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## AUTOMATA, ALGEBRAICITY AND DISTRIBUTION OF SEQUENCES OF POWERS

### by J.-P. ALLOUCHE, J.-M. DESHOUILLERS, T. KAMAE, T. KOYANAGI

#### 1. Introduction.

Let K be a finite field of characteristic p. Let K((x)) be the field of formal Laurent series in x. We call  $f \in K((x))$  algebraic if it is algebraic over the rational function field K(x). We say that

$$f(x) = \sum_{n = -\infty}^{\infty} f_n x^n \in K((x)),$$

where  $f_n = 0$  if *n* is sufficiently small, is *p*-automatic (see for example [4] and the references therein), if there exists a finite automaton  $M = (\Sigma, \phi, \sigma_0, \tau)$  over the alphabet  $[p] := \{0, 1, \dots, p-1\}$  such that

(1) 
$$f_n = \tau(\phi(\cdots \phi(\phi(\sigma_0, n_0), n_1) \cdots, n_L))$$

for any nonnegative integers n and L with

(2) 
$$n = \sum_{i=0}^{\infty} n_i p^i = \sum_{i=0}^{L} n_i p^i, \ n_i \in [p],$$

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where  $\Sigma$  is a finite set,  $\sigma_0 \in \Sigma$ ,  $\phi : \Sigma \times [p] \to \Sigma$  and  $\tau : \Sigma \to K$ . In this case, we say that M recognizes f. The elements in  $\Sigma$  are called the **states** and  $\sigma_0$  is called the **initial state** of M. We call  $\tau$  the **output function** of M.

Remark 1. — The usual definition for that M recognizes f is different from ours, but is that (1) holds for any nonnegative integers n and L with (2) together with  $n_L \neq 0$ . If we have a finite automaton M like this, then we can modify it to have a finite automaton  $M' = (\Sigma \times \Sigma, \phi', (\sigma_0, \sigma_0), \tau')$ which recognizes f in our sense. In fact, we define

$$\phi'((\sigma,\sigma'),k) = egin{cases} (\phi(\sigma,k),\sigma') & (k=0)\ (\phi(\sigma,k),\phi(\sigma,k)) & (k
eq 0) \end{cases}$$

and

 $\tau'(\sigma, \sigma') = \tau(\sigma').$ 

Thus, f is recognized by some automaton in our sense if and only if f is recognized by some automaton in the usual sense.

The notion of "(p-)automaticity" does not change if automata read the highest digit first, i.e., if we replace (1) by

$$f_n = \tau(\phi(\cdots \phi(\phi(\sigma_0, n_L), n_{L-1}) \cdots, n_0)).$$

In this case, we say that M dually recognizes f. If M recognizes f, then the dual automaton  $M^*$  dually recognizes f (Section 6).

It holds that

THEOREM 1 ([3], [4]). — The series  $f \in K((x))$  is algebraic if and only if it is p-automatic.

This theorem was generalized to the multi-dimensional case by Salon:

THEOREM 2 ([12], [13]). — The formal power series  $F(x, y) \in K[[x, y]]$  is algebraic if and only if it is p-automatic.

F. von Haeseler and A. Petersen [8] and F. von Haeseler [9] also discussed the multi-dimensional generalization. In fact, they proved the equivalence between finite kernel property and automaticity in a general setting which essentially implies our Theorem 6, which generalizes the "only if" part of Theorem 2 for

$$F(x,y) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} F_{n,m} x^n y^m \in K((x))[[y]].$$

Here,  $F_{n,m} \in K$  for any  $n, m \in \mathbb{Z}$  with  $m \ge 0$  and it holds that for any  $m \ge 0$ , there exists  $n_0(m)$  such that  $F_{n,m} = 0$  for any  $n < n_0(m)$ . The meaning of "*p*-automatic" for such an F(x, y) is that there exists a finite automaton  $M = (\Sigma, \phi, \sigma_0, \tau)$  over  $[p] \times [p]$  such that

(3) 
$$F_{n,m} = \tau(\phi(\cdots \phi(\phi(\sigma_0, n_0, m_0), n_1, m_1) \cdots, n_L, m_L))$$

for any nonnegative integers n, m and L with the following (4):

(4) 
$$n = \sum_{i=0}^{\infty} n_i p^i = \sum_{i=0}^{L} n_i p^i, \quad m = \sum_{i=0}^{\infty} m_i p^i = \sum_{i=0}^{L} m_i p^i,$$
where  $n_i \in [p]$  and  $m_i \in [p]$ .

The reader may compare the definition with [1] and [9]. Our definition of p-automaticity does not involve the part of  $F_{n,m}$  with n < 0.

We apply this theorem to discuss the distribution of the sequence  $({f^m})_{m \ge 0}$  for  $f \in K((x))$ , where  $\{f\}$  is the **nonnegative part** of f, i.e.,

$$\{f\} = \sum_{n=0}^{\infty} f_n x^n \in K[[x]].$$

The following result was proved by Allouche and Deshouillers [2] (see Deshouillers [5], [6], [7] for more precise results if f is rational).

THEOREM 3 ([2]). — For any algebraic  $f \in K((x))$ , the logarithmic distribution of  $(\{f^m\})_{m\geq 0}$  exists and its support has Hausdorff dimension zero.

In the above, a Borel probability measure  $\mu$  on K[[x]] is called the **logarithmic distribution** of a sequence  $(f^{(m)})_{m \ge 0}$  in K[[x]] if for any finite sequence  $(c_i)_{0 \le i < b}$  (b > 0) of elements in K, it holds that

$$\lim_{M \to \infty} \frac{1}{\log M} \sum_{\substack{m=0 \\ f_i^{(m)} = c_i, \ \forall i \in [0,b)}}^{M-1} \frac{1}{m+1} = \mu\{\omega \in K[[x]]; \ \omega_i = c_i, \ \forall i \in [0,b)\}.$$

Here, we call  $\mu$  simply the **distribution** of a sequence  $(f^{(m)})_{m\geq 0}$  in K[[x]] if for any finite sequence  $(c_i)_{0\leq i< b}$  of elements in K, it holds that

$$\lim_{M \to \infty} \frac{1}{M} \sum_{\substack{m=0 \\ f_i^{(m)} = c_i, \ \forall i \in [0,b)}}^{M-1} 1 = \mu\{\omega \in K[[x]]; \ \omega_i = c_i, \ \forall i \in [0,b)\}.$$

It is clear that, if a sequence has a distribution, then it has a logarithmic distribution and both distributions coincide.

