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Semi-infinite cohomology and superconformal algebras


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SEMI-INFINITE COHOMOLOGY
AND SUPERCONFORMAL ALGEBRAS

by Elena POLETAEVA

1. Introduction.

B. Feigin and E. Frenkel have introduced a semi-infinite analogue of the Weil complex based on the space

\[ W^{\infty, +}(g) = S^{\infty, +}(g) \otimes \Lambda^{\infty, +}(g). \]

In their construction \( g = \bigoplus_{n \in \mathbb{Z}} g_n \) is a graded Lie algebra, \( S^{\infty, +}(g) \) and \( \Lambda^{\infty, +}(g) \) are some semi-infinite analogues of the symmetric and exterior power modules. [FF]. As in the classical case, two differentials, \( d \) and \( h \), are defined on \( W^{\infty, +}(g) \). They are analogous to the differential in Lie algebra (co)homology and the Koszul differential, respectively. The semi-infinite Weil complex

\[ \{W^{\infty, +}(g), \ d + h\} \]

is acyclic similarly to the classical Weil complex. The cohomology of the complex

\[ \{W^{\infty, +}(g), \ d\} \]

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is called the semi-infinite cohomology of $\mathfrak{g}$ with coefficients in its “adjoint semi-infinite symmetric powers” $H^{\infty,*}(\mathfrak{g}, S^{\infty,*}(\mathfrak{g}))$. One can also define the relative semi-infinite Weil complex $W^{\infty,*}_{rel}(\mathfrak{g})$ (relatively $\mathfrak{g}_0$), and the relative semi-infinite cohomology $H^{\infty,*}(\mathfrak{g}, \mathfrak{g}_0, S^{\infty,*}(\mathfrak{g}))$, [FF].

E. Getzler has shown that the semi-infinite Weil complex of the Virasoro algebra admits an action of the $N = 2$ superconformal algebra, [G].

Recall that a superconformal algebra (SCA) is a simple complex Lie superalgebra such that it contains the centerless Virasoro algebra (i.e. the Witt algebra) $\text{Witt} = \oplus_{n \in \mathbb{Z}} CL_n$ as a subalgebra, and has growth 1. The $\mathbb{Z}$-graded superconformal algebras are ones for which $\text{ad}L_0$ is diagonalizable with finite-dimensional eigenspaces, [KL]:

$$s = \oplus_j s_j, s_j = \{x \in s \mid [L_0, x] = jx\}. \quad (1.4)$$

In this work we consider the semi-infinite Weil complex constructed for the next natural (after the Virasoro algebra) class of graded Lie algebras: the loop algebras of the complex finite-dimensional Lie algebras. The action of the Virasoro algebra on such complex is ensured by the fact that it has a structure of a vertex operator superalgebra (see [Ak]).

Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra, and $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the corresponding loop algebra. We obtain a representation of the $N = 2$ SCA in the semi-infinite Weil complex $W^{\infty,*}(\tilde{\mathfrak{g}})$ and in the semi-infinite cohomology $H^{\infty,*}(\tilde{\mathfrak{g}}, S^{\infty,*}(\tilde{\mathfrak{g}}))$ with central charge $3\dim \mathfrak{g}$. We extend the representation of the $N = 2$ SCA in $W^{\infty,*}(\tilde{\mathfrak{g}})$ to a representation of the one-parameter family $\tilde{S}'(2, \alpha)$ of deformations of the $N = 4$ SCA (see [Ad] and [KL]). In the case, when $\mathfrak{g}$ is endowed with a non-degenerate invariant symmetric bilinear form, we obtain a representation of $\tilde{S}'(2, 0)$ in $H^{\infty,*}(\tilde{\mathfrak{g}}, S^{\infty,*}(\tilde{\mathfrak{g}}))$. Finally, there exists a representation of a central extension of the Lie superalgebra of all derivations of $S'(2, 0)$ in the relative semi-infinite cohomology $H^{\infty,*}(\tilde{\mathfrak{g}}, \mathfrak{g}_0, S^{\infty,*}(\tilde{\mathfrak{g}}))$.

It was shown in [FGZ] that the cohomology of the relative semi-infinite complex $C^*_{\infty}(\mathfrak{l}, \mathfrak{l}_0, V)$, where $\mathfrak{l}$ is a complex graded Lie algebra, and $V$ is a graded Hermitian $\mathfrak{l}$-module, has (under certain conditions) a structure analogous to that of the de Rham cohomology in Kähler geometry.

Recall that given a compact Kähler manifold $M$, there exists a number of classical operators on the space of differential forms on $M$, such as the differentials $\partial, \bar{\partial}, d, d_c$, their corresponding adjoint operators and the associated Laplacians (see [GH]). There also exists an action of $\mathfrak{sl}(2)$ on...
$H^\ast(M)$ according to the Lefschetz theorem. All these operators satisfy a series of identities known as Hodge identities, [GH]. Naturally, the classical operators form a finite-dimensional Lie superalgebra.

We show that given a complex finite-dimensional Lie algebra $\mathfrak{g}$ endowed with a non-degenerate invariant symmetric bilinear form, there exist the analogues of the classical operators on the complex $W_{rel}^{\infty (+)}(\mathfrak{g})$. We prove that the exterior derivations of $S'(2, 0)$ form an $\mathfrak{sl}(2)$, and observe that they define an $\mathfrak{sl}(2)$-module structure on $H_{\infty}^{\infty (+)}(\mathfrak{g}, \mathfrak{g}_0, S_{\infty}^{\infty (+)}(\mathfrak{g}))$, which is the analogue of the $\mathfrak{sl}(2)$-module structure on the de Rham cohomology in Kähler geometry.

The action of $\hat{S}'(2, 0)$ provides $H_{\infty}^{\infty (+)}(\mathfrak{g}, \mathfrak{g}_0, S_{\infty}^{\infty (+)}(\mathfrak{g}))$ with eight series of quadratic operators. In particular, they include the semi-infinite Koszul differential $h$, and the semi-infinite analogue of the homotopy operator (cf. [Fu]). We prove that the degree zero part of the $\mathbb{Z}$-grading of $S'(2, 0)$ defined by the element $L_0 \in \text{Witt}$, is isomorphic to the Lie superalgebra of classical operators in Kähler geometry.

It would be interesting to interpret the superconformal algebra $S'(2, 0)$ as “affinization” of the classical operators in the case of an infinite-dimensional manifold.

This work is partly based on [P1]-[P3].

2. Semi-infinite Weil complex.

The semi-infinite Weil complex of a graded Lie algebra was introduced by B. Feigin and E. Frenkel in [FF]. Recall the necessary definitions. More generally, let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be a graded vector space over $\mathbb{C}$, such that $\dim V_n < \infty$. Let $V' = \bigoplus_{n \in \mathbb{Z}} V'_n$ be the restricted dual of $V$. The linear space $V \oplus V'$ carries non-degenerate skew-symmetric and symmetric bilinear forms: $(\cdot, \cdot)$ and $\{\cdot, \cdot\}$. Let $H(V)$ and $C(V)$ be the quotients of the tensor algebra $T^* (V \otimes V')$ by the ideals generated by the elements of the form $xy - yx - (x, y)$ and $xy + yx - \{x, y\}$, respectively, where $x, y \in V \oplus V'$. We fix $K \in \mathbb{Z}$. Let $V = V_+ \oplus V_-$ be the corresponding polarization of $V$: $V_+ = \bigoplus_{n > K} V_n$, $V_- = \bigoplus_{n < K} V_n$.

