is constructed via a two-dimensional Farey series derived from the coordinates of the vertices of Q_k. This geometric interpretation is analogous to the partitioning of the unit interval according to m-Farey fractions.

In this paper we study various diophantine properties of this algorithm including its approximation qualities. It is convenient to make a change of coordinates $(\alpha, \beta) \mapsto (\alpha + \beta, \beta - \alpha)$. As in the regular continued fraction algorithm, truncating the iteration of T at stage k gives rise to a pair of rational numbers $(p_k/q_k, r_k/q_k)$ with the same denominator, and with p_k/q_k approximating $\alpha + \beta$ and r_k/q_k approximating $\beta - \alpha$. Geometrically $(p_k/q_k, r_k/q_k)$ corresponds to the barycenter of the Farey quadrilateral Q_k. We show that

$$\left|(\alpha + \beta) - \frac{p_k}{q_k}\right| < \frac{1}{q_k^2}$$

for infinitely many k. In this privileged direction, we obtain a so-called *semi-regular continued fraction* giving an approximation which is about as good as the regular continued fraction. But it is shown that the approximation of $\alpha + \beta$ depends on the values of both α and β; and hence the negative slope algorithm is not a skew product of the regular continued fraction, or for that matter other higher-dimensional continued fractions found in the literature. In contrast the approximation of $\beta - \alpha$ is only linear in $1/q$.

As a consequence of the approximation qualities mentioned above, we show that the transformation T satisfies a full Lagrange type theorem: The iteration $T^k(\alpha, \beta)$ (or equivalently the expansion (ϵ_k, m_k, n_k)) is ultimately periodic if and only if α and β belong to the same quadratic extension of $\mathbb{Q}$.

The negative slope algorithm is intimately connected with the dynamics of 3-interval exchange transformations; in fact, it may be independently reformulated in terms of the dynamical and symbolic properties of 3-interval exchange transformations. Though these connections are not necessary to the understanding of the present paper, the last section is devoted to a short presentation of these ideas. In the sequels [16], [17] we use the symbolic/combinatorial/arithmetic interaction to solve several long standing problems on the symbolic, spectral and ergodic properties of 3-interval exchanges. In particular, in [17] the diophantine properties developed in the present paper are used to characterize possible eigenvalues of a 3-interval exchange and to obtain necessary and sufficient conditions for weak mixing, solving two questions posed by Veech in [42]. In a forthcoming fourth paper we apply our methods to study the joinings of 3-interval exchanges.

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2. The negative slope algorithm.

Let I denote the interval $[0,1[$. We assume that $(\alpha, \beta) \in I \times I$. We define by recursion a sequence $(\alpha_k, \beta_k, \delta_k)_{1 \le k < K}$ with K finite or $K = +\infty$, with

$$0 < \alpha_k, \beta_k < 1 \quad \text{and} \quad 0 < |\delta_k| < \min\{\alpha_k, \beta_k\}.$$

For $k = 1$ set

$$\delta_1 = 1 - (\alpha + \beta).$$

If either α, β, or δ_1 is equal to 0, then the algorithm stops. In this case we take $K = 1$. Otherwise

- if $\delta_1 < 0$, put $\alpha_1 = \alpha$ and $\beta_1 = \beta$,
- while if $\delta_1 > 0$, put $\alpha_1 = 1 - \alpha$ and $\beta_1 = 1 - \beta$.

It is readily verified that α_1, β_1 and δ_1, if they are defined, satisfy the above inequalities.

Having defined $(\alpha_k, \beta_k, \delta_k)$ with the required properties, we write

$$\begin{cases} \alpha_k = n_k|\delta_k| + \alpha'_k & \text{with } 0 \le \alpha'_k < |\delta_k|, \\ \beta_k = m_k|\delta_k| + \beta'_k & \text{with } 0 \le \beta'_k < |\delta_k|, \end{cases} \tag{2.1}$$

where n_k, m_k are each positive integers. Set

$$\delta_{k+1} = |\delta_k| - (\alpha'_k + \beta'_k). \tag{2.2}$$

If either α'_k, β'_k, or δ_{k+1} is equal to 0, the algorithm stops and we set $K = k+1$. Otherwise

- if $\delta_{k+1} < 0$, put

$$\alpha_{k+1} = \beta'_k \text{ and } \beta_{k+1} = \alpha'_k \tag{2.2}$$

- while if $\delta_{k+1} > 0$, put

$$\alpha_{k+1} = |\delta_k| - \beta'_k \text{ and } \beta_{k+1} = |\delta_k| - \alpha'_k. \tag{2.3}$$

It is readily verified that $\alpha_{k+1}, \beta_{k+1}$, and $|\delta_{k+1}|$ satisfy the required inequalities.

The above arithmetic algorithm determines a sequence

$$E^-(\alpha, \beta) = (\epsilon_k, n_k, m_k)_{1 \leq k < K} \tag{2.5}$$

we call the *negative slope expansion*[1] *of* (α, β) where $\epsilon_k = \operatorname{sgn}(\delta_k)$. We put $\epsilon_K = 0$. It also defines the sequence $(\alpha'_k/|\delta_k|, \beta'_k/|\delta_k|)_{0 \leq k < K}$ with values in $I \times I$ where for $k = 0$ we take $\delta_0 = 1$, $\alpha'_0 = \beta$ and $\beta'_0 = \alpha$. As we shall see in what follows, the sequence $(\alpha'_k/|\delta_k|, \beta'_k/|\delta_k|)_{0 \leq k < K}$ plays an important role while $(\alpha_k, \beta_k, \delta_k)_{1 \leq k < K}$ is mainly an auxiliary sequence.

A geometric interpretation of this algorithm is given in the last section of the paper in connection with interval exchange transformations. The referee suggested the following alternative geometric description: at stage k we consider the interval $[0, |\delta_k|]$ and the two subintervals $[0, \beta'_k]$ and $[|\delta_k| - \alpha'_k, |\delta_k|]$, of respective lengths β'_k and α'_k. Except in the cases in which $\alpha'_k = 0$ or $\beta'_k = 0$ or $\alpha'_k + \beta'_k = |\delta_k|$ (in which case the process stops), this naturally defines a partition of $[0, |\delta_k|]$ into three subintervals. The middle interval is of length $|\delta_{k+1}|$. If the first and third intervals do not intersect, *i.e.*, $\delta_{k+1} > 0$, then we replace their lengths by the lengths of the complementary intervals, otherwise we keep the lengths α'_k and β'_k. This defines the auxiliary quantities $(\alpha_{k+1}, \beta_{k+1})$. Finally we subtract the length of the middle interval as many times as possible from the lengths

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[1] A geometric motivation for the name given to this algorithm is described in the following section.

of the first and last intervals (or their complementaries if $\delta_{k+1} > 0$); by construction we can always make the subtraction at least once. What is left over from the first interval has length α'_{k+1}, and what is left over from the third interval has length β'_{k+1}. After renormalization, we recover the formula for the map T given in the introduction.

For each $0 \leq k < K$ we define the quantities

$$S_k = \alpha'_k + \beta'_k, \qquad D_k = \alpha'_k - \beta'_k,$$
$$s_k = \frac{\alpha'_k}{|\delta_k|} + \frac{\beta'_k}{|\delta_k|}, \qquad d_k = \frac{\alpha'_k}{|\delta_k|} - \frac{\beta'_k}{|\delta_k|}.$$

The following inequalities are easily verified:

$$0 < S_k < 2|\delta_k| \quad \text{and hence} \quad 0 < s_k < 2, \tag{2.6}$$

$$-|\delta_k| < D_k < |\delta_k| \quad \text{and hence} \quad -1 < d_k < 1, \tag{2.7}$$

$$\big|S_k - |\delta_k|\big| + |D_k| < |\delta_k| \quad \text{and hence} \quad |s_k - 1| + |d_k| < 1. \tag{2.8}$$

LEMMA 2.1. — *For each $1 \leq k < K$ we have*

$$T^k(\beta, \alpha) = \Big(\frac{\alpha'_k}{|\delta_k|}, \frac{\beta'_k}{|\delta_k|}\Big)$$

where $T : I \times I \to I \times I$ is defined by

$$T(x,y) = \begin{cases} \Big(\Big\{\dfrac{y}{(x+y)-1}\Big\}, \Big\{\dfrac{x}{(x+y)-1}\Big\}\Big) & \text{if } (x+y) > 1, \\ \Big(\Big\{\dfrac{1-y}{1-(x+y)}\Big\}, \Big\{\dfrac{1-x}{1-(x+y)}\Big\}\Big) & \text{if } (x+y) < 1, \end{cases}$$

where $\{a\}$ denotes the fractional part of a.

Proof. — We will show that for each $0 \leq k < K-1$ we have

$$T\Big(\frac{\alpha'_k}{|\delta_k|}, \frac{\beta'_k}{|\delta_k|}\Big) = \Big(\frac{\alpha'_{k+1}}{|\delta_{k+1}|}, \frac{\beta'_{k+1}}{|\delta_{k+1}|}\Big).$$

By equation (2.2) we have

$$\frac{\delta_{k+1}}{|\delta_k|} = 1 - \Big(\frac{\alpha'_k}{|\delta_k|} + \frac{\beta'_k}{|\delta_k|}\Big).$$

If $\delta_{k+1} < 0$ then (2.1) and (2.3) give

$$\alpha_{k+1} = \beta'_k = n_{k+1}|\delta_{k+1}| + \alpha'_{k+1} \quad \text{with} \quad 0 < \alpha'_{k+1} < |\delta_{k+1}|,$$
$$\beta_{k+1} = \alpha'_k = m_{k+1}|\delta_{k+1}| + \beta'_{k+1} \quad \text{with} \quad 0 < \beta'_{k+1} < |\delta_{k+1}|,$$

whence

$$\frac{\beta'_k}{|\delta_{k+1}|} = n_{k+1} + \frac{\alpha'_{k+1}}{|\delta_{k+1}|}, \qquad \frac{\alpha'_k}{|\delta_{k+1}|} = m_{k+1} + \frac{\beta'_{k+1}}{|\delta_{k+1}|}.$$

Thus

$$\frac{\alpha'_{k+1}}{|\delta_{k+1}|} = \left\{\frac{\beta'_k}{|\delta_{k+1}|}\right\} = \left\{\frac{\beta'_k/|\delta_k|}{|\delta_{k+1}|/|\delta_k|}\right\} = \left\{\frac{\beta'_k/|\delta_k|}{(\alpha'_k/|\delta_k| + \beta'_k/|\delta_k|) - 1}\right\}$$

and similarly

$$\frac{\beta'_{k+1}}{|\delta_{k+1}|} = \left\{\frac{\alpha'_k/|\delta_k|}{(\alpha'_k/|\delta_k| + \beta'_k/|\delta_k|) - 1}\right\}$$

as required.

On the other hand if $\delta_{k+1} > 0$ then (2.1) and (2.4) yield

$$\alpha_{k+1} = |\delta_k| - \beta'_k = n_{k+1}|\delta_{k+1}| + \alpha'_{k+1} \quad \text{with} \quad 0 < \alpha'_{k+1} < |\delta_{k+1}|,$$
$$\beta_{k+1} = |\delta_k| - \alpha'_k = m_{k+1}|\delta_{k+1}| + \beta'_{k+1} \quad \text{with} \quad 0 < \beta'_{k+1} < |\delta_{k+1}|,$$

whence

$$\frac{\alpha'_{k+1}}{|\delta_{k+1}|} = \left\{\frac{|\delta_k| - \beta'_k}{|\delta_{k+1}|}\right\} = \left\{\frac{1 - \beta'_k/|\delta_k|}{|\delta_{k+1}|/|\delta_k|}\right\} = \left\{\frac{1 - \beta'_k/|\delta_k|}{1 - (\alpha'_k/|\delta_k| + \beta'_k/|\delta_k|)}\right\}$$

and similarly

$$\frac{\beta'_{k+1}}{|\delta_{k+1}|} = \left\{\frac{1 - \alpha'_k/|\delta_k|}{1 - (\alpha'_k/|\delta_k| + \beta'_k/|\delta_k|)}\right\}. \qquad \square$$

The lemma shows that the negative slope expansion $E^-(\alpha, \beta)$ in (2.5) may be obtained directly from the sequence $(T^k(\beta, \alpha))_{1 \le k < K} = (\alpha'_k/|\delta_k|, \beta'_k/|\delta_k|)_{k=0}^{\infty}$. The quantity ϵ_k codes which of the two defining rules for T is used at stage k of the iteration, while n_k and m_k record the integer parts in each coordinate.

