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ON THE EMBEDDING OF 1-CONVEX MANIFOLDS
WITH 1-DIMENSIONAL EXCEPTIONAL SET

by L. ALESSANDRINI & G. BASSANELLI *

Introduction.

Any 1-convex (i.e. strongly pseudoconvex) space $X$ is a proper modification at finitely many points of a Stein space $Y$; $X$ is called embeddable if it is biholomorphic to a closed analytic subspace of $\mathbb{C}^m \times \mathbb{CP}_n$, for suitable $m$ and $n$. C. Banica [2] and Vo Van Tan [15] had shown that every 1-convex surface is embeddable; the case when $\dim X > 2$ has been studied in some papers of M. Coltoiu and Vo Van Tan (see [5] and [16] for the references), mainly when $X$ is smooth and its exceptional set $S$ is 1-dimensional. Up to now, the best result is the following:

**THEOREM.** — Let $X$ be a 1-convex manifold with 1-dimensional exceptional set $S$. Then $X$ is embeddable, with a possible exception given by

$$\dim X = 3\text{ and } S\text{ contains a rational curve of type } (-1, -1)\text{ or } (0, -2)$$

or $(1, -3)$.

In these three exceptional cases, there are some examples of non-embeddable manifolds, only one of which is completely clarified (see [5]); anyway, the above result gives no criterion in order to check the embedding

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property in these situations. In this paper, we give a precise criterion, which turns out to be topological, namely:

**Theorem I.** — Let $X$ be a 1-convex manifold with 1-dimensional exceptional set $S$. The following statements are equivalent:

(i) $X$ is Kähler,

(ii) $S$ does not contain any effective curve $C$ which is a boundary, i.e. such that $[C] = 0$ in $H_2(X, \mathbb{Z})$.

**Theorem II.** — Let $X$ be a 1-convex manifold with 1-dimensional exceptional set $S$. If $H_2(X, \mathbb{Z})$ is finitely generated, then $X$ is embeddable if and only if it is Kähler.

The proofs do not depend on the results of M. Coltoiu and Vo Van Tan, nevertheless using them one can show that condition (ii) in Theorem I is equivalent to

(ii') $\dim X \neq 3$ or $S$ does not contain any effective curve which is a boundary and such that its irreducible components are rational curves of type $(-1, -1)$ or $(0, -2)$ or $(1, -3)$.

Theorems I and II give a partial answer to several questions of [16] (in particular to Question 2.6, Problem 2.7 and Problem 3.9), and support the idea of a parallelism between Moishezon and 1-convex manifolds (see the Introduction of [15]); more precisely, one should compare Theorem II with the fact that a Moishezon manifold is projective if and only if it is Kähler. Finally we remark that all the known examples of non-embeddable 1-convex manifolds with 1-dimensional exceptional set satisfy our hypothesis, that is $H_2(X, \mathbb{Z})$ is finitely generated. We also hope that our result will give some suggestion in order to study the general case.

Let us conclude with a remark on techniques; as to the author’s knowledge, the duality between forms and currents on a non-compact manifold is used here for the first time (in the compact case, see [9], [10], [1] et al.), and this new tool provides quite elementary proofs.

**Results.**

First of all, we fix some notations and recall the characterization of 1-convex and embeddable manifolds which we need in what follows.
DEFINITION 1. — Let $X$ be a complex manifold.

A real $(1,1)$-form $\varphi \in \mathcal{E}^{1,1}_{\mathbb{R}}(X)$ is called positive (semi-positive) at $x \in X$ if the hermitian form induced by $\varphi_x$ on the holomorphic tangent space $T_x^h X$ is positive definite (semi-definite); we shall write $\varphi_x > 0$ ($\varphi_x \geq 0$). If $\varphi_x > 0$ ($\varphi_x \geq 0$) for every $x \in X$, $\varphi$ is called positive (semi-positive), and we shall write $\varphi > 0$ ($\varphi \geq 0$).

A current $T \in \mathcal{E}^{1,1}_{\mathbb{R}}(X)'$ is positive (in the classical sense of Lelong) if $T(\varphi) > 0$, for every $\varphi \geq 0$.

An effective curve is a finite combination $\sum_{j=1}^{N} n_j C_j$ of irreducible curves $C_j$ with integral coefficients $n_j > 0$.

Let $L$ be a holomorphic line bundle on $X$ endowed with a hermitian metric $h$; if $\Theta$ is the associated curvature form, we denote by $c_1(L, h) = \frac{i}{2\pi} \Theta$ its Chern form. $L$ is positive if and only if there is a metric $h$ such that $c_1(L, h) > 0$.

