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EXISTENCE OF PERMANENT AND BREAKING WAVES FOR A SHALLOW WATER EQUATION: A GEOMETRIC APPROACH

by Adrian CONSTANTIN

Introduction.

There are several models describing the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity.

We have the celebrated Korteweg-de Vries (KdV) equation [22]

$$\begin{cases} u_t + 6uu_x + u_{xxx} = 0, \qquad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Here and below $u(t,x)$ represents the wave height above a flat bottom, x is proportional to distance in the direction of propagation and t is proportional to elapsed time. The Cauchy problem (1.1) has been studied extensively, cf. [20], [21], and citations therein. A very interesting aspect of the KdV equation is that it admits traveling wave solutions, i.e. solutions of the form $u(t,x) = \phi(x - ct)$ which travel with fixed speed c and vanish at infinity. Further, these traveling wave solutions are solitons: two traveling waves reconstitute their shape and size after interacting with each other [15]. KdV is integrable[(1)] (for a discussion of this aspect, we refer to [25]). An astonishing plentitude of structures is tied into the KdV equation

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(1) Integrability is meant in the sense of the infinite-dimensional extension of a classical completely integrable Hamiltonian system: there is a transformation which converts the equation into an infinite sequence of linear ordinary differential equations which can be trivially integrated [25].

which explains the many interesting (and physically relevant) phenomena modeled by (1.1). However, the KdV equation does not model the occurence of breaking for shallow water waves: as soon as $u_0 \in H^1(\mathbb{R})^{(2)}$, the solutions of (1.1) are global, cf. [21], and it is known that some shallow water waves break! Under wave breaking we understand [30] the phenomenon that a wave remains bounded, but its slope becomes unbounded in finite time.

An alternative model for KdV is the regularized long wave equation

$$\tag{1.2} \begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

proposed by Benjamin, Bona and Mahony [3]. Equation (1.2) has better analytical properties than the KdV model but it is not integrable and numerical work suggests that its traveling waves are not solitons [14]. As any initial profile $u_0 \in H^2(\mathbb{R})$ for (1.2) develops into a solution of permanent form cf. [3], the regularized long wave equation does not model wave breaking.

Whitham [30] emphasized that the breaking phenomenon is one of the most intriguing long standing problems of water wave theory. He suggested the equation

$$\tag{1.3} \begin{cases} u_t + uu_x + \displaystyle\int_{\mathbb{R}} k_0(x-\xi)\, u_x(t,\xi)\, d\xi = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

with the singular kernel

$$k_0(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\frac{\tanh \xi}{\xi}\Big)^{\frac{1}{2}} e^{i\xi x}\, d\xi,$$

as a relative simple model equation combining full linear dispersion with long wave nonlinearity to describe the breaking of waves. The numerical calculations carried out for the Whitham equation (1.3) do not support the hypothesis that soliton interaction occurs for its traveling waves, cf. [14].

Recently, Camassa and Holm [4] derived a new equation describing unidirectional propagation of surface waves on a shallow layer of water which is at rest at infinity:

$$\tag{1.4} \begin{cases} v_t - v_{txx} + 2\kappa v_x + 3vv_x = 2v_xv_{xx} + vv_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ v(0,x) = v_0(x), \quad x \in \mathbb{R}. \end{cases}$$

(2) $H^k(\mathbb{R})$, $k \in \mathbb{N}$, stands for the Sobolev space of functions with derivatives up to order k having finite $L^2(\mathbb{R})$ norm.

The constant κ is related to the critical shallow water speed. The term vv_{xxx} makes (1.4) a nonlinear dispersive wave model(3): the transition to full nonlinearity (compared with the weakly nonlinear regime of the other models) is motivated by the search of a single model describing, at the same time, as many as possible of physically interesting phenomena observed in the propagation of shallow water waves. Note that we can get rid of κ in (1.4) by the substitution $u(t,x) = v(t, x - \kappa t) + \kappa$, obtaining the Cauchy problem

$$\begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{1.5}$$

Equation (1.5) was found earlier by Fuchssteiner and Fokas (see [18], [19]) as a bi-Hamiltonian generalization of KdV.

A quite intensive study of equation (1.4) started with the discoveries of Camassa and Holm [4]: besides deriving the equation from physical principles, they obtained the associated isospectral problem and found that the equation has solitary waves that interact like solitons. Numerical simulations [5] support their results. The study of the associated isospectral problem proves the integrability of (1.4) in the periodic case for a large class of initial data [12]. The well-posedness of the shallow water equation in $H^3(\mathbb{R})$ and results on the existence of global solutions and wave breaking were obtained in [8] and [9].

As noted by Whitham [30], it is intriguing to know which mathematical models for shallow water waves exhibit both, phenomena of soliton interaction and wave breaking. Equation (1.5) is the first such equation found and "has the potential to become the new master equation for shallow water wave theory", cf. [19], modeling the soliton interaction of peaked traveling waves, wave breaking, admitting as solutions permanent waves, and being an integrable Hamiltonian system.

Let us now turn to a geometrical interpretation of the equation (1.5). Following the seminal paper of Arnold [1] and subsequent work by Ebin and Marsden [16] for the Euler equation in hydrodynamics, equation (1.5) can be associated with the geodesic flow on the infinite dimensional Hilbert manifold $\mathcal{D}^3(\mathbb{R})$ of diffeomorphisms of the line satisfying certain asymptotic conditions at infinity, equipped with the right invariant metric, which, at

(3) In the case of KdV, the linear dispersion term balances the breaking effect of the nonlinear term (cf. Burgers equation [30]).

the identity, is given by the $H^1(\mathbb{R})$ inner product (see Section 2). With this metric $\mathcal{D}^3(\mathbb{R})$ becomes a weak Riemannian manifold[(4)]. The connection of the shallow water flow with infinite dimensional geometry was announced in [12] and [27] - for a more detailed discussion we refer to Section 2.

Let us now present a brief overview of the contents of this paper.

In Section 2, we review the manifold structure of $\mathcal{D}^3(\mathbb{R})$ as analyzed in [6] and (following [12], [27], [23], with some additions) the connection between the shallow water equation (1.5) and the geodesic flow[(5)] on $\mathcal{D}^3(\mathbb{R})$ with respect to the weak Riemannian structure induced by the right-invariant metric which at the identity is given by the $H^1(\mathbb{R})$ inner product on the tangent space. We prove that the Riemannian exponential map is a local diffeomorphism and deduce from this that two points on $\mathcal{D}^3(\mathbb{R})$ which are close enough can be joined by a unique geodesic - intuitively, this says that one state of the surface of shallow water is connected to another nearby state through a uniquely determined solution of equation (1.5).

The study of the local geometry on $\mathcal{D}^3(\mathbb{R})$ leads us to introduce in Section 3 some useful tools needed in Section 4 and Section 5 where we deal with the existence of global solutions and the phenomenon of blow-up of solutions for the shallow water equation (1.5), respectively. We describe in detail the wave-breaking mechanism for solutions of (1.5) with certain initial profiles and find the exact blow-up rate. For a large class of initial profiles we also determine the blow-up set.

In the last part of this paper we apply the results on the shallow water equation obtained in Sections 4 and 5 to the study of geodesics on the diffeomorphism group $\mathcal{D}^3(\mathbb{R})$: while there are geodesics which can be continued indefinitely in time, we also exhibit geodesics with a finite life-span[(6)].

Finally, let us mention that it is interesting to consider the problems studied in this paper for spatially periodic solutions of the shallow water

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(4) Since $\mathcal{D}^3(\mathbb{R})$ is not a complete metric space with respect to the distance obtained from the Riemannian metric.

(5) The fundamental theorem of classical Riemannian geometry stating that every Riemannian metric admits a unique smooth Levi-Civita connection fails in general for weak Riemannian manifolds (see [16]). In the case of $\mathcal{D}^3(\mathbb{R})$ the existence of geodesics follows from the existence of a smooth metric spray (see Section 2).

(6) It is interesting to note that it is not possible to study qualitative properties of the KdV equation looking at the geodesic flow on the diffeomorphism group; one has to consider a larger group that includes the group of diffeomorphisms, the Bott-Virasoro group [2]. The geodesic equation on the diffeomorphism group with respect to the right-invariant $L^2(\mathbb{R})$ inner product is the nonviscous Burgers equation, cf. [2].

equation (1.5). It is reasonable to expect that most of our results are valid in the periodic case as well. For some investigations treating the blow-up of solutions with special odd initial profiles we refer to [11].

2. Diffeomorphism group.

There are two standard coordinate systems used in classical fluid dynamics. In material (Lagrangian) coordinates, one describes the fluid as seen from one of the particles of the fluid (the observer follows the fluid). In spatial (Eulerian) coordinates, one describes the fluid from the viewpoint of a fixed observer. In this section we present the connection between the shallow water flow given by (1.5) and the geodesic motion on the group of diffeomorphisms of the line satisfying certain asymptotic conditions at infinity, endowed with a weak Riemannian structure - working on the diffeomorphism group corresponds to using Lagrangian coordinates while working with the equation in $u(t,x)$ means working in Eulerian coordinates.

Following Cantor [6], we define for $k, s \in \mathbb{N}$ the weighted Hilbert space $\mathcal{M}_s^k$ as the completion of the space of smooth real functions $f : \mathbb{R} \to \mathbb{R}$, compactly supported on the line, with respect to the norm

$$||f||^2 = \int_{\mathbb{R}} \sum_{j\leqslant k} (1+x^2)^{j+s} \, (\partial_x^j f)^2(x) \, dx.$$

Define $\mathcal{M} := \{\eta : \mathbb{R} \to \mathbb{R}, \quad (\eta - \mathrm{Id}) \in \mathcal{M}_1^3\}$, and consider the group of orientation-preserving diffeomorphisms of the line modeled on $\mathcal{M}_1^3$,

$$\mathcal{D}^3(\mathbb{R}) = \big\{\eta : \mathbb{R} \to \mathbb{R}\,,\ \eta \text{ bijective increasing and } \eta, \eta^{-1} \in \mathcal{M}\big\}.$$

The conditions at infinity are imposed on the diffeomorphisms in $\mathcal{D}^3(\mathbb{R})$ for technical reasons [6] in order to obtain a manifold.

$\mathcal{D}^3(\mathbb{R})$ is an infinite dimensional manifold, which locally, around each of its points η, looks like a Hilbert space. Indeed, $\mathcal{M}$ is a translate of the Hilbert space $\mathcal{M}_1^3$ and since $\mathcal{D}^3(\mathbb{R})$ is open in $\mathcal{M}$ [6], $\mathcal{D}^3(\mathbb{R})$ is an infinite dimensional manifold modeled on a Hilbert space.

The group $\mathcal{D}^3(\mathbb{R})$ admits a 'Lie group'-like structure which allows to extend some of the results valid for finite-dimensional Lie groups (see [28]) to the infinite-dimensional case.

$\mathcal{D}^3(\mathbb{R})$ can be given a group structure with multiplication being defined as the composition of two such diffeomorphisms. Right multiplication

$r_\eta(\phi) := \phi \circ \eta$ is smooth (C^∞), but left multiplication and inversion are only continuous so that $\mathcal{D}^3(\mathbb{R})$ is not a Lie group in a strict sense. However, it shares some important properties of a Lie group.

The Lie algebra of a Lie group G, consisting of all vector fields on G which are invariant under the group multiplication, may be identified with the tangent space to G at the identity, cf. [28].