LEMMA 2.2. — *For each* $0 \le k < K$ *set*

$$A_k = \begin{pmatrix} 1 & 0 & m_k + n_k - \epsilon_k - 1 \\ 0 & \epsilon_k & (n_k - m_k)\epsilon_k \\ 1 & 0 & m_k + n_k - 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} S_{k-1} \\ D_{k-1} \\ |\delta_{k-1}| \end{pmatrix} = A_k \begin{pmatrix} S_k \\ D_k \\ |\delta_k| \end{pmatrix}. \tag{2.9}$$

Proof. — We express the quantities $\alpha'_{k-1}, \beta'_{k-1}$ and $|\delta_{k-1}|$ in terms of α'_k, β'_k and $|\delta_k|$.

- If $\epsilon_k = 1$ then (2.1), (2.2) and (2.4) imply

$$\alpha_k = |\delta_{k-1}| - \beta'_{k-1} = n_k|\delta_k| + \alpha'_k,$$
$$\beta_k = |\delta_{k-1}| - \alpha'_{k-1} = m_k|\delta_k| + \beta'_k,$$
$$\delta_k = |\delta_{k-1}| - (\alpha'_{k-1} + \beta'_{k-1}),$$

whence

$$\alpha'_{k-1} = (n_k - 1)|\delta_k| + \alpha'_k, \quad \beta'_{k-1} = (m_k - 1)|\delta_k| + \beta'_k, \tag{2.10}$$

$$|\delta_{k-1}| = |\delta_k| + (\alpha'_{k-1} + \beta'_{k-1}) = (n_k + m_k - 1)|\delta_k| + (\alpha'_k + \beta'_k). \tag{2.11}$$

- If $\epsilon_k = -1$ then (2.1) , (2.2) and (2.3) imply

$$\alpha_k = \beta'_{k-1} = n_k|\delta_k| + \alpha'_k, \quad \beta_k = \alpha'_{k-1} = m_k|\delta_k| + \beta'_k$$

and

$$|\delta_{k-1}| = \epsilon_k|\delta_k| + (\alpha'_{k-1} + \beta'_{k-1}) = (n_k + m_k - 1)|\delta_k| + (\alpha'_k + \beta'_k).$$

The lemma now follows immediately from the above expressions. □

We next look at the "improper" case when the algorithm stops (this situation is called "Stőrungen" by Perron and others).

PROPOSITION 2.3. — *The algorithm stops* ($K < +\infty$) *if and only if* (α, β) *lies in the intersection of* $I \times I$ *and one of the following rational lines:*

$$q\alpha + q\beta = p, \quad q\alpha + (q+1)\beta = p, \quad (q+1)\alpha + q\beta = p$$

for any integers $0 \le p \le 2q$.

Proof. — The algorithm stops if and only if $\delta_K = 0$ or $\alpha'_{K-1} = 0$ or $\beta'_{K-1} = 0$, hence if and only if the triple $(S_{K-1}, D_{K-1}, |\delta_{K-1}|)$ satisfies one of the relations

- $S_{K-1} = |\delta_{K-1}|$,
- $S_{K-1} = D_{K-1}$,
- $S_{K-1} = -D_{K-1}$.

Suppose $S_{K-1} = eD_{K-1}$ with $e = \pm 1$; then, by applying the matrix formula in Lemma 2.2, we get that for each $0 \le k \le K-1$ there exist integers B_k and C_k such that

$$B_k S_k + e_k D_k = C_k |\delta_k| \tag{2.2}$$

where $e_k = \pm 1$. We note that since α'_k and β'_k are nonnegative, B_k and C_k can each be assumed to be nonnegative. By writing each of S_k, D_k, and $|\delta_k|$ in terms of $S_{k-1}, D_{k-1}, |\delta_{k-1}|$ (using Lemma 2.2), we find that

$$B_{k-1} = \big|\epsilon_k B_k(m_k + n_k - 1) + \epsilon_k e_k(n_k - m_k) + \epsilon_k C_k\big|,$$
$$C_{k-1} = |B_{k-1} - B_k|$$

and $e_{k-1} = \pm e_k$.

Looking at the possible parities of m_k and n_k, we see by induction on k that B_k is an odd positive integer while C_k is a nonnegative even integer. Moreover since

$$C_k|\delta_k| \le B_k S_k + |D_k| = (B_k - 1)S_k + S_k + |D_k|$$

we obtain

$$\begin{aligned}(C_k - 1)|\delta_k| &\le (B_k - 1)S_k + S_k - |\delta_k| + |D_k| \\ &\le (B_k - 1)S_k + \big|S_k - |\delta_k|\big| + |D_k| \\ &< (B_k - 1)S_k + |\delta_k| < 2(B_k - 1)|\delta_k| + |\delta_k|\end{aligned}$$

where the last two inequalities follow from (2.8) and (2.6) respectively. Hence $C_k - 1 < 2(B_k - 1) + 1$ and thus

$$C_k < 2B_k \implies C_k \le 2B_k - 1 \implies C_k \le 2B_k - 2. \tag{2.13}$$

Equation (2.12) when $k = 0$ gives a linear relation of the form

$$q\alpha + (q+1)\beta = p \quad \text{or} \quad (q+1)\alpha + q\beta = p \quad \text{with} \quad p \le 2q.$$

In fact, $q = \frac{1}{2}(B_0 - 1)$ and $p = \frac{1}{2}C_0$.

Conversely, if α and β satisfy such a relation, we can write it in the form $B_0S_0 + e_0D_0 = C_0|\delta_0|$, where B_0 is an odd positive integer, C_0 is a nonnegative even integer, and $e_0 = \pm 1$. Suppose for $k \geq 1$ we have

$$B_{k-1}S_{k-1} + e_{k-1}D_{k-1} = C_{k-1}|\delta_{k-1}|,$$

where B_{k-1} is an odd positive integer, C_{k-1} is a nonnegative even integer, and $e_{k-1} = \pm 1$. If $C_{k-1} = 0$, then we must have $B_{k-1} = 1$ as $|D_{k-1}| \leq |S_{k-1}|$, and hence the algorithm stops. On the other hand if $C_{k-1} > 0$, then we apply the matrix formula in Lemma 2.2 as above to get

$$B_kS_k + e_kD_k = C_k|\delta_k|,$$

with the same parity properties, and with $B_k = |B_{k-1} - C_{k-1}|$. Since $C_{k-1} < 2B_{k-1}$ by (2.13), we obtain that $B_k < B_{k-1}$. Hence by this process we obtain a K for which either $C_{K-1} = 0$ or $B_{K-1} = 1$. If $C_{K-1} = 0$ we saw that the algorithm stops. While if $B_{K-1} = 1$ either $\alpha'_{K-1} = C_{K-1}|\delta_{K-1}|$ or $\beta'_{K-1} = C_{K-1}|\delta_{K-1}|$, and in either case this implies $C_{K-1} = 0$ since C_{K-1} is even and α'_{K-1} and β'_{K-1} are each less than $|\delta_K|$. Thus in all cases the algorithm stops.

A similar reasoning takes care of the case $S_{K-1} = |\delta_{K-1}|$, which corresponds to the rational values of $\alpha + \beta$ and relations

$$B_kS_k = C_k|\delta_k|. \qquad \square$$

LEMMA 2.4. — *If the algorithm does not stop* ($K = +\infty$), *then* $(n_k, \epsilon_k) \neq (1, +1)$ *for infinitely many* k *and* $(m_k, \epsilon_k) \neq (1, +1)$ *for infinitely many* k.

Proof. — Suppose the algorithm does not stop and $(n_k, \epsilon_k) = (1, +1)$ for $k > k_0$; then it follows from (2.10) that $\alpha'_{k+1} = \alpha'_k = C$ for some constant $C > 0$ and for all $k > k_0$; but (2.11) implies that $|\delta_{k-1}| \geq |\delta_k| + \alpha'_k + \beta'_k \geq C$. Since $|\delta_k| \to 0$, it follows that $C = 0$, a contradiction. $\square$

In terms of 3-interval exchange transformations, Proposition 2.3 gives necessary and sufficient conditions for the corresponding 3-interval exchange to have periodic orbits, while Lemma 2.4 implies that in the absence of periodic orbits the exchange is minimal (each point has a dense orbit).

3. Farey quadrilaterals.

We begin by recalling a well-known fact about regular continued fractions. For an arbitrary sequence of positive integers $(n_k)_{k=1}^{\infty}$, the set I_k of numbers in $[0,1)$ whose first k partial quotients are $n_1, n_2, \ldots, n_k$ is a half-open interval having endpoints $a_k/b_k < a'_k/b'_k$ with $b_k a'_k - a_k b'_k = 1$. Indeed, the I_k are given by $a_0 = 0, b_0 = a'_0 = b'_0 = 1$, and inductively,

$$I_{k+1} = \begin{cases} \left[\dfrac{a_k + (n_k - 1)a'_k}{b_k + (n_k - 1)b'_k}, \dfrac{a_k + n_k a'_k}{b_k + n_k b'_k}\right) & \text{if } k \text{ is odd,} \\ \left(\dfrac{n_k a_k + a'_k}{n_k b_k + b'_k}, \dfrac{(n_k - 1)a_k + a'_k}{(n_k - 1)b_k + b'_k}\right] & \text{if } k \text{ is even.} \end{cases} \tag{3.1}$$

The endpoints of each interval are successive Farey approximations of the numbers in that interval.

For a sequence $(\epsilon_k, n_k, m_k)_{k \geq 1}$ with $n_k, m_k \geq 1$ and $\epsilon_k \in \{\pm 1\}$ we shall formulate a similar description for the set Q_k of points $(\alpha, \beta) \in I \times I$ having $(\epsilon_1, n_1, m_1), (\epsilon_2, n_2, m_2), \ldots, (\epsilon_k, n_k, m_k)$ as their first k "negative slope coefficients". It is useful to regard $\alpha_k, \beta_k, \delta_k, \alpha'_k$ and β'_k as functions of the variables (α, β), defined inductively by setting $\alpha'_0 = \beta$, $\beta'_0 = \alpha$, $\delta_0 = 1$ and $\epsilon_0 = 1$, and, for all $k \geq 0$,

$$\begin{cases} \delta_{k+1} = \epsilon_k \delta_k - \alpha'_k - \beta'_k, \\ \alpha_{k+1} = \begin{cases} \beta'_k & \text{if } \epsilon_{k+1} = -1, \\ \epsilon_k \delta_k - \beta'_k & \text{if } \epsilon_{k+1} = +1, \end{cases} \\ \beta_{k+1} = \begin{cases} \alpha'_k & \text{if } \epsilon_{k+1} = -1, \\ \epsilon_k \delta_k - \alpha'_k & \text{if } \epsilon_{k+1} = +1, \end{cases} \\ \alpha'_{k+1} = \alpha_{k+1} - n_{k+1}\epsilon_{k+1}\delta_{k+1} \\ \beta'_{k+1} = \beta_{k+1} - m_{k+1}\epsilon_{k+1}\delta_{k+1}. \end{cases} \tag{3.2}$$

We record a few simple facts, the proofs of which are elementary.

LEMMA 3.1. — *If $E^-(\alpha, \beta)$ begins with $(\epsilon_k, n_k, m_k)_{1 \leq k \leq N}$ then the values at (α, β) of the functions $\alpha'_k, \beta'_k, \alpha_{k+1}, \beta_{k+1}, \delta_{k+1}$ for $0 \leq k \leq N$ are exactly those of their constant namesakes in the negative slope algorithm of Section 2.*

LEMMA 3.2. — *The functions* $\alpha'_k, \beta'_k, \delta_{k+1}$ *are of the form*

$$\alpha'_k = \varepsilon_k\Big(p_k^{(\alpha)}\alpha + (p_k^{(\alpha)}+1)\beta - q_k^{(\alpha)}\Big),$$

$$\beta'_k = \varepsilon_k\Big((p_k^{(\beta)}+1)\alpha + p_k^{(\beta)}\beta - q_k^{(\beta)}\Big),$$

$$\delta_{k+1} = \varepsilon_k\Big(-p_{k+1}^{(\delta)}\alpha - p_{k+1}^{(\delta)}\beta + q_{k+1}^{(\delta)}\Big),$$

where $\varepsilon_k = \epsilon_0\epsilon_1\epsilon_2\cdots\epsilon_k$ *and* $p_k^{(\alpha)}, q_k^{(\alpha)}, p_k^{(\beta)}, q_k^{(\beta)}, p_{k+1}^{(\delta)}, q_{k+1}^{(\delta)}$ *are nonnegative integers such that* $p_{k+1}^{(\delta)} > \max(p_k^{(\alpha)}, p_k^{(\beta)})$ *and* $q_{k+1}^{(\delta)} > \max(q_k^{(\alpha)}, q_k^{(\beta)})$.