The symmetric algebra $S^\ast (V_+ \oplus V_-)$ is a subalgebra of $H(V)$ and the exterior algebra $\Lambda^\ast (V_+ \oplus V_-)$ is a subalgebra of $C(V)$. Let $S_{\infty}^{\infty (+)}(V)$, $\Lambda_{\infty}^{\infty (+)}(V)$ be the representations of $H(V)$ and $C(V)$ induced from the trivial representations $1_S >$ and $1_A >$ of $S^\ast (V_+ \oplus V_-)$ and of $\Lambda^\ast (V_+ \oplus V_-)$,
respectively. Thus we obtain some semi-infinite analogues of symmetric and exterior power modules. Denote the actions of $H(V)$ and $C(V)$ on these modules by $\beta(x)$, $\gamma(x')$ and $\epsilon(x)$, $\epsilon(x')$, respectively, for $x \in V$, $x' \in V'$. Notice that each element of $S^{\infty+*}(V)$ and of $\Lambda^{\infty+*}(V)$ is a finite linear combination of the monomials of the type $\gamma(x'_1) \cdots \gamma(x'_k) \beta(y_1) \cdots \beta(y_m) 1_S$ and of the type $\epsilon(x'_1) \cdots \epsilon(x'_k) \tau(y_1) \cdots \tau(y_m) 1_A$, respectively, where $x'_1, \ldots, x'_k \in V_+'$, $y_1, \ldots, y_m \in V_-$. Let $\text{Deg}\epsilon(x') = \text{Deg} \gamma(x') = 1$, and $\text{Deg} \tau(x) = \text{Deg} \beta(x) = -1$. Correspondingly, we obtain $\mathbb{Z}$-gradings on the spaces of semi-infinite power modules: $S^{\infty+*}(V) = \oplus_{i \in \mathbb{Z}} S^{\infty+i}(V)$, $\Lambda^{\infty+*}(V) = \oplus_{i \in \mathbb{Z}} \Lambda^{\infty+i}(V)$.

Let $\{e_i\}_{i \in \mathbb{Z}}$ be a homogeneous basis of $V$ so that if $i \in \mathbb{Z}$, then $e_i \in V_n$ for some $n \in \mathbb{Z}$, and if $e_i \in V_n$, then $e_{i+1} \in V_n$ or $e_{i+1} \in V_{n+1}$. Let $\{e'_i\}_{i \in \mathbb{Z}}$ be the dual basis. Let $i_0 \in \mathbb{Z}$ be such that $e^{i_0} \in V_K$ and $e^{i_0+1} \in V_{K+1}$.

Notice that one can think of $\Lambda^{\infty+*}(V)$ as the vector space spanned by the elements $w = e'_{i_1} \wedge e'_{i_2} \wedge \ldots$ such that there exists $N(w) \in \mathbb{Z}$ such that $i_{n+1} = i_n - 1$ for $n > N(w)$. Then $1_A = e'_{i_0} \wedge e'_{i_0-1} \wedge \ldots$ is a vacuum vector in this space. The actions of $\epsilon(x')$, $\tau(x)$ are, respectively, the exterior multiplication and contraction in the space of semi-infinite exterior products.

Let $\mathfrak{g} = \oplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ be a graded Lie algebra over $\mathbb{C}$, such that $\dim \mathfrak{g}_n < \infty$. Let $\phi$ be a representation of $\mathfrak{g}$ in $V$ so that

\begin{equation}
\phi(\mathfrak{g}_n)V_k \subset V_{k+n}.
\end{equation}

One can define the projective representations $\rho$ and $\pi$ of $\mathfrak{g}$ in $\Lambda^{\infty+*}(V)$ and $S^{\infty+*}(V)$, respectively

\begin{align}
\rho(x) &= \sum_{i \in \mathbb{Z}} \tau(\phi(x)e_i)e(e'_i) ; , \\
\pi(x) &= \sum_{i \in \mathbb{Z}} \beta(\phi(x)e_i)\gamma(e'_i) ; ,
\end{align}

where $x \in \mathfrak{g}$, and where the double colons $: :$ denote a normal ordering operation:

\begin{align}
: \tau(e_j)e(e'_i) : &= \begin{cases} 
\tau(e_j)e(e'_i) & \text{if } i \leq i_0 \\
-\epsilon(e'_i)\tau(e_j) & \text{if } i > i_0
\end{cases} , \\
: \beta(e_j)\gamma(e'_i) : &= \begin{cases} 
\beta(e_j)\gamma(e'_i) & \text{if } i \leq i_0 \\
\gamma(e'_i)\beta(e_j) & \text{if } i > i_0
\end{cases} .
\end{align}
Thus
\begin{equation}
\rho(x)1_A = \pi(x)1_S = 0 \text{ for } x \in g_0
\end{equation}
and
\begin{equation}
\begin{aligned}
&[\rho(x), \rho(y)] = \rho([x, y]) + c_A(x, y), \\
&[\pi(x), \pi(y)] = \pi([x, y]) + c_S(x, y),
\end{aligned}
\end{equation}
where \( x, y \in g \) and \( c_A, c_S \) are 2-cocycles. Notice that \( -c_S \). Let
\begin{equation}
W_{\infty}^{\infty} + * (V) = S_{\infty}^{\infty} + * (V) \otimes \Lambda_{\infty}^{\infty} + * (V).
\end{equation}
Since the cocycles corresponding to the projective representations cancel, the representation \( \theta(x) = \rho(x) + \pi(x) \) of \( g \) in \( W_{\infty}^{\infty} + * (V) \) is well-defined. We define a \( \mathbb{Z} \)-grading on \( W_{\infty}^{\infty} + * (V) \) setting
\begin{equation}
W_{\infty}^{\infty} + i (V) = \bigoplus_{2l + j = i} S_{\infty}^{\infty} + l (V) \otimes \Lambda_{\infty}^{\infty} + j (V).
\end{equation}
Let \( V = g = \oplus_{n \in \mathbb{Z}} g_n \) and \( \phi \) be the adjoint representation of \( g \). We define two differentials on the space \( W_{\infty}^{\infty} + * (g) \):
\begin{equation}
\begin{aligned}
d &= \sum_{i < j} : \tau([e_i, e_j])e_i'(e_j')e_i e_j : + \sum_{i, j} : \beta([e_j, e_i]) \gamma(e_i')e_j : , \\
h &= \sum_i \gamma(e_i') \tau(e_i).
\end{aligned}
\end{equation}
We obtain the semi-infinite Weil complex
\begin{equation}
\{ W_{\infty}^{\infty} + * (g), \ d + h \}.
\end{equation}
The differential \( d \) is the analogue of the classical differential for the Lie algebra (co)homology, and \( h \) is the analogue of the Koszul differential. Notice that
\begin{equation}
d^2 = 0, h^2 = 0, [d, h] = 0, (d + h)^2 = 0.
\end{equation}
Notice also that if \( g \) is a finite-dimensional Lie algebra, then applying the definitions given above to the polarization \( g = g_+ \oplus g_- \), where \( g_+ = g, g_- = 0 \), we obtain the classical Weil complex.

As in the case of the classical Weil complex, one can construct two filtrations, \( F_1^p \) and \( F_2^p \), on \( W_{\infty}^{\infty} + * (g) \):
\begin{equation}
F_1^p = \bigoplus_{l + j \geq p} S_{\infty}^{\infty} + l (g) \otimes \Lambda_{\infty}^{\infty} + j (g), \quad F_2^p = \bigoplus_{2l \geq p} S_{\infty}^{\infty} + l (g) \otimes \Lambda_{\infty}^{\infty} + * (g).
\end{equation}
For filtration \( F_1^* \) the complex is acyclic, the second term of the spectral sequence associated to filtration \( F_2^* \) is the semi-infinite cohomology of Lie algebra \( \mathfrak{g} \) with coefficients in its “adjoint semi-infinite symmetric powers” \( \mathcal{H}^{\infty, +*}(\mathfrak{g}, S^{\infty, +*}(\mathfrak{g})) \) (see [FF]). Let

\[
W^{\infty, +*}_{\text{rel}}(V) = \{ w \in W^{\infty, +*}(V) \mid \tau(x)w = 0 \text{ for all } x \in V_0, \theta(x)w = 0 \text{ for all } x \in \mathfrak{g}_0 \}.
\]

The differential \( d \) preserves the space \( W^{\infty, +*}_{\text{rel}}(\mathfrak{g}) \) since

\[
[d, \tau(x)] = d\tau(x) + \tau(x)d = \theta(x),
\]

and

\[
[d, \theta(x)] = 0,
\]

for any \( x \in \mathfrak{g} \). The complex \( \{ W^{\infty, +*}_{\text{rel}}(\mathfrak{g}), d \} \) is called the relative semi-infinite Weil complex. Its cohomology is called the relative semi-infinite cohomology \( \mathcal{H}^{\infty, +*}(\mathfrak{g}, \mathfrak{g}_0, S^{\infty, +*}(\mathfrak{g})) \).