In Section 2, we obtain the generic distribution of  $({f^m})_{m\geq 0}$  for random  $f \in K((x))$  such that  $\min\{n; f_n \neq 0\} < 0$ . The generic distribution is not the Haar measure on K[[x]] but is equivalent to it, which is proved in Theorem 4 in Section 2.

In Section 3, we consider  $(\{f^m\})_{m\geq 0} = (f^m)_{m\geq 0}$  when  $f \in K[[x]]$ . In this case, there always exists a continuous distribution if  $f_0 \neq 0$  and  $f \neq f_0$ . Moreover, the distributions of  $f^{-1}$  and  $f_0^{-2}f$  coincide. In particular, if  $f_0 = 1$ , then f and  $f^{-1}$  have the same distribution.

In the further sections, we consider

(5) 
$$F(x,y) := \sum_{m=0}^{\infty} f(x)^m y^m = \frac{1}{1 - f(x)y} \in K((x))[[y]]$$

for an algebraic  $f(x) \in K((x))$ . We give an alternative proof of Theorem 3 using the fact that F(x, y) is algebraic, and hence, *p*-automatic. In fact, we prove that the support of the logarithmic distribution is not only of Hausdorff dimension zero, but also of sublinear (block-)complexity. We construct a finite automaton which recognizes F(x, y) for a rational  $f(x) \in K((x))$  to discuss the distribution of the sequence  $(\{f^m\})_{m\geq 0}$ . Using it, we obtain a sufficient condition for the distribution to be the Dirac measure at 0 in the case where either the denominator or the numerator is a monomial. This generalizes results by Houndonougbo [10] and by Deshouillers [6] as well as simplifies the proofs.

#### 2. Generic distribution.

For any  $n \in \mathbb{Z}$ , denote  $\mathbf{K}_n = \{f \in K((x)); f_i = 0 \text{ for any } i < n\}$ , which is identified with the product space  $K^{\{n,n+1,n+2,\cdots\}}$ . Let  $\lambda_n$  be the uniform distribution on  $\mathbf{K}_n$ . That is,  $\lambda_n$  is the product measure  $(\lambda_K)^{\{n,n+1,n+2,\cdots\}}$ , where  $\lambda_K$  is the uniform probability measure on K. The following theorem is essentially due to De Mathan [11] (see Théorème 3 bis, p. 40).

THEOREM 4. — For any n < 0 and for almost all  $f \in \mathbf{K}_n \setminus \mathbf{K}_{n+1}$  with respect to  $\lambda_n$ , the distribution of  $(\{f^m\})_{m \ge 0}$  exists and is equal to

$$\mu = (p-1) \sum_{k=1}^{\infty} p^{-k} \lambda_0 \circ T^{1-k},$$

where  $T : K[[x]] \to K[[x]]$  is defined by  $T(f) = \sum_{i=0}^{\infty} f_i x^{p_i}$ . Hence,  $\mu$  is equivalent to  $\lambda_0$  and the support is the whole space K[[x]].

COROLLARY 1. — The logarithmic distribution of  $(\{f^m\})_{m\geq 0}$  for  $f \in K[[x]]$  or algebraic  $f \in K((x))$ , for which we know that the support has Hausdorff dimension 0, is singular with respect to this generic distribution  $\mu$ .

Remark 2.— The uniform distribution  $\lambda_0$  cannot be a logarithmic distribution of the sequence  $(\{f^m\})_{m\geq 0}$  for any  $f \in K((x))$ , since the relative frequency of m such that  $(f^m)_1 = (f^m)_{p+1} = 0$  is at least 1/p as  $(f^{jp})_1 = (f^{jp})_{p+1} = 0$   $(j = 1, 2, \cdots)$ . On the other hand, the  $\lambda_0$ -measure of the set of  $g \in K[[x]]$  such that  $g_1 = g_{p+1} = 0$  is at most  $1/p^2$ .

Proof of Theorem 4. — Let  $f = \sum_{i \ge n} Z_i x^i$  be a random variable on  $\mathbf{K}_n \setminus \mathbf{K}_{n+1}$ , where  $Z_{n+1}, Z_{n+2}, Z_{n+3}, \cdots$  are independent random variables uniformly distributed on K and  $Z_n$  is a uniformly distributed random variable on  $K \setminus \{0\}$  which is independent of  $Z_{n+1}, Z_{n+2}, Z_{n+3}, \cdots$ . Take any  $m \ge 0$  which is not a multiple of p. Then, for any  $i \ge 0$ , we have

$$(f^m)_i = A_{m,i} + mZ_n^{m-1}Z_{i-n(m-1)}$$
  
=  $A_{m,i} + B_mZ_{i-n(m-1)}$ ,

where  $A_{m,i} \in K$  and  $B_m \in K \setminus \{0\}$  are random variables determined by  $Z_n, Z_{n+1}, \dots, Z_{i-n(m-1)-1}$ . Therefore, for any k and  $(c_0, c_1, \dots, c_{k-1}) \in K^k$ , we have