LEMMA 3.3. — *The lines* $\alpha'_k = 0$ *and* $\alpha'_k = \epsilon_k\delta_k$ *have negative slope* > -1 *while the lines* $\beta'_k = 0$ *and* $\beta'_k = \epsilon_k\delta_k$ *have slope* < -1 *and the line* $\delta_{k+1} = 0$ *has slope* -1. *The intersection of the lines* $\alpha'_k = 0$ *and* $\beta'_k = \epsilon_k\delta_k$ *lies on the line* $\delta_{k+1} = 0$, *as does the intersection of the lines* $\beta'_k = 0$ *and* $\alpha'_k = \epsilon_k\delta_k$.

COROLLARY 3.4. — *The four lines* $\alpha'_k = 0$, $\alpha'_k = \epsilon_k\delta_k$, $\beta'_k = 0$ *and* $\beta'_k = \epsilon_k\delta_k$ *bound a quadrilateral.*

3.1. Geometric formulation.

We claim that Q_k is the semi-open[(2)] quadrilateral of Corollary 3.4, excluding the lines $\alpha'_k = \epsilon_k\delta_k$ and $\beta'_k = \epsilon_k\delta_k$, and that the positions of the bounding lines are as shown in Figure 1. This holds trivially in the case $k = 0$. Assume the assertion holds for some k.

It follows from Lemma 3.1 that Q_{k+1} is the subset of Q_k on which

$$\begin{cases} n_{k+1}\epsilon_{k+1}\delta_{k+1} \le \alpha_{k+1} < (n_{k+1}+1)\epsilon_{k+1}\delta_{k+1}, \\ m_{k+1}\epsilon_{k+1}\delta_{k+1} \le \beta_{k+1} < (m_{k+1}+1)\epsilon_{k+1}\delta_{k+1}. \end{cases} \tag{3.3}$$

We first consider the case $\epsilon_{k+1} = +1$. Equations (3.2) give $\alpha_{k+1} = \delta_{k+1} + \alpha'_k$ and $\beta_{k+1} = \delta_{k+1} + \beta'_k$, whence inequalities (3.3) may be rewritten as

$$(n_{k+1}-1)\delta_{k+1} \le \alpha'_k < n_{k+1}\delta_{k+1},$$

$$(m_{k+1}-1)\delta_{k+1} \le \beta'_k < m_{k+1}\delta_{k+1}.$$

(2) If n_k or m_k is 1 then we must also exclude the lines $\alpha'_k = 0$ and $\beta'_k = 0$ from Q_k, in which case Q_k is actually an open quadrilateral. This is (partly) due to our convention of stopping the negative slope algorithm when $\alpha'_k = 0$ or $\beta'_k = 0$.

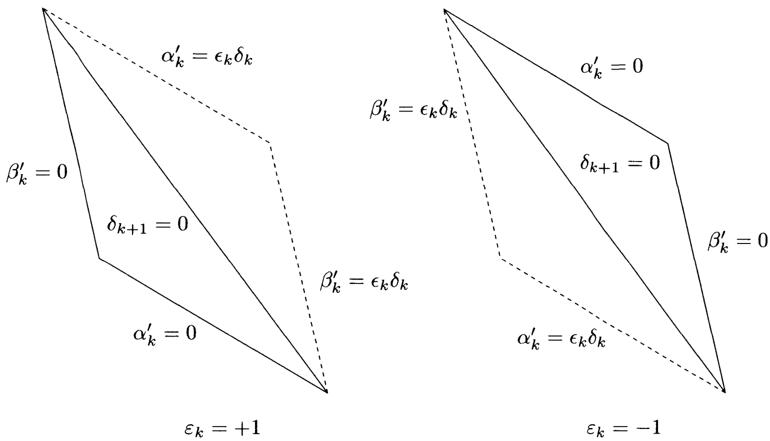

Figure 1. *The quadrilateral Q_k. The two configurations correspond to the possible values of $\varepsilon_k = \epsilon_1\epsilon_2\cdots\epsilon_k$.*

Each of the lines $\alpha'_k = n\delta_{k+1}$, $n \geq 0$, contains the intersection of the lines $\delta_{k+1} = 0$ and $\alpha'_k = 0$, and their slopes strictly decrease to -1 as $n \to \infty$. Similarly, each of the lines $\beta'_k = m\delta_{k+1}$ contains the intersection of the lines $\delta_{k+1} = 0$ and $\beta'_k = 0$, and their slopes (strictly) increase to -1 as $m \to \infty$. Figure 2 shows how the four lines bound Q_{k+1} in the desired manner.

Now let us assume that $\epsilon_{k+1} = -1$. Using (3.2) we may rewrite the inequalities (3.3) as

$$n_{k+1}\delta_{k+1} + \epsilon_k\delta_k < \alpha'_k \leq (n_{k+1} - 1)\delta_{k+1} + \epsilon_k\delta_k,$$

$$m_{k+1}\delta_{k+1} + \epsilon_k\delta_k < \beta'_k \leq (m_{k+1} - 1)\delta_{k+1} + \epsilon_k\delta_k.$$

Each of the lines $\alpha'_k = n\delta_{k+1} + \epsilon_k\delta_k$, $n \geq 0$ contains the intersection of the lines $\delta_{k+1} = 0$ and $\alpha'_k = \epsilon_k\delta_k$, and their slopes (strictly) decrease to -1 as $n \to \infty$. (To see this last part, use (3.2) to write the equation of the line as $\beta'_k + (n+1)\delta_{k+1} = 0$.) Similarly the slopes of the lines $\beta'_k = m\delta_{k+1} + \epsilon_k\delta_k$ (strictly) increase to -1 and each line contains the intersection of the lines $\delta_{k+1} = 0$ and $\beta'_k = \epsilon_k\delta_k$. Figure 3 shows that these lines bound Q_{k+1} as claimed. This completes the induction.

Remark 3.5. — The line $\delta_{k+1} = 0$ divides the quadrilateral Q_k into two triangular regions, corresponding to the possible values of ϵ_{k+1}: If $\epsilon_{k+1} = -1$ then Q_{k+1} is contained in the triangle bounded by $\delta_{k+1} = 0$, $\alpha'_k = \epsilon_k\delta_k$ and $\beta'_k = \epsilon_k\delta_k$, while if $\epsilon_{k+1} = +1$ then Q_{k+1} is contained in the triangle bounded by $\delta_{k+1} = 0$, $\alpha'_k = 0$ and $\beta'_k = 0$.

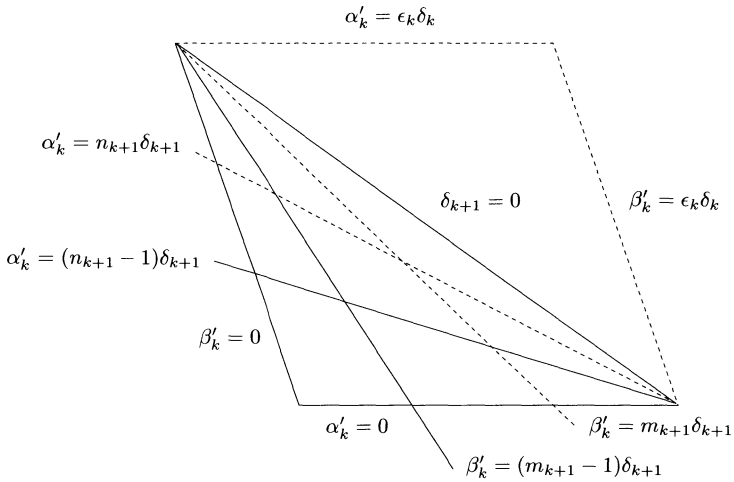

Figure 2. The position of Q_{k+1} in Q_k when $\epsilon_{k+1} = +1$ and $\varepsilon_k = +1$. The horizontal aspect is exaggerated to exhibit greater detail. The lines $\alpha'_k = (n_{k+1}-1)\delta_{k+1}, \alpha'_k = n_{k+1}\delta_{k+1}, \beta'_k = (m_{k+1}-1)\delta_{k+1}, \beta'_k = m_{k+1}\delta_{k+1}$ are the lines $\alpha'_{k+1} = 0$, $\alpha'_{k+1} = \epsilon_{k+1}\delta_{k+1}$, $\beta'_{k+1} = 0$, $\beta'_{k+1} = \epsilon_{k+1}\delta_{k+1}$, respectively, which bound the quadrilateral Q_{k+1}.

3.2. Arithmetic formulation.

We shall give a formula similar in flavor to (3.1) for the vertices of Q_k. First we need a specialized result.

LEMMA 3.6. — *Let $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ be nonnegative (or nonpositive) integers such that a_i and b_i are not both zero, $i = 1, 2, 3$. Suppose that the lines*

$$\ell_i := \{(x, y) : a_i x + b_i y - c_i = 0\}$$

are mutually nonparallel, that the slope of ℓ_3 lies between the slopes of ℓ_1 and ℓ_2, and that each intersection $\ell_i \cap \ell_j$, $i \neq j$, lies in the closure of the first quadrant. Let $n, m \geq 0$ and set

$$a_4 = a_1 + na_3, \quad b_4 = b_1 + nb_3, \quad c_4 = c_1 + nc_3,$$
$$a_5 = a_2 + ma_3, \quad b_5 = b_2 + mb_3, \quad c_5 = c_2 + mc_3,$$
$$\ell_4 = \{(x, y) : a_4 x + b_4 y - c_4 = 0\}, \quad \ell_5 = \{(x, y) : a_5 x + b_5 y - c_5 = 0\}.$$

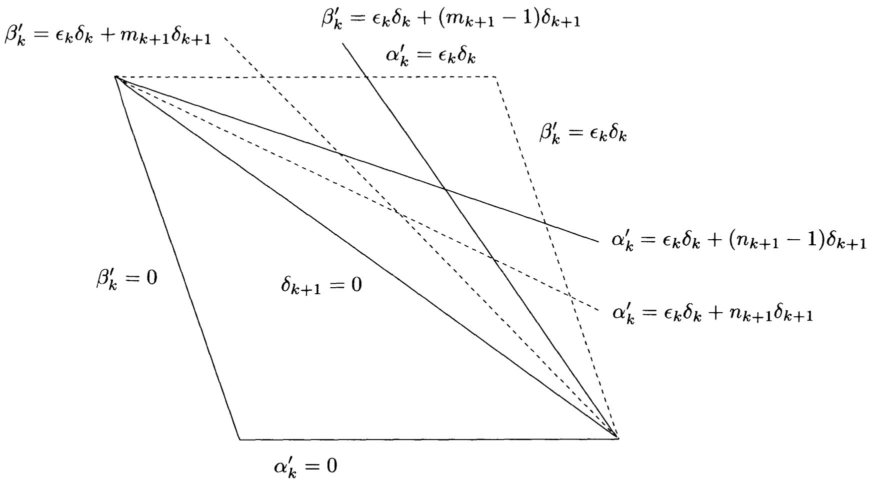

Figure 3. Location of Q_{k+1} in Q_k in the case $\epsilon_{k+1} = -1$ and $\varepsilon_k = +1$. The horizontal direction is exaggerated; the slopes of all the lines are typically quite close to -1. The lines $\alpha'_k = \epsilon_k\delta_k + (n_{k+1}-1)\delta_{k+1}$, $\alpha'_k = \epsilon_k\delta_k + n_{k+1}\delta_{k+1}$, $\beta'_k = \epsilon_k\delta_k + (m_{k+1}-1)\delta_{k+1}$, $\beta'_k = \epsilon_k\delta_k + m_{k+1}\delta_{k+1}$ are the lines $\alpha'_{k+1} = 0$, $\alpha'_{k+1} = \epsilon_{k+1}\delta_{k+1}$, $\beta'_{k+1} = 0$, $\beta'_{k+1} = \epsilon_{k+1}\delta_{k+1}$, respectively, which bound the quadrilateral Q_{k+1}.