Denote by $\mathcal{H}^3(\mathbb{R})$ the vector space of all $\mathcal{M}_1^3$-vector fields on $\mathbb{R}$. Any tangent vector X_η to $\mathcal{D}^3(\mathbb{R})$ at η is of the form $X \circ \eta$ with some $X \in \mathcal{H}^3(\mathbb{R})$. For a given $X \in \mathcal{H}^3(\mathbb{R})$, let $X^r(\eta) = X \circ \eta$ denote the right-invariant vector field on $\mathcal{D}^3(\mathbb{R})$[7] whose value at the identity Id is X. $\mathcal{H}^3(\mathbb{R})$ can be thought of as the Lie algebra of $\mathcal{D}^3(\mathbb{R})$. The (right) Lie algebra bracket on $\mathcal{H}^3(\mathbb{R})$ is defined as

$$(\mathcal{L}_{X^r} Y^r)(\eta) := [X, Y] \circ \eta, \quad X, Y \in \mathcal{H}^3(\mathbb{R}),\ \eta \in \mathcal{D}^3(\mathbb{R}),$$

where $[X, Y]$ denotes the Lie bracket of the vector fields X and Y on $\mathbb{R}$, given in local coordinates by

$$[X, Y] = \Big(f \frac{\partial g}{\partial x} - g \frac{\partial f}{\partial x}\Big) \frac{\partial}{\partial x}$$

if $X = f(x) \frac{\partial}{\partial x}$ and $Y = g(x) \frac{\partial}{\partial x}$. Note that $\mathcal{H}^3(\mathbb{R})$ is not a Lie algebra in the strict sense since it is not closed under the bracket operation due to loss of smoothness.

Let us now describe the weak Riemannian structure with which we endow $\mathcal{D}^3(\mathbb{R})$ in order to recover the shallow water equation as the metric spray on the diffeomorphism group.

Recall that a Riemannian metric on a Lie group is called right-invariant if it is preserved by all right multiplications [24]. If the Lie group is connected, it suffices to prescribe such a metric at the identity (the metric can be carried over to the remaining points by right multiplications).

Consider the $H^1(\mathbb{R})$-inner product

$$\langle f, g \rangle_{H^1(\mathbb{R})} = \int_{\mathbb{R}} f(x) g(x)\, dx + \int_{\mathbb{R}} f'(x) g'(x)\, dx, \quad f, g \in H^1(\mathbb{R}),$$

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[7] Right translation being smooth, we can talk about right-invariant vector fields on $\mathcal{D}^3(\mathbb{R})$.

on $T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R}) \equiv \mathcal{H}^3(\mathbb{R})$. It induces a metric on the whole tangent space $T\mathcal{D}^3(\mathbb{R})$ by right multiplication, i.e. for $V, W \in T_\eta\mathcal{D}^3(\mathbb{R})$,

$$\langle V, W\rangle_{H^1(\mathbb{R})} := \left\langle V \circ \eta^{-1}, W \circ \eta^{-1}\right\rangle_{H^1(\mathbb{R})}.$$

This metric is right-invariant (by definition) but not left-invariant. As the topology induced by this metric is weaker than the topology of $\mathcal{D}^3(\mathbb{R})$, $\mathcal{D}^3(\mathbb{R})$ is said to be a weak Riemannian manifold.

It turns out that $\mathcal{D}^3(\mathbb{R})$, endowed with this metric, is the appropriate configuration space for the shallow water equation

$$\text{(2.1)} \qquad \begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

in the sense that (2.1) is a re-expression of the geodesic flow on $\mathcal{D}^3(\mathbb{R})$ with the above described (right-invariant) metric. More precisely, if u solves (2.1) with $u_0 \in \mathcal{M}_1^3$ on the time interval $[0,T)$, $u \in C^1([0,T);\mathcal{M}_1^3)$, and if $q \in C^2([0,T_1);\mathcal{D}^3(\mathbb{R}))$ solves

$$\text{(2.2)} \qquad \begin{cases} q_t = u(t,q) \\ q(0,x) = x, \quad x \in \mathbb{R}, \end{cases}$$

on the time interval $[0,T_1)$ with $0 < T_1 \leqslant T$, then the curve $\{q(t,\cdot) : t \in [0,T_1)\}$ is a geodesic in $\mathcal{D}^3(\mathbb{R})$, starting at the identity in the direction $u_0 \in \mathcal{M}_1^3$. Conversely, if $q \in C^2([0,T_1);\mathcal{D}^3(\mathbb{R}))$ is a geodesic, then $u = q_t \circ q^{-1}$ solves (2.1) for $0 < t < T_1$. This interpretation of (2.1), announced in [12], [27], and also discussed in [23], resembles the situation for Euler's equation in hydrodynamics [1], [16].

For a Riemannian metric on a Hilbert manifold a Levi-Civita (metric) derivative can be defined and the local existence of geodesics is ensured [24]. However, for infinite dimensional weak Riemannian manifolds the latter is not always true, cf. [16], and we have to prove the existence of geodesics on $\mathcal{D}^3(\mathbb{R})$. We will do this below(8).

Recall that, in local coordinates, the equations for a geodesic on a n-dimensional Riemannian manifold are given by

$$\frac{d^2 x_i}{dt^2} + \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{dx_k}{dt}\frac{dx_j}{dt} = 0, \quad i = 1,\ldots,n,$$

(8) In [23] the realization of the shallow water equation as metric spray on $\mathcal{D}^3(\mathbb{R})$ is explained but the problem of the existence of geodesics is not dealt with.

where Γ^i_{jk} are the Christoffel symbols. In some cases, cf. [24], it is profitable to find geodesics on a finite dimensional Riemannian manifold by following a different approach: the metric gives rise to a spray and one can recover geodesics using this coordinate independent formalism. The spray of a metric is the natural way to deal with geodesics in infinite dimensional Riemannian geometry, cf. [24].

For an infinite dimensional manifold M modeled on a Hilbert space H, denote by $T(M)$ its tangent bundle and by $T(T(M))$ the tangent bundle of the tangent bundle. Let U be open in the Hilbert space H, so that $T(U) = U \times H$ and $T(T(U)) = (U \times H) \times (H \times H)$. A **second order vector field** on $U \times H$ has a local representation

$$F : U \times H \to H \times H, \quad F(u, X) = (u, f(u, X)), \quad (u, X) \in U \times H,$$

with $f : U \times H \to H$. Following [24], we define a **spray** to be a second order vector field F over M (that is, a vector field on the tangent bundle $T(M)$ with a chart representation as above) which satisfies a homogeneous quadratic condition which is, in local coordinates as above, of the form

$$f(u, sX) = s^2 f(u, X) \quad \text{for} \quad s \in \mathbb{R},\ (u, X) \in H \times H.$$

A C^2-curve $\alpha : J_X \to M$, defined on an open interval J_X containing zero, is said to be a **geodesic with respect to the spray** F with initial condition $\frac{d\alpha}{dt}\Big|_{t=0} = X \in T_{\alpha(0)}M$ if the curve $\frac{d\alpha}{dt} : J_X \to T(M)$ is an integral curve of the vector field F. The homogeneity condition for the spray translates into the following property for the geodesic flow

$$\alpha(t, sX) = \alpha(st, X) \quad \text{for} \quad t \in J_{sX} \quad \text{with} \quad st \in J_X.$$

For a weak Riemannian manifold M modeled on a Hilbert space H with the property that M is at the same time a topological group with C^1-right multiplication, a spray depending on the right-invariant metric induced by the scalar product $\langle \cdot, \cdot \rangle$ on the tangent space at the neutral element e of M, T_eM, can be constructed as follows (for more details, we refer to [23]): Assume that a bilinear map B on $T_eM \times T_eM$ can be defined implicitely by [9]

$$\langle B(X, Y), Z \rangle = \langle X, [Y, Z] \rangle, \quad X, Y, Z \in T_eM,$$

(9) The existence of such a bilinear map must be proven in each individual case as we deal with a weak Riemannian structure (its existence is not ensured by a general argumentation).

where $[\cdot,\cdot]$ stands for the Lie bracket. The metric spray F is then locally given by

$$F(u,X) = (u, B(X,X)), \quad u \in U,\ X \in H,$$

where U is open in the Hilbert space H satisfying $T(U) = U \times H$. In other words, considering an integral curve $V(t) \in T_{\eta(t)}M$ of the metric spray, we have for its pullback $u = V \circ \eta^{-1}$ (where $\circ$ stands for the group multiplication) the equation

$$\frac{du}{dt} = B(u,u). \tag{2.3}$$

Note that $V(t) = \frac{d\eta}{dt}(t)$ where $t \mapsto \eta(t)$ is the geodesic of the metric spray.

For $M = \mathcal{D}^3(\mathbb{R})$, the existence of the bilinear map can be easily proved. As $T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R}) = \mathcal{M}^3_1$, we have, using integration by parts,

$$\langle X, [Y,Z]\rangle_{H^1(\mathbb{R})} = -\int_{\mathbb{R}} \Big([2Y_x(1-\partial_x^2)X + Y(1-\partial_x^2)X_x] Z \Big),$$

while

$$\langle B(X,Y), Z\rangle_{H^1(\mathbb{R})} = \int_{\mathbb{R}} \Big([(1-\partial_x^2)\, B(X,Y)]\, Z \Big).$$

Thus $B(X,Y)$ is given by

$$B(X,Y) = -\,(1-\partial_x^2)^{-1}\Big(2Y_x(1-\partial_x^2)X + Y(1-\partial_x^2)X_x\Big). \tag{2.4}$$

Actually, in order to justify these integrations by parts, we have to assume decay properties of X, Y, Z, and $B(X,Y)$ at infinity. The following auxiliary result is needed.

LEMMA 2.1.— *If $f \in \mathcal{M}^3_1$, then all three functions*

$$x \mapsto (1+x^2)^{\frac{3}{4}} f(x), \quad x \mapsto (1+x^2)^{\frac{5}{4}} f'(x) \quad \textit{and} \quad x \mapsto (1+x^2)^{\frac{7}{4}} f''(x)$$

belong to the space $L^\infty(\mathbb{R})$.

Indeed, assuming that this lemma holds, we have no problems with the integration by parts involving $X, Y, Z \in \mathcal{M}^3_1$. To complete the argumentation, note that $Q := (1-\partial_x^2)^{-1}$ is an isomorphism between $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$ and by Lemma 2.1, the function $x \mapsto 2Y_x(1-\partial_x^2)X + Y(1-\partial_x^2)X_x$ belongs to $L^2(\mathbb{R})$ if $X, Y \in \mathcal{M}^3_1$.

Proof of Lemma 2.1.— Since all three cases are similar, we consider only the function $g(x) := (1+x^2)^{\frac{5}{4}} f'(x)$, $x \in \mathbb{R}$. From $f \in \mathcal{M}^3_1$ we infer that the function $x \mapsto \partial_x g^2(x)$ belongs to $L^1(\mathbb{R})$ and therefore $g^2 \in L^\infty(\mathbb{R})$ as

$$g^2(x) - g^2(y) = \int_x^y \partial_r g^2(r)\, dr \quad \text{for} \quad x < y$$

by the absolute continuity of the function g^2. □

In conclusion, combining relations (2.3) and (2.4), we deduce that $t \mapsto q(t)$ is a geodesic on $\mathcal{D}^3(\mathbb{R})$ if and only if $u := q_t \circ q^{-1}$ solves the equation

$$\frac{du}{dt} = -(1-\partial_x^2)^{-1}[3uu_x - 2u_x u_{xx} + uu_{xxx}]$$

or, applying the operator $(1-\partial_x^2)$ to both sides,

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} - uu_{xxx}$$

which is exactly equation (2.1).