$$E\left[\prod_{0\leqslant i\leqslant k-1} 1_{(f^{m})_{i}=c_{i}}\right]$$
  
=  $E\left[E\left[\prod_{0\leqslant i\leqslant k-1} 1_{(f^{m})_{i}=c_{i}} | Z_{n}, Z_{n+1}, \cdots, Z_{k-n(m-1)-2}\right]\right]$   
=  $E\left[\prod_{0\leqslant i\leqslant k-2} 1_{(f^{m})_{i}=c_{i}}\right]$   
 $P[A_{m,k-1} + B_{m}Z_{k-n(m-1)-1} = c_{k}|Z_{n}, Z_{n+1}, \cdots, Z_{k-n(m-1)-2}]$   
=  $E\left[\prod_{0\leqslant i\leqslant k-2} 1_{(f^{m})_{i}=c_{i}}\right](\sharp K)^{-1} = \cdots = (\sharp K)^{-k},$ 

where  $\sharp K$  denotes the number of elements in K. Now let us estimate the variance of  $(1/M) \sum_{m \in a(M)} \prod_{0 \leq i \leq k-1} 1_{(f^m)_i = c_i}$ , where we denote by a(M) the set of the least M positive integers not divisible by p. We denote  $A = (\sharp K)^{-1}$  and  $B = \frac{k}{-n}$ . Then we have

$$\begin{split} & \left| E \Big[ \Big( \sum_{m \in a(M)} \prod_{0 \leqslant i \leqslant k-1} 1_{(f^m)_i = c_i} - MA^k \Big)^2 \Big] \right| \\ &= \Big| \sum_{m,h \in a(M)} E \Big[ \Big( \prod_{0 \leqslant i \leqslant k-1} 1_{(f^m)_i = c_i} - A^k \Big) \Big( \prod_{0 \leqslant i \leqslant k-1} (1_{(f^h)_i = c_i} - A^k \Big) \Big] \Big| \\ &\leqslant (2B+1)M + 2 \Big| \sum_{m,h \in a(M); \ m-h > B} \\ & E \Big[ \Big( \prod_{0 \leqslant i \leqslant k-1} 1_{(f^m)_i = c_i} - A^k \Big) \Big( \prod_{0 \leqslant i \leqslant k-1} (1_{(f^h)_i = c_i} - A^k \Big) \Big] \Big| \\ &= (2B+1)M. \end{split}$$

The last equality in the above holds since for any  $m, h \in a(M)$  with m-h > B, the term

$$\Big(\prod_{0\leqslant i\leqslant k-1} 1_{(f^m)_i=c_i} - A^k\Big)\Big(\prod_{0\leqslant i\leqslant k-1} (1_{(f^h)_i=c_i} - A^k\Big)$$

can be written as the sum of terms:

$$A^{2k-2-j-j'}(1_{(f^m)_j=c_j}-A)(1_{(f^h)_{j'}=c_{j'}}-A) \times \prod_{0 \le i \le j-1} 1_{(f^m)_i=c_i} \prod_{0 \le i \le j'-1} 1_{(f^h)_i=c_i},$$

which has 0 expectation since all the terms but  $(1_{(f^m)_j=c_j} - A)$  are determined by  $Z_n, Z_{n+1}, \dots, Z_{j-n(m-1)-1}$ , while as above

$$E[1_{(f^m)_j=c_j} - A|Z_n, Z_{n+1}, \cdots, Z_{j-n(m-1)-1}]$$
  
=  $P[A_{m,j} + B_m Z_{j-n(m-1)} = c_j | Z_n, Z_{n+1}, \cdots, Z_{j-n(m-1)-1}] - A$   
= 0.

Thus the variance of  $(1/M) \sum_{m \in a(M)} \prod_{0 \leq i \leq k-1} 1_{(f^m)_i = c_i}$  is at most (2B+1)/M and we have the law of large numbers. That is, with probability 1,  $(1/M) \sum_{m \in a(M)} \prod_{0 \leq i \leq k-1} 1_{(f^m)_i = c_i}$  converges to  $A^k$ . Since this holds for any finite sequence  $(c_0, c_1, \dots, c_{k-1}) \in K^k$ , it holds with probability 1 that the distribution of  $(\{f^m\})_{m \in a(\infty)}$  is  $\lambda_0$ , where  $a(\infty)$  is the set

of positive integers which are not multiples of p. Since

$$(f^{pm})_n = \begin{cases} (f^m)_{n/p} & \text{(if } p|n) \\ 0 & \text{(otherwise)}, \end{cases}$$

the distribution of  $({f^m})_{m \in pa(\infty)}$  is  $\lambda_0 \circ T^{-1}$  with probability 1. In the same way, the distribution of  $({f^m})_{m \in p^2 a(\infty)}$  is  $\lambda_0 \circ T^{-2}$  with probability 1. Hence, the distribution of  $({f^m})_{m \ge 0}$  is

$$\frac{p-1}{p}\lambda_0 + \frac{p-1}{p^2}\lambda_0 \circ T^{-1} + \frac{p-1}{p^3}\lambda_0 \circ T^{-2} + \cdots,$$

which completes the poof.

#### **3.** Case K[[x]].

In this section, we consider the case where  $f \in K[[x]]$ . That is,

$$f = \sum_{n=0}^{\infty} f_n x^n.$$

For a positive integer N, let  $f|_N := \sum_{n=0}^{N-1} f_n x^n$ . Let G be a transformation on the finite set  $K_{[0,N)} := \{g|_N; g \in K[[x]]\}$  defined by  $G(g) := (gf)|_N$ . Then, since we have  $f^m|_N = G^m(1)$  for  $m = 0, 1, 2, \cdots$ , the sequence  $f^m|_N \in K_{[0,N)}$  in m is ultimately periodic. We have 3 cases.

If  $f_0 = 0$ , then  $f^m|_N = 0$  for any  $m \ge N$ .

If  $f = f_0 \neq 0$ , then since  $G^{p-1}(1) = 1$ ,  $f^m|_N$  is purely periodic in m with period p - 1.