Let us denote by $\tilde{H}^p(X, \mathbb{Z})$ the range of the map $i^* : H^p(X, \mathbb{Z}) \to H^p(X, \mathbb{R})$ induced by the inclusion $i : \mathbb{Z} \to \mathbb{R}$. So a real, closed $p$-form $\psi$ is called integral if its class $[\psi] \in H^p(X, \mathbb{R})$ belongs to $\tilde{H}^p(X, \mathbb{Z})$.

Consider the map

$$\delta : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}),$$

arising from the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0.$$

It is well known that if $L$ is a holomorphic line bundle, for every hermitian metric $h$ on $L$, $c_1(L, h) \in i^*(\delta(L))$.

DEFINITION 2. — A complex space $X$ is said 1-convex (or strongly pseudoconvex) if it satisfies one of the following equivalent conditions:

(i) there exists a smooth exhaustion function $F : X \to \mathbb{R}$ which is strictly plurisubharmonic outside a compact subset of $X$,

(ii) there exists a proper surjective holomorphic map $f : X \to Y$ onto a Stein space $Y$, and a finite subset $B \subset Y$ such that, if $S := f^{-1}(B)$, the induced map $X \setminus S \to Y \setminus B$ is biholomorphic and $\mathcal{O}_Y \simeq f_* \mathcal{O}_X$.

Remark 3. — The equivalence of the above conditions was proved in [11]. The map $f$ is called the Remmert reduction, and $S$, which is the
maximal compact analytic set of positive dimension in $X$, is called the exceptional set of $X$.

**DEFINITION 4.** — A 1-convex space $X$ is called embeddable if there is a holomorphic embedding $j : X \to \mathbb{C}^m \times \mathbb{CP}_n$, for some $m$ and $n$.

Roughly speaking, a 1-convex space $X$ “differs” from a Stein space $Y$ (its Remmert reduction) in its exceptional set $S$; $Y$ is embeddable in some $\mathbb{C}^m$, and $S$ is Moishezon and sometimes projective. Therefore it seems natural to ask when $X$ is embeddable in $\mathbb{C}^m \times \mathbb{CP}_n$, for some $m$ and $n$ (this is not always the case, see [5]). As for the compact/projective case, to prove that a 1-convex space is embeddable, it is useful to search “positive” holomorphic line bundles on it: see [14] and [13]. Since we shall assume that $X$ is a manifold, we use a simpler result, due to Eto, Kazama and Watanabe:

**PROPOSITION 5 ([8], Theorem III).** — Let $X$ be a 1-convex manifold carrying a positive holomorphic line bundle $L$. Then $X$ is embeddable.

To prove Theorems I and II, we need a couple of Lemmata; the first one is an elementary result in group theory.

**LEMMA 6.** — If $g_1, \ldots, g_q$ are elements of an abelian group $G$ such that, for suitable real numbers $r_1, \ldots, r_q > 0$, $g_1 \otimes r_1 + \cdots + g_q \otimes r_q = 0$ in $G \otimes \mathbb{R}$, then there exist positive integers $n_1, \ldots, n_q$ such that $n_1 g_1 + \cdots + n_q g_q = 0$ in $G$.

**Proof.** — Let $G'$ be the subgroup of $G$ generated by $g_1, \ldots, g_q$. Since $G' \otimes \mathbb{R}$ is a subgroup of $G \otimes \mathbb{R}$, $G'$ satisfies the hypotheses too; so there is no restriction in assuming that $G$ is finitely generated. Therefore $G = F \oplus T$, where $F$ is free and $T$ is the torsion part, and $g_i = f_j + t_j, j = 1, \ldots, q$, with $f_j \in F$ and $t_j \in T$. Thus by the hypothesis $f_1 \otimes r_1 + \cdots + f_q \otimes r_q = 0$; identifying $F$ with $\mathbb{Z}^n \subset \mathbb{R}^n$, for a suitable $n$, we get, in $\mathbb{R}^n$,

$$r_1 f_1 + \cdots + r_q f_q = 0.$$ 

This turns out to be a linear system with integral coefficients: therefore there are positive integers $m_1, \ldots, m_q$, such that $m_1 f_1 + \cdots + m_q f_q = 0$. For a suitable positive integer $m$: $m t_1 = \ldots = m t_q = 0$; thus

$$m m_1 g_1 + \cdots + m m_q g_q = 0$$

Q.E.D.
LEMMA 7. — Let $\varphi \in \mathcal{E}^{1,1}_R(X)$ be a closed $(1,1)$-form on $X$ which is positive on $S$ (i.e. $\varphi_x > 0$, for every $x \in S$). Then there exists a smooth function $F : X \to \mathbb{R}$ such that $\varphi + \frac{i}{2\pi} \partial \bar{\partial} F$ is positive on $X$.