For further use, let us derive an equivalent form of the differential equation that must be fulfilled by a geodesic $t \mapsto q(t)$ on $\mathcal{D}^3(\mathbb{R})$. Differentiating $q_t = u \circ q$ with respect to time, we obtain

$$\begin{aligned} q_{tt} &= u_t \circ q + u_x \circ q \cdot q_t \\ &= (u_t + uu_x) \circ q \\ &= -\Big((1-\partial_x^2)^{-1}\, \partial_x(u^2 + \tfrac{1}{2}u_x^2)\Big) \circ q := S(q, q_t). \end{aligned} \tag{2.5}$$

To obtain the last identity, note that by applying $(1-\partial_x^2)^{-1}$ to both sides of (2.1), we obtain the equivalent form of the equation

$$u_t + uu_x + \left((1-\partial_x^2)^{-1}\, \partial_x\left(u^2 + \frac{1}{2}u_x^2\right)\right) = 0.$$

We now investigate some local properties of the Riemannian exponential map $\exp_{\mathrm{Id}}$, defined as the exponential map associated with the metric spray. This will enable us to obtain information about the local geometry of $\mathcal{D}^3(\mathbb{R})$.

For the moment, let us assume (cf. Theorem 2.2 below for a proof) that for every $u \in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$ sufficiently small, there is a unique geodesic $\alpha(t,u)$ starting at the identity $\mathrm{Id} \in \mathcal{D}^3(\mathbb{R})$ in the tangent direction u[10]. The homogeneous quadratic property of the metric spray ensures that

[10] Note that by right-translation, a similar statement will hold at any point $q \in \mathcal{D}^3(\mathbb{R})$.

$\alpha(ts,u)=\alpha(t,su)$ for those $t,s\in\mathbb{R}$ for which both expressions are well-defined. This is used to prove that there exists a small neighborhood $\mathcal{O}$ of zero in $T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})=\mathcal{M}_1^3$ so that for any $u\in\mathcal{O}$, $\alpha(t,u)$ is defined on the interval $[0,1]$ (for details we refer to [24]). We define the Riemannian exponential map by

$$\exp_{\mathrm{Id}}:\mathcal{O}\subset T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})\to\mathcal{D}^3(\mathbb{R}),\qquad \exp_{\mathrm{Id}}(u)=\alpha(1,u),\quad u\in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R}).$$

Let us now prove

THEOREM 2.2. — *There exists an open neighborhood $\mathcal{U}$ of the identity in $\mathcal{D}^3(\mathbb{R})$ and an open neighborhood $\mathcal{V}$ of zero in $T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$ such that the Riemannian exponential map is a diffeomorphism from $\mathcal{V}$ onto $\mathcal{U}$. Furthermore, any two elements in $\mathcal{U}$ can be joined by a unique geodesic inside $\mathcal{U}$.*

In particular, Theorem 2.2 says that the Riemannian exponential map is well-defined in a neighborhood of zero in $T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})\equiv\mathcal{M}_1^3$.

To make the proof of Theorem 2.2 more transparent, we collect some auxiliary results in the following.

LEMMA 2.3. — *Let $p(x):=\frac{1}{2}e^{-|x|}$, $x\in\mathbb{R}$. For every $\alpha\geqslant 0$ we can find a constant $c(\alpha)>0$ such that*[11]

$$\|(1+x^2)^\alpha\,[p*f](x)\|_{L^2(\mathbb{R})}\leqslant c(\alpha)\,\|f(x)[1+x^2]^\alpha\|_{L^2(\mathbb{R})},\quad f\in L^2(\mathbb{R}),$$

and

$$\|(1+x^2)^\alpha\,[p_x*f](x)\|_{L^2(\mathbb{R})}\leqslant c(\alpha)\,\|f(x)[1+x^2]^\alpha\|_{L^2(\mathbb{R})},\quad f\in L^2(\mathbb{R}).$$

Proof. — We first note the elementary inequality

$$1+x^2\leqslant 2\,[1+(x-y)^2]\,(1+y^2),\quad x,y\in\mathbb{R}, \tag{2.6}$$

obtained from the identity $4y^2-4xy+x^2=(x-2y)^2$.

(11) Here $*$ stands for the convolution.

Let $f \in L^2(\mathbb{R})$ and $\alpha \geqslant 0$ be fixed. We have that

$$\begin{aligned}
&\|(1+x^2)^\alpha\,[p*f](x)\|^2_{L^2(\mathbb{R})}\\
&\qquad= \frac{1}{4}\|(1+x^2)^\alpha\Big(\int_{\mathbb{R}} e^{-|y|} f(x-y)\,dy\Big)\|^2_{L^2(\mathbb{R})}\\
&\qquad= \frac{1}{4}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-|y|} f(x-y)[1+(x-y)^2]^\alpha\,\frac{(1+x^2)^\alpha}{[1+(x-y)^2]^\alpha}\,dy\Big)^2 dx\\
&\qquad\leqslant 2^{2\alpha-2}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-|y|}(1+y^2)^\alpha\, f(x-y)[1+(x-y)^2]^\alpha\,dy\Big)^2 dx\\
&\qquad= 2^{2\alpha-2}\|\Big(e^{-|y|}(1+y^2)^\alpha\Big)*\Big(f(x)[1+x^2]^\alpha\Big)\|^2_{L^2(\mathbb{R})}\\
&\qquad\leqslant 2^{2\alpha-2}\|e^{-|y|}(1+y^2)^\alpha\|^2_{L^1(\mathbb{R})}\,\|f(x)[1+x^2]^\alpha\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where (2.6) was used to obtain the inequality on line 4 and where in the end we applied Young's inequality [29]. In a similar way we deal with $(p_x * f)$ with the result that we may choose $c(\alpha) = 2^{\alpha-1}\|e^{-|x|}(1+x^2)^\alpha\|_{L^1(\mathbb{R})}$. □

Proof of Theorem 2.2. — We can write the differential equation (2.5) satisfied by a geodesic $t \mapsto q(t)$ on $\mathcal{D}^3(\mathbb{R})$ starting at the identity Id in the direction $u_0 \in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$ as a first order system on $\mathcal{D}^3(\mathbb{R}) \times \mathcal{M}^3_1$:

$$\begin{cases} q_t = X, \\ X_t = S(q,X) \\ q(0) = \mathrm{Id} \\ X(0) = u_0, \end{cases} \tag{2.7}$$

with $S : \mathcal{D}^3(\mathbb{R}) \times \mathcal{M}^3_1 \to \mathcal{M}^3_1$ defined by

$$S(q,X) := -\left((1-\partial_x^2)^{-1}\,\partial_x\left[(X\circ q^{-1})^2 + \frac{1}{2}(\partial_x X\circ q^{-1})^2\right]\right)\circ q.$$

Let us first prove that the map

$$G : u \mapsto \partial_x\left(f^2 + 1/2\, f_x^2\right)$$

is smooth from $\mathcal{M}^3_1$ to $\mathcal{M}^1_3$.

For $f \in \mathcal{M}^3_1$, we have by Lemma 2.1 that the function $[x \mapsto (1+x^2)f'(x)]$ belongs to $L^\infty(\mathbb{R})$ and since the definition of $\mathcal{M}^3_1$ ensures that

$$[x \mapsto (1+x^2)^{\frac{1}{2}} f(x)],\ [x \mapsto (1+x^2)^{\frac{1}{2}} f''(x)] \in L^2(\mathbb{R}),$$

we deduce that the function $[x \mapsto (1+x^2)^{\frac{3}{2}}\Big(2f(x)f'(x)+f'(x)f''(x)\Big)]$ is in the space $L^2(\mathbb{R})$. Along the same lines one can check that

$$[x \mapsto (1+x^2)^2\left(\partial_x^2\,(f^2+\frac{1}{2}f_x^2)\right)(x)] \in L^2(\mathbb{R}),$$

thus we have showed that $[x \mapsto \partial_x(f^2+\frac{1}{2}f_x^2)] \in \mathcal{M}_3^1$.

We claim now that the operator $Q := (1-\partial_x^2)^{-1}$, defined on $L^2(\mathbb{R})$, is a bounded linear map from $\mathcal{M}_3^1$ into $\mathcal{M}_1^3$.

Indeed, Q is an isomorphism from $H^1(\mathbb{R})$ to $H^3(\mathbb{R})$ and we need only to worry about the weights. Note that

$$Qf = p * f, \quad f \in L^2(\mathbb{R}),$$

where, as before, $p(x) := \frac{1}{2}\,e^{-|x|}$, $x \in \mathbb{R}$.

Let $f \in \mathcal{M}_3^1$. Using Lemma 2.3 we deduce that

$$\|(1+x^2)^{\frac{1}{2}}\,[p*f](x)\|_{L^2(\mathbb{R})} \leqslant c\Big(\frac{1}{2}\Big)\,\|f(x)[1+x^2]^{\frac{1}{2}}\|_{L^2(\mathbb{R})},$$

while, in view of $\partial_x(p*f) = p*f_x$ and $\partial_x^2(p*f) = p_x * f_x$, we have

$$\|(1+x^2)\,[\partial_x(p*f)](x)\|_{L^2(\mathbb{R})} \leqslant c(1)\,\|f'(x)[1+x^2]\|_{L^2(\mathbb{R})},$$

respectively

$$\|(1+x^2)^{\frac{3}{2}}\,[\partial_x^2(p*f)](x)\|_{L^2(\mathbb{R})} \leqslant c\Big(\frac{3}{2}\Big)\,\|f'(x)[1+x^2]\|_{L^2(\mathbb{R})}.$$

Finally, note that by definition $(1-\partial_x^2)Qf = f$ for $f \in L^2(\mathbb{R})$ so that $\partial_x^2(p*f) = p*f-f$ and therefore

$$\partial_x^3(p*f) = p*f_x - f_x, \quad f \in H^1(\mathbb{R}).$$

Combining this with the result from Lemma 2.3, we obtain

$$\begin{aligned}\|(1+x^2)^2\,[\partial_x^3(p*f)](x)\|_{L^2(\mathbb{R})} &= \|(1+x^2)^2\,[p*f_x-f_x](x)\|_{L^2(\mathbb{R})}\\ &\leqslant \|(1+x^2)^2\,[p*f_x](x)\|_{L^2(\mathbb{R})} + \|(1+x^2)^2\,f'(x)\|_{L^2(\mathbb{R})}\\ &\leqslant (c(2)+1)\,\|f'(x)(1+x^2)^2\|_{L^2(\mathbb{R})}\end{aligned}$$

and we proved that Q is a bounded linear map from $\mathcal{M}_3^1$ into $\mathcal{M}_1^3$.

At this point, observe that we can write

$$S(q,X) = -\,r_q \circ Q \circ G \circ r_{q^{-1}}(X)$$

where r_q stands for right multiplication by q.

The smoothness of conjugation (see [16], [6]) combined with the above information on G and Q proves that S is of class C^∞ on $\mathcal{D}^3(\mathbb{R}) \times \mathcal{M}_1^3$. Its derivative being continuous it is necessarily locally bounded so that S is locally Lipschitz by the mean value theorem (see [13]). The basic existence and uniqueness theorem of ordinary differential equations on Hilbert manifolds [24] applied to (2.7) shows that, given an arbitrary $u_0 \in \mathcal{M}_1^3$, the initial value problem (2.7) has a unique solution $(q, q_t) \in C^1\big([0,T); \mathcal{D}^3(\mathbb{R}) \times \mathcal{M}_1^3\big)$ for some $T > 0$. Moreover, the solution depends smoothly on the initial data.