We fix \( K = 0 \) from this point on. Correspondingly, \( V = V_+ \oplus V_- \), where \( V_+ = \bigoplus_{n > 0} V_n, V_- = \bigoplus_{n \leq 0} V_n \).

### 3. The \( N = 2 \) superconformal algebra.

Recall that the \( N = 2 \) SCA is spanned by the Virasoro generators \( \mathcal{L}_n \), the Heisenberg generators \( H_n \), two fermionic fields \( G^+_r \), and a central element \( C \), where \( n \in \mathbb{Z}, r \in \mathbb{Z} + 1/2 \), and where the non-vanishing commutation relations are as follows, [FST]:

\[
[\mathcal{L}_n, \mathcal{L}_m] = (n - m)\mathcal{L}_{n+m} + \frac{C}{12}(n^3 - n)\delta_{n,-m},
\]

\[
[\mathcal{L}_n, H_m] = -mH_{n+m}, [\mathcal{L}_n, G^+_r] = \left( \frac{n}{2} - r \right) G^+_{n+r},
\]

\[
[G^+_r, G^-_s] = 2\mathcal{L}_{r+s} + (r - s)H_{r+s} + \frac{C}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r,-s},
\]

\[
[H_n, H_m] = \frac{C}{3} n\delta_{n,-m}, [H_n, G^+_r] = \pm G^+_n G^+_{n+r}.
\]

Let \( \text{Witt} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}L_i \) be the Witt algebra:

\[
[L_i, L_j] = (i - j)L_{i+j}.
\]
Let $\lambda, \mu \in \mathbb{C}$. Let $\mathcal{F}_{\lambda, \mu} = \oplus_{m \in \mathbb{Z}} \mathbb{C} u_m$ be a module over Witt defined as follows:

\begin{equation}
\phi(L_n) u_m = \left( -m + \mu - (n - 1)\lambda \right) u_{n+m}.
\end{equation}

**Remark 3.1.** — The module $\mathcal{F}_{\lambda, \mu} = \oplus_{m \in \mathbb{Z}} \mathbb{C} u_m$ is isomorphic to the module $\mathcal{F}_{-\lambda, \mu+1} = \oplus_{j \in \mathbb{Z}} \mathbb{C} f_j$ over the Witt algebra defined in [Fu]. The isomorphism is given by the correspondence $u_m \leftrightarrow f_{-m-1}$.

**Theorem 3.1.** — The space $W^{\infty,+}((\mathcal{F}_{\lambda, \mu})$ is a module over the $N = 2$ SCA with central charge $3 - 6\lambda$.

**Proof.** — Set

\begin{equation}
h_n = \frac{1}{\sqrt{2}} G^+_{n-\frac{1}{2}}, p_n = \frac{1}{\sqrt{2}} G^-_{n+\frac{1}{2}}.
\end{equation}

We define a representation of Witt in $W^{\infty,+}((\mathcal{F}_{\lambda, \mu})$ as follows:

\begin{equation}
\theta(L_n) = \sum_{m \in \mathbb{Z}} \left( -m + \mu - n\lambda + \lambda \right) \left( \tau(u_{m+n}) \varepsilon(u'_m) : + \beta(u_{m+n}) \gamma(u'_m) : \right).
\end{equation}

Let us extend $\theta$ to a representation of the $N = 2$ SCA in $W^{\infty,+}((\mathcal{F}_{\lambda, \mu})$:

\begin{equation}
\theta(H_n) = \lambda \sum_{m \in \mathbb{Z}} \tau(u_m) \varepsilon(u'_{m+n}) : \\
+ (\lambda - 1) \sum_{m \in \mathbb{Z}} \beta(u_m) \gamma(u'_{m+n}) : + \mu \delta_n,0,
\end{equation}

\begin{align*}
\theta(h_n) &= \sum_{m \in \mathbb{Z}} \gamma(u'_{m+n}) \tau(u_m), \\
\theta(p_n) &= \sum_{m \in \mathbb{Z}} (m - \mu - (n + 1)\lambda) \beta(u_{m-n}) \varepsilon(u'_m), \\
\theta(l_n) &= -\theta(L_n) + \frac{n + 1}{2} \theta(H_n).
\end{align*}

We calculate the central charge by checking the commutation relations on the vacuum vector $1 = 1_S \otimes 1_{\lambda}$. Let $n > 0$. Then

\begin{equation}
\theta([H_n, H_{-n}]) 1 = -\theta(H_{-n}) \theta(H_n) 1 \\
= -\theta(H_{-n}) \left( \lambda \sum_{m = 1-n}^{0} \tau(u_m) \varepsilon(u'_{m+n}) \right)
\end{equation}
Thus the central charge is $3 - 6\lambda$. The other commutation relations on the vacuum vector $1$ are calculated in the same way.

$\square$

Remark 3.2. — In the case $\lambda = 1$, the module $\mathcal{F}_{\lambda,\mu}$ is the adjoint representation of Witt. Thus we obtain a representation of the $N = 2$ SCA in the semi-infinite Weil complex of the Witt algebra (cf. [G]).

**Theorem 3.2.** — Let $V$ be a complex finite-dimensional vector space, $\tilde{V} = V \otimes \mathbb{C}[t, t^{-1}]$. There exists a representation of the $N = 2$ SCA in $W^{2\mathbb{Z}}(\tilde{V})$ with central charge $3\dim V$.

**Proof.** — There is the natural $\mathbb{Z}$-grading $\tilde{V} = \oplus_{n \in \mathbb{Z}} \tilde{V}_n$, where $\tilde{V}_n = V \otimes t^n$. Let $u$ run through a fixed basis of $V$, $u_n$ stand for $u \otimes t^n$, and let $\{u'_n\}$ be the dual basis of $V'$. Define the following quadratic expansions by analogy with (3.5) and (3.6), where $\lambda = 0, \mu = 0$:

$$L_n = - \sum_u \sum_{m \in \mathbb{Z}} (m : \tau(u_{m+n}) \varepsilon(u'_m) : + m : \beta(u_{m+n}) \gamma(u'_m) : )$$

$$H_n = - \sum_u \sum_{m \in \mathbb{Z}} : \gamma(u'_{m+n}) \beta(u_m) :$$

$$(3.9) \quad h_n = \sum_u \sum_{m \in \mathbb{Z}} \gamma(u'_{m+n}) \tau(u_m),$$

$$p_n = \sum_u \sum_{m \in \mathbb{Z}} m \beta(u_{m-n}) \varepsilon(u'_m).$$

Set

$$\mathcal{Q}_n = -L_{-n} + \frac{n + 1}{2} H_n.$$
Then $\mathfrak{L}_n, H_n, \mathfrak{h}_n,$ and $\mathfrak{p}_n$ span the centerless $N = 2$ SCA.

Let $n > 0$. Then $H_{-n} \mathbf{1} = 0$. Hence

\begin{equation}
\tag{3.11}
[H_n, H_{-n}] \mathbf{1} = -H_{-n} \left( \sum_{u, m=1}^{0} \gamma(u'_{m+n})\beta(u_m) \right) \mathbf{1}
\end{equation}

\begin{align*}
&= \left( -\sum_{u, m=1}^{n} \gamma(u'_{m-n})\beta(u_m) \right) \left( \sum_{u, m=1}^{0} \gamma(u'_{m+n})\beta(u_m) \right) \mathbf{1} \\
&= -\sum_{u, m=1}^{0} \gamma(u'_{m})\beta(u_{m+n})\gamma(u'_{m+n})\beta(u_m) \mathbf{1} \\
&= -\text{dim}V(-n) \mathbf{1},
\end{align*}

since $\gamma(u'_i)\beta(u_i) - \beta(u_i)\gamma(u'_i) = 1$. Notice that

\begin{equation}
\tag{3.12}
[H_n, H_m] \mathbf{1} = 0, \text{ if } m \neq -n.
\end{equation}

Hence

\begin{equation}
\tag{3.13}
[H_n, H_m] \mathbf{1} = n\text{dim}V \delta_{n,-m} \mathbf{1}.
\end{equation}

Thus the central charge is $3\text{dim}V$. \hfill \Box

**Corollary 3.1.** — Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra, let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$. There exists a representation of the $N = 2$ SCA in $H^{\infty + \ast}((\tilde{\mathfrak{g}}), S^{\infty + \ast}((\tilde{\mathfrak{g}})))$ with central charge $3\text{dim}g$.