Proof. — Let $g : X \to \mathbb{C}^N$ be the proper holomorphic map given by the composition of the Remmert reduction $f : X \to Y$ and an embedding $j : Y \to \mathbb{C}^N$. Denote by $z = (z_1, \ldots, z_N)$ coordinates in $\mathbb{C}^N$ and let $\alpha := \frac{i}{2\pi} \partial \bar{\partial}|z|^2$; then $g^*\alpha \geq 0$ on $X$ and $g^*\alpha > 0$ on $X \setminus S$. Since $\varphi > 0$ in a neighborhood of $S$, for every compact subset $K$ of $X$ there exists a positive constant $M_K$ such that $\varphi + M_K g^*\alpha > 0$ on $K$.

Let $K_1 := \overline{B(1)} = \{z \in \mathbb{C}^N; |z| \leq 1\}$ and for every $n \geq 1$, $K_{n+1} := \overline{B(n+1)} \setminus \overline{B(n)}$. Therefore, for suitable $M_n > 0$:

$$\varphi + M_n g^* \alpha > 0,$$

on $g^{-1}(K_n)$.

Choose a smooth function $u : [0, +\infty) \to \mathbb{R}$ such that $u'(t) > 0$, for every $t \geq 0$, and $u(|z|^2) > M_n$ on $K_n$, for $n \geq 1$.

Let $v(s) := \int_0^s u(t)dt$, $s \geq 0$. We get

$$i\partial \bar{\partial} v(|z|^2) = v''(|z|^2) i\partial |z|^2 \wedge \bar{\partial}|z|^2 + v'(|z|^2) i\partial \bar{\partial}|z|^2.$$

The first term in the right hand side is $\geq 0$, hence on $K_n$ we have

$$i \frac{1}{2\pi} \partial \bar{\partial} v(|z|^2) > M_n \frac{i}{2\pi} \partial \bar{\partial}|z|^2;$$

therefore we get on $X$:

$$\varphi + g^* \frac{i}{2\pi} \partial \bar{\partial} v(|z|^2) > 0.$$

Q.E.D.

Let us go to the proofs of Theorem I and Theorem II.

THEOREM I. — Let $X$ be a 1-convex manifold with 1-dimensional exceptional set $S$. The following statements are equivalent:

(i) $X$ is Kähler,

(ii) $S$ does not contain any effective curve $C$ which is a boundary, i.e. such that $[C] = 0$ in $H_2(X, \mathbb{Z})$.

Proof. — (i) $\Rightarrow$ (ii): In a Kähler manifold no positive combination of compact analytic subsets can be a boundary.

(ii) $\Rightarrow$ (i): Let

$$A := \{\varphi \in \mathcal{E}^{1,1}_R(X); \varphi_x > 0, \text{ for every } x \in S\}$$

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Let us argue by contradiction and assume that $A \cap B = \emptyset$. Since $A$ is a non-empty open cone in the natural topology of $\mathcal{E}^{1,1}_\mathbb{R}(X)$ and $B$ is a linear subspace, by the Hahn-Banach Theorem ([12], II.3.1), we get a closed hyperplane in the topological vector space $\mathcal{E}^{1,1}_\mathbb{R}(X)$, containing $B$ and not intersecting $A$. That is, there exists a current $T \in \mathcal{E}^{1,1}_\mathbb{R}(X)'$ such that $T(\varphi) > 0$ for every $\varphi \in A$, and $T(\varphi) = 0$ for every $\varphi \in B$.

**Claim.** Let $T \in \mathcal{E}^{1,1}_\mathbb{R}(X)'$.

(i) If $T(\varphi) > 0$ for every $\varphi \in A$, then $T$ is positive and $\text{supp}(T) \subset S$;

(ii) If $T(\varphi) = 0$ for every $\varphi \in B$, then $i\partial \bar{\partial} T = 0$.