The smooth dependence of q on the initial data shows that $\exp_{\mathrm{Id}}$ is a smooth map, cf. [24]. It now follows from the inverse function theorem that $\exp_{\mathrm{Id}}$ is a local diffeomorphism near zero. Further, on the Hilbert space $\mathcal{M}_1^3$ we can join two points u_1 and u_2 near zero by a straight line. The image of this straight line by the Riemannian exponential map is a geodesic connecting $\exp_{\mathrm{Id}}(u_1)$ and $\exp_{\mathrm{Id}}(u_2)$. □

Let us observe that a curve $t \mapsto q(t)$ in $\mathcal{D}^3(\mathbb{R})$ is a geodesic with respect to the metric spray starting at the identity in $\mathcal{D}^3(\mathbb{R})$ if and only if it coincides with $t \mapsto \exp_{\mathrm{Id}}(tv)$ for some $v \in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$. One can easily see this by using the result on the local uniqueness of the geodesic flow (see the proof of Theorem 2.2) combined with the fact that $\alpha(ts, u) = \alpha(t, su)$ for all $t, s \in \mathbb{R}$ such that both expressions occuring in the equality are well-defined, where $\alpha(t,u)$ stands for the geodesic on $\mathcal{D}^3(\mathbb{R})$ starting at the identity in the direction $u \in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$.

Remark 2.4. — The result proved in Theorem 2.2 has the following interpretation for the shallow water equation: A surface configuration (state) of shallow water is connected to a nearby surface configuration by a uniquely determined solution of equation (2.1). □

We conclude this section with a discussion of the 'Lie group' exponential map of $\mathcal{D}^3(\mathbb{R})$.

For a Lie group G one can define the Lie group exponential map exp from T_eG onto a neighborhood of the neutral element e of G. If $X \in T_eG$, let $\tilde{X}$ be the right-invariant vector field on G whose value at e is X. Then $\exp(X)$ is given by $\exp(X) = \eta(1)$ where $\{\eta(t) : t \in \mathbb{R}\}$ is the one-parameter subgroup of G defined by $\eta(0) = e$ and $\frac{d\eta}{dt} = \tilde{X}(\eta)$.

If G has also a Riemannian structure induced by a bi-invariant metric, it is known (see [26]) that the Lie group exponential map coincides with the Riemannian exponential map. Note that a compact group always has a bi-invariant metric, cf. [26].

For the diffeomorphism group $\mathcal{D}^3(\mathbb{R})$ we can define a 'Lie group' exponential map by

$$\exp : T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R}) \to \mathcal{D}^3(\mathbb{R}), \quad \exp(u_0) = \eta_{u_0}(1), \quad u_0 \in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R}) = \mathcal{M}_1^3,$$

where $\{\eta_{u_0}(t) : t \in \mathbb{R}\}$ is the flow of the right-invariant vector field on $\mathcal{D}^3(\mathbb{R})$ whose value at the identity is u_0. In other words, η_{u_0} solves

$$\begin{cases} \eta_t = u_0(\eta), & t \in \mathbb{R}, \\ \eta(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{2.8}$$

Note that the metric we consider on $\mathcal{D}^3(\mathbb{R})$ is right-invariant by construction but is not left-invariant, as one can easily check. Therefore the next result is not that surprising:

PROPOSITION 2.5. — *On $\mathcal{D}^3(\mathbb{R})$ the Riemannian exponential map differs from the 'Lie group' exponential map.*

Proof. — Assume that the two exponential maps are equal on some small neighborhood $\mathcal{O}$ of zero in $T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$. We may assume that $\mathcal{O}$ is an open ball centered at zero and identify $T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$ with the Hilbert space $\mathcal{M}_1^3$.

We know that the geodesic starting at the identity $\mathrm{Id} \in \mathcal{D}^3(\mathbb{R})$ in the direction $u_0 \in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$ is simply $t \mapsto \exp_{\mathrm{Id}}(tu_0)$, where, as before, $\exp_{\mathrm{Id}}$ stands for the Riemannian exponential map.

On the other hand, for the one-parameter subgroup of $\mathcal{D}^3(\mathbb{R})$ defined by η_v for $v \in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$, one can easily check that

$$\eta_v(st) = \eta_{tv}(s), \qquad s,t \in \mathbb{R},\ v \in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R})$$

so that

$$\eta_v(t) = \exp(tv) \quad \text{for} \quad t \in \mathbb{R},\ v \in T_{\mathrm{Id}}\mathcal{D}^3(\mathbb{R}),$$

where exp stands for the 'Lie group' exponential map.

We conclude that the equality of the two exponential maps forces the two flows to be equal. We now show that they are not equal, obtaining the desired contradiction.

Indeed, if $q(t) := \exp_{\mathrm{Id}}(tu_0)$ for $u_0 \in \mathcal{O}$, we have, by the results of the previous subsection, that

$$q_t = u(t,q), \quad t \in [0,1], \tag{2.9}$$

where $u(t,x)$ solves (2.1) with the initial condition u_0. On the other hand, using (2.8) and the equality of the flows, we would also have that

$$q_t = u_0(q), \quad t \in [0,1]. \tag{2.10}$$

As $q(t,\cdot)$, for any $t \in [0,1]$, is a diffeomorphism of $\mathbb{R}$, relations (2.9) and (2.10) show that $u(t,x) = u_0(x)$ for $t \in [0,1]$, $x \in \mathbb{R}$. This would mean that for $u_0 \in \mathcal{M}_1^3 \subset H^3(\mathbb{R})$ small enough, the only solutions of (2.1) with initial data u_0 are stationary solutions.

We complete the proof of the proposition by showing that in the space $H^3(\mathbb{R})$ the identical zero function is the only stationary solution of (2.1).

Let $u_0 \in H^3(\mathbb{R})$ be a stationary solution to (2.1). Multiplying the relation

$$3u_0u_0' = 2u_0'u_0'' + u_0u_0'''$$

by u_0 and integrating on $(-\infty, x]$, an integration by parts in the last integral term leads us to

$$u_0^2(u_0 - u_0'') = 0.$$

If u_0 is not identical zero, there is some open interval $(a,b) \subset \mathbb{R}$ where u_0 has no zeros and $u_0 - u_0'' = 0$. By (possibly) extending this interval we may assume $u_0(a) = u_0(b) = 0$. Here we do not exclude that one or both endpoints are infinite, in which case the equality has to be understood as a limit. We infer that $u_0(x) = c_1e^x + c_2e^{-x}$ on (a,b) for some constants $c_1, c_2 \in \mathbb{R}$. One verifies that in all possible cases (endpoints finite or not) we cannot have $u_0(a) = u_0(b) = 0$ with $u_0 \in H^3(\mathbb{R})$ without $c_1 = c_2 = 0$. Thus $u_0 \equiv 0$ on $\mathbb{R}$ is the only stationary solution of (2.1). □

3. A family of diffeomorphisms of the line.

The geometric considerations of the previous section serve, in this paper, as a main tool to investigate the question of existence of permanent and breaking waves for the model (2.1).

We associate to (2.1) a new equation

$$\begin{cases} q_t = u(t,q), & t > 0,\ x \in \mathbb{R}, \\ q(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $u(t,x)$ solves (2.1). It is useful to consider solutions for the shallow water equation (2.1) in the Sobolev space $H^3(\mathbb{R})$ instead of the weighted spaces $\mathcal{M}_1^3 \subset H^3(\mathbb{R})$ defined in Section 2.

Assume $u_0 \in H^3(\mathbb{R})$. Associating to a solution of (2.1) the potential $y := u - u_{xx}$, one can write equation (2.1) in the following equivalent form:

$$\begin{cases} y_t = -y_x u - 2yu_x, & t > 0,\ x \in \mathbb{R}, \\ y(0,x) = y_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.2}$$

Equation (3.2) can be analyzed with Kato's semigroup approach to the Cauchy problem for quasi-linear hyperbolic evolution equations [20]. We have the following well-posedness result:

THEOREM A [9].— *Given an initial data $u_0 \in H^3(\mathbb{R})$, there exists a maximal time $T = T(u_0) > 0$ so that, on $[0,T)$, equation (2.1) admits a unique solution*

$$u = u(\cdot, u_0) \in C([0,T); H^3(\mathbb{R})) \ \cap \ C^1([0,T); H^2(\mathbb{R})).$$

Further, if $T < \infty$, then $\limsup_{t\uparrow T} |u(t)|_{H^3(\mathbb{R})} = \infty$.

If $u_0 \in H^4(\mathbb{R})$ then the solution possesses additional regularity,

$$u \in C([0,T); H^4(\mathbb{R})) \ \cap \ C^1([0,T); H^3(\mathbb{R})).$$

The solution depends continuously on the initial data, i.e. the mapping

$$u_0 \mapsto u(\cdot, u_0) : H^3(\mathbb{R}) \to C([0,T); H^3(\mathbb{R})) \ \cap \ C^1([0,T); H^2(\mathbb{R}))$$

is continuous. Moreover, the $H^1(\mathbb{R})$-norm of the solution $u(t,x)$ is conserved on $[0,T)$.

We prove now that for $u_0 \in H^3(\mathbb{R})$, equation (3.1) defines, for some time, a curve of orientation-preserving diffeomorphisms $q(t,\cdot)$ of the line. By enlarging the class of diffeomorphisms we loose the manifold and Riemannian structure but we can derive useful qualitative information about the solutions to the shallow water equation (see Section 4 and Section 5) for a wider class of initial profiles.

THEOREM 3.1.— *For $u_0 \in H^3(\mathbb{R})$, let $[0,T)$ be the maximal interval of existence of the corresponding solution to* (2.1), *as given by Theorem A. Then* (3.1) *has a unique solution $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. Moreover, for every fixed $t \in [0,T)$, $q(t,\cdot)$ is an increasing diffeomorphism of the line.*

Proof.— For a fixed $x \in \mathbb{R}$ let us consider the ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}\, v^x = u(t, v^x), & t \in (0,T), \\ v^x(0) = x, \end{cases} \tag{3.3}$$

where $u(t,x)$ is the solution to (2.1) with prescribed initial data u_0. Since $u \in C^1([0,T); H^2(\mathbb{R}))$ and $H^1(\mathbb{R})$ is continuously imbedded in $L^\infty(\mathbb{R})$, we see that both functions $u(t,x)$ and $u_x(t,x)$ are bounded, Lipschitz in the second variable, and of class C^1 in time. The basic theory of ordinary differential equations concerning existence on some maximal time interval and dependence on the initial data guarantees that (3.3) has a unique solution $v^x(t)$ defined on the whole interval $[0,T)$. Moreover, the map $q : [0,T) \times \mathbb{R} \to \mathbb{R}$ defined by $q(t,x) := v^x(t)$ belongs to the space $C^1([0,T) \times \mathbb{R}, \mathbb{R})$.

Integrate relation (3.3) with respect to time on $[0,t)$ with $t \in (0,T)$, then differentiate with respect to space and finally with respect to time to obtain

$$\frac{d}{dt}\, q_x = u_x(t,q)\, q_x, \quad t \in (0,T), \ x \in \mathbb{R}.$$

As $q(0,x) = x$ on $\mathbb{R}$, we have $q_x(0,x) = 1$ on $\mathbb{R}$ and thus, by continuity, $q_x(t,x) > 0$ for $t > 0$ small enough.