**Proof.** — We will show that the expansions (3.9) commute with the differential $d$. Recall that

\begin{equation}
\tag{3.14}
d = d^{(1)} + d^{(2)},
\end{equation}

where

\begin{align*}
d^{(1)} &= \frac{1}{2} \sum_{u,v,i,j} \tau([u_i,v_j]) \varepsilon(v'_j) \varepsilon(u'_i) : \\
d^{(2)} &= \sum_{u,v,i,j} \beta([u_i,v_j]) \gamma(v'_j) \varepsilon(u'_i) :
\end{align*}

$u, v$ run through a fixed basis of $\mathfrak{g}$, and $i, j \in \mathbb{Z}$. Then

\begin{align*}
\tag{3.16}
[L_n, d^{(1)}] &= \frac{1}{2} \sum_{u,v,i,j} \tau([u, v]_{i+j+n}) \varepsilon(v'_j) \varepsilon(u'_i) : \\
&+ \tau([u_i, v_j])(j - n) \varepsilon(v'_{j-n}) \varepsilon(u'_i) : \\
&+ \tau([u_i, v_j]) \varepsilon(v'_j)(i - n) \varepsilon(u'_{i-n}) := 0
\end{align*}

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and

\[(3.17) \quad [L_n, d^{(2)}] = \sum_{u,v,i,j} : -(i+j)\beta([u,v]_{i+j+n})\gamma(v'_j)\varepsilon(u'_i) : + \beta([u,v]_{i})\gamma(v'_j)(j-n)\varepsilon(u'_i) : + \beta([u,v]_{j})\gamma(v'_j)(i-n)\varepsilon(u'_i) :) = 0.\]

Clearly,

\[(3.18) \quad [H_n, d^{(1)}] = 0,\]

and

\[(3.19) \quad [H_n, d^{(2)}] = \sum_{u,v,i,j} : -\beta([u,v]_{i+j-n})\gamma(v'_j)\varepsilon(u'_i) : + \beta([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) : = 0.\]

Next,

\[(3.20) \quad [h_n, d^{(1)}] = (1/2) \sum_{u,v,i,j} : -\tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) : + \tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) :) = - \sum_{u,v,i,j} : \tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) :,\]

\[(3.21) \quad [h_n, d^{(2)}] = \sum_{u,v,i,j} : \tau([u,v]_{i+j-n})\gamma(v'_j)\varepsilon(u'_i) : + \beta([u,v]_{i+j})\gamma(v'_j)\gamma(u'_{i+n}) : = \sum_{u,v,i,j} : \tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) :,\]

since \(\sum_{u,v,i,j} : \beta([u,v]_{i+j})\gamma(v'_j)\gamma(u'_{i+n}) :) = 0.\) Hence

\[(3.22) \quad [h_n, d^{(2)}] = -[h_n, d^{(1)}].\]

Finally,

\[(3.23) \quad [p_n, d^{(1)}] = (1/2) \sum_{u,v,i,j} : (i+j)\beta([u,v]_{i+j-n})\varepsilon(v'_j)\varepsilon(u'_i) :,\]

\[(3.24) \quad [p_n, d^{(2)}] = \sum_{u,v,i,j} : -\beta([u,v]_{i+j}) (j+n)\varepsilon(v'_{j+n})\varepsilon(u'_i) :,\]
4. The superconformal algebras $S'(2, \alpha)$. 

Recall the necessary definitions, [KL]. Let $W(N)$ be the superalgebra of all derivations of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $\Lambda(N)$ is the Grassmann algebra in $N$ variables $\theta_1, \ldots, \theta_N$, and $p(t) = 0$, $p(\theta_i) = 1$ for $i = 1, \ldots, N$. Let $\partial_i$ stand for $\partial/\partial \theta_i$, and $\partial_t$ stand for $\partial/\partial t$. Let

\begin{equation}
S(N, \alpha) = \{D \in W(N) \mid \text{Div}(t^\alpha D) = 0\} \quad \text{for } \alpha \in \mathbb{C}.
\end{equation}

Recall that

\begin{equation}
\text{Div}\left( f \partial_t + \sum_{i=1}^{N} f_i \partial_i \right) = \partial_t f + \sum_{i=1}^{N} (-1)^{p(f_i)} \partial_i f_i
\end{equation}

where $f, f_i \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, and

\begin{equation}
\text{Div}(fD) = DF + f \text{Div}D,
\end{equation}

where $f$ is an even function. Let $S'(N, \alpha) = [S(N, \alpha), S(N, \alpha)]$ be the derived superalgebra. Assume that $N > 1$. If $\alpha \notin \mathbb{Z}$, then $S(N, \alpha)$ is simple, and if $\alpha \in \mathbb{Z}$, then $S'(N, \alpha)$ is a simple ideal of $S(N, \alpha)$ of codimension 1:

\begin{equation}
0 \rightarrow S'(N, \alpha) \rightarrow S(N, \alpha) \rightarrow \mathbb{C}t^{-\alpha} \theta_1 \cdots \theta_N \partial_t \rightarrow 0.
\end{equation}

Notice that

\begin{equation}
S(N, \alpha) \cong S(N, \alpha + n) \quad \text{for } n \in \mathbb{Z}.
\end{equation}

The superalgebra $S'(N, \alpha)$ has, up to equivalence, only one non-trivial 2-cocycle if and only if $N = 2$, which is important for our task. Let

\begin{equation}
\{\mathcal{L}_n^\alpha, E_n, H_n, F_n, h_n^\alpha, p_n, x_n, y_n^\alpha\}_{n \in \mathbb{Z}}
\end{equation}
be the basis of $S'(2, \alpha)$ defined as follows:

\begin{align}
\mathcal{L}_n^\alpha &= -t^n (t\partial_t + \frac{1}{2} (n + \alpha + 1) (\theta_1 \partial_1 + \theta_2 \partial_2)), \\
E_n &= t^n \theta_2 \partial_1, \\
H_n &= t^n (\theta_2 \partial_2 - \theta_1 \partial_1), \\
F_n &= t^n \theta_1 \partial_2, \\
h_n^\alpha &= t^n \theta_2 \partial_t - (n + \alpha) t^{-1} \theta_1 \theta_2 \partial_1, \\
p_n &= -t^{n+1} \partial_2, \\
x_n &= t^{n+1} \partial_1, \\
y_n^\alpha &= t^n \theta_1 \partial_t + (n + \alpha) t^{-1} \theta_1 \theta_2 \partial_2.
\end{align}