**Proof of the claim.** Let $\psi \in \mathcal{E}^{1,1}_\mathbb{R}(X)$, $\psi \geq 0$, and let $\varphi \in A$: then for every $\varepsilon > 0$, $\psi + \varepsilon \varphi \in A$, hence

$$T(\psi) = \lim_{\varepsilon \to 0} T(\psi + \varepsilon \varphi) \geq 0.$$ 

Now let $\psi \in \mathcal{E}^{1,1}_\mathbb{R}(X)$, $\text{supp}(\psi) \subset X \setminus S$, and let $\varphi \in A$. For every $c \in \mathbb{R}$, $c\psi + \varphi \in A$, so that

$$cT(\psi) + T(\varphi) > 0 \quad \forall c \in \mathbb{R};$$

this implies $T(\psi) = 0$.

To prove the second assertion, notice that for every $f \in \mathcal{E}^0_\mathbb{R}(X)$, $i\partial \bar{\partial} f \in B$ so that $i\partial \bar{\partial} T(f) = T(i\partial \bar{\partial} f) = 0$. Q.E.D.

The claim allows to apply Theorem 4.10 in [3] to the current $T$ arising from the Hahn-Banach Theorem: therefore, there are some irreducible components of $S$, say $C_1, \ldots, C_q$, and some positive real numbers $r_1, \ldots, r_q$ such that

$$T = \sum_{j=1}^q r_j C_j$$

(in particular, $T$ is closed). Let $\varphi = \varphi_{02} + \varphi_{11} + \varphi_{02}$ be a closed real 2-form on $X$; define $\pi(\varphi) := \varphi_{02}$. Then $\pi$ induces a linear map $\Pi : H^2(X, \mathbb{R}) \to H^{0,2}_B(X) = H^2(X, \mathcal{O})$. But $S$ is 1-dimensional, thus ([4], Lemma 1) $H^2(X, \mathcal{O}) = 0$; since $\Pi$ is the null map, every class belonging to $H^2(X, \mathbb{R})$ can be represented by means of an element of $B$. Hence “$T$ vanishes on $B$” means that $\sum_{j=1}^q r_j C_j$ vanishes on every class of $H^2(X, \mathbb{R})$. But $H^2(X, \mathbb{R}) = \text{Hom}(H_2(X, \mathbb{R}), \mathbb{R})$, thus $\sum_{j=1}^q r_j [C_j] = 0$. 

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$H_2(X, \mathbb{R}) = H_2(X, \mathbb{Z}) \otimes \mathbb{R}$. Applying Lemma 6 we get a contradiction with (ii). Therefore there exists $\varphi \in A \cap B$; by means of Lemma 7 the form $\varphi$ can be modified in order to obtain a closed Kähler form for $X$. Q.E.D.

Remark 8. — When $S$ is irreducible, condition (ii) in Theorem I means that the class of $S$ in $H_2(X, \mathbb{Z})$ is not a torsion class: therefore the map $H_2(S, \mathbb{R}) \to H_2(X, \mathbb{R})$ turns out to be injective and $\gamma : H^2(X, \mathbb{R}) \to H^2(S, \mathbb{R})$ is surjective. In this case there is another proof\footnote{Private communication of M. Coltoiu.} of Theorem I. Denote by $\mathcal{H}$ the sheaf of germs of real pluriharmonic functions; as in [9], we get easily that $0 \to \mathbb{R} \to \mathcal{O} \to \mathcal{H} \to 0$ is exact and that $H^1(X, \mathcal{H})$ is the space of real closed $(1,1)$-forms, modulo the space of $i\partial\bar{\partial}$-exact forms. Now for a suitable embeddable neighborhood $U$ of $S$ (see [6], p. 563) we get that in the following commutative diagram with exact lines:

\[
\begin{array}{cccccc}
\cdots & \to & H^1(X, \mathcal{O}) & \to & H^1(X, \mathcal{H}) & \to & H^2(X, \mathbb{R}) & \to & H^2(X, \mathcal{O}) & = 0 \\
& \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & & & \\
\cdots & \to & H^1(U, \mathcal{O}) & \to & H^1(U, \mathcal{H}) & \to & H^2(U, \mathbb{R}) & \to & H^2(U, \mathcal{O}) & = 0
\end{array}
\]

$\alpha$ is an isomorphism, $H^2(U, \mathbb{R}) = H^2(S, \mathbb{R})$, $\gamma$ is surjective. Hence $\beta$ is surjective, thus we can extend a Kähler form on $U$ to a closed $(1,1)$-form on $X$; by Lemma 7, $X$ is Kähler.

Theorem II. — Let $X$ be a 1-convex manifold with 1-dimensional exceptional set $S$. If $H_2(X, \mathbb{Z})$ is finitely generated, the following statements are equivalent:

(i) $X$ is Kähler,

(ii) $S$ does not contain any effective curve $C$ which is a boundary, i.e. such that $[C] = 0$ in $H_2(X, \mathbb{Z})$,

(iii) $X$ is embeddable.