Defining for every fixed $x \in \mathbb{R}$, $t(x) := \sup\{t \in [0,T) : \ q_x(t,x) > 0\}$, observe that

$$\frac{\frac{d}{dt}\, q_x(t,x)}{q_x(t,x)} = u_x(t, q(t,x)), \quad t \in [0, t(x)).$$

Integrating, we obtain

$$q_x(t,x) = e^{\int_0^t u_x(s,q)\, ds}, \quad t \in [0, t(x)).$$

If for some $x \in \mathbb{R}$, we had $t(x) < T$ we could deduce by continuity and the way we defined $t(x)$ that $q_x(t(x),x) = 0$. However, the previous relation ensures that this can not hold. Therefore we have that $t(x) = T$ for all $x \in \mathbb{R}$, that is,

$$q_x(t,x) = e^{\int_0^t u_x(s,q)\,ds}, \quad t \in [0,T),\ x \in \mathbb{R}. \tag{3.4}$$

Recalling that $u(t,\cdot) \in H^3(\mathbb{R})$ for all $t \in [0,T)$ and using Sobolev imbedding results (see [17]) to ensure the uniform boundedness of $u_x(s,z)$, for $(s,z) \in [0,t] \times \mathbb{R}$ with $t \in [0,T)$, we obtain for every $t \in [0,T)$ a constant $K(t) > 0$ such that

$$e^{-K(t)} \leqslant q_x(t,x) \leqslant e^{K(t)}, \quad x \in \mathbb{R}. \tag{3.5}$$

We conclude from (3.5) that the function $q(t,\cdot)$ is stricly increasing on $\mathbb{R}$ with $\lim_{x\to \pm\infty} q(t,x) = \pm\infty$ as long as $t \in [0,T)$. □

The following result plays a key role in our further considerations. It roughly says that the form of $y(t,\cdot)$ does not change on the time interval where it is well-defined. We therefore found by means of the geometric interpretation a very important invariant for the solutions to the shallow water equation.

LEMMA 3.2.— *Assume $u_0 \in H^3(\mathbb{R})$ and let $T > 0$ be given as in Theorem A. We then have, with $y := u - u_{xx}$,*

$$y_0(x) = y(t,q(t,x))\, q_x^2(t,x), \quad t \in [0,T),\ x \in \mathbb{R}.$$

Proof.— For $t = 0$, the claimed identity holds. Thus, it suffices to show that the right-hand side is independent of t. For $t \in (0,T)$, differentiate the right-hand side with respect to time and use equations (3.1) and (3.2) to conclude that $\frac{d}{dt}\Big(y(t,q(t,x))\,q_x^2(t,x)\Big) = 0$. □

Remark 3.3.— The evolution problem (3.2) admits the conservation laws

$$\int_{\mathbb{R}} \sqrt{y_+(t,x)}\,dx = \int_{\mathbb{R}} \sqrt{(y_0)_+(x)}\,dx, \quad t \in [0,T),$$

and

$$\int_{\mathbb{R}} \sqrt{y_-(t,x)}\,dx = \int_{\mathbb{R}} \sqrt{(y_0)_-(x)}\,dx, \quad t \in [0,T),$$

where f_+, f_- stand for the positive, respectively the negative part of the function f and $[0,T)$ is the maximal interval of existence for the solution to (2.1) with initial data u_0, as given by Theorem A.

The proof in [9] that these quantities are conserved is quite technical.

Lemma 3.2 provides an alternative proof of the validity of these conservation laws for the flow defined by (1.5).

Indeed, note that $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ as long as $t \in [0,T)$. Assume that $y_0(x) \geqslant 0$ on $[a,b]$. Then by Lemma 3.2, for $0 < t < T$ fixed, $y(t,x) \geqslant 0$ on $[q(t,a), q(t,b)]$ and thus

$$\int_{q(t,a)}^{q(t,b)} \sqrt{y_+(t,x)}\,dx = \int_a^b \sqrt{y_+(t,q(t,\xi))}\; q_x(t,\xi)\,d\xi = \int_a^b \sqrt{y_0(\xi)}\,d\xi. \quad \square$$

4. Permanent waves.

In this section we consider the problem of the existence of permanent waves for the model (2.1). Using the continuous family of diffeomorphisms of the line associated to an arbitrary initial profile $u_0 \in H^3(\mathbb{R})$ of the shallow water equation described above, we show that for a large class of initial profiles the corresponding solutions to (2.1) exist globally in time.

THEOREM 4.1.— *Assume $u_0 \in H^3(\mathbb{R})$ is such that $y_0 := u_0 - u_{0,xx}$ does not change sign on $\mathbb{R}$. Then the corresponding solution $u(t,x)$ to the initial-value problem* (2.1)*, given by Theorem A, exists globally in time. If, in addition, $u_0 \in H^4(\mathbb{R})$, we obtain a global classical solution to* (2.1).

Proof.— Let $T > 0$ be the maximal existence time of the solution of (2.1) for the initial profile u_0, as given by Theorem A. For each $t \in [0,T)$, let $q(t,\cdot)$ be the increasing diffeomorphism of the line given by Theorem 3.1.

As $y(t,x) := u(t,x) - u_{xx}(t,x)$, $u(t,x)$ is given by the convolution $u(t,x) = p * y$ with $p(x) := \frac{1}{2}\,e^{-|x|}$, $\in \mathbb{R}$, and therefore

$$\begin{aligned} u(t,x) &= \frac{1}{2}\,e^{-x}\int_{-\infty}^{x} e^{\xi} y(t,\xi)\,d\xi \\ &+\frac{1}{2}\,e^{x}\int_{x}^{\infty} e^{-\xi} y(t,\xi)\,d\xi, \quad t \in [0,T),\ x \in \mathbb{R}, \end{aligned} \tag{4.1}$$

from where we infer that

$$
\begin{aligned}
u_x(t,x) &= -\frac{1}{2}\,e^{-x}\int_{-\infty}^{x} e^{\xi}y(t,\xi)\,d\xi \\
&\quad +\frac{1}{2}\,e^{x}\int_{x}^{\infty} e^{-\xi}y(t,\xi)\,d\xi, \quad t\in[0,T),\ x\in\mathbb{R}.
\end{aligned} \tag{4.2}
$$

(a) Consider first the case where $y_0(x) \geqslant 0$ on $\mathbb{R}$.

Relation (3.4) shows that $q_x(t,x) \neq 0$ for $(t,x) \in [0,T)\times\mathbb{R}$ so that by Lemma 3.2 we get $y(t,x) \geqslant 0$ on $[0,T)\times\mathbb{R}$ since $q(t,\cdot)$ is a diffeomorphism of $\mathbb{R}$. Using (4.1) and (4.2), we deduce

$$u_x(t,x) \geqslant -u(t,x), \quad (t,x)\in[0,T)\times\mathbb{R}. \tag{4.3}$$

We now prove that (4.3) yields a uniform bound from below for $u_x(t,x)$ on $[0,T)\times\mathbb{R}$.

Indeed, for $x\in\mathbb{R}$ and $t\in[0,T)$,

$$u^2(t,x) \leqslant \int_{\mathbb{R}} \left[u^2(t,\xi)+u_x^2(t,\xi)\right] d\xi = |u(t,\cdot)|^2_{H^1(\mathbb{R})} = |u_0|^2_{H^1(\mathbb{R})},$$

using the conservation law given by Theorem A. Thus,

$$|u(t,x)| \leqslant ||u_0||_{H^1(\mathbb{R})}, \qquad x\in\mathbb{R}, \quad t\in[0,T). \tag{4.4}$$

Combining (4.4) with (4.3) we obtain

$$u_x(t,x) \geqslant -||u_0||_{H^1(\mathbb{R})}, \quad (t,x)\in[0,T)\times\mathbb{R}. \tag{4.5}$$

Going carefully through the steps of the proof of Theorem 3.5 in [9], we deduce from (4.5) that $T=\infty$.

(b) Now consider the case $y_0(x) \leqslant 0$ on $\mathbb{R}$.

Since, by Theorem 3.1, $q(t,\cdot)$ is a diffeomorphism of $\mathbb{R}$ for all $t\in[0,T)$, and $q_x(t,x)\neq 0$ for $(t,x)\in[0,T)\times\mathbb{R}$ in view of relation (3.4), we obtain from Lemma 3.2 that $y(t,x)\leqslant 0$ as $(t,x)\in[0,T)\times\mathbb{R}$. A combination of this fact with (4.1)-(4.2) yields

$$u_x(t,x) \geqslant u(t,x), \quad (t,x)\in[0,T)\times\mathbb{R},$$

which guarantees that (4.5) holds - recalling the conservation law from Theorem A and using a Sobolev inequality. We infer again by the methods used in Theorem 3.5 [9] that $T=\infty$. □

Remark 4.2.— Theorem 4.1 improves the global existence result obtained in [9] where the additional assumption $y_0 \in L^1(\mathbb{R})$ is needed. We are able to eliminate the condition $y_0 \in L^1(\mathbb{R})$ since by Lemma 3.2 we know a new interesting feature of the shallow water equation: the form of $y(t,\cdot)$ does not change as $t \in [0,T)$. □

Example 4.3.— In view of Theorem 4.1, the initial profile $u_0 = p * p$ develops into a permanent wave. □

The next result shows that there are initial potentials which change sign on $\mathbb{R}$ such that the corresponding solution of (2.1) still exists globally in time.

THEOREM 4.4.— *Assume $u_0 \in H^3(\mathbb{R})$ is such that the associated potential $y_0 = u_0 - u_{0,xx}$ satisfies $y_0(x) \leqslant 0$ on $(-\infty, x_0]$ and $y_0(x) \geqslant 0$ on $[x_0, \infty)$ for some point $x_0 \in \mathbb{R}$. Then the corresponding solution $u(t,x)$ to the initial-value problem* (2.1)*, as given by Theorem A, exists globally in time.*

Proof.— Let $T > 0$ be the maximal existence time of the corresponding solution of (2.1), as given by Theorem A. We associate to (2.1) the equation (3.1). For $t \in [0,T)$, let $q(t,\cdot)$ be the increasing diffeomorphism of the line whose existence is guaranteed by Theorem 3.1.

Since $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ as long as $t \in [0,T)$, we deduce from Lemma 3.2 (relation (3.5) guarantees $q_x(t,x) > 0$ on $[0,T) \times \mathbb{R}$) that for $t \in [0,T)$, we have

$$\begin{cases} y(t,x) \leqslant 0 & \text{if} \quad x \leqslant q(t,x_0), \\ y(t,x) \geqslant 0 & \text{if} \quad x \geqslant q(t,x_0). \end{cases} \tag{4.6}$$

We infer from (4.6) and the formulas (4.1) and (4.2) that

$$\begin{aligned} u_x(t,x) &= -u(t,x) + e^x \int_x^\infty e^{-\xi} y(t,\xi)\, d\xi \\ &\geqslant -u(t,x) \quad \text{for} \quad x \geqslant q(t,x_0), \end{aligned} \tag{4.7}$$

while

$$\begin{aligned} u_x(t,x) &= u(t,x) - e^{-x} \int_{-\infty}^x e^{\xi} y(t,\xi)\, d\xi \\ &\geqslant u(t,x) \quad \text{for} \quad x \leqslant q(t,x_0). \end{aligned} \tag{4.8}$$

The relations (4.6)-(4.8) show that

$$u_x(t,x) \geqslant -||u(t,\cdot)||_{L^\infty(\mathbb{R})}, \quad (t,x) \in [0,T) \times \mathbb{R},$$

and this guarantees $T = \infty$ as relation (4.5) is again fulfilled (in view of a Sobolev inequality and the conservation law from Theorem A). □

Let us recall the following blow-up result for (2.1):

THEOREM B [9]. — *Assume that $u_0 \in H^3(\mathbb{R})$ is odd and $u_0'(0) < 0$. Then the corresponding solution of (2.1) does not exist globally. The maximal time of existence is estimated from above by $1/(2|u_0'(0)|)$.*

Remark 4.5. — Assume that $y_0 \in H^1(\mathbb{R})$, $y_0 \not\equiv 0$, is odd, $y_0(x) \leqslant 0$ on $\mathbb{R}_-$ and $y_0(x) \geqslant 0$ on $\mathbb{R}_+$. By Theorem 4.4, the solution $u(t,x)$ to the initial-value problem (2.1) correponding to $u_0 := Qy_0$ exists globally in time. This case is of interest if we compare it with the blow-up result from Theorem B: since $u_0 = p * y_0$ with $p(x) := \frac{1}{2}\, e^{-|x|}$, $x \in \mathbb{R}$, one verifies that u_0 is odd as well. However, the representation formula (4.2) shows that $u_0'(0) > 0$. □

Example 4.6. — By Theorem 4.4, the initial profile $u_0(x) = p * [x\, e^{-|x|}]$ on $\mathbb{R}$ develops into a permanent wave. □

5. Wave breaking.

In the present section we use the existence of a continuous family of diffeomorphisms of the line associated above to each initial data $u_0 \in H^3(\mathbb{R})$ to analyze in detail the possible blow-up phenomena for solutions for the shallow water equation.