Assume that  $f \neq f_0 \neq 0$ . Let  $n \in \mathbb{N}$  satisfy that  $p^{n-1} < N \leq p^n$ . Then since  $f^{(p-1)p^n}|_N = 1$ ,  $f^m|_N$  is purely periodic in m with period  $(p-1)p^n$ . Let  $c_N$  be the least period of the sequence  $f^m|_N$  in m. Then we have  $c_N \leq (p-1)p^n < p(p-1)N$ . It is clear that  $f^m|_N \neq f^{m'}|_N$  if  $|m-m'| < c_N$ . Hence, f has a distribution, say  $\mu_f$ , and  $\mu_f$  is continuous if  $c_N \to \infty$  as  $N \to \infty$ .

We prove that  $\mu_f$  is continuous if  $f \neq f_0 \neq 0$ . Assume that  $f \neq f_0 \neq 0$ . Let  $n_0$  be the least positive integer such that  $f_{n_0} \neq 0$ . Let  $c_N = p^L c'$  with c' which is not a multiple of p. Since  $(f^{c_N})_{p^L n_0} = (f^{c'})_{n_0} \neq 0$ , and  $f^{c_N}|_N = 1$ , we have  $p^L n_0 \geq N$ . Hence,  $c_N \geq p^L \geq N/n_0$  and  $c_N \to \infty$  as  $N \to \infty$ . Thus,  $\mu_f$  is continuous.

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The complexity  $C_N(\Omega)$  of a closed subset  $\Omega$  of K[[x]] is defined by

$$C_{N}(\Omega) = \sharp \left\{ \begin{array}{l} (H_{0}, H_{1}, \cdots, H_{N-1}) \in K^{N}; \text{ there exists } \omega \in \Omega \\ \text{ such that } \omega_{i} = H_{i}, \quad \forall i = 0, 1, \cdots, N-1 \end{array} \right\}$$

Let  $\Omega(f)$  be the topological support of the measure  $\mu_f$  on K[[x]]. Then it is clear that  $C_N(\Omega(f)) = c_N$  for any  $N = 1, 2, \cdots$ .

When we discuss the Hausdorff dimension of subsets in K[[x]], it is with respect to the metric  $\rho$  defined by

$$\rho(\omega,\omega'):=p^{-\min\{n\ge 0;\ \omega_n\neq\omega'_n\}}$$

for any  $\omega \neq \omega' \in K[[x]]$ . For the  $\alpha$ -Hausdorff measure  $\Lambda_{\alpha}$  of  $\Omega(f)$ , we have

$$\Lambda_{\alpha}(\Omega(f)) \leq \lim_{n \to \infty} \sum_{\substack{(H_0, \dots, H_{n-1}) \in K^n \\ \exists \omega \in \Omega(f), \ \omega_i = H_i, \ i = 0, \dots, n-1}} p^{-n\alpha}$$
$$= \lim_{n \to \infty} C_n(\Omega(f)) p^{-n\alpha}$$
$$\leq \lim_{n \to \infty} p(p-1)n \cdot p^{-n\alpha}$$
$$= 0$$

for any  $\alpha > 0$ . Thus, dim  $\Omega(f) = 0$ .

THEOREM 5. — For  $f \in K[[x]]$ , the sequence  $(\{f^m\})_{m=0,1,\cdots}$  has a distribution  $\mu_f$ . If  $f_0 = 0$ , then  $\mu_f$  is the Dirac measure at  $0 \in K((x))$ . If  $f_0 \neq 0$  and  $f \neq f_0$ , then  $\mu_f$  is a continuous distribution supported by  $\Omega(f)$  while  $\Omega(f)$  has a sublinear complexity and hence 0-Hausdorff dimension. In fact,  $C_N(\Omega(f)) < p(p-1)N$  for any  $N = 1, 2, \cdots$ . Moreover, in this case, it holds that  $\mu_{f^{-1}} = \mu_{f^{-2}_{\sigma}f}$ .

*Proof.* — We only have to prove that  $\mu_{f^{-1}} = \mu_{f_0^{-2}f}$ . It suffices to prove this in the case  $f_0 = 1$ . Since  $f^{p^k}|_{p^k} = 1$ ,  $f^{p^k-m}|_{p^k} = f^{-m}|_{p^k}$  for any  $k = 1, 2, \cdots$  and m with  $0 \leq m < p^k$ . This implies that  $\mu_{f^{-1}} = \mu_f$ .  $\Box$ 

#### 4. Construction of automata.

For  $i \in [p]$ , define the linear operators  $X_i$  and  $Y_i$  on K((x))[[y]] by

$$X_i\left(\sum_{n,m=-\infty}^{\infty} H_{n,m} \ x^n y^m\right) := \sum_{n,m=-\infty}^{\infty} H_{np+i,m} \ x^n y^m$$

and

$$Y_i\left(\sum_{n,m=-\infty}^{\infty}H_{n,m}\ x^ny^m\right):=\sum_{n,m=-\infty}^{\infty}H_{n,mp+i}\ x^ny^m.$$

LEMMA 1.

(i) 
$$X_i Y_j(x^n y^m) = \begin{cases} x^{(n-i)/p} y^{(m-j)/p} & \text{if } n \equiv i \text{ and } m \equiv j \mod p, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For any  $i, j \in [p]$ , we have  $X_i Y_j = Y_j X_i$ .

(iii) For any  $i, j \in [p]$  and for any  $H, G \in K((x))[[y]]$ , we have  $X_i Y_j(HG^p) = X_i Y_j(H)G$ .

Proof. — Assertions (i) and (ii) are clear from the definition. For the proof of (iii), it is sufficient to remark that  $G(x, y)^p = G(x^p, y^p)$  holds for any  $G \in K((x))[[y]]$ .

We state now a theorem to be compared with [3], [4], [9], [12], [13]. The proof either follows from them or at least is essentially the same. But for the readers' convenience, we give the proof.

THEOREM 6. — If  $F \in K((x))[[y]]$  is algebraic, then it is p-automatic.