Then

$$\ell_i \cap \ell_j = \left\{\left(\frac{|b_i c_j - b_j c_i|}{|a_j b_i - a_i b_j|}, \frac{|a_i c_j - a_j c_i|}{|a_j b_i - a_i b_j|}\right)\right\} \tag{3.4}$$

and

$$\begin{aligned}
|b_4c_5 - b_5c_4| &= |b_1c_2 - b_2c_1| + n|b_2c_3 - b_3c_2| + m|b_1c_3 - b_3c_1|,\\
|a_4c_5 - a_5c_4| &= |a_1c_2 - a_2c_1| + n|a_2c_3 - a_3c_2| + m|a_1c_3 - a_3c_1|,\\
|a_4b_5 - b_5a_4| &= |a_2b_1 - a_1b_2| + n|a_3b_2 - a_2b_3| + m|a_1b_3 - b_3a_1|,
\end{aligned}$$

whence

$$\begin{aligned}
\ell_4 \cap \ell_5 = \Bigg\{\Bigg(&\frac{|b_1c_2 - b_2c_1| + n|b_2c_3 - b_3c_2| + m|b_1c_3 - b_3c_1|}{|a_2b_1 - a_1b_2| + n|a_3b_2 - a_2b_3| + m|a_1b_3 - b_3a_1|},\\
&\qquad\frac{|a_1c_2 - a_2c_1| + n|a_2c_3 - a_3c_2| + m|a_1c_3 - a_3c_1|}{|a_2b_1 - a_1b_2| + n|a_3b_2 - a_2b_3| + m|a_1b_3 - b_3a_1|}\Bigg)\Bigg\}.
\end{aligned}$$

Note that formula (3.4) for the intersection $\ell_4 \cap \ell_5$ gives the same numerators and denominator as the last line of the lemma.

3.2.1. *Equations for bounding lines.* — We must be careful how we write our lines if we wish to apply Lemma 3.6. We write the bounding lines of Q_k as $\alpha'_k = 0$, $\alpha'_k - \epsilon_k\delta_k = 0$, $\beta'_k = 0$ and $\beta'_k - \epsilon_k\delta_k = 0$, while the diagonal is written $-\delta_{k+1} = 0$. The coefficients of α, β and 1 in each expression are given by Lemma 3.2. For example, the line $\alpha'_k - \epsilon_k\delta_k = 0$ becomes

$$\varepsilon_k\left[(p_{k+1}^{(\delta)} - p_k^{(\beta)} - 1)\alpha + (p_{k+1}^{(\delta)} - p_k^{(\beta)})\beta - (q_{k+1}^{(\delta)}) - q_k^{(\beta)})\right] = 0,$$

and the coefficient of α is $\varepsilon_k(p_{k+1}^{(\delta)} - p_k^{(\beta)} - 1)$.

3.2.2. *Labeling vertices.* — The vertices of Q_k are the intersections of one of the lines $\alpha'_k = 0$ and $\alpha'_k - \epsilon_k\delta_k = 0$ with one of the lines $\beta'_k = 0$ and $\beta'_k - \epsilon_k\delta_k = 0$. Let P_k^1 be the point of intersection of the lines $\alpha'_k = 0$ and $\beta'_k - \epsilon_k\delta_k = 0$, and label the remaining vertices P_k^2, P_k^3 and P_k^4 in a clockwise manner. Let

$$P_k^i = \left(\frac{p_k^i}{q_k^i}, \frac{r_k^i}{q_k^i}\right) \tag{3.5}$$

be the vertices of Q_k as obtained from (3.4) using the coefficients of 3.2.1 and without reducing the fractions. We see from Figure 1 that if $\varepsilon_k = +1$ then P_k^1 and P_k^3 are the lower right and upper left vertices of Q_k, respectively, and if $\varepsilon_k = -1$ then these roles are reversed.

Remark 3.7. — Two of the vertices could be obtained using formula (3.4) in other ways, as the intersection of $-\delta_{k+1} = 0$ with any one of the four lines bounding Q_k is either P_k^1 or P_k^3. However, it follows from the first equation of (3.2) that formula (3.4) gives the same expression for the vertices in each case.

3.2.3. *Application of Lemma 3.6.* — We are finally ready to show how the lemma may be used to find the vertices of Q_{k+1} from those of Q_k. The details depend on ϵ_{k+1}.

- $\epsilon_{k+1} = +1$. Let ℓ_1, ℓ_2 and ℓ_3 be the lines $\alpha'_k = 0$, $\beta'_k = 0$ and $-\delta_{k+1} = 0$, respectively. These (with the coefficients of 3.2.1) satisfy the hypotheses of Lemma 3.6. The vertices of Q_{k+1} are the intersections of the bounding lines $\alpha'_k - n\delta_{k+1} = 0$ and $\beta'_k - m\delta_{k+1} = 0$, $n \in \{n_{k+1} - 1, n_{k+1}\}$, $m \in \{m_{k+1} - 1, m_{k+1}\}$, given by Lemma 3.6 as

$$\left(\frac{p_k^2 + np_k^3 + mp_k^1}{q_k^2 + nq_k^3 + mq_k^1}, \frac{r_k^2 + nr_k^3 + mr_k^1}{q_k^2 + nq_k^3 + mq_k^1}\right).$$

See Figure 4.

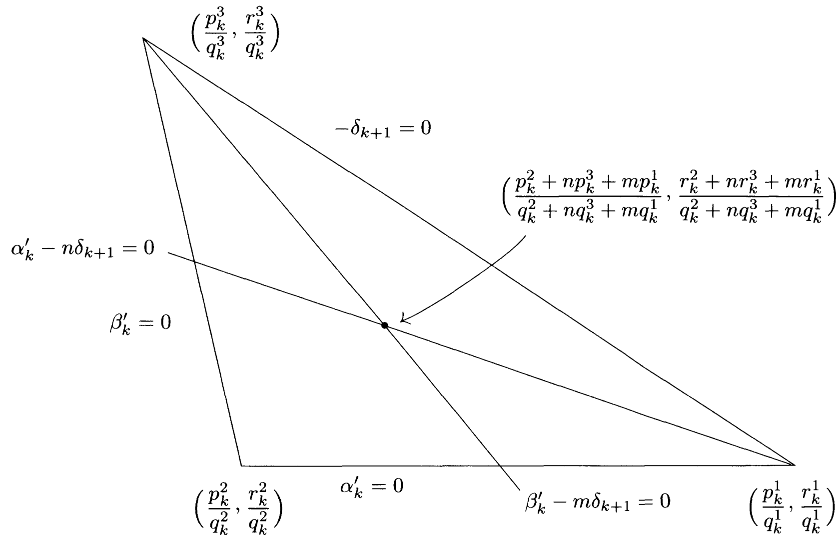

Figure 4. Formula for a vertex of Q_{k+1} in the case $\varepsilon_k = +1$, $\epsilon_{k+1} = +1$. The other cases are similar.

- $\epsilon_{k+1} = -1$. Let ℓ_1, ℓ_2 and ℓ_3 be the lines $\alpha'_k - \epsilon_k\delta_k = 0$, $\beta'_k - \epsilon_k\delta_k = 0$ and $-\delta_{k+1} = 0$, respectively. As before, these satisfy the hypotheses of Lemma 3.6. The vertices of Q_{k+1} are the intersections of the lines $\alpha'_k - \epsilon_k\delta_k - n\delta_{k+1} = 0$ and $\beta'_k - \epsilon_k\delta_k - m\delta_{k+1} = 0$, $n \in \{n_{k+1} - 1, n_{k+1}\}$, $m \in \{m_{k+1} - 1, m_{k+1}\}$, given by Lemma 3.6 as

$$\left(\frac{p_k^4 + np_k^1 + mp_k^3}{q_k^4 + nq_k^1 + mq_k^3}, \frac{r_k^4 + nr_k^1 + mr_k^3}{q_k^4 + nq_k^1 + mq_k^3}\right).$$

Note that the expressions for the bounding lines are the same as those of 3.2.1 in the first case, while in the second case they differ from 3.2.1 by sign. In either case, the expressions obtained for the vertices of Q_{k+1} are identical to those given by 3.2.2. This is important because it means we can iterate the process.

3.3. Algorithm.

Assembling our observations thus far leads to the following recursive formula for the vertices of the Q_k. Put

$$\begin{aligned} &p_0^1 = 1, \quad q_0^1 = 1, \quad r_0^1 = 0, \quad p_0^2 = 0, \quad q_0^2 = 1, \quad r_0^1 = 0, \\ &p_0^3 = 0, \quad q_0^3 = 1, \quad r_0^3 = 1, \quad p_0^4 = 1, \quad q_0^4 = 1, \quad r_0^4 = 1. \end{aligned}$$

These are the numerators and denominators from 3.2.2 in the case $k=0$. We see from Subsection 3.2.3 that

- if $\epsilon_{k+1} = +1$ then

$$\begin{aligned}
(p^1_{k+1}, q^1_{k+1}, r^1_{k+1}) &= (p^2_k, q^2_k, r^2_k) + (n_{k+1}-1)(p^3_k, q^3_k, r^3_k) \\
&\qquad + m_{k+1}(p^1_k, q^1_k, r^1_k),\\
(p^2_{k+1}, q^2_{k+1}, r^2_{k+1}) &= (p^2_k, q^2_k, r^2_k) + (n_{k+1}-1)(p^3_k, q^3_k, r^3_k) \\
&\qquad + (m_{k+1}-1)(p^1_k, q^1_k, r^1_k),\\
(p^3_{k+1}, q^3_{k+1}, r^3_{k+1}) &= (p^2_k, q^2_k, r^2_k) + n_{k+1}(p^3_k, q^3_k, r^3_k) \\
&\qquad + (m_{k+1}-1)(p^1_k, q^1_k, r^1_k),\\
(p^4_{k+1}, q^4_{k+1}, r^4_{k+1}) &= (p^2_k, q^2_k, r^2_k) + n_{k+1}(p^3_k, q^3_k, r^3_k) \\
&\qquad + m_{k+1}(p^1_k, q^1_k, r^1_k),
\end{aligned}$$

- while if $\epsilon_{k+1} = -1$ then

$$\begin{aligned}
(p^1_{k+1}, q^1_{k+1}, r^1_{k+1}) &= (p^4_k, q^4_k, r^4_k) + (n_{k+1}-1)(p^1_k, q^1_k, r^1_k) \\
&\qquad + m_{k+1}(p^3_k, q^3_k, r^3_k),\\
(p^2_{k+1}, q^2_{k+1}, r^2_{k+1}) &= (p^4_k, q^4_k, r^4_k) + (n_{k+1}-1)(p^1_k, q^1_k, r^1_k) \\
&\qquad + (m_{k+1}-1)(p^3_k, q^3_k, r^3_k),\\
(p^3_{k+1}, q^3_{k+1}, r^3_{k+1}) &= (p^4_k, q^4_k, r^4_k) + n_{k+1}(p^1_k, q^1_k, r^1_k) \\
&\qquad + (m_{k+1}-1)(p^3_k, q^3_k, r^3_k),\\
(p^4_{k+1}, q^4_{k+1}, r^4_{k+1}) &= (p^4_k, q^4_k, r^4_k) + n_{k+1}(p^1_k, q^1_k, r^1_k) \\
&\qquad + m_{k+1}(p^3_k, q^3_k, r^3_k).
\end{aligned}$$

One easily verifies

LEMMA 3.8. — *For each k*

$$(p^1_k, q^1_k, r^1_k) - (p^3_k, q^3_k, r^3_k) = \varepsilon_k(1, 0, -1),$$

$$(p^1_k, q^1_k, r^1_k) + (p^3_k, q^3_k, r^3_k) = (p^2_k, q^2_k, r^2_k) + (p^4_k, q^4_k, r^4_k).$$

In summary we have proved

PROPOSITION 3.9. — *The set of points whose negative slope expansion begins with a given finite sequence $(\epsilon_1, n_1, m_1) \dots (\epsilon_k, n_k, m_k)$ is a nonempty semi-open (or open, if $1 \in \{n_k, m_k\}$) quadrilateral Q_k as shown in Figure 1. The vertices of Q_k are given by Algorithm 3.3.*

Now we can prove a converse of Lemma 2.4:

COROLLARY 3.10. — *If $(\epsilon_k, n_k, m_k)_{k\geq 1}$ is such that $(n_k, \epsilon_k) \neq (1, +1)$ for infinitely many k and $(m_k, \epsilon_k) \neq (1, +1)$ for infinitely many k then there is a unique point $(\alpha, \beta) \in I \times I$ with negative slope expansion $(\epsilon_k, n_k, m_k)_{k\geq 1}$.*

Proof. — For $k = 1, 2, \dots$ let Q_k be the quadrilateral determined by $(\epsilon_1, n_1, m_1) \dots (\epsilon_k, n_k, m_k)$. Obviously, $Q_{k+1} \subset Q_k$ holds for all k. We just need to show that $\bigcap_k Q_k$ consists of a single point. The geometry of Q_k and Lemma 3.8 together imply $\mathrm{diam}(Q_k) = \sqrt{2}/q_k^1$. It is evident from our algorithm 3.3 that q_k^1 increases as k increases, whence the intersection $\bigcap_k Q_k$ contains at most one point.