Let us recall

THEOREM C [10]. — *Let $T > 0$ and $v \in C^1([0,T); H^2(\mathbb{R}))$. Then for every $t \in [0,T)$ there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x\in\mathbb{R}} \left[v_x(t,x)\right] = v_x(t,\xi(t)),$$

and the function m is almost everywhere differentiable on $(0,T)$ with

$$\frac{dm}{dt}(t) = v_{tx}(t,\xi(t)) \qquad \text{a.e. on } (0,T).$$

We will use Theorem C and the connection with the diffeomorphism group of the line in order to investigate the breaking of waves for the model (2.1).

Let us first show that a classical solution to (1.5) can only have singularities which correspond to wave breaking. Note that the conservation law given by Theorem A, implies that every solution is uniformly bounded as long as it is defined.

THEOREM 5.1. — *Let $u_0 \in H^3(\mathbb{R})$. The maximal existence time $T > 0$ of the solution $u(t,x)$ to (2.1) with initial profile u_0 is finite if and only if the slope of the solution becomes unbounded from below in finite time.*

Proof. — Let $T < \infty$ and assume that for some constant $K > 0$ the solution satisfies

$$u_x(t,x) \geqslant -K, \quad (t,x) \in [0,T) \times \mathbb{R}.$$

By Sobolev's imbedding theorem and the conservation of the $H^1(\mathbb{R})$ norm stated in Theorem A we deduce that u satisfies

$$\sup_{t \in [0,T) \times \mathbb{R}} |u(t,x)| < \infty. \tag{5.1}$$

Using (3.2) and integration by parts we find for $t \in (0,T)$ that

$$\frac{d}{dt} \int_{\mathbb{R}} [y(t,x)]^2\, dx = -3 \int_{\mathbb{R}} u_x(t,x)\, [y(t,x)]^2\, dx \leqslant 3K \int_{\mathbb{R}} [y(t,x)]^2\, dx.$$

Gronwall's inequality yields

$$\int_{\mathbb{R}} [y(t,x)]^2\, dx \leqslant e^{3Kt} \int_{\mathbb{R}} [y_0(x)]^2\, dx, \quad t \in (0,T). \tag{5.2}$$

Let us now approximate $u_0 \in H^3(\mathbb{R})$ in the space $H^3(\mathbb{R})$ with a sequence $u_0^n \in H^4(\mathbb{R})$, $n \geqslant 1$. We denote by u^n the solution of (2.1) with initial data u_0^n, defined on the maximal interval of existence $[0,T_n)$ given by Theorem A, and let $y^n := u^n - u^n_{xx}$ for $n \geqslant 1$.

The additional regularity of u^n (ensured by Theorem A) enables us to differentiate (3.2). This leads to

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} [y^n_x(t,x)]^2\, dx &= -5 \int_{\mathbb{R}} u^n_x(t,x)\, [y^n_x(t,x)]^2\, dx \\ &\quad -4 \int_{\mathbb{R}} u^n(t,x)\, y^n(t,x)\, y^n_x(t,x)\, dx, \quad t \in (0,T_n). \end{aligned} \tag{5.3}$$

As before, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}}[y^n(t,x)]^2\,dx = -3\int_{\mathbb{R}} u_x^n(t,x)\,[y^n(t,x)]^2\,dx, \quad t\in(0,T_n). \tag{5.4}$$

We first claim that there is a sequence $n_k \to \infty$ with

$$\inf_{t\in[0,T_{n_k})}\left[\inf_{x\in\mathbb{R}} u_x^{n_k}(t,x)\right] = -\infty. \tag{5.5}$$

Otherwise, we find that $\inf_{t\in[0,T_n)}\left[\inf_{x\in\mathbb{R}} u_x^n(t,x)\right] > -\infty$ for $n \geqslant 1$ large enough and by (5.4)-(5.5), taking into account (5.1), we would obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}&\Big([y^n(t,x)]^2+[y_x^n(t,x)]^2\Big)\,dx\\ &\leqslant K_n\int_{\mathbb{R}}\Big([y^n(t,x)]^2+[y_x^n(t,x)]^2\Big)\,dx, \quad t\in(0,T_n),\end{aligned}$$

for some $K_n > 0$. But then, Gronwall's inequality gives

$$\begin{aligned}\int_{\mathbb{R}}&\Big([y^n(t,x)]^2+[y_x^n(t,x)]^2\Big)\,dx\\ &\leqslant e^{K_nt}\int_{\mathbb{R}}\Big([y_0^n(t,x)]^2+[y_{0,x}^n(t,x)]^2\Big)\,dx, \quad t\in(0,T_n).\end{aligned}$$

By Theorem A, we would obtain $T_n = \infty$ for all $n \geqslant 1$ large enough which is in contradiction to the continuous dependence on initial data (we assumed $T < \infty$).

Therefore (5.5) holds and we obtain $\sup_{t\in[0,T_{n_k})} |u_x^{n_k}(t,\cdot)|_{L^\infty(\mathbb{R})} = \infty$, which implies on its turn that $\sup_{t\in[0,T_{n_k})} |u^{n_k}(t,\cdot)|_{H^2(\mathbb{R})} = \infty$. Taking into account that $y^{n_k} := u^{n_k} - u_{xx}^{n_k}$, we find

$$\sup_{t\in[0,T_{n_k})} |y^{n_k}(t,\cdot)|_{L^2(\mathbb{R})} = \infty.$$

The latter relation and (5.2) cannot hold simultaneously in view of the continuous dependence on initial data. The obtained contradiction shows that our assumption on the boundedness from below of the x-derivative of the solution is false. The converse of the claimed statement is immediate and therefore the proof is complete. □

We will give now sufficient conditions to ensure wave breaking.

THEOREM 5.2. — *Assume $u_0 \in H^3(\mathbb{R})$ is such that the associated potential $y_0 = u_0 - u_{0,xx}$ satisfies $y_0(x) \geqslant 0$ on $(-\infty, x_0]$ and $y_0(x) \leqslant 0$ on $[x_0, \infty)$ for some point $x_0 \in \mathbb{R}$ and y_0 changes sign. Then the corresponding solution $u(t,x)$ to the initial-value problem (2.1) has a finite existence time.*

Proof. — Let $u \in C([0,T); H^3(\mathbb{R})) \cap C^1([0,T); H^2(\mathbb{R}))$ be the solution of the initial value problem (2.1), as given by Theorem A. We associate to (2.1) the equation (3.1). For $t \geqslant 0$, let $q(t,\cdot)$ be the increasing diffeomorphism of the line whose existence is guaranteed by Theorem 3.1.

The idea of the proof is to obtain a differential inequality for the time evolution of $u_x(t, q(t,x_0))$ which can be used to prove that $T < \infty$.

With $p(x) := \frac{1}{2}\exp(-|x|)$, $x \in \mathbb{R}$, the resolvent $(1-\partial_x^2)^{-1}$ can be represented as the convolution operator

$$Q^{-1}f := (1-\partial_x^2)^{-1} f = p * f, \qquad f \in L^2(\mathbb{R}),$$

where Q denotes the operator $(1-\partial_x^2)$ acting in $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. We can write (2.1) as (see Section 2)

$$u_t + uu_x = -\partial_x\left(p * \left(u^2 + \frac{1}{2}u_x^2\right)\right).$$

Differentiating this relation with respect to x, we find

$$\begin{aligned} u_{tx} + uu_{xx} + u_x^2 &= -\partial_x^2\left(p * \left(u^2 + \frac{1}{2}u_x^2\right)\right) \\ &= (Q - Id)\left(p * \left(u^2 + \frac{1}{2}u_x^2\right)\right) \\ &= u^2 + \frac{1}{2}u_x^2 - p * \left(u^2 + \frac{1}{2}u_x^2\right), \end{aligned}$$

that is,

$$u_{tx} + uu_{xx} + \frac{1}{2}u_x^2 = u^2 - p * \left(u^2 + \frac{1}{2}u_x^2\right). \tag{5.6}$$

Combining (5.6) with (3.1), we obtain

$$\begin{aligned} \frac{d}{dt}\, u_x(t, q(t,x_0)) &= u_{tx}(t, q(t,x_0)) + u_{xx}(t, q(t,x_0))\left(\frac{d}{dt}q(t,x_0)\right) \\ &= u_{tx}(t, q(t,x_0)) + u_{xx}(t, q(t,x_0))u(t, q(t,x_0)) \\ &= -\frac{1}{2}u_x^2(t, q(t,x_0)) + u^2(t, q(t,x_0)) \\ &\quad - \left(p * \left[u^2 + \frac{1}{2}u_x^2\right]\right)(t, q(t,x_0)). \end{aligned} \tag{5.7}$$

Observe that the inequality

$$e^{-x}\int_{-\infty}^{x} e^{\eta}[u^2(t,\eta)+u_x^2(t,\eta)]\,d\eta \geqslant 2e^{-x}\int_{-\infty}^{x} e^{\eta}u(t,\eta)u_x(t,\eta)\,d\eta$$
$$= u^2(t,x) - e^{-x}\int_{-\infty}^{x} e^{\eta}u^2(t,\eta)\,d\eta$$

yields

$$e^{-x}\int_{-\infty}^{x} e^{\eta}[2u^2(t,\eta)+u_x^2(t,\eta)]\,d\eta \geqslant u^2(t,x), \tag{5.8}$$

whereas

$$e^{x}\int_{x}^{\infty} e^{-\eta}[u^2(t,\eta)+u_x^2(t,\eta)]\,d\eta \geqslant -2e^{x}\int_{x}^{\infty} e^{-\eta}u(t,\eta)u_x(t,\eta)\,d\eta$$
$$= u^2(t,x) - e^{x}\int_{x}^{\infty} e^{-\eta}u^2(t,\eta)\,d\eta$$

leads to

$$e^{x}\int_{x}^{\infty} e^{-\eta}[2u^2(t,\eta)+u_x^2(t,\eta)]\,d\eta \geqslant u^2(t,x). \tag{5.9}$$

Using (5.8)-(5.9) and taking into account that $p(x)=\frac{1}{2}\,e^{-|x|}$, $x\in\mathbb{R}$, we obtain

$$\Big(p*\big[u^2+\frac{1}{2}u_x^2\big]\Big)\,(t,x)\geqslant \frac{1}{2}u^2(t,x),\quad (t,x)\in[0,T)\times\mathbb{R}.$$

Combining this inequality with (5.7) we deduce that on $(0,T)$,

$$\frac{d}{dt}\,u_x(t,q(t,x_0))\leqslant \frac{1}{2}u^2(t,q(t,x_0))-\frac{1}{2}u_x^2(t,q(t,x_0)). \tag{5.10}$$

For $t\in[0,T)$ note that the function $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with $q_x(t,x)\neq 0$ on $[0,T)\times\mathbb{R}$, in view of relation (3.5). We deduce from Lemma 3.2 that as long as $t\in[0,T)$ we have

$$\begin{cases} y(t,x) & \geqslant 0 \quad \text{if} \quad x\leqslant q(t,x_0),\\ y(t,x) & \leqslant 0 \quad \text{if} \quad x\geqslant q(t,x_0). \end{cases} \tag{5.11}$$

Here $y:=u-u_{xx}$ is the potential associated to u.