Proof. — Assume that a nonzero element  $F \in K((x))[[y]]$  is algebraic over K(x, y) with degree  $h_0$ . Then, the elements  $F, F^p, F^{p^2}, \dots, F^{p^{h_0}}$  are linearly dependent over K(x, y). Let h be the least integer such that  $F, F^p, F^{p^2}, \dots, F^{p^h}$  are linearly dependent over K(x, y). Then, there exist  $A_0, A_1, A_2, \dots, A_h \in K[x, y]$  with at least one of them nonzero such that

(7) 
$$A_0F + A_1F^p + A_2F^{p^2} + \dots + A_hF^{p^h} = 0.$$

We may also assume that  $A_0, A_1, A_2, \dots, A_h$  have no nontrivial common factor.

We prove that  $A_0 \neq 0$ . Suppose that  $A_0 = 0$ . Then we have

$$A_1F^p + A_2F^{p^2} + \dots + A_hF^{p^h} = 0.$$

Since at least one of  $A_1, A_2, \dots, A_h$  is nonzero, there exist  $i, j \in [p]$  such that at least one of  $X_i Y_j(A_1), X_i Y_j(A_2), \dots, X_i Y_j(A_h)$  is nonzero. Then, by Lemma 1,

$$0 = X_i Y_j (A_1 F^p + A_2 F^{p^2} + \dots + A_h F^{p^h})$$
  
=  $X_i Y_j (A_1) F + X_i Y_j (A_2) F^{p^2} + \dots + X_i Y_j (A_h) F^{p^{h-1}},$ 

which contradicts the minimality of h.

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Thus, we have (7) with  $A_0 \neq 0$ . Let  $G := F/A_0 \in K((x))[[y]]$ . Then, it holds that

$$G = -A_0^{p-2}A_1G^p - A_0^{p^2-2}A_2G^{p^2} - \dots - A_0^{p^h-2}A_hG^{p^h}$$
  
=:  $B_1G^p + B_2G^{p^2} + \dots + B_hG^{p^h}$ 

and  $F = A_0 G$  with  $A_0, B_1, B_2, \dots, B_h \in K[x, y]$ .

Let  $d := \max\{\deg A_0, \deg B_1, \deg B_2, \cdots, \deg B_h\}$  and

$$\overline{\mathbf{S}}(f) := \{a_0 G + a_1 G^p + \dots + a_{h-1} G^{p^{h-1}} \in K((x))[[y]]; \\ a_i \in K[x, y] \text{ and } \deg a_i \leqslant d, \ i = 0, 1, \dots, h-1\}.$$

Note that  $\overline{\mathbf{S}}(f)$  is a finite set containing F. For any  $i, j \in [p]$  and  $H \in \overline{\mathbf{S}}(f)$  with

$$H = a_0 G + a_1 G^p + a_2 G^{p^2} + \dots + a_{h-1} G^{p^{h-1}},$$

it holds by Lemma 1 that

$$\begin{aligned} X_i Y_j(H) &= X_i Y_j(a_0 G + a_1 G^p + a_2 G^{p^2} + \dots + a_{h-1} G^{p^{h-1}}) \\ &= X_i Y_j(a_0 (B_1 G^p + \dots + B_h G^{p^h}) + \\ &a_1 G^p + a_2 G^{p^2} + \dots + a_{h-1} G^{p^{h-1}}) \\ &= X_i Y_j(a_0 B_1 + a_1) G + X_i Y_j(a_0 B_2 + a_2) G^p + \dots + \\ &X_i Y_j(a_0 B_h) G^{p^{h-1}} \\ &\in \overline{\mathbf{S}}(f), \end{aligned}$$

since, for any  $k = 0, 1, \dots, h - 1$ ,

$$\deg X_i Y_j(a_0 B_k) \leqslant rac{1}{p} (\deg a_0 + \deg B_k) \leqslant rac{2d}{p} \leqslant d.$$

Let

(8) 
$$\overline{\mathbf{M}}(f) := (\overline{\mathbf{S}}(f), \phi, F, \eta)$$

be the finite automaton over  $[p] \times [p]$  such that

$$\phi(H, i, j) := X_i Y_j(H)$$
 and  $\eta(H) = H_{0,0}$ 

for any  $H = \sum H_{n,m} x^n y^m \in \overline{\mathbf{S}}(f)$  and  $i, j \in [p]$ . Let  $\mathbf{S}(f)$  be the set of states in  $\overline{\mathbf{S}}(f)$  which are **attainable** from the initial state F in  $\overline{\mathbf{M}}(f)$ , i.e.,

the set of states  $S \in \overline{\mathbf{S}}(f)$  such that there exists a finite sequence of inputs in  $[p] \times [p]$  which sends the state F to S. Let  $\mathbf{M}(f) := (\mathbf{S}(f), \phi, F, \eta)$  be the automaton obtained from  $\overline{\mathbf{M}}(f)$  by restricting the set of states to be  $\mathbf{S}(f)$ .

We prove that  $\mathbf{M}(f)$  recognizes F. Take any nonnegative integers n, m and L with (4). It holds that

$$F_{n,m} = (X_{n_L} Y_{m_L} \cdots X_{n_1} Y_{m_1} X_{n_0} Y_{m_0}(F))_{0,0}$$
  
=  $\eta(\phi(\cdots \phi(\phi(F, n_0, m_0), n_1, m_1) \cdots, n_L, m_L)),$ 

which completes the proof.

#### 5. Rational functions.

Let

(9) 
$$f(x) = \frac{P(x)}{Q(x)}$$
, where  $P, Q \in K[x]$ , are coprime

be a rational function in K((x)). Then, F(x, y) defined in (5) satisfies

$$F(x,y) = \frac{1}{1 - f(x)y} = \frac{Q(x)}{Q(x) - P(x)y}$$

Let

$$G(x,y) = \frac{1}{Q(x) - P(x)y}.$$

Let  $\overline{S}(f)$  be the set of all  $H \in K[x]$  with deg  $H \leq \max\{\deg P, \deg Q\}$ . Define  $\phi : \Sigma \times [p] \times [p] \to \Sigma$  by

(10) 
$$\phi(H, i, j) = X_i (HQ^{p-1-j}P^j).$$

Let  $\tau : \Sigma \to K$  be  $\tau(H) = (\frac{H}{Q})_0$ , i.e., the coefficient of  $\frac{H}{Q} \in K((x))$  of degree 0. Thus, we define a finite automaton  $(\overline{S}(f), \phi, Q, \tau)$  over  $[p] \times [p]$ . Let S(f) be the set of states in  $\overline{S}(f)$  which are attainable from the initial state Q in this automaton. Let  $M(f) := (S(f), \phi, Q, \tau)$  be the automaton obtained from  $(\overline{S}(f), \phi, Q, \tau)$  by restricting the set of states to be S(f).