An easy geometric proof shows that if we have two indices $k_0 < k_1$ such that $(n_k, \epsilon_k) \neq (1, +1)$ for two distinct $k \in (k_0, k_1)$ and $(m_k, \epsilon_k) \neq (1, +1)$ for two distinct $k \in (k_0, k_1)$ then $\overline{Q_{k_1}} \subset Q_{k_0}$. It follows from this that $\bigcap_k Q_k \neq \emptyset$. □

Allowing the coefficients $(\epsilon_1, n_1, m_1), \dots, (\epsilon_k, n_k, m_k)$ to vary yields a collection $\mathcal{Q}_k$ of disjoint quadrilaterals. The union of $\mathcal{Q}_k$ is the set of initial points for which the negative slope algorithm does not terminate within the first $k - 1$ steps. It follows from Lemma 2.1 that if $E^-(\alpha, \beta) = (\epsilon_k, n_k, m_k)_{k\geq 1}$ then $E^-(T(\alpha, \beta)) = (\epsilon_{k+1}, n_{k+1}, m_{k+1})_{k\geq 1}$. This, together with Corollary 3.10 gives

COROLLARY 3.11. — *For $k \geq 2$ and $Q \in \mathcal{Q}_k$, the map T of Lemma 2.1 restricts to a homeomorphism $Q \to T(Q) \in \mathcal{Q}_{k-1}$.*

Remark 3.12. — A similar statement holds for $k = 1$. Let $Q \in \mathcal{Q}_1$. If Q is open then $T_{|Q}$ is a homeomorphism $Q \to (0, 1) \times (0, 1)$, and if Q is semi-open then $T_{|Q}$ is a homeomorphism $Q \to I \times I$.

We thus have an alternate characterization for the quadrilaterals:

COROLLARY 3.13. — *The interior of each quadrilateral of $\mathcal{Q}_k$ is a maximal domain of continuity for T^k.*

Figure 5 illustrates the quadrilaterals which constitute the natural Markov partition into the sets of continuity of the map T; each of them is in one-to-one correspondence with the entire set Q_0. As was pointed out

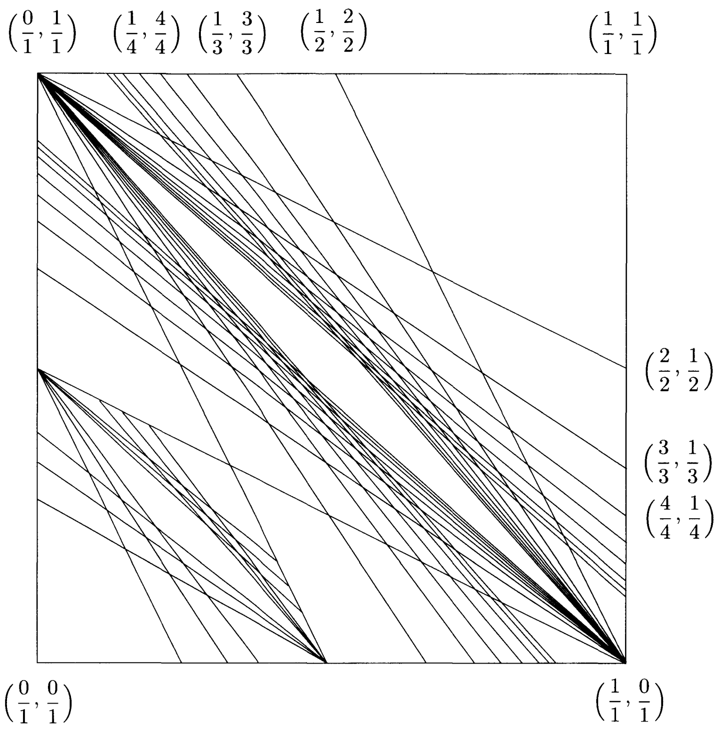

Figure 5. The unit square Q_0 is divided into quadrilaterals by the lines $\alpha'_0 = n\delta_1$ and $\beta'_0 = m\delta_1$, $n, m \in \mathbb{Z}$. The figure shows these lines for $|n|, |m| \leq 8$. Also shown are the first few lines subdividing the lower-leftmost quadrilateral (Q_1 when $n_1 = m_1 = 1, \epsilon = +1$).

by the referee, this explains why there is no restriction of finite type on the possible negative slope expansions, which is not the case, for example, for the Jacobi-Perron algorithm. The figure also shows that $(0,0)$ is an indifferent fixed point (the map T can be extended there by continuity), which explains why the sequence $(+1,1,1)^\infty$ plays such a special role. There are other fixed points on the boundary, corresponding to forbidden infinite sequences $(+1,1,m)^\infty$ or $(+1,n,1)^\infty$, but they are not indifferent.

4. Simultaneous rational approximations.

For the regular continued fraction approximation of an irrational α, there are general bounds on the distance between α and its convergents, and precise estimates of the quality of approximation in terms of the partial quotients. We explore analogous results for the negative slope algorithm; it is convenient to make the change of coordinates $(\alpha,\beta)\mapsto(\alpha+\beta,\beta-\alpha)$.

PROPOSITION 4.1. — *For $1\le k<K$ set $M_k=A_1A_2\cdots A_k$. Then*

$$M_k=\begin{pmatrix} p_{k-1} & 0 & p_k-p_{k-1}\\ r_{k-1} & \epsilon_1\cdots\epsilon_k & r_k-r_{k-1}\\ q_{k-1} & 0 & q_k-q_{k-1}\end{pmatrix}$$

where p_k, q_k and r_k are integers defined recursively by the relations

$$\begin{aligned} p_{k+1}&=(m_{k+1}+n_{k+1})p_k-\epsilon_{k+1}p_{k-1},\\ q_{k+1}&=(m_{k+1}+n_{k+1})q_k-\epsilon_{k+1}q_{k-1},\\ r_{k+1}&=(m_{k+1}+n_{k+1})r_k-\epsilon_{k+1}r_{k-1}+\epsilon_1\cdots\epsilon_{k+1}(n_{k+1}-m_{k+1}),\end{aligned}$$

starting from $p_{-1}=1$, $q_{-1}=0$, $r_{-1}=0$, $p_0=1$, $q_0=1$, $r_0=0$. Moreover

$$\det M_k=1,\quad p_{k-1}q_k-p_kq_{k-1}=\epsilon_1\cdots\epsilon_k,$$

$$q_{k+1}-q_k\ge q_k-q_{k-1}\ge\cdots\ge q_1-q_0\ge 1$$

and

$$\begin{pmatrix}\alpha+\beta\\ \beta-\alpha\\ 1\end{pmatrix}=M_k\begin{pmatrix}S_k\\ D_k\\ |\delta_k|\end{pmatrix}. \tag{4.1}$$

Proof. — We start from $\delta_0=1$, $\alpha_0'=\beta$, $\beta_0'=\alpha$, so the matrix formula is a straightforward consequence of Lemma 2.2. The recursive relations for p_k, q_k and r_k follow from $M_1=A_1$ and $M_{k+1}=M_kA_{k+1}$. Since $\det A_k=1$ for all k, it follows that $\det M_k=1$ which in turn implies $p_{k-1}q_k-p_kq_{k-1}=\epsilon_1\cdots\epsilon_k$. Finally the last inequality follows from the recursive relation for q_k and the fact that $m_k\ge 1$ and $n_k\ge 1$ for each k. □

The matrix formula (4.1) above allows us to approximate the quantities $\alpha+\beta$ and $\beta-\alpha$ (and hence α and β) by two rationals with the same denominator. Namely one approximates s_k and d_k by integers, which may be 0, 1 or 2 in the case of s_k, and -1, 0 or 1 in the case of d_k. If we use the central values $s_k=1$, $d_k=0$, we obtain the rationals p_k/q_k and r_k/q_k; other values would involve the integers $p_k+\epsilon p_{k-1}$, $q_k+\epsilon q_{k-1}$, $r_k+\epsilon r_{k-1}+\epsilon'$, with $\epsilon=\pm1$ and $\epsilon'=\pm1$. The next proposition and corollary concern the rational approximation of $\alpha+\beta$.

PROPOSITION 4.2. — *Let $(\epsilon_k, n_k, m_k)_{1\le k<K}$ be the negative slope expansion of (α,β). Then we have the following "semi-regular continued fraction expansion" of $\alpha+\beta$ (see* [1], [26], [32]) :

$$\alpha+\beta-1=-\cfrac{\epsilon_1}{m_1+n_1-\cfrac{\epsilon_2}{m_2+n_2-\cfrac{\epsilon_3}{m_3+n_3-\cfrac{\epsilon_4}{m_4+n_4-\cfrac{\epsilon_5}{\ddots}}}}}. \tag{4.2}$$

In case $K<\infty$, this formula stops with m_{K-1}, n_{K-1} and $\epsilon_K=0$; otherwise it is infinite. Moreover for $k<K-1$ we have

$$\alpha+\beta-\frac{p_k}{q_k}=\frac{-\epsilon_1\cdots\epsilon_{k+1}}{q_k}\frac{1}{q_{k+1}+(s_{k+1}-1)q_k}, \tag{4.3}$$

where

$$s_{k-1}-1=\frac{-\epsilon_k}{m_k+n_k+s_k-1}, \tag{4.4}$$

$$-1<s_{k+1}-1=\cfrac{-\epsilon_{k+2}}{m_{k+2}+n_{k+2}-\cfrac{\epsilon_{k+3}}{\ddots}}<1. \tag{4.5}$$

Proof. — The matrix equation (2.9) of Lemma 2.2 implies

$$s_{k-1}=\frac{S_k+(m_k+n_k-\epsilon_k-1)|\delta_k|}{S_k+(m_k+n_k-1)|\delta_k|}$$

which in turn yields (4.4) and (4.2) since $\alpha+\beta=s_0$.