Introduce

$$V(t):=e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)} e^{\xi}y(t,\xi)\,d\xi,\quad t\in[0,T),$$

and

$$W(t) := e^{q(t,x_0)} \int_{q(t,x_0)}^{\infty} e^{-\xi} y(t,\xi)\, d\xi, \quad t \in [0,T).$$

Since $y(t,q(t,x_0)) = 0$ for $t \in [0,T)$, we have

$$\begin{aligned}\frac{d}{dt} V(t) &= -\Big(\frac{d}{dt} q(t,x_0)\Big) V(t) \\ &\quad + e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\xi} y_t(t,\xi)\, d\xi, \quad t \in (0,T).\end{aligned} \tag{5.12}$$

From (3.2), using $y = u - u_{xx}$, we obtain, integrating by parts,

$$\begin{aligned}\int_{-\infty}^{q(t,x_0)} e^{\xi} y_t(t,\xi)\, d\xi &= -\int_{-\infty}^{q(t,x_0)} e^{\xi} \Big(y(t,\xi)u(t,\xi)\Big)_x\, d\xi \\ &- \int_{-\infty}^{q(t,x_0)} e^{\xi} y(t,\xi)u_x(t,\xi)\, d\xi = \int_{-\infty}^{q(t,x_0)} e^{\xi} y(t,\xi)u(t,\xi)\, d\xi \\ &- \int_{-\infty}^{q(t,x_0)} e^{\xi} u(t,\xi)u_x(t,\xi)\, d\xi + \int_{-\infty}^{q(t,x_0)} e^{\xi} u_x(t,\xi)u_{xx}(t,\xi)\, d\xi \\ &= \int_{-\infty}^{q(t,x_0)} e^{\xi} u^2(t,\xi)\, d\xi + \frac{1}{2}\int_{-\infty}^{q(t,x_0)} e^{\xi} u_x^2(t,\xi)\, d\xi \\ &- \Big[e^{\eta} u(t,\eta)u_x(t,\eta) - \frac{1}{2} e^{\eta} u_x^2(t,\eta)\Big]_{\eta=q(t,x_0)}.\end{aligned}$$

Recall that $y(t,q(t,x_0)) = 0$ on $[0,T)$. Substituting the above obtained expression into (5.12) and using (3.1), we obtain that

$$\begin{aligned}\frac{d}{dt} V(t) = -u(t,q(t,x_0))\, V(t) + e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\xi} [u^2(t,\xi) + \frac{1}{2} u_x^2(t,\xi)]\, d\xi \\ - u(t,q(t,x_0))u_x(t,q(t,x_0)) + \frac{1}{2} u_x^2(t,q(t,x_0)), \quad t \in (0,T).\end{aligned}$$

We therefore infer from (5.8) the inequality

$$\begin{aligned}\frac{d}{dt} V(t) &\geqslant -u(t,q(t,x_0))\, V(t) - u(t,q(t,x_0))u_x(t,q(t,x_0)) \\ &\quad + \frac{1}{2} u_x^2(t,q(t,x_0)) + \frac{1}{2} u^2(t,q(t,x_0)) \\ &= \frac{1}{2} u_x^2(t,q(t,x_0)) - \frac{1}{2} u^2(t,q(t,x_0)),\end{aligned} \tag{5.13}$$

for all $t \in (0,T)$, since the representation formulas (4.1) and (4.2) yield

$$V(t) + u_x(t, q(t,x_0)) = u(t, q(t,x_0)) \quad \text{for} \quad t \in [0,T).$$

In an analogous way we obtain

$$\begin{aligned} \frac{d}{dt} W(t) = u(t,q(t,x_0))\, W(t) - e^{q(t,x_0)} \int_{q(t,x_0)}^{\infty} e^{-\xi} [u^2(t,\xi) + \frac{1}{2} u_x^2(t,\xi)]\, d\xi \\ - u(t,q(t,x_0)) u_x(t,q(t,x_0)) - \frac{1}{2} u_x^2(t,q(t,x_0)) \end{aligned}$$

and, using (5.9), we get

$$\begin{aligned} \frac{d}{dt} W(t) &\leqslant u(t,q(t,x_0))\, W(t) - u(t,q(t,x_0)) u_x(t,q(t,x_0)) \\ &\quad - \frac{1}{2} u_x^2(t,q(t,x_0)) - \frac{1}{2} u^2(t,q(t,x_0)) \\ &= \frac{1}{2} u^2(t,q(t,x_0)) - \frac{1}{2} u_x^2(t,q(t,x_0)), \quad t \in (0,T), \end{aligned} \tag{5.14}$$

since $W(t) - u_x(t,q(t,x_0)) = u(t,q(t,x_0))$ by (4.1) and (4.2).

Taking into account the inequalities (5.11) and the representation formulas (4.1)-(4.2), we observe that

$$u_x^2(t,q(t,x_0)) > u^2(t,q(t,x_0)), \quad t \in [0,T).$$

The differential inequalities (5.13) and (5.14) show therefore that $V(t)$ is strictly increasing while $W(t)$ is strictly decreasing on $[0,T)$. The hypotheses ensure $V(0) > 0$ and $W(0) < 0$ so that

$$V(t)W(t) \leqslant V(0)W(0) < 0, \quad t \in [0,T). \tag{5.15}$$

By (4.1)-(4.2) we see that

$$u^2(t,q(t,x_0)) - u_x^2(t,q(t,x_0)) = V(t)W(t) \quad \text{on} \quad [0,T)$$

so that from (5.10) and (5.15) we obtain

$$\frac{d}{dt} g(t) \leqslant \frac{1}{2} V(t)W(t) \leqslant \frac{1}{2} V(0)W(0), \quad t \in (0,T), \tag{5.16}$$

where we defined

$$g(t) := u_x(t,q(t,x_0)) \quad \text{for} \quad t \in [0,T).$$

Assume now that $T = \infty$, i.e. that the solution exists globally in time. We now show that this leads to a contradiction.

From (5.16) we would obtain, by integration,

$$g(t) \leqslant g(0) + \frac{1}{2} V(0) W(0)\, t, \quad t \in [0, \infty). \tag{5.17}$$

Since $V(0)W(0) < 0$ and $|u(t,\cdot)|_{L^\infty(\mathbb{R})}$ is bounded on $\mathbb{R}_+$, as the $H^1(\mathbb{R})$-norm of the solution of (2.1) is a conservation law, there exists certainly some $t_0 > 0$ such that

$$g^2(t) \geqslant 2\, ||u(t,\cdot)||^2_{L^\infty(\mathbb{R})}, \quad t \geqslant t_0.$$

Combining the latter inequality with (5.11) yields

$$\frac{d}{dt}\, g(t) \leqslant -\frac{1}{4}\, g^2(t), \quad t \in (t_0, \infty).$$

By (4.2), $g(0) < 0$ and thus by (5.17) $g(t) < 0$ for $t \geqslant 0$. Thus we can divide both sides of the above inequality by $g^2(t)$ and integrating, we get

$$\frac{1}{g(t_0)} - \frac{1}{g(t)} + \frac{1}{4}(t - t_0) \leqslant 0, \quad t \geqslant t_0.$$

Taking into account that $-\frac{1}{g(t)} > 0$ and $\frac{1}{4}(t - t_0) \to \infty$ as $t \to \infty$, we obtain a contradiction. This proves that $T < \infty$. □

We are now concerned with the rate of blow-up of the slope of a breaking wave for the shallow water equation (2.1).

THEOREM 5.3.— *Let $T < \infty$ be the blow-up time of the solution corresponding to some initial data $u_0 \in H^3(\mathbb{R})$. We have*

$$\lim_{t \to T} \left(\inf_{x \in \mathbb{R}} \{u_x(t,x)\}\, (T - t) \right) = -2$$

while the solution remains uniformly bounded.

Proof.— The solution is uniformly bounded on $[0,T) \times \mathbb{R}$ by the $H^1(\mathbb{R})$ conservation law (cf. Theorem A). Moreover, by Theorem 5.1 we know that

$$\liminf_{t \to T}\, m(t) = -\infty \tag{5.18}$$

where $m(t) := \inf_{x\in\mathbb{R}} \{u_x(t,x)\}$ for $t \in [0,T)$. It is not hard to check that the function m is locally Lipschitz (see [10]) with $m(t) < 0$ for $t \in [0,T)$. Moreover, if $\xi(t) \in \mathbb{R}$ is such that

$$m(t) = u_x(t,\xi(t)), \quad t \in [0,T),$$

by Theorem C and relation (5.6) we deduce that for a.e. $t \in (0,T)$,

$$\frac{d}{dt}\, m(t) = u^2(t,\xi(t)) - \frac{1}{2}\, m^2(t) - \left(p * \left[u^2 + \frac{1}{2}u_x^2\right]\right)(t,\xi(t)), \tag{5.19}$$

since $u_{xx}(t,\xi(t)) = 0$ for $t \in (0,T)$ - we deal with a minimum.