THEOREM 7. — The finite automaton M(f) recognizes F(x, y).

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*Proof.*— For  $H \in S(f)$  and  $i, j \in [p]$ , it holds by Lemma 1 that

$$\begin{aligned} X_i Y_j (HG) &= X_i Y_j (H(Q - Py)^{p-1} G^p) \\ &= X_i Y_j (H(Q - Py)^{p-1}) G \\ &= X_i (H \begin{pmatrix} p-1 \\ j \end{pmatrix} Q^{p-1-j} (-P)^j) G \\ &= X_i (HQ^{p-1-j} P^j) G = \phi(H,i,j) G \end{aligned}$$

Take any nonnegative integers n, m and L with (4). Then it holds that

$$F_{n,m} = X_{n_L} Y_{m_L} \cdots X_{n_1} Y_{m_1} X_{n_0} Y_{m_0}(F)_{0,0}$$
  

$$= X_{n_L} Y_{m_L} \cdots X_{n_1} Y_{m_1} X_{n_0} Y_{m_0}(QG)_{0,0}$$
  

$$= X_{n_L} Y_{m_L} \cdots X_{n_1} Y_{m_1} (\phi(Q, n_0, m_0)G)_{0,0}$$
  

$$= \cdots$$
  

$$= (\phi(\cdots \phi(\phi(Q, n_0, m_0), n_1, m_1) \cdots, n_L, m_L)G)_{0,0}$$
  

$$= \tau(\phi(\cdots \phi(\phi(Q, n_0, m_0), n_1, m_1) \cdots, n_L, m_L)),$$

which completes the proof.

Let f be as in (9), F be as in (5) for this f, and the finite automaton M := M(f) be as above. For each  $i \in [p]$ , let  $M_i := (S(f), \phi_i, Q, \tau)$  be the finite automaton over [p] such that  $\phi_i(H, j) = \phi(H, i, j)$  for any  $j \in [p]$  and  $H \in S(f)$ . Then, the sequence  $(F_{n,m})_{m \ge 0}$  in K for a fixed nonnegative integer n with (2) is "recognizable" by the sequence of automata related to n:

$$M_{n_0}, M_{n_1}, \cdots, M_{n_L}, M_0, M_0, \cdots$$

in the sense that

$$F_{n,m} = \tau(\phi_{n_N}(\cdots\phi_{n_1}(\phi_{n_0}(Q,m_0),m_1)\cdots,m_N))$$

for any nonnegative integers m and  $N \ge L$  with

$$m = \sum_{i=0}^{\infty} m_i p^i = \sum_{i=0}^{N} m_i p^i, \ m_i \in [p].$$

Theorem 8.

(i) The distribution of the sequence  $(\{f^m\})_{m\geq 0}$  is equal to  $\delta_0$ , the Dirac measure at  $0 \in K((x))$  if in the finite automaton  $M_0$  as above, 0 is attainable from any state in S(f).

(ii) If P = 1 and  $Q \neq 0$  satisfies Q(0) = 0, then the distribution of the sequence  $(\{f^m\})_{m \ge 0}$  is equal to  $\delta_0$ .

(iii) If  $Q = x^u$  with  $u \ge 1$ ,  $P(0) \ne 0$  and for some  $k = 1, 2, \dots, P^k$  lacks the term  $x^{ku}$ , i.e.,  $(P^k)_{ku} = 0$ , then the distribution of the sequence  $(\{f^m\})_{m\ge 0}$  is equal to  $\delta_0$ .

Proof. — (i) Assume that 0 is attainable from any state in S(f) in  $M_0$ . By the above consideration, 0 is the only "sink" of the sequence of automata related to any  $n \ge 0$ . Since  $\tau(0) = 0$ , this implies that for any  $n \ge 0$  the frequency of 0 in the sequence  $(F_{n,m})_{m\ge 0}$  is equal to 1. Thus, the distribution of the sequence  $(\{f^m\})_{m\ge 0}$  is equal to  $\delta_0$ .

(ii) Since 
$$\phi_0(x^c, p-1)$$
 is  $x^{c/p}$  if  $p \mid c$  and 0 otherwise,  

$$\underbrace{\phi_0(\cdots \phi_0(\phi_0(x^c, p-1), p-1) \cdots, p-1) \neq 0}_{k \text{ times}}$$

only if  $p^k \mid c$ . Therefore, for any  $H \in S(f)$  and for any sufficiently large integer k, it holds that

$$\underbrace{\phi_0(\cdots\phi_0(\phi_0}_k(H,p-1),p-1)\cdots,p-1)=H(0).$$
  
*k* times

Assume that H = C (constant). Then, since

$$\phi_0(H, p-2) = CX_0(Q) =: J,$$

the relation J(0) = 0 follows from the assumption Q(0) = 0.

Thus, 0 is attainable from any element H in S(f) in  $M_0$  by reading (p-1) sufficiently many times followed by reading (p-2) once and again (p-1) sufficiently many times.

(iii) Assume that  $(P^k)_{ku} = 0$  for some  $k = 1, 2, \cdots$ . Since  $\phi_0(x^c, 0)$  is  $x^{u+(c-u)/p}$  if  $p \mid c-u$  and 0 otherwise,

$$\underbrace{\phi_0(\cdots\phi_0(\phi_0}_{j \text{ times}}(x^c,0),0)\cdots,0)\neq 0$$

only if  $p^j | c-u$ . Therefore, for any  $H \in S(f)$  and for any sufficiently large integer j, it holds that

$$\underbrace{\phi_0(\cdots\phi_0(\phi_0}_{j \text{ times}}(H,0),0)\cdots,0) = H_u x^u.$$

Therefore, for any state in S(f), there exists  $C \in K$  such that  $Cx^u$  is attainable from it. Hence, it suffices to prove that 0 is attainable from  $x^u$ .