Using the matrix relation (4.1) of Proposition 4.1 we obtain

$$\alpha+\beta=\frac{p_{k-1}S_k+(p_k-p_{k-1})|\delta_k|}{q_{k-1}S_k+(q_k-q_{k-1})|\delta_k|}=\frac{p_{k-1}s_k+p_k-p_{k-1}}{q_{k-1}s_k+q_k-q_{k-1}}$$

hence

$$\alpha+\beta-\frac{p_k}{q_k}=\frac{(s_k-1)(p_{k-1}q_k-p_kq_{k-1})}{q_k(q_k+(s_k-1)q_{k-1})}.$$

From Proposition 4.1 we deduce that

$$\begin{aligned}
\alpha+\beta-\frac{p_k}{q_k} &= \frac{\epsilon_1\cdots\epsilon_k}{q_k}\,\frac{1}{\frac{q_k}{s_k-1}+q_{k-1}}\\
&= \frac{\epsilon_1\cdots\epsilon_k}{q_k}\,\frac{1}{-\epsilon_{k+1}q_k(m_{k+1}+n_{k+1}+s_{k+1}-1)+q_{k-1}}\\
&= \frac{-\epsilon_1\cdots\epsilon_{k+1}}{q_k}\,\frac{1}{(m_{k+1}+n_{k+1})q_k+(s_{k+1}-1)q_k-\epsilon_{k+1}q_{k-1}}\\
&= \frac{-\epsilon_1\cdots\epsilon_{k+1}}{q_k}\,\frac{1}{q_{k+1}+(s_{k+1}-1)q_k}
\end{aligned}$$

where the last equality is a consequence of the recursive relation for q_k. □

COROLLARY 4.3. — *If $k<K$ and $(m_k,n_k,\epsilon_{k+1})\neq(1,1,+1)$, then*

$$\left|\alpha+\beta-\frac{p_{k-1}}{q_{k-1}}\right|<\frac{1}{q_{k-1}^2}, \tag{4.6}$$

$$\left|\alpha+\beta-\frac{p_{k-1}}{q_{k-1}}\right|<\frac{2}{q_{k-1}q_k}, \tag{4.7}$$

$$\left|\alpha+\beta-\frac{p_k}{q_k}\right|<\frac{2}{q_k^2}. \tag{4.8}$$

If every string of consecutive k for which $(m_k,n_k,\epsilon_{k+1})=(1,1,+1)$ has length at most M, then for all $k<K$,

$$\left|\alpha+\beta-\frac{p_k}{q_k}\right|<\frac{M+2}{q_k^2}.$$

Suppose the negative slope expansion of (α,β) does not stop. If the lengths of strings of consecutive k for which $(m_k,n_k,\epsilon_{k+1})=(1,1,+1)$ are unbounded, then there exists a sequence J of integers such that for $k\in J$,

$$q_k^2\left|\alpha+\beta-\frac{p_k}{q_k}\right|\longrightarrow+\infty$$

(but then we shall see later that $(p_k-p_{k-1})/(q_k-q_{k-1})$ provides a better approximation).

Proof. — If $(m_k,n_k,\epsilon_{k+1})\neq(1,1,+1)$, then either $\epsilon_{k+1}=-1$, or equivalently $s_k-1>0$, or $m_k+n_k\geq 3$, in which case by Proposition 4.1 we have that $q_k>2q_{k-1}$, which in turn implies $q_{k-1}/(q_k-q_{k-1})<1$ and $q_k/(q_k-q_{k-1})<2$. In either case, (4.6) and (4.7) are readily verified using equation (4.3) of Proposition 4.2. The estimate (4.8) stems from the intermediate equality

$$\alpha + \beta - \frac{p_k}{q_k} = \frac{\epsilon_1 \cdots \epsilon_k}{q_k} \frac{s_k - 1}{q_{k-1}(s_k - 1) + q_k}$$

in the proof of Proposition 4.2.

If $(m_p, n_p, \epsilon_{p+1}) = (1, 1, +1)$ for every $k' + 1 \leq p \leq k + 1$ but $(m_{k'}, n_{k'}, \epsilon_{k'+1}) \neq (1, 1, +1)$, then the recursive relation for q_k in Proposition 4.1 yields

$$\frac{q_k}{q_{k+1} - q_k} = \frac{q_k}{q_k - q_{k-1}} = \cdots = \frac{(k - k_1)q_{k_1+1} - (k - k_1 - 1)q_{k_1}}{q_{k_1+1} - q_{k_1}}$$

with $q_{k_1+1} > 2q_{k_1}$, by taking either $k_1 = k'$ or $k_1 = k' - 1$. The fourth claim now follows from (4.3) of Proposition 4.2.

Suppose now $(m_p, n_p, \epsilon_{p+1}) = (1, 1, +1)$ for every $k_0 \leq p \leq k_0 + M + N + 1$. Let $k = k_0 + N$, $P = q_{k_0-1}$, $Q = q_{k_0}$. Then the recursion formulas give $q_k = NQ - (N - 1)P$, $q_{k+1} = (N + 1)Q - NP$, while the semi-regular continued fraction expansion of $s_{k+1} - 1$ begins in

$$-\cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{\ddots}}}}}$$

hence $s_{k+1} = \phi^M(s_{k+1+M})$, where ϕ^M denotes the M-th iterate of the function $\phi(x) = x/(x + 1)$; hence

$$0 < s_{k+1} < \phi^M(2) < \frac{1}{M}\cdot$$

By Proposition 4.2

$$q_k^2 \left| \alpha + \beta - \frac{p_k}{q_k} \right| = \frac{q_k}{q_{k+1} + (s_{k+1} - 1)q_k} > \frac{NQ - (N - 1)P}{Q - P + \frac{1}{M}(NQ - (N - 1)P)},$$

which is large whenever N is large and N/M is small, independently of P and Q. The last claim now follows. $\square$

Remark. — The semi-regular expansion of $\alpha+\beta$ is in general not unique, but in our case it is completely determined by the negative slope algorithm; in fact the expansion depends on both α and β and not only on the value of $\alpha+\beta$. In particular, the negative slope expansion may stop because $q\alpha+(q+1)\beta=p$, while $\alpha+\beta$ admits an infinite semi-regular expansion. If for each k we have $m_k+n_k-\epsilon_{k+1}\geq 2$, that is if $m_k=1$, $n_k=1$, and $\epsilon_{k+1}=+1$ never occurs, then the semi-regular expansion of $\alpha+\beta$ is the "nearest integer continued fraction expansion" defined by Hurwitz in [22]. In this case the p_k/q_k are a subsequence of the convergents of $\alpha+\beta$ (see [32], Part I, Chap. 5, § 40).

We next look at the induced rational approximation of $\beta-\alpha$.

PROPOSITION 4.4. — *Suppose $k<K$. Then*

$$\beta-\alpha-\frac{r_k}{q_k}=\frac{1}{q_k}\frac{(s_k-1)U_k+q_k\epsilon_1\cdots\epsilon_k d_k}{q_k+(s_k-1)q_{k-1}}, \tag{4.9}$$

where

$$-1<\frac{U_k}{q_k-q_{k-1}}<1, \tag{4.10}$$

and

$$\frac{d_k}{s_k-1}=-\epsilon_{k+1}d_{k+1}+(m_{k+1}-n_{k+1})\epsilon_{k+1}.$$

Proof. — By Proposition 4.1 we have

$$\beta-\alpha=\frac{r_{k-1}S_k+\epsilon_1\cdots\epsilon_k D_k+(r_k-r_{k-1})|\delta_k|}{q_{k-1}S_k+(q_k-q_{k-1})|\delta_k|},$$

$$\beta-\alpha-\frac{r_k}{q_k}=\frac{1}{q_k}\frac{(s_k-1)(q_kr_{k-1}-q_{k-1}r_k)+q_k\epsilon_1\cdots\epsilon_k d_k}{q_k+(s_k-1)q_{k-1}}.$$

Define

$$U_k=q_kr_{k-1}-q_{k-1}r_k.$$

We establish (4.10) by induction on k. For $k=1$ we have

$$|U_1|=|n_1-m_1|<n_1+m_1-1=q_1-q_0.$$

Next suppose that

$$|U_k|<q_k-q_{k-1}.$$

By Proposition 4.1

$$U_{k+1} = \epsilon_{k+1}U_k - q_k\epsilon_1 \cdots \epsilon_{k+1}(n_{k+1} - m_{k+1})$$

hence, if $n_{k+1} = m_{k+1}$, then

$$|U_{k+1}| = |U_k| < q_k - q_{k-1} \leq q_{k+1} - q_k;$$

if $n_{k+1} \neq m_{k+1}$, then

$$\begin{aligned}|U_{k+1}| &\leq |n_{k+1} - m_{k+1}|q_k + |U_k| \\ &< (m_{k+1} + n_{k+1} - 2)q_k + q_k - q_{k-1} \leq q_{k+1} - q_k\end{aligned}$$

as required.

Finally the last claim is a straightforward consequence of the matrix equation in Lemma 2.2. □

COROLLARY 4.5. — *For every k,*

$$\left|\beta - \alpha - \frac{r_k}{q_k}\right| < \frac{1}{q_k}.$$

Suppose the negative slope expansion of (α, β) does not stop. If, on a sequence J of integers, we have $m_{k+1} + n_{k+1} \to +\infty$ and $|n_{k+1} - m_{k+1}|/(m_{k+1} + n_{k+1}) \to 0$, then for $k \in J$,

$$q_k\left|\beta - \alpha - \frac{r_k}{q_k}\right| \longrightarrow 0.$$

There exist α and β with an infinite expansion, and a constant $C > 0$, such that for every k,

$$\frac{C}{q_k} < \left|\beta - \alpha - \frac{r_k}{q_k}\right| < \frac{1-C}{q_k}.$$

Proof. — We use (4.9) of Proposition 4.4; it implies

$$\left|\beta - \alpha - \frac{r_k}{q_k}\right| \leq \frac{1}{q_k}\frac{|xU_k| + q_k|y|}{q_k + xq_{k-1}},$$

with

$$|x| + |y| < 1$$

by the inequality (2.8). It is readily verified that this function of two variables has no extremum (except $(0,0)$) inside the domain $|x| + |y| \leq 1$, hence its maximum on this domain is reached on $|x| + |y| = 1$.

If $|x| + |y| = 1$ and $x \geq 0$, then

$$\frac{1}{q_k} \frac{|xU_k| + q_k|y|}{q_k + xq_{k-1}} \leq \frac{1}{q_k} \frac{x|U_k| + q_k(1-x)}{q_k} < \frac{1}{q_k}$$

where the last inequality is a consequence of (4.10) in Proposition 4.4.

If $|x| + |y| = 1$ and $x \leq 0$, then

$$\frac{1}{q_k} \frac{|xU_k| + q_k|y|}{q_k + xq_{k-1}} = \frac{1}{q_k} \frac{z|U_k| + q_k(1-z)}{q_k - zq_{k-1}},$$

with $0 \leq z \leq 1$. Again using (4.10) one verifies that the maximum of this quantity is reached either for $z = 0$ or $z = 1$, and so is either $1/q_k$ or $(1/q_k)|U_k|/(q_k - q_{k-1})$, hence at most $1/q_k$ as required.

The second claim follows immediately from Proposition 4.4, the matrix equation in Lemma 2.2, and the equality

$$\begin{aligned}\frac{(s_k - 1)(q_k r_{k-1} - q_{k-1} r_k)}{q_k + (s_k - 1)q_{k-1}} &= \frac{U_k}{q_{k-1} + \frac{q_k}{s_k - 1}} \\ &= \frac{U_k}{q_{k-1} + q_k(-\epsilon_{k+1}(m_{k+1} + n_{k+1} + s_{k+1} - 1))}.\end{aligned}$$

We can build an infinite sequence $(m_k, n_k, \epsilon_{k+1})$ such that the m_k and n_k are bounded, $|n_k - m_k| \geq 2$ for every $n \geq 1$, and ϵ_{k+1} and $U_k \epsilon_1 \cdots \epsilon_k$ have opposite signs for every $n \geq 1$. In view of Proposition 4.4 this sequence defines α and β satisfying the third claim. □

5. More diophantine properties.

In this section we express in terms of the negative slope algorithm some diophantine properties of $\alpha + \beta$, and some properties of simultaneous approximation of $\alpha + \beta$ and α (or β). The following proposition was proved by del Junco [15] for some particular semi-regular expansions; the last part of our proof is similar to the one in del Junco's paper.

PROPOSITION 5.1. — *Suppose the negative slope expansion of* (α, β) *does not stop. Then the two following properties are equivalent:*

- *in the negative slope expansion of* (α, β), *the* $n_k + m_k$ *are unbounded or the lengths of the strings of* $(1, 1, +1)$ *are unbounded*;

- *for every $\epsilon > 0$, there exist integers p and q such that*

$$\left|\alpha + \beta - \frac{p}{q}\right| \leq \frac{\epsilon}{q^2}.$$

Proof. — If the $n_k + m_k$ are unbounded, the second assertion follows from

$$\alpha + \beta - \frac{p_k}{q_k} = \frac{-\epsilon_1 \cdots \epsilon_{k+1}}{q_k} \frac{1}{(m_{k+1} + n_{k+1})q_k + (s_{k+1} - 1)q_k - \epsilon_{k+1}q_{k-1}}$$

by putting $p = p_k$, $q = q_k$ for suitable k.