The non-vanishing commutation relations between these elements are

\begin{align}
[\mathcal{L}_n^\alpha, \mathcal{L}_k^\alpha] &= (n-k) \mathcal{L}_{n+k}^\alpha, \\
[E_n, F_k] &= H_{n+k}, [H_n, E_k] = 2E_{n+k}, [H_n, F_k] = -2F_{n+k}, \\
[\mathcal{L}_n^\alpha, E_k] &= -kE_{n+k}, [\mathcal{L}_n^\alpha, H_k] = -kH_{n+k}, [\mathcal{L}_n^\alpha, F_k] = -kF_{n+k}, \\
[\mathcal{L}_n^\alpha, h_k^\alpha] &= \frac{1}{2} (n - 2k + 1 - \alpha) h_{n+k}^\alpha, [\mathcal{L}_n^\alpha, p_k] \\
&= \frac{1}{2} (n - 2k - 1 + \alpha) p_{n+k}, \\
[\mathcal{L}_n^\alpha, x_k] &= \frac{1}{2} (n - 2k - 1 + \alpha) x_{n+k}, [\mathcal{L}_n^\alpha, y_k^\alpha] \\
&= \frac{1}{2} (n - 2k + 1 - \alpha) y_{n+k}, \\
[E_n, y_k^\alpha] &= h_{n+k}^\alpha, [F_n, h_k^\alpha] = y_{n+k}^\alpha, [E_n, p_k] \\
&= x_{n+k}, [F_n, x_k] = p_{n+k}, \\
[H_n, h_k^\alpha] &= h_{n+k}^\alpha, [H_n, y_k^\alpha] = -y_{n+k}^\alpha, [H_n, x_k] \\
&= x_{n+k}, [H_n, p_k] = -p_{n+k}, \\
h_n^\alpha, x_k &= (k + 1 - n - \alpha) E_{n+k}, [p_n, y_k^\alpha] \\
&= (k - n - 1 + \alpha) F_{n+k}, \\
h_n^\alpha, p_k &= \mathcal{L}_{n+k}^\alpha - \frac{1}{2} (k - n + 1 - \alpha) H_{n+k}, \\
x_n, y_k^\alpha &= -\mathcal{L}_{n+k}^\alpha + \frac{1}{2} (k - n - 1 + \alpha) H_{n+k}.
\end{align}

A non-trivial 2-cocycle on $S'(2, \alpha)$ is

\begin{align}
c(\mathcal{L}_n^\alpha, \mathcal{L}_k^\alpha) &= \frac{C}{12} n(n^2 - 1) \delta_{n, -k},
\end{align}
Let \( \hat{S}'(2, \alpha) \) be the corresponding central extension of \( S'(2, \alpha) \). In particular, \( \hat{S}'(2, 0) \) is isomorphic to the \( N = 4 \) SCA (see [Ad]).

**Remark 4.1** — Notice that

\[
S'(2, \alpha) = \text{Witt} \times \mathfrak{sl}(2), \quad \text{where} \quad \text{Witt} = \langle \mathfrak{L}^\alpha_{n} \rangle_{n \in \mathbb{Z}} \oplus \mathfrak{sl}(2) = \langle E_{n}, H_{n}, F_{n} \rangle_{n \in \mathbb{Z}},
\]

and

\[
S'(2, \alpha)_{1} = \langle h_{n}^\alpha, y_{n}^\alpha \rangle_{n \in \mathbb{Z}} \oplus \langle p_{n}, x_{n} \rangle_{n \in \mathbb{Z}}
\]

is a direct sum of two standard (odd) \( \mathfrak{sl}(2) \)-modules.

**Remark 4.2** — For any \( \alpha \in \mathbb{C} \) one can consider the subalgebra of \( \hat{S}'(2, \alpha) \), spanned by \( \mathfrak{L}^\alpha_{n}, H_{n}, h_{n}^\alpha, p_{n} \), and \( \mathbb{C} \). Thus we obtain a one-parameter family of superalgebras, which are isomorphic to the \( N = 2 \) SCA. The isomorphism

\[
\varphi : \langle \mathfrak{L}^\alpha_{n}, H_{n}, h_{n}^\alpha, p_{n}, \mathbb{C} \rangle \longrightarrow \langle \mathfrak{L}_{n}, H_{n}, h_{n}, p_{n}, \mathbb{C} \rangle
\]

is given as follows:

\[
\varphi(\mathfrak{L}^\alpha_{n}) = \mathfrak{L}_{n} - \frac{\alpha}{2} H_{n} + \frac{\alpha^2}{24} \delta_{n,0} \mathbb{C},
\]

\[
\varphi(H_{n}) = H_{n} - \frac{\alpha}{6} \delta_{n,0} \mathbb{C},
\]

\[
\varphi(h_{n}^\alpha) = h_{n}, \quad \varphi(p_{n}) = p_{n}, \quad \varphi(\mathbb{C}) = \mathbb{C}.
\]

Notice that formulae (4.13) correspond to the spectral flow transformation for the \( N = 2 \) SCA (cf. [FST]).

Let \( \text{Der}S'(2, \alpha) \) be the Lie superalgebra of all derivations of \( S'(2, \alpha) \), and \( \text{Der}_{\text{ext}}S'(2, \alpha) \) be the exterior derivations of \( S'(2, \alpha) \) (see [Fu]).
THEOREM 4.1.

1) If \( \alpha \in \mathbb{Z} \), then \( \text{Der}_{\text{ext}}S'(2, \alpha) \cong \mathfrak{sl}(2) = \langle \mathcal{E}, \mathcal{H}, \mathcal{F} \rangle \), where

\[
[\mathcal{H}, \mathcal{E}] = 2\mathcal{E}, [\mathcal{H}, \mathcal{F}] = -2\mathcal{F}, [\mathcal{E}, \mathcal{F}] = \mathcal{H}.
\]

The action of \( \mathfrak{sl}(2) \) is given as follows:

\[
[\mathcal{E}, h_k] = x_{k-1+\alpha}, [\mathcal{E}, y_k] = p_{k-1+\alpha};
[\mathcal{F}, x_k] = h_{k+1-\alpha}, [\mathcal{F}, p_k] = y_{k+1-\alpha};
[\mathcal{H}, x_k] = x_k, [\mathcal{H}, h_k] = -h_k,
[\mathcal{H}, p_k] = p_k, [\mathcal{H}, y_k] = -y_k.
\]

2) If \( \alpha \in \mathbb{C} \setminus \mathbb{Z} \), then \( \text{Der}_{\text{ext}}S'(2, \alpha) = \langle \mathcal{H} \rangle \).

Proof. — Recall that the exterior derivations of a Lie (super) algebra can be identified with its first cohomology with coefficients in the adjoint representation (see [Fu]). Thus

\[
\text{Der}_{\text{ext}}S'(2, \alpha) \cong H^1(S'(2, \alpha), S'(2, \alpha)).
\]

The superalgebra \( S'(2, \alpha) \) has the following \( \mathbb{Z} \pm \alpha \)-grading \( \text{deg} \):

\[
\text{deg} e_n^\alpha = n, \text{deg} e_n = n + 1 - \alpha, \text{deg} f_n = n - 1 + \alpha,
\text{deg} h_n = n, \text{deg} h_n^\alpha = n, \text{deg} p_n = n, \text{deg} x_n = n + 1 - \alpha,
\text{deg} y_n^\alpha = n - 1 + \alpha.
\]

Let

\[
L_0 = -\mathcal{E}_0^\alpha + \frac{1}{2} (1 - \alpha) H_0.
\]

Then

\[
[L_0, s] = (\text{deg} s) s
\]

for a homogeneous \( s \in S'(2, \alpha) \). Accordingly,

\[
[L_0, D] = (\text{deg} D) D
\]

for a homogeneous \( D \in \text{Der}_{\text{ext}}S'(2, \alpha) \). On the other hand, since the action of a Lie superalgebra on its cohomology is trivial (see [Fu]), then one must have

\[
[L_0, D] = 0.
\]
Hence the non-zero elements of $\text{Der}_{\text{ext}}S'(2,\alpha)$ have $\text{deg} = 0$, and they preserve the superalgebra $S'(2,\alpha)_{\text{deg}=0}$. Let $\alpha \in \mathbb{Z}$. Then one can check that the exterior derivations of $S'(2,\alpha)_{\text{deg}=0}$ form an $\mathfrak{sl}(2)$, and extend them to the exterior derivations of $S'(2,\alpha)$ as in (4.15). One should also note that if the restriction of a derivation of $S'(2,\alpha)$ to $S'(2,\alpha)_{\text{deg}=0}$ is zero, then this derivation is inner.