Proof. — (iii) $\Rightarrow$ (i) is obvious, so by the previous theorem it is enough to show (ii) $\Rightarrow$ (iii). Let

\[B' := \{ \varphi \in \mathcal{E}_{\mathbb{R}}^{1,1}(X); \varphi = c_1(L, h), \text{ for some holomorphic line bundle } L \}
\]

on $X$ and some hermitian metric $h$ on $L$.

Let us assume $A \cap B' = \emptyset$. In order to argue exactly as in Theorem I, with $B'$ instead of $B$, only two remarks are necessary. First, from $A \cap B' = \emptyset$ it follows $A \cap (B' \otimes \mathbb{R}) = \emptyset$ where $B' \otimes \mathbb{R}$ is the linear subspace generated by $B'$; so $A$ and $B'$ can be separated as above. Second, let $f$ be a real smooth
function on $X$ and $L$ a holomorphic line bundle with a hermitian metric $h$; then
\[ \frac{i}{2\pi} \partial \bar{\partial} f = c_1(L, h) - c_1(L, e^f h) \in B' \otimes \mathbb{R}. \]

Thus Claim (ii) in the proof of Theorem I holds with $B'$ instead of $B$. Therefore there is a current $T = \sum_{j=1}^{q} r_j C_j$, with $r_j > 0$, which vanishes on $B'$.

As we have just noted, $H^2(X, \mathcal{O}) = 0$, thus the map $\delta : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ is surjective. Therefore each integral class can be represented by an element of $B'$, thus $\sum_{j=1}^{q} r_j [C_j] \in H_2(X, \mathbb{R})$ vanishes on $H^2(X, \mathbb{Z})$. Since $H_2(X, \mathbb{Z})$ is finitely generated, $H^2(X, \mathbb{R}) = H^2(X, \mathbb{Z}) \otimes \mathbb{R}$; thus $\sum_{j=1}^{q} r_j [C_j]$ vanishes on $H^2(X, \mathbb{R})$. As we have just seen in the previous proof, this gives a contradiction, so we conclude that there are $L$ and $h$ such that $c_1(L, h) \in A \cap B'$. Finally, applying Lemma 7, it follows that $c_1(L, e^{-F} h)$ is positive on $X$. Q.E.D.

**Corollary 9.** — In the hypotheses of Theorem I, the statements (i), (ii) are also equivalent to

(ii') $\dim X \neq 3$ or $S$ does not contain any effective curve which is a boundary and such that its irreducible components are rational curves of type $(-1, -1)$ or $(0, -2)$ or $(1, -3)$.

**Proof.** — Obviously (ii) $\Rightarrow$ (ii'). Assume that (ii) is false, that is, there are positive integers $n_1, \ldots, n_q$ and irreducible components $C_1, \ldots, C_q$ of $S$ such that $T = \sum_{j=1}^{q} n_j C_j$ is a boundary. Then the canonical bundle $K_X$ vanishes on $T$, that is, $0 = \sum_{j=1}^{q} n_j K_X.C_j$, so $K_X.C_j = 0$ for every $j = 1, \ldots, q$. By the Step 1 of Theorem 1.5 in [16], every $C_j$ is the exceptional set of a 1-convex neighborhood $U_j$ of it. Now $0 = K_X.C_j = K_{U_j}.C_j$; so $K_{U_j}$ is not ample and, by means of Theorem 3 in [5], this implies that $C_j$ is rational, $\dim X = 3$ and (see [5], Remark p. 463) $C_j$ must be of type $(-1, -1), (0, -2)$ or $(1, -3)$. Q.E.D.

**Remark 10.** — All the examples given in the literature in connection to the embedding problem satisfy the property "$H_2(X, \mathbb{Z})$ is finitely generated"; nevertheless, using the example given in [7], it is easy to build a 1-convex threefold whose second homology group is not finitely generated.

Now assume that the exceptional curve $S$ is irreducible. One can use the proof of Theorem II or apply the argument of Remark 8 to the sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$, to see that (without the hypothesis...
"$H_2(X, \mathbb{Z})$ is finitely generated") $X$ is embeddable if and only if there exists a homomorphism in $\text{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$ which does not vanish on $S$, that is, the map $H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is not the zero map.

Therefore a comparison between this last condition and Theorem I may suggest the way to a counterexample (if it exists!) of a 1-convex Kähler manifold, with 1-dimensional exceptional set, which is non-embeddable.

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