If $(m_p, n_p, \epsilon_{p+1}) = (1, 1, +1)$ for every $k + 1 \leq p \leq k + M$ and $\epsilon_{k+1} = +1$; then, as in the proof of Corollary 4.3, we have $0 < s_k < 1/M$. The relation $\alpha + \beta = (p_{k-1}s_k + p_k - p_{k-1})/(q_{k-1}s_k + q_k - q_{k-1})$ coming from Proposition 4.1 implies that

$$\alpha + \beta - \frac{p_k - p_{k-1}}{q_k - q_{k-1}} = \frac{\pm s_k}{(q_k - q_{k-1})(q_k - q_{k-1} + q_{k-1}s_k)}.$$

The second assertion now follows, if M is large enough, by taking $p = p_k - p_{k-1}$, $q = q_k - q_{k-1}$ for suitable k.

If $(m_p, n_p, \epsilon_{p+1}) = (1, 1, +1)$ for every $k + 1 \leq p \leq k + M$ and $\epsilon_{k+1} = -1$; then we get $2 - 1/M < s_k < 2$, and the second assertion follows, if M is large enough, by taking $p = p_k + p_{k-1}$, $q = q_k + q_{k-1}$ for suitable k.

Suppose now that α and β satisfy the second assertion. For $-1 \leq x \leq 1$, let $\|x\|$ denote the distance of x to the nearest integer.

For a given ϵ, we put

$$v_0 = \frac{\epsilon}{2}\Big(\prod_{j=2}^{+\infty} \frac{j^2}{j^2 - 1}\Big)^{-1}$$

and apply the hypothesis on α and β to get

$$s_0 - 1 = \frac{b_0}{a_0} + \frac{t_0}{a_0^2}$$

for a positive integer a_0, an integer b_0 with $-a_0 \leq b_0 \leq a_0$, and some $-v_0 < t_0 < v_0$.

Suppose that a_i is a positive integer, b_i is an integer with $-a_i \leq b_i \leq a_i$, t_i satisfies $-1 < t_i < 1$ and

$$s_i - 1 = \frac{b_i}{a_i} + \frac{t_i}{a_i^2}.$$

Then either $b_i = \pm a_i$ and $\|s_i - 1\| \leq |t_i|$, or

$$\left|\frac{1}{s_i - 1} - \frac{a_i}{b_i}\right| = \left|\frac{t_i}{b_i(b_i + \frac{t_i}{a_i})}\right| \leq \frac{|t_i|}{b_i^2 - \frac{|b_i|}{a_i}} \leq \frac{1}{b_i^2} \frac{b_i^2 |t_i|}{b_i^2 - 1}.$$

Hence, because of the formula (4.4), which says that $s_{i+1} - 1$ is $\pm 1/(s_i - 1)$, up to the addition of an integer, we get

$$s_{i+1} - 1 = \frac{b_{i+1}}{a_{i+1}} + \frac{t_{i+1}}{a_{i+1}^2}$$

with $a_{i+1} = |b_i| < a_i$, $-a_{i+1} \leq b_{i+1} \leq a_{i+1}$, $|t_{i+1}| \leq |t_i| \frac{b_i^2}{b_i^2 - 1} < 1$, and we can continue.

So this process will give some k such that $b_k = \pm a_k$ or $a_k = \pm 1$, and

$$\|s_k - 1\| < |t_k| < \epsilon.$$

But this is possible only if $s_k - 1$ is close to 0, which means $m_k + n_k$ is big, or close to ± 1, which means there is a long string of $(1, 1, +1)$. □

The following proposition is expressed in terms of $\alpha + \beta$ and $\alpha - \beta$, but, because of the good approximation of $\alpha + \beta$, the good (or bad) approximation of $\alpha - \beta$ is equivalent to the good (or bad) approximation of α, or β. Of course, it is not irrelevant that this kind of properties is used in the theory of three-interval exchanges [23].

PROPOSITION 5.2. — *Suppose the negative slope expansion of (α, β) does not stop. Then the two following properties are equivalent:*

- *In the negative slope expansion of (α, β), the lengths of the strings of $(1, 1, +1)$ are unbounded or for every $\epsilon > 0$, there exists k such that either*

$$0 < \left|\frac{m_k - n_k}{m_k + n_k}\right| < \epsilon \quad \textit{or} \quad \left|\frac{m_k - n_k}{m_k + n_k}\right| > 1 - \epsilon \quad \textit{or} \quad m_k = n_k > \frac{1}{\epsilon}.$$

- *For every $\epsilon > 0$, there exist integers p, q, r such that*

$$\left|\alpha + \beta - \frac{p}{q}\right| \leq \frac{\epsilon}{q^2} \quad \textit{and} \quad \left|\alpha - \beta - \frac{r}{q}\right| \leq \frac{\epsilon}{q}.$$

And the two following properties are equivalent:

• *In the negative slope expansion of* (α, β), *there exist* $C > 0$ *such that for every* $M > 0$ *there exists* k *such that* $m_k + n_k > M$ *and*

$$C < \left| \frac{m_k - n_k}{m_k + n_k} \right| < 1 - C.$$

• *There exist* $C' > 0$ *such that for every* $\epsilon > 0$, *there exists integers* p, q *such that*

$$\left| \alpha + \beta - \frac{p}{q} \right| \leq \frac{\epsilon}{q^2}$$

and for any integer r

$$\left| \alpha - \beta - \frac{r}{q} \right| > \frac{C'}{q}.$$

Proof. — We prove first that the first assertion implies the second one, then that the third one implies the fourth one, then the two converses simultaneously.

If $0 < |(m_{k+1} - n_{k+1})/(m_{k+1} + n_{k+1})| < \epsilon$ or $m_{k+1} = n_{k+1} > 1/\epsilon$, by Corollary 4.5 p_k/q_k and r_k/q_k provide the simultaneous approximation of the second assertion.

If $|(m_{k+1} - n_{k+1})/(m_{k+1} + n_{k+1})| > 1 - \epsilon$, the computations in Corollary 4.5 show that p_k/q_k and either $(r_k + 1)/q_k$ or $(r_k - 1)/q_k$ provide the simultaneous approximation of the second assertion.

If $(m_p, n_p, \epsilon_{p+1}) = (1, 1, +1)$ for every $k + 1 \leq p \leq k + M$, and $\epsilon_{k+1} = +1$; then $0 < s_k < 1/M$; but also $|d_k| \leq s_k < 1/M$. Then if M is large enough $(p_k - p_{k-1})/(q_k - q_{k-1})$ and $(r_k - r_{k-1})/(q_k - q_{k-1})$ provide the simultaneous approximation of the second assertion.

If $(m_p, n_p, \epsilon_{p+1}) = (1, 1, +1)$ for every $k + 1 \leq p \leq k + M$, and $\epsilon_{k+1} = -1$; then $2 - 1/M < s_k < 2$, and $|d_k| \leq 1/M$. Then if M is large enough $(p_k + p_{k-1})/(q_k + q_{k-1})$ and $(r_k + r_{k-1})/(q_k + q_{k-1})$ provide the simultaneous approximation of the second assertion.

If $m_{k+1} + n_{k+1}$ is large and $C < |(m_{k+1} - n_{k+1})/(m_{k+1} + n_{k+1})| < 1 - C$, then p_k/q_k gives a good approximation of $\alpha + \beta$; but the computations in Corollary 4.5 show that neither r_k/q_k, nor $(r_k + 1)/q_k$, nor $(r_k - 1)/q_k$ provide an approximation of $\alpha - \beta$ better than in $C/2q_k$; hence, because of the first assertion of Corollary 4.5, our third assertion implies our fourth one.

Suppose now α and β satisfy our second or our fourth assertion. We can then make the construction of the proof of last proposition, to get $a_1, \dots a_k$, and $\|s_k - 1\| < \epsilon$, hence s_k is close to 1, 0 or 2.

Suppose that $|s_k - 1| < \epsilon$; then, by construction, b_{k-1}/a_{k-1} is the value that formula (4.4) gives to s_{k-1} if we replace s_k by 0, and b_0/a_0 is the value that k iterations of formula (4.4) give to s_0 if we replace s_k by 1; hence, with the notations of Section 4, $b_0/a_0 = S_0'/\delta_0'$ where

$$\begin{pmatrix} S_0' \\ D_0' \\ \delta_0' \end{pmatrix} = M_k \begin{pmatrix} |\delta_k| \\ D_k \\ |\delta_k| \end{pmatrix},$$

hence

$$q = a_0 = q_k.$$

But then $m_{k+1} + n_{k+1}$ must be large, and, because of the above discussion, if the first assertion is not satisfied there is no good approximation of $\alpha - \beta$ with denominator q_k, while if the third assertion is not satisfied there is a good approximation of $\alpha - \beta$ with denominator q_k.

Suppose that $|s_k| < \epsilon$; then, by construction, b_0/a_0 is the value that k iterations of formula (4.4) gives to s_0 if we replace s_k by 0; and we get

$$q = a_0 = q_k - q_{k-1}.$$

But then we must have $\epsilon_{k+1} = +1$ and there must be a long string of $(m_p, n_p, \epsilon_{p+1}) = (1, 1, +1)$ starting at $p = k+1$; hence our first assertion is satisfied, by the discussion above there is a good approximation of $\alpha - \beta$ with denominator $q_k - q_{k-1}$, and we must have started from our second assertion.

Suppose that in fact $|s_k - 2| < \epsilon$; then, by construction, b_0/a_0 is the value that k iterations of formula (4.4) gives to s_0 if we replace s_k by 2; and we get

$$q = a_0 = q_k + q_{k-1}.$$

But then we must have $\epsilon_{k+1} = -1$ and there must be a long string of $(m_p, n_p, \epsilon_{p+1}) = (1, 1, +1)$ starting at $p = k+1$; hence our first assertion is satisfied, by the discussion above there is a good approximation of $\alpha - \beta$ with denominator $q_k + q_{k-1}$, and we must have started from our second assertion. $\square$

6. Periodic negative slope expansions.

We show that the negative slope algorithm satisfies the following Lagrange type theorem:

THEOREM 6.1. — *Suppose (α,β) does not lie on one of the rational lines of Proposition 2.3. Then the sequence $(m_k, n_k, \epsilon_{k+1})$ is ultimately periodic (or equivalently the sequence $T^k(\alpha,\beta)$ is ultimately periodic), if and only if α and β are in the same quadratic extension of $\mathbb{Q}$.*

Proof. — If the sequence $(m_k, n_k, \epsilon_{k+1})$ is ultimately periodic, it follows from the definition of the algorithm that for some $k > \ell > 0$ we have $T^k(\alpha,\beta) = T^\ell(\alpha,\beta)$, and hence $s_k = s_\ell$, $d_k = d_\ell$. But we have

$$\begin{pmatrix} \alpha+\beta \\ \beta-\alpha \\ 1 \end{pmatrix} = |\delta_k| M_k \begin{pmatrix} s_k \\ d_k \\ 1 \end{pmatrix},$$

and the same relation with ℓ, hence

$$\delta \begin{pmatrix} \alpha+\beta \\ \beta-\alpha \\ 1 \end{pmatrix} = M_\ell M_k^{-1} \begin{pmatrix} \alpha+\beta \\ \beta-\alpha \\ 1 \end{pmatrix},$$

with $\delta = |\delta_k|/|\delta_\ell|$. As $\det M_k = 1$, we check that

$$M_k^{-1} = \begin{pmatrix} \epsilon_1\cdots\epsilon_k(q_k - q_{k-1}) & 0 & -\epsilon_1\cdots\epsilon_k(p_k - p_{k-1}) \\ q_{k-1}r_k - q_k r_{k-1} & p_{k-1}q_k - p_k q_{k-1} & p_k r_{k-1} - p_{k-1} r_k \\ -\epsilon_1\cdots\epsilon_k q_{k-1} & 0 & \epsilon_1\cdots\epsilon_k p_{k-1} \end{pmatrix}.$$

Hence the matrix $M_l M_k^{-1}$ has integer coefficients, and the entries in its second column are 0, $+1$ or -1, and 0. We deduce δ from the third line of the above relation; the first line gives then a non-trivial algebraic relation of degree 2 satisfied by $\alpha+\beta$, from which we deduce

$$\alpha+\beta \in \mathbb{Q}(\sqrt{d})$$

for an integer d; then the second line implies

$$\beta-\alpha \in \mathbb{Q}(\sqrt{d})$$

hence α and β are in $\mathbb{Q}(\sqrt{d})$.

We suppose now that $\alpha \in \mathbb{Q}(\sqrt{d})$ and $\beta \in \mathbb{Q}(\sqrt{d})$.