Finally, notice that the exterior derivations $\mathcal{E}$ and $\mathcal{F}$ interchange $\{h_k^a\}$ with $\{x_k\}$. If $\alpha \notin \mathbb{Z}$, then $\text{deg} h_k^\alpha - \text{deg} x_n \notin \mathbb{Z}$ for any $k, n \in \mathbb{Z}$. Hence $\mathcal{E}$ and $\mathcal{F}$ cannot have $\text{deg} = 0$. By this reason, $\text{Der}_{\text{ext}}S'(2,\alpha) = \langle \mathcal{H} \rangle$ for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$.

Remark 4.3. — If $\alpha \in \mathbb{Z}$, then one can identify $\mathcal{F}$ with $-t^{-\alpha}\theta_1\theta_2\partial_t$ (see (4.4)).

5. An action of $\hat{S}'(2,\alpha)$ on the semi-infinite Weil complex of a loop algebra.

We will consider a more general case, i.e. when $V$ is a complex finite-dimensional vector space, and $\tilde{V} = V \otimes \mathbb{C}[t, t^{-1}]$. Let $\text{Der}S'(2,\alpha)$ be a non-trivial central extension of $\text{Der}S'(2,\alpha)$.

Theorem 5.1.

1) The space $W_{\mathbb{F}}^{\alpha,+}(\tilde{V})$, where $\alpha \in \mathbb{C}$, is a module over $\hat{S}'(2,\alpha)$ with central charge $3\dim V$;

2) if $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, then $W_{\mathbb{F}}^{\alpha,+}(\tilde{V})$ is a module over $\text{Der}S'(2,\alpha)$.

Proof. — Let $u$ run through a fixed basis of $V$, $u_n$ stand for $u \otimes t^n$, and $\{u'_n\}$ be the dual basis of $\tilde{V}$. One can define a representation of Witt in $W_{\mathbb{F}}^{\alpha,+}(\tilde{V})$ by analogy with (3.5), where $\lambda = 0, \mu = \alpha/2$:

$$\theta(L_n) = -\sum_u \sum_m \left( m - \frac{\alpha}{2} \right) : \tau(u_{m+n})\epsilon(u'_m) : + : \beta(u_{m+n})\gamma(u'_m) : ,$$

then extend it to a representation of the $N = 2$ SCA, and apply (4.13). We obtain the following representation of $\hat{S}'(2,\alpha)$:

$$\theta(H_n) = -\sum_u \sum_m : \beta(u_m)\gamma(u'_{m+n}) : ,$$

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One can check that the central charge is $3\dim V$ in the same way as in Theorem 3.2.

**THEOREM 5.2.** Let $g$ be a complex finite-dimensional Lie algebra endowed with a non-degenerate invariant symmetric bilinear form. Then $H^{\infty,+ +}(\tilde{g}, S^{\infty,+ +}(\tilde{g}))$ is a module over $S'(2,0)$ with central charge $3\dim g$.

**Proof.** Let $\{v_i\}$ be a basis of $g$ so that with respect to the given form $\langle v_i, v_j \rangle = \delta_{i,j}$. Let $u$ run through this basis. Then by Theorem 5.1, there is a representation of $S'(2,0)$ in $W^{\infty,+ +}(\tilde{g})$. Notice that we can identify the elements of $S'(2,0)$ with the quadratic expansions obtained by putting $\alpha = 0$ in the equations (5.2). One can check that the commutation relations (4.8) (where $\alpha = 0$) are fulfilled. One can notice that

Recall that

\[
\theta(L_n) = -\theta(L_{-n}) + \frac{n+1 - \alpha}{2} \theta(H_n) + \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right) \dim V \epsilon_{n,0},
\]

\[
\theta(h_n^0) = \sum_u \sum_m \gamma(u_{m+n}) \tau(u_m),
\]

\[
\theta(p_n) = \sum_u \sum_m \left(m - \frac{\alpha}{2}\right) \beta(u_{m-n}) \epsilon(u_m'),
\]

\[
\theta(E_n) = -(1/2)i \sum_u \sum_m \gamma(u_m') \gamma(u_{1-m+n}'),
\]

\[
\theta(F_n) = -(1/2)i \sum_u \sum_m \beta(u_m) \beta(u_{1-m-n}),
\]

\[
\theta(y_n^0) = i \sum_u \sum_m \beta(u_m) \tau(u_{1-m-n}),
\]

\[
\theta(x_n) = -i \sum_u \sum_m \left(m - \frac{\alpha}{2}\right) \gamma(u_{1-m+n}) \epsilon(u_m'),
\]

\[
\theta(H) = -\sum_u \sum_m : \tau(u_m) \epsilon(u_m'):.
\]

One can check that the central charge is $3\dim V$ in the same way as in Theorem 3.2. \hfill \Box

**Theorem 5.2.** Let $g$ be a complex finite-dimensional Lie algebra endowed with a non-degenerate invariant symmetric bilinear form. Then $H^{\infty,+ +}(\tilde{g}, S^{\infty,+ +}(\tilde{g}))$ is a module over $S'(2,0)$ with central charge $3\dim g$.

**Proof.** Let $\{v_i\}$ be a basis of $g$ so that with respect to the given form $\langle v_i, v_j \rangle = \delta_{i,j}$. Let $u$ run through this basis. Then by Theorem 5.1, there is a representation of $S'(2,0)$ in $W^{\infty,+ +}(\tilde{g})$. Notice that we can identify the elements of $S'(2,0)$ with the quadratic expansions obtained by putting $\alpha = 0$ in the equations (5.2). One can check that the commutation relations (4.8) (where $\alpha = 0$) are fulfilled. One can notice that

\[
[S'(2,0), d] = 0.
\]

In fact, since $\langle \cdot, \cdot \rangle$ is an invariant symmetric bilinear form on $g$, then the elements $E_n, H_n,$ and $F_n$ commute with $\pi(g)$ for any $g \in \tilde{g}$. Hence they commute with $d$. According to Corollary 3.1,

\[
[h_n^0, d] = [p_n, d] = 0.
\]

Recall that

\[
S'(2,0)_1 = \{h_n^0, y_n^0, p_n, x_n\}_{n \in \mathbb{Z}}.
\]
Since

\[ [E_n, p_k] = \eta_{n+k}, [F_n, h^0_k] = y^0_{n+k}, \]

then

\[ [S'(2,0)\bar{\alpha}, d] = 0. \]

Since

\[ S'(2,0)\bar{\alpha} = [S'(2,0)\bar{\alpha}, S'(2,0)\bar{\alpha}], \]

then (5.3) follows. \( \square \)

To define an action of Der\(S'(2,0)\), one should consider a relative semi-infinite Weil complex.

Let \( g \) be a complex finite-dimensional Lie algebra, \( \phi \) be a representation of \( g \) in \( V \), \( \langle \cdot, \cdot \rangle \) be a non-degenerate \( g \)-invariant symmetric bilinear form on \( V \). One can naturally extend \( \phi \) to a representation of \( \bar{g} \) in \( \bar{V} \):

\[ \phi(g \otimes t^n)(v \otimes t^k) = (\phi(g)v) \otimes t^{n+k}, \quad \text{for } g \in g, v \in V. \]

**Theorem 5.3.** — The space \( W^{\otimes \cdot \cdot}_{\text{rel} \cdot \cdot} (\bar{V}) \) is a module over Der\(S'(2,0)\) with central charge \( 3\dim V \).

**Proof.** — Let \( \{v_i\} \) be a basis of \( V \) so that \( \langle v_i, v_j \rangle = \delta_{i,j} \). Let \( u \) run through this basis. Then by Theorem 5.1, there is a representation of \( \hat{S}'(2,0) \) in \( W^{\otimes \cdot \cdot}_{\text{rel} \cdot \cdot} (\bar{V}) \). We can identify the elements of \( S'(2,0) \) with the expansions (5.2) where \( \alpha = 0 \).