Let 
$$k = \sum_{i=0}^{j-1} k_i p^i$$
 with  $k_i \in [p]$ . Then we have  

$$H := \underbrace{\phi_0(\cdots \phi_0(\phi_0(x^u, k_0), k_1) \cdots, k_{j-1})}_{j \text{ times}}$$

$$= X_0(\cdots X_0(X_0(x^{(p-k_0)u}P^{k_0})x^{(p-k_1-1)u}P^{k_1}) \cdots x^{(p-k_{j-1}-1)u}P^{k_{j-1}})$$

$$= X_0(\cdots X_0(X_0(x^{(p-k_0)u}P^{k_0}(x^{(p-k_1-1)u}P^{k_1})^p) \cdots x^{(p-k_{j-1}-1)u}P^{k_{j-1}})$$

$$= X_0(\cdots X_0(X_0(x^{(p^2-k_0-k_1p)u}P^{k_0+k_1p})) \cdots x^{(p-k_{j-1}-1)u}P^{k_{j-1}})$$

$$= X_0(\cdots X_0(X_0(x^{(p^2-k_0-k_1p-\dots-k_{j-1}p^{j-1})u}P^{k_0+k_1p+\dots+k_{j-1}p^{j-1}})) \cdots)$$

$$= x^u X_0^j(x^{-ku}P^k).$$

Therefore,  $H_u = 0$  follows from the assumption  $(P^k)_{ku} = 0$ . Thus, 0 is attainable by applying the preceding procedure again, which completes the proof.

Remark 3. — To cover the case where one of P or Q is a monomial, we have to consider the following subcases in addition to (ii) and (iii) in Theorem 8:

- (iv) P = 1 and  $Q(0) \neq 0$ ,
- (v)  $P = x^u$  with  $u \ge 1$  and  $Q(0) \ne 0$ ,
- (vi) Q = 1 and  $P(0) \neq 0$ ,
- (vii) Q = 1 and P(0) = 0, and

(viii)  $Q = x^u$  with  $u \ge 1$  and  $(P^k)_{ku} \ne 0$  for any  $k = 1, 2, \cdots$ .

The distribution is  $\delta_0$  in the cases (v) and (vii), since  $(f^m)_n = 0$ if m > n. In the cases (iv) and (vi), the distributions are continuous by Theorem 5 if f is nonconstant. In the case (viii), the distribution is always continuous by [6]

The case (iii) in Theorem 8 is due to Deshouillers [6]. Here we gave an alternative and simpler proof.

Example 1 (Pascal triangle). — Let p = 2,  $K = \{0, 1\}$  and f = 1 + x, (P = 1 + x, Q = 1). Then, the table  $(F_{n,m})_{n,m \ge 0}$  is the Pascal triangle modulo 2. In the automaton M = M(f), the initial state is 1,  $S(f) = \{0, 1\}$ , and it holds that

$$\phi(0,i,j)=0, \ \phi(1,i,j)= egin{cases} 1 & ext{if } i\leqslant j \ 0 & ext{otherwise} \end{cases}$$



Figure 1. – Automaton in Example 1.

for any  $i, j \in [2]$ . Therefore,  $M_0$  has two sinks 0 and 1. Furthermore we have  $\tau(0) = 0$  and  $\tau(1) = 1$ .

The distribution  $\mu$  for this f is determined using the automaton. In fact, we have

(11) 
$$F_{n,m} = \begin{cases} 1 & \text{if } n_i \leq m_i \quad \forall i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Define a partial order  $\leq$  on the nonnegative integers by

 $n \leq m$  if and only if  $n_i \leq m_i$  for all i,

where we use the notation in (4). Then, for any fixed  $m \ge 0$ , the function  $F_m$  defined by  $F_m(n) = F_{n,m}$  is monotone decreasing with respect to the partial order  $\preceq$  on the set of nonnegative integers. It is not difficult to see that  $\Omega(f)$  consists of all  $\sum_{n\ge 0} g_n x^n$  such that the function  $n \mapsto g_n$  is monotone decreasing in this sense. The distribution  $\mu$  is the uniform distribution on  $\Omega(f)$  in some sense.

By the arguments in Section 3, the function  $m \mapsto F_m|_{2^k}$  is purely periodic with least period at most  $2^k$  for  $k = 1, 2, \cdots$ . In our case, it is exactly  $2^k$  since otherwise, there exists m with  $0 < m < 2^k$  such that  $F_m|_{2^k} = F_0|_{2^k} = \delta_0$ . But this is impossible since  $F_m(m) = F_{m,m} = 1$ by (11). The  $\mu$ -measure of the cylinder determined by  $F_m|_{2^k}$  is  $2^{-k}$  for  $m = 0, 1, \cdots, 2^k - 1$  using the periodicity.



Figure 2. – Automaton in Example 2.

Example 2. — Let p = 2,  $K = \{0, 1\}$  and  $f = (1+x^2)/x$ ,  $(P = 1+x^2, Q = x)$ . Then, we have

$$\begin{split} \phi(x,i,0) &= X_i(x^2) = \begin{cases} x & \text{if } i = 0, \\ 0 & \text{if } i = 1, \end{cases} \\ \phi(x,i,1) &= X_i(x+x^3) = \begin{cases} 0 & \text{if } i = 0, \\ 1+x & \text{if } i = 1, \end{cases} \\ \phi(0,i,j) &= 0 & \forall i, j \in [2] \\ \phi(1+x,i,0) &= X_i(x+x^2) = \begin{cases} x & \text{if } i = 0, \\ 1 & \text{if } i = 1, \end{cases} \\ \phi(1+x,i,1) &= X_i(1+x+x^2+x^3) = 1+x & \forall i \in [2] \\ \phi(1,i,0) &= X_i(x) = \begin{cases} 0 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \end{cases} \\ \phi(1,i,1) &= X_i(1+x^2) = \begin{cases} 1+x & \text{if } i = 0, \\ 0 & \text{if } i = 1. \end{cases} \end{split}$$

In this case, f has a distribution equal to  $\delta_0$ .