Then $\alpha + \beta \in \mathbb{Q}(\sqrt{d})$, and there exist integers a, b, c such that

$$c(\alpha+\beta)^2 = a(\alpha+\beta) + b.$$

We write this relation

$$c(\alpha+\beta)\begin{pmatrix}\alpha+\beta\\ 1\end{pmatrix} = B\begin{pmatrix}\alpha+\beta\\ 1\end{pmatrix}$$

where

$$B = \begin{pmatrix} a & b\\ c & 0\end{pmatrix}.$$

From Proposition 4.1 we deduce that

$$\begin{pmatrix}\alpha+\beta\\ 1\end{pmatrix} = |\delta_k| C_k \begin{pmatrix} s_k - 1\\ 1\end{pmatrix}$$

where

$$C_k = \begin{pmatrix} p_{k-1} & p_k\\ q_{k-1} & q_k\end{pmatrix}.$$

These two relations imply

$$c(\alpha+\beta)\begin{pmatrix} s_k-1\\ 1\end{pmatrix} = C_k^{-1} B C_k \begin{pmatrix} s_k-1\\ 1\end{pmatrix}$$

and hence

$$z_k(s_k-1)^2 + (t_k - x_k)(s_k-1) - y_k = 0$$

where

$$C_k^{-1} B C_k = \begin{pmatrix} x_k & y_k\\ z_k & t_k\end{pmatrix};$$

x_k, y_k, z_k, t_k are integers as B and C_k have integer coefficients and C_k has determinant 1 or -1; let us show these coefficients are universally bounded on those k for which $(m_k, n_k, \epsilon_{k+1}) \neq (1,1,+1)$.

For a matrix M, we denote by $\|M\|$ the maximum of the absolute value of its entries. Let

$$F_k = \begin{pmatrix} p_{k-1} - q_{k-1}(\alpha+\beta) & p_k - q_k(\alpha+\beta)\\ 0 & 0\end{pmatrix}.$$

We check that

$$B(C_k - F_k) = c(\alpha+\beta)(C_k - F_k)$$

hence

$$C_k^{-1} B C_k = c(\alpha+\beta) + C_k^{-1}(B - c(\alpha+\beta)I_2)F_k$$

where I_2 is the identity matrix. Hence

$$\|C_k^{-1} B C_k\| \le c_1 + c_2 \|C_k^{-1}\| \, \|F_k\|$$

for universal constants c_1 and c_2. We check that

$$\|C_k^{-1}\| \le 2q_k$$

while

$$\|F_k\| = \big|p_{k-1} - q_{k-1}(\alpha+\beta)\big| \vee \big|p_k - q_k(\alpha+\beta)\big|.$$

Hence, by Corollary 4.3, for those n for which $(m_k, n_k, \epsilon_{k+1}) \neq (1, 1, +1)$,

$$\|F_k\| < \frac{2}{q_k}$$

and the coefficients of $C_k^{-1} B C_k$, which are x_k, y_k, z_k, t_k, are bounded by $c_1 + 8c_2$.

Now, α and β being in $\mathbb{Q}(\sqrt{d}\,)$ and $\alpha+\beta$ being irrational, there exist integers a', b' c' such that

$$c'(\beta-\alpha) = a'(\alpha+\beta) + b'.$$

By the same reasoning as above, the relation

$$\begin{pmatrix} \alpha+\beta \\ \beta-\alpha \\ 1 \end{pmatrix} = |\delta_k| M_k \begin{pmatrix} s_k \\ d_k \\ 1 \end{pmatrix}$$

translates into

$$c' \begin{pmatrix} s_k \\ d_k \\ 1 \end{pmatrix} = M_k^{-1} E M_k \begin{pmatrix} s_k \\ d_k \\ 1 \end{pmatrix}$$

where

$$E = \begin{pmatrix} c' & 0 & 0 \\ a' & 0 & b' \\ 0 & 0 & c' \end{pmatrix}.$$

Using the expression of M_k^{-1} given above, an explicit computation of the second line of the previous equality gives

$$c' d_k = x'_k s_k + y'_k$$

with

$$\begin{aligned} x'_k &= (p_{k-1}q_k - p_k q_{k-1})(a' p_{k-1} + b' q_{k-1} - c' r_{k-1}), \\ y'_k &= (p_{k-1}q_k - p_k q_{k-1})(a' p_{k-1} + b' q_{k-1} - c' r_{k-1}) \\ &\qquad - (p_{k-1}q_k - p_k q_{k-1})(a' p_k + b' q_k - c' r_k) \end{aligned}$$

hence

$$|x'_k| \vee |y'_k| \leq |a' p_{k-1} + b' q_{k-1} - c' r_{k-1}| + |a' p_k + b' q_k - c' r_k|,$$

$$\begin{aligned} a' p_k + b' q_k - c' r_k &= q_k \Big(a' \frac{p_k}{q_k} + b' - c' \frac{r_k}{q_k} \Big) \\ &= a' q_k \Big(\frac{p_k}{q_k} - (\alpha + \beta) \Big) - c' q_k \Big(\frac{r_k}{q_k} - (\beta - \alpha) \Big), \end{aligned}$$

and similarly with k replaced by $k-1$; it follows then from Proposition 4.2 and Corollary 4.3 that the integers x'_k and y'_k are universally bounded.

Hence for the k for which $(m_k, n_k, \epsilon_{k+1}) \neq (1, 1, +1)$, the quantity $s_k - 1$ satisfies only a finite number of equations of degree two with integer coefficients, and the pair (s_k, d_k) satisfies only a finite number of relations $c' d_k = x' s_k + y'$ with integer coefficients; as these k are infinitely many because $\alpha + \beta$ is irrational, there exists $k' > k$ such that $s_{k'} = s_k$ and $d_{k'} = d_k$. But because of the definition of the algorithm, this implies

$$(m_{k+\ell}, n_{k+\ell}, \epsilon_{k+\ell+1}) = (m_{k'+\ell}, n_{k'+\ell}, \epsilon_{k'+\ell+1})$$

for every $\ell \geq 1$. □

Remark 6.2. — In [11] Burger gives necessary and sufficient conditions on the partial quotients of two quadratic irrationals to insure that they are elements of the same quadratic number field. It would be interesting to understand these conditions in the context of the negative slope algorithm.

7. Connection with interval exchange transformations.

The regular continued fraction algorithm provides a link between the arithmetic properties of an irrational number α, the ergodic and spectral properties of a circle rotation by angle α, and the combinatorial properties of a class of binary sequences called the Sturmian infinite words (see [5], [14], [30], [31], [33]). A fundamental problem is to generalize and extend this rich interaction to dimension two or greater, starting either from a dynamical system or a specified class of sequences. A primary motivation is that such a generalization could yield a satisfying algorithm of simultaneous rational approximation in $\mathbb{R}^n$.

In case $n = 2$ there are a number of different dynamical systems all of which are natural candidates to play the role of a rotation in dimension one. One such system is a rotation on the 2-torus [3], [5], [33], [34], [35]. In this case, the corresponding symbolic counterpart is a class of sequences of complexity $2n + 1$ introduced by Arnoux and Rauzy in [5] which are a natural generalization of the Sturmian sequences; the arithmetic component is given by a 2-dimensional division algorithm originally defined by Arnoux and Rauzy in [5] and later studied in greater generality in [13], [43], [44]. Though the resulting arithmetic/ergodic/combinatorial interaction is very satisfying in the special case of the so-called Tribonacci system, as pioneered by work of Arnoux and Rauzy [3], [5], [35], a more general canonical equivalence (through what is called a natural coding) between two-dimensional rotations and Arnoux-Rauzy sequences is not always verified (see [12]).

Berthé and Vuillon [7] studied the dynamics resulting from two rotations on the circle (see also work of Arnoux, Berthé and Ito in [4]). They found in this case the symbolic counterpart is given by a family of $\mathbb{Z}^2$-shifts, and that the Jacobi-Perron algorithm provides a suitable arithmetic tool for studying this class of dynamical systems.

Since a circle rotation is equivalent to an exchange of two intervals on $[0,1]$, another possible generalization is the dynamical system arising from an exchange on three intervals. Associated to each $0 < \alpha < 1$ and $0 < \beta < 1$ with $\alpha + \beta < 1$, is a dynamical system on the interval $[0,1[$ given by

$$f(x) = \begin{cases} x + 1 - \alpha & \text{if } x \in [0, \alpha[, \\ x + 1 - 2\alpha - \beta & \text{if } x \in [\alpha, \alpha + \beta[, \\ x - \alpha - \beta & \text{if } x \in [\alpha + \beta, 1[. \end{cases}$$

That is, the unit interval is partitioned according to three subintervals, $[0,1] = [0,\alpha] \cup [\alpha, \alpha+\beta] \cup [\alpha+\beta, 1]$ which are then rearranged according to the permutation $(3,2,1)$. In [16] we show that the negative slope algorithm is intimately connected with the dynamics of 3-interval exchanges, and in fact bears the same connection with this class of dynamical systems as regular continued fraction with circle rotations.

The negative slope algorithms may be reformulated combinatorially in terms of the structure of the so-called *bispecial words* of the symbolic subshift obtained by symbolically coding the trajectories of points under a 3-interval exchange transformation according to the above partition into three subintervals. This connection relies on a generalization of a combinatorial construction originally developped in [RisZam] to study the evolution of bispecial factors in Arnoux-Rauzy sequences. More precisely, let

$$p = \alpha, \quad q = \alpha + \beta, \quad p' = 1 - (\alpha + \beta), \quad q' = 1 - \alpha,$$

so that p, q are the points of discontinuity of f and p', q' the points of discontinuity of f^{-1}. We call a subinterval I *bispecial* if I is an interval of continuity of f^n for some $n \geq 1$, and I contains either p' or q', and its image $I' = f^n(I)$ contains either p or q. Among the bispecial intervals, we denote by $\{I_k\}_{k\geq 1}$ (respectively $\{J_k\}_{k\geq 1}$) those bispecial intervals for which $p' \in I_k$, $p \in I'_k$, respectively $q' \in J_k$, $q \in J'_k$, ordered so that $I_1 \supset I_2 \supset \cdots$, respectively $J_1 \supset J_2 \supset \cdots$ Then in [16] we show that there are infinitely many I_k and J_k and that (under some initial conditions on the initial lengths α, β), $|\delta_k| = |I_k| = |J_k|$ for each $k \geq 1$. The sign of δ_{k+1} determines on which side (left or right) the intervals I_k and J_k are "cut" to produce the next bispecial intervals U_k, V_k; that is U_k, respectively V_k, is the largest bispecial interval properly contained in I_k, respectively J_k. It can be shown that U_k contains p' and its image U'_k contains q, while V_k contains q' and its image V'_k contains p; in particular U_k is not I_{k+1} and V_k is not J_{k+1}. The quantities n_{k+1} and m_{k+1} count the number of bispecial intervals between I_k and I_{k+1} and J_k and J_{k+1} respectively. If $I_k \supset E_1 \supset E_2 \supset \cdots \supset E_{n_k} \supset I_{k+1}$ are the n_{k+1} bispecial intervals between I_k and I_{k+1} (so $E_1 = U_k$), then $|E_i| = \alpha_{k+1} - (i-1)|\delta_{k+1}|$. For each fixed k, E_{i+1} is obtained from E_i by cutting $|\delta_{k+1}|$ on the same side, while I_{k+1} is obtained by cutting $E_{n_{k+1}}$ by α'_{k+1} on the opposite side.

From this point of view, the negative slope algorithm corresponds to a double renormalization process: unlike Rauzy induction (see [34], [41]) or the Boshernitzan-Carroll induction process [8], we induce according to the first return map simultaneously on two subintervals. In some stages the two inductions proceed independent of one another and generate the quantities n_k, m_k (of the expansion of $T^k(\alpha, \beta)$), while in other instances the induction process depends on an inequality involving both intervals, and gives rise to the quantity ϵ_k.

This combinatorial construction gives a recursive method of generating three sequences of nested Rokhlin stacks which describe the system from a measure-theoretic point of view and which in turn gives an explicit characterization of the eigenvalues of the associated unitary operator. In [17] we obtain necessary and sufficient conditions for weak mixing which, in addition to unifying all previously known examples, allow us to exhibit new interesting examples of weakly mixing three-interval exchanges. Finally our methods provide affirmative answers to two long standing questions posed by W.A. Veech in [42] on the existence of three-interval exchanges having irrational eigenvalues and discrete spectrum.

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