Since the form \( \langle \cdot, \cdot \rangle \) is \( g \)-invariant, then there is an action of \( \hat{S}'(2,0) \) on \( W^{\otimes \cdot \cdot}_{\text{rel} \cdot \cdot} (\bar{V}) \). To extend this representation to Der\(S'(2,0)\), we have to define it on \( SL(2) = (F, H, E) \). Let

\[ E = i \sum_{u, m > 0} m \varepsilon(u'_m) \varepsilon(u'_m), \]
\[ H = - \sum_{u, m \neq 0} (\tau(u_m) \varepsilon(u'_m)), \]
\[ F = -i \sum_{u, m > 0} (1/m) \tau(u_m) \tau(u_m). \]

Notice that \( SL(2) \) acts on \( W^{\otimes \cdot \cdot}_{\text{rel} \cdot \cdot} (\bar{V}) \). The commutation relations between \( E, H, F \) and the elements of \( S'(2,0) \) coincide with the relations (4.15),
where $a = 0$, up to some terms which contain elements $\tau(u_0)$. Since the action of $\tau(u_0)$ on $W_{rel}^{\infty,+*}(\bar{V})$ is trivial, then a representation of $\mathcal{D}er S'(2,0)$ in $W_{rel}^{\infty,+*}(\bar{V})$ is well-defined.

**Corollary 5.1.** $H^{\infty,+*}_T(\mathfrak{g}, \mathfrak{g}_0, S^{\infty,+*}(\mathfrak{g}))$ is a module over $\tilde{S}'(2,0)$ with central charge $3\dim \mathfrak{g}$.

**Proof.** — Follows from Theorem 5.2.

### 6. Relative semi-infinite cohomology and Kähler geometry.

Let $M$ be a compact Kähler manifold with associated $(1,1)$-form $\omega$, let $\dim_{\mathbb{C}} M = n$. There exists a number of operators on the space $A^*(M)$ of differential forms on $M$ such as $\partial, \bar{\partial}, d, d_c$, their corresponding adjoint operators and the associated Laplacians (see [GH]). Recall that

\begin{equation}
\begin{align*}
\partial : A^{p,q}(M) &\rightarrow A^{p+1,q}(M), \\
\bar{\partial} : A^{p,q}(M) &\rightarrow A^{p,q+1}(M), \\
d & = \partial + \bar{\partial}, \\
d_c & = i(\partial - \bar{\partial}), \\
\Delta & = dd^* + d^*d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.
\end{align*}
\end{equation}

The Hodge $\ast$-operator maps

\begin{equation}
\ast : A^{p,q}(M) \rightarrow A^{n-q,n-p}(M),
\end{equation}

so that $\ast^2 = (-1)^{p+q}$ on $A^{p,q}(M)$. Correspondingly, the Hodge inner product is defined on each of $A^{p,q}(M)$:

\begin{equation}
(\varphi, \psi) = \int_M \varphi \wedge \ast \bar{\psi}.
\end{equation}

In addition, $A^*(M)$ admits an $\mathfrak{sl}(2)$-module structure. Namely, $\mathfrak{sl}(2) = \langle L, H, \Lambda \rangle$, where

\begin{equation}
\end{equation}

The operator

\begin{equation}
L : A^{p,q}(M) \rightarrow A^{p+1,q+1}(M),
\end{equation}

\begin{equation}
\mathfrak{sl}(2)
\end{equation}
is defined by

\( L(\varphi) = \varphi \wedge \omega. \)

Let \( \Lambda = L^* \) be its adjoint operator:

\( \Lambda : A^{p,q}(M) \to A^{p-1,q-1}(M), \)

and

\( H \mid_{A^{p,q}(M)} = p + q - n. \)

According to the Lefschetz theorem, there exists the corresponding action of \( \mathfrak{sl}(2) \) on \( H^*(M) \). These operators satisfy a series of identities, known as the Hodge identities (see [GH]). Consider the Lie superalgebra spanned by the classical operators:

\( S := \langle \triangle, L, H, \Lambda, d, d^*, d_c, d_c^* \rangle. \)

The non-vanishing commutation relations in \( S \) are as follows:

\[
\begin{align*}
[L, \Lambda] &= H, [H, L] = 2L, [H, \Lambda] = -2\Lambda, \\
[d, d^*] &= dd^* + d^*d = \triangle, \\
[d_c, d_c^*] &= d_c d_c^* + d_c^*d_c = \triangle, \\
[H, d] &= d, [H, d^*] = -d^*, \\
[H, d_c] &= d_c, [H, d_c^*] = -d_c^*, \\
[L, d^*] &= -d_c, [L, d_c^*] = d, \\
[\Lambda, d] &= d_c^*, [\Lambda, d_c] = -d^*.
\end{align*}
\]

**Theorem 6.1.** — Let \( \mathfrak{g} \) be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form. Then there exist operators on \( W_{rel}^{\infty,+,*}(\mathfrak{g}) \), which are analogous to the classical operators in Kähler geometry.

**Proof.** — It was shown in [FGZ] that a relative semi-infinite complex \( C^*_\infty(I, l_0, V) \), where \( I = \bigoplus_{n \in \mathbb{Z}} l_n \) is a complex \( \mathbb{Z} \)-graded Lie algebra, and \( V \) is a graded Hermitian \( l \)-module, has a structure, which is similar to that of the de Rham complex in Kähler geometry. It is assumed that there exists a 2-cocycle \( \gamma \) on \( I \) such that \( \gamma|_{l_n \times l_{-n}} \) is non-degenerate if \( n \in \mathbb{Z} \setminus 0 \) and it is zero otherwise. Then there exist operators on \( C^*_\infty(I, l_0, V) \) analogous to the classical ones.
We will define analogues of the classical operators on $W_{\text{rel}}^{\leq +}$. Using the form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ we obtain the 2-cocycle $\gamma$ on $\tilde{\mathfrak{g}}$:

\[(6.11) \quad \gamma(g_1 \otimes t^n, g_2 \otimes t^m) = n(g_1, g_2) \delta_{n,-m}, \text{ for } g_1, g_2 \in \mathfrak{g}.\]

Notice that $\gamma|_{\tilde{\mathfrak{g}}_n \times \tilde{\mathfrak{g}}_{-n}}$ is non-degenerate if $n \in \mathbb{Z} \setminus 0$ and zero otherwise. Let

\[(6.12) \quad \Lambda_{\text{rel}}^{\leq +} = \bigoplus_{a,b \geq 0} \Lambda^a(n_+^r) \wedge \Lambda^b(n_-^l).\]

For a homogeneous element in $\Lambda^a(n_+^r) \wedge \Lambda^b(n_-^l)$, $a$ is the number of added elements, and $b$ is the number of missing elements with respect to the vacuum vector $1_{\text{rel}}$. Let

\[(6.13) \quad C^{a,b}(\tilde{\mathfrak{g}}) = [S_{\text{rel}}^{\leq +} \otimes \Lambda^a(n_+^r) \wedge \Lambda^b(n_-^l)]\tilde{\mathfrak{g}}_0.\]

We obtain a bigrading on the relative semi-infinite Weil complex, such that

\[(6.14) \quad W_{\text{rel}}^{\leq +} = \bigoplus_{a-b=i} C^{a,b}(\tilde{\mathfrak{g}}).\]

Let $d$ be the restriction of the differential to the relative subcomplex. Notice that

\[(6.15) \quad d : C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a+1,b}(\tilde{\mathfrak{g}}) \oplus C^{a,b-1}(\tilde{\mathfrak{g}}).\]

Define $d_1$ and $d_2$ such that

\[(6.16) \quad d = d_1 + d_2, \quad d_1 : C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a+1,b}(\tilde{\mathfrak{g}}), \quad d_2 : C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a,b-1}(\tilde{\mathfrak{g}}).\]

Let

\[(6.17) \quad d_c = i(d_1 - d_2).\]

To define the adjoint operators, we have to introduce a Hermitian form on $W_{\text{rel}}^{\leq +}$. It was shown in [FGZ] that if a $\mathbb{Z}$-graded Lie algebra $\mathfrak{l}$ admits an antilinear automorphism $\sigma$ of order 2 such that $\sigma(l_n) = l_{-n}$, then there exists a Hermitian form on $\Lambda^{\leq +}(\mathfrak{l})$ such that

\[(6.18) \quad \varepsilon(x')^* = -\varepsilon(\sigma(x')), \quad \tau(x)^* = -\tau(\sigma(x)),\]

where $x \in \mathfrak{l}, x' \in \mathfrak{l}'$. 