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#### 6. Dual automata and complexity.

Let  $\mathbf{M}(f) := (\mathbf{S}(f), \phi, F, \eta)$  be the automaton constructed in Section 4 which recognizes F in (5) for an algebraic  $f \in K((x))$ . We construct the dual automaton  $\mathbf{M}(f)^* := (\mathbf{S}(f)^*, \phi^*, \eta, F^*)$  over  $[p] \times [p]$  which dually recognizes F.

Let

$$\overline{\mathbf{S}(f)}^{*} := \{ \xi : \mathbf{S}(f) \to K \}$$

$$\phi^{*}(\xi, i, j) := \xi \circ \phi_{i,j} \ (\forall i, j \in [p], \ \xi \in \overline{\mathbf{S}(f)}^{*})$$

$$F^{*}(\xi) := \xi(F) \ (\forall \xi \in \overline{\mathbf{S}(f)}^{*}),$$

where  $\phi_{i,j} : \mathbf{S}(f) \to \mathbf{S}(f)$  is defined by  $\phi_{i,j}(H) = \phi(H, i, j), \forall H \in \mathbf{S}(f)$ . Let  $\mathbf{S}(f)^*$  be the set of all states which are attainable from the initial state  $\eta$  in the automaton  $(\overline{\mathbf{S}(f)}^*, \phi^*, \eta, F^*)$  over  $[p] \times [p]$ . Let  $\mathbf{M}(f)^* := (\mathbf{S}(f)^*, \phi^*, \eta, F^*)$  be the restriction of this automaton.

Then for any nonnegative integers n, m and L with (4), we have

$$F^{*}(\phi^{*}(\cdots\phi^{*}(\phi^{*}(\eta, n_{L}, m_{L}), n_{L-1}, m_{L-1}) \cdots, n_{0}, m_{0}))$$

$$= F^{*}(\phi^{*}(\cdots\phi^{*}(\eta \circ \phi_{n_{L}, m_{L}}, n_{L-1}, m_{L-1}) \cdots, n_{0}, m_{0}))$$

$$= \cdots$$

$$= F^{*}(\eta \circ \phi_{n_{L}, m_{L}} \circ \phi_{n_{L-1}, m_{L-1}} \circ \cdots \circ \phi_{n_{0}, m_{0}})$$

$$= \eta \circ \phi_{n_{L}, m_{L}} \circ \phi_{n_{L-1}, m_{L-1}} \circ \cdots \circ \phi_{n_{0}, m_{0}}(F)$$

$$= \eta(\phi(\cdots\phi(\phi(F, n_{0}, m_{0}), n_{1}, m_{1}) \cdots, n_{L}, m_{L})))$$

$$= F_{n, m}.$$

Thus,  $\mathbf{M}(f)^*$  dually recognizes F.

THEOREM 9. — If f is algebraic, then it holds that

$$C_n(\Omega(f)) \leq pn \sharp \mathbf{S}(f)^*$$

for any  $n = 1, 2, 3, \dots$ , where the notation is as in (6). In particular, the logarithmic distribution of the sequence  $(\{f^m\})_{m \ge 0}$  is supported by  $\Omega(f)$  which has Hausdorff dimension zero.

Proof. — Since the table  $(F_{u,v})_{0 \leq u < p^k, mp^k \leq v < (m+1)p^k}$  for  $m \geq 0$  with

$$m = \sum_{i=0}^{\infty} m_i p^i = \sum_{i=0}^{L} m_i p^i \ m_i \in [p], \ m_L \neq 0$$

is determined by

$$\phi^*(\cdots \phi^*(\phi^*(\eta, 0, m_L), 0, m_{L-1}) \cdots, 0, m_0) \in \mathbf{S}(f)^*,$$

there exist at most  $\sharp \mathbf{S}(f)^*$  different tables as above. Hence, there exist at most  $p^k \sharp \mathbf{S}(f)^*$  different sequences among  $(F_{u,v})_{0 \leq u < p^k}$   $(v = 0, 1, 2, \cdots)$ . Take any positive integer n. Then, there are at most  $pn \sharp \mathbf{S}(f)^*$  different sequences among  $(F_{u,v})_{0 \leq u \leq n-1}$   $(v = 0, 1, 2, \cdots)$ , since there exists a positive integer k such that  $p^{k-1} \leq n < p^k \leq pn$ . Thus, we have

$$C_n(\Omega(f)) \leqslant pn \sharp \mathbf{S}(f)^*$$

for any  $n = 1, 2, 3, \dots$ . For the  $\alpha$ -Hausdorff measure  $\Lambda_{\alpha}$  of  $\Omega(f)$ , we have

$$\Lambda_{\alpha}(\Omega(f)) \leq \lim_{n \to \infty} \sum_{\substack{(H_0, \cdots, H_{n-1}) \in K^n \\ \exists \omega \in \Omega(f), \ \omega_i = H_i, i = 0, \cdots, n-1}} p^{-n\alpha}$$
$$= \lim_{n \to \infty} C_n(\Omega(f)) p^{-n\alpha}$$
$$\leq \lim_{n \to \infty} pn \sharp \mathbf{S}(f)^* p^{-n\alpha}$$
$$= 0$$

for any  $\alpha > 0$ . Thus, dim  $\Omega(f) = 0$ .

Problem. — It seems to be true that if  $f \in K((x))$  is algebraic, then the sequence  $(\{f^m\})_{m \ge 0}$  has a distribution which is either  $\delta_0$  or continuous. We do not have a proof of this assertion.

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