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To define a Hermitian form \{\cdot, \cdot\} on \( \Lambda_{rel}^{\infty, +*}(\tilde{\mathfrak{g}}) \), we set \( \{1_{rel}, 1_{rel}\} = 1 \).

We fix a basis \( \{v_i\} \) of \( \mathfrak{g} \) so that \( \langle v_i, v_j \rangle = \delta_{i,j} \). Let \( u \) run through this basis.

We define an antilinear automorphism \( \sigma \) of \( \tilde{\mathfrak{g}} \) as follows:

\[
\sigma(u_n) = iu_{-n}.
\]

Correspondingly,

\[
\sigma(u'_n) = -iu'_{-n}.
\]

We introduce a Hermitian form on \( \Lambda_{rel}^{\infty, +*}(\tilde{\mathfrak{g}}) \) so that the relations (6.18), where

\[
x \in \tilde{\mathfrak{g}}_n, x' \in \tilde{\mathfrak{g}}'_n \text{ for } n \neq 0
\]

hold. In the similar way we introduce a Hermitian form on \( S_{rel}^{\infty, +*}(\tilde{\mathfrak{g}}) \), such that

\[
\gamma(x')^* = \gamma(\sigma(x')), \quad \beta(x)^* = -\beta(\sigma(x)).
\]

Then we obtain a Hermitian form \( \{\cdot, \cdot\} \) on \( W_{rel}^{\infty, +*}(\tilde{\mathfrak{g}}) \) by tensoring these two forms. It gives a pairing: \( C^{a,b}(\tilde{\mathfrak{g}}) \to C^{b,a}(\tilde{\mathfrak{g}}) \). To define a Hermitian form on \( C^{a,b}(\tilde{\mathfrak{g}}) \), we use the linear map

\[
*: C^{a,b}(\tilde{\mathfrak{g}}) \to C^{b,a}(\tilde{\mathfrak{g}}),
\]

defined as follows:

\[
* \left( v \otimes (\varepsilon(u'_{n_1}) \cdots \varepsilon(u'_{n_a}) \tau(u_{m_1}) \cdots \tau(u_{m_b}) 1_{rel}) \right)
\]

\[
= v \otimes (\varepsilon(u'_{-n_1}) \cdots \varepsilon(u'_{-m_b}) \tau(u_{-n_1}) \cdots \tau(u_{-n_a}) 1_{rel}),
\]

where \( v \in S_{rel}^{\infty, +*}(\tilde{\mathfrak{g}}) \), \( \{n_i\}_{i=1}^a \geq 0 \) and \( \{m_i\}_{i=1}^b \leq 0 \). Finally, the Hermitian form on \( C^{a,b}(\tilde{\mathfrak{g}}) \) is defined by \( (w_1, w_2) = \{i^{a+b} \ast w_1, w_2\} \) (cf. [FGZ]).

We introduce the adjoint operators \( d^*, d_c^* \) and the Laplace operator \( \Delta = dd^* + d^*d \).

It was pointed out in [FGZ] that as in the classical theory (see [GH]), there exists an action of \( \mathfrak{sl}(2) \) on \( H^*_\infty(I, l_0, V) \). One can identify \( l_n^l \) with \( \Lambda_{-n} \) by means of the cocycle \( \gamma \). If \( \{e_i\} \) is a homogeneous basis in \( I \), then \( \mathfrak{sl}(2) = \langle L, H, \Lambda \rangle \) is defined as follows:

\[
L = (i/2) \sum_{m \in \mathbb{Z} \setminus 0} \varepsilon(e_m) \varepsilon(e'_m),
\]
We identify \( \mathfrak{g}_n \) with \( \mathfrak{g}_{-n} \) by means of the cocycle \( \gamma \) (see (6.11)), and set

\[
\Lambda = (i/2) \sum_{m \in \mathbb{Z} \setminus 0} \tau(e_m)\tau(e'_m).
\]

Then we obtain the \( \mathfrak{s}\mathfrak{l}(2) = (E, \mathcal{H}, \mathcal{F}) \) defined in (5.10). The operators

\[
\{ \triangle, E, \mathcal{H}, \mathcal{F}, d, d^*, d_c, d_c^* \}
\]

are the analogues of the classical operators (6.9).

**Theorem 6.2.** — Let \( \mathfrak{g} \) be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form. Then \( H^{\mathfrak{g}^{++}}(\mathfrak{g}, \bar{\mathfrak{g}}, \mathcal{S}^{\mathfrak{g}^{++}}(\bar{\mathfrak{g}})) \) is a module over \( \operatorname{Der}\mathcal{S}'(2,0) \) with central charge \( 3\dim \mathfrak{g} \).

**Proof.** — By Theorem 5.3, \( W^{\mathfrak{g}^{++}}(\mathfrak{g}) \) is a module over \( \operatorname{Der}\mathcal{S}'(2,0) \) with central charge \( 3\dim \mathfrak{g} \). By Corollary 5.1, there is an action of \( \mathcal{S}'(2,0) \) on \( H^{\mathfrak{g}^{++}}(\mathfrak{g}, \bar{\mathfrak{g}}, \mathcal{S}^{\mathfrak{g}^{++}}(\bar{\mathfrak{g}})) \). We have proved that

\[
\operatorname{Der}_{\text{ext}}\mathcal{S}'(2,0) = \mathcal{S}\mathcal{L}(2) = \langle E, \mathcal{H}, \mathcal{F} \rangle,
\]

see (5.10). Notice that as in the classical case, the element \( \mathcal{F} \) and the differential \( d \) do not commute. Nevertheless, there exists an action of \( \mathcal{S}\mathcal{L}(2) \) on the relative semi-infinite cohomology according to [FGZ].

**Theorem 6.3.** — The degree zero part of the Z-grading \( \deg \) of \( \mathcal{S}'(2,0) \) is isomorphic to the Lie superalgebra of classical operators in Kähler geometry.

**Proof.** — Recall that the Z-grading \( \deg \) of \( \mathcal{S}'(2,0) \) is defined by the element \( L_0 \in \text{Witt} \), see (4.17)-(4.19). One can easily check that

\[
\mathcal{S}'(2,0)_{\deg=0} = \langle L_0, E_{-1}, H_0, F_1, h^0_0, p_0, x_{-1}, y_1^0 \rangle.
\]

The isomorphism of Lie superalgebras

\[
\psi: \mathcal{S} \longrightarrow \mathcal{S}'(2,0)_{\deg=0}
\]
is given as follows:

\begin{align}
\psi(\Delta) &= L_0, \psi(L) = E_{-1}, \psi(H) = H_0, \psi(\Lambda) = F_1, \\
\psi(d) &= h_0^0, \psi(d^*) = -p_0, \psi(d_c) = x_{-1}, \psi(d_c^*) = y_1^0.
\end{align}

\[\square\]

**Corollary 6.1.** — The action of \( S'(2, 0)_{\text{deg}=0} \) defines a set of quadratic operators on \( W_{\text{rel}}^{\infty} + *(\widehat{\mathfrak{g}}) \) (correspondingly, on \( H_{\text{rel}}^{\infty} + *(\widehat{\mathfrak{g}}, \mathfrak{g}_0, S_{\text{rel}}^{\infty} + *(\widehat{\mathfrak{g}})) \)), which are analogues of the classical ones, and include the semi-infinite Koszul differential \( \mathcal{h} = h_{0}^0 \) and the semi-infinite homotopy operator \( p_{0} \).

**Remark 6.1.** — In this work we have realized superconformal algebras by means of quadratic expansions on the generators of the Heisenberg and Clifford algebras related to \( \mathfrak{g} \). Note that the differentials on a semi-infinite Weil complex are represented by cubic expansions. One can possibly define an additional (to the already known) action of the \( N = 2 \) SCA on \( W_{\text{rel}}^{\infty} + *(\widehat{\mathfrak{g}}) \), considering Fourier components of the differentials \( d \) and \( d^* \), [Fe].

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