Young's inequality yields

$$\begin{aligned} ||(p * [u^2 + \frac{1}{2}u_x^2])(t,\cdot)||_{L^\infty(\mathbb{R})} &\leqslant ||p||_{L^\infty(\mathbb{R})}\, ||u^2 + \frac{1}{2}u_x^2||_{L^1(\mathbb{R})} \\ &\leqslant ||u(t,\cdot)||^2_{H^1(\mathbb{R})} = ||u_0||^2_{H^1(\mathbb{R})}. \end{aligned}$$

Since the solution itself is uniformly bounded, we can find a constant $K > 0$ such that

$$|u^2(t,\xi(t)) - \left(p * \left[u^2 + \frac{1}{2}u_x^2\right]\right)(t,\xi(t))| \leqslant K, \quad t \in [0,T). \tag{5.20}$$

Choose now $\epsilon \in (0,\frac{1}{2})$. Using the inequality (5.20) combined with relation (5.19), we deduce that

$$\frac{d}{dt}\, m(t) \leqslant -\frac{1}{2}m^2(t) + K \quad \text{for a.e.} \quad t \in (0,T).$$

Using (5.18), we find $t_0 \in (0,T)$ such that $m(t_0) < -\sqrt{2K + \frac{K}{\epsilon}}$. Note now that m is locally Lipschitz and therefore absolutely continuous, cf. [17]. By integrating the above differential inequality on intervals $[t_0,t)$ with $t_0 < t < T$ and using the absolute continuity of m, we infer that m is decreasing on $[t_0,T)$. Therefore,

$$m(t) < -\sqrt{\frac{K}{\epsilon}}, \quad t \in [t_0,T). \tag{5.22}$$

We saw that m is decreasing on $[t_0,T)$ and by (5.18) we obtain

$$\lim_{t\to T} m(t) = -\infty.$$

Since m is locally Lipschitz and less than $m(t_0) < 0$ on (t_0, T), one can easily check that $\frac{1}{m}$ is also locally Lipschitz on (t_0, T). Moreover, differentiating the relation $m(t) \cdot \frac{1}{m(t)} = 1$ on (t_0, T) yields

$$\frac{d}{dt}\frac{1}{m(t)} = -\frac{\frac{d}{dt}m(t)}{m^2(t)} \quad \text{for a.e.} \quad t \in (t_0, T).$$

From (5.20) and (5.21) we deduce that

$$\frac{1}{2} + \epsilon \geqslant \frac{d}{dt}\frac{1}{m(t)} \geqslant \frac{1}{2} - \epsilon \quad \text{for a.e.} \quad t \in (t_0, T).$$

Integrating this relation on (t, T) with $t \in (t_0, T)$ and taking into account that $\lim_{t \to T} m(t) = -\infty$, we obtain

$$\left(\frac{1}{2} + \epsilon\right)(T - t) \geqslant -\frac{1}{m(t)} \geqslant \left(\frac{1}{2} - \epsilon\right)(T - t), \quad t \in (t_0, T),$$

or, since $m(t) < 0$ on $[t_0, T)$,

$$\frac{1}{\frac{1}{2} + \epsilon} \leqslant -m(t)\,(T - t) \leqslant \frac{1}{\frac{1}{2} - \epsilon}, \quad t \in (t_0, T). \tag{5.22}$$

In view of the definition of $m(t)$ and the fact that $\epsilon \in (0, \frac{1}{2})$ was arbitrary, (5.22) implies the statement of the theorem. □

In the case of breaking waves corresponding to initial profiles satisfying the hypotheses of Theorem 5.2, we have

THEOREM 5.4.— *Let $T < \infty$ be the blow-up time of the solution corresponding to some initial data $u_0 \in H^3(\mathbb{R})$ such that the associated potential $y_0 = u_0 - u_{0,xx}$ satisfies $y_0(x) \geqslant 0$ on $(-\infty, x_0]$ and $y_0(x) \leqslant 0$ on $[x_0, \infty)$ for some point $x_0 \in \mathbb{R}$ and y_0 does not have a constant sign. Then*

$$\lim_{t \to T}\left(\sup_{x \in \mathbb{R}}\{|u_x(t, x)|\}\,(T - t)\right) = 2$$

while the solution remains uniformly bounded.

Proof. — It has been already established that u is uniformly bounded,

$$\sup_{t \in [0,T),\ x \in \mathbb{R}} |u(t, x)| < \infty.$$

By Theorem 5.3 we also know that

$$\lim_{t\to T}\left(\inf_{x\in\mathbb{R}}\{u_x(t,x)\}\,(T-t)\right)=-2.$$

We associate to (2.1) equation (3.1). For $t\in[0,T)$, let $q(t,\cdot)$ be the increasing diffeomorphism of the line given by Theorem 3.1.

Taking into account relation (5.12), we obtain from (4.1)-(4.2) that

$$u_x(t,x)=u(t,x)-e^{-x}\int_{-\infty}^{x}e^{\xi}y(t,\xi)\,d\xi\leqslant u(t,x)\quad\text{if}\quad x\leqslant q(t,x_0),$$

whereas

$$u_x(t,x)=-u(t,x)+e^{x}\int_{x}^{\infty}e^{-\xi}y(t,\xi)\,d\xi\leqslant -u(t,x)\quad\text{if}\quad x\geqslant q(t,x_0).$$

We deduce that

$$\sup_{x\in\mathbb{R}}u_x(t,x)\leqslant\sup_{x\in\mathbb{R}}|u(t,x)|,\quad t\in[0,T).$$

The solution u being uniformly bounded we infer from the latter inequality a uniform upper bound for the x-derivative of the solution to complete the proof. □

Let us now provide some information about the blow-up set of a breaking wave for the shallow water equation (2.1).

The first result gives a more precise description of the blow-up mechanism that occurs in the case of Theorem 5.4: while the solution remains bounded (as known), there is at least one point where the slope of the wave becomes infinite exactly at breaking time.

THEOREM 5.5. — *Assume $u_0\in H^3(\mathbb{R})\cap L^1(\mathbb{R})$ is such that the associated potential $y_0:=u_0-u_{0,xx}$ satisfies $y_0(x)\geqslant 0$ on $(-\infty,x_0]$ and $y_0(x)\leqslant 0$ on $[x_0,\infty)$ for some point $x_0\in\mathbb{R}$ and y_0 does not have a constant sign. With $T<\infty$ denoting the finite blow-up time of the corresponding solution of (2.1), we have*

$$u_x(t,q(t,x_0))\to-\infty\quad\text{as}\quad t\to T,$$

where $q(t,\cdot)$ are the diffeomorphisms of the line given by (3.1).

Proof. — By Theorem 5.2 we know that we have finite time blow-up and from the proof of Theorem 5.4 we have a uniform bound from above for the x-derivative of the solution.

Assume the statement of the theorem is false. Then there exists $M > 0$ such that

$$|u_x(t, q(t, x_0))| \leqslant M, \quad t \in [0, T).$$

As already noted in the first part of the proof of Theorem 5.2, we can write (2.1) in the equivalent form

$$u_t + uu_x + \partial_x \left(p * \left[u^2 + \frac{1}{2} u_x^2 \right] \right) = 0.$$

Use this equation to conclude that for any $0 < t < T$ and $-\infty < a < b < \infty$,

$$\frac{d}{dt} \int_a^b u(t, x)\, dx = -\frac{1}{2} u^2(t, b) + \frac{1}{2} u^2(t, a) - p * \left[u^2 + \frac{1}{2} u_x^2 \right] \Big|_a^b .$$

Inequality (5.20) and the uniform boundedness of u imply that there exists a constant $K > 0$ independent of a, b, and t, with

$$\left| \frac{d}{dt} \int_a^b u(t, x)\, dx \right| \leqslant K, \quad a, b \in \mathbb{R},\ t \in (0, T).$$

Integrating over the time interval $[0, t]$, this estimate yields

$$\left| \int_a^b u(t, x)\, dx - \int_a^b u_0(x)\, dx \right| \leqslant KT, \quad a, b \in R,\ t \in [0, T). \tag{5.23}$$

Fix $t \in [0, T)$.

For $x \geqslant q(t, x_0)$ we have by relation (5.11) that

$$\frac{d}{dx} u_x(t, x) = u_{xx}(t, x) = u(t, x) - y(t, x) \geqslant u(t, x).$$

Integrating on $[q(t, x_0), x]$, this inequality leads to

$$u_x(t, x) \geqslant u_x(t, q(t, x_0)) + \int_{q(t, x_0)}^x u(t, \xi)\, d\xi, \quad x \geqslant q(t, x_0). \tag{5.24}$$

If $x \leqslant q(t, x_0)$ we have again by (5.11) that

$$\frac{d}{dx} u_x(t, x) = u(t, x) - y(t, x) \leqslant u(t, x),$$

and integrating on $[x, q(t, x_0)]$, we obtain in this case

$$u_x(t,x) \geqslant u_x(t,q(t,x_0)) - \int_x^{q(t,x_0)} u(t,\xi)\,d\xi, \quad x \leqslant q(t,x_0). \tag{5.25}$$

Combining (5.23)-(5.25) we get

$$u_x(t,x) \geqslant -M - KT - |u_0|_{L^1(\mathbb{R})}, \quad (t,x) \in [0,T) \times \mathbb{R},$$

or

$$\liminf_{t \to T} \left(\inf_{x \in R} \{u_x(t,x)\} \right) > -\infty$$

and, according to Theorem 5.1, the solution does not blow-up as $t \to T$. The obtained contradiction completes the proof. □

Remark 5.6. — Assume $u_0 \in H^3(\mathbb{R}) \cap L^1(\mathbb{R})$ is odd and satisfies the hypotheses of Theorem 5.5. In this case, we have that $q(t,0) = 0$ for $t \in [0,T)$. Indeed, according to (3.1), $f(t) := q(t,0)$, $t \in [0,T)$ satisfies the ordinary differential equation

$$\frac{d}{dt} f(t) = u(t, f(t)), \quad t \in (0,T).$$

Note that

$$v(t,x) := -u(t,-x), \quad t \in [0,T), \quad x \in \mathbb{R},$$

is also a solution of (2.1) in $C([0,T); H^3(\mathbb{R})) \cap C^1([0,T); H^2(\mathbb{R}))$ with initial data u_0. By uniqueness we conclude that $v \equiv u$ and therefore $u(t,\cdot)$ is odd on $\mathbb{R}$ for any $t \in [0,T)$. As $u(t,0) = 0$, we have that the zero function is a solution of the differential equation for f. Taking into account the fact that $u(t,\cdot)$ is locally Lipschitz on $\mathbb{R}$, as one can easily see, we conclude from the uniqueness theorem for ODE's that $f(t) = 0$ for all $t \in [0,T)$ since $f(0) = 0$. Therefore, by Theorem 5.5 we obtain that at breaking time $T < \infty$ we have $\lim_{t \to T} u_x(t,0) = -\infty$. □

Remark 5.7. — For initial data $u_0 \in H^3(\mathbb{R})$ such that $y_0 := u_0 - u_{0,xx}$ changes sign we may have global existence or blow-up of the solution to (2.1) according to Theorem 4.4, respectively Theorem 5.2. Note the contrast with the periodic case where as soon as $y_0 \in H^1(\mathbb{S})$, $y_0 \not\equiv 0$, ($\mathbb{S}$ being the unit circle) satisfies $\int_{\mathbb{S}} y_0(x)\,dx = 0$, we have finite time blow-up (cf. [8] - see also [7] for the special case of odd initial data). □

For a large class of initial data, the blow-up set consists of one point:

THEOREM 5.8.— *Let $u_0 \in H^3(\mathbb{R})$, $u_0 \not\equiv 0$ be odd, and such that the associated potential $y_0 := u_0 - u_{0,xx}$ is nonnegative on $\mathbb{R}_-$. Then the solution to* (2.1) *with initial profile u_0 breaks in finite time at zero but nowhere else.*

Proof.— Let $T > 0$ be the maximal existence time of the solution $u(t,x)$ to (2.1) with initial data u_0, as given by Theorem A. We associate to (2.1) equation (3.1). For $t \in [0,T)$, let $q(t,\cdot)$ be the increasing diffeomorphism of the line whose existence is guaranteed by Theorem 3.1.

Note that $u(t,\cdot)$ is odd on $\mathbb{R}$ for any $t \in [0,T)$, cf. Remark 5.6.

Let $s(t) := u_x(t,0)$. Due to the form of the initial profile, $s(0) \leqslant 0$ and, setting $x = 0$ in (5.6),

$$s_t + \frac{1}{2}s^2 \leqslant 0, \quad t \in [0,T),$$

we conclude that $T < \infty$: the solution $u(t,x)$ of (2.1) blows-up in finite time.

We give now a precise description of the blow-up mechanism. As noted before, we have a uniform bound on $u(t,x)$ for $t \in [0,T)$ and $x \in \mathbb{R}$. We will see below that at any $x \neq 0$, the slope $u_x(t,x)$ remains bounded on $[0,T)$ while $u_x(t,0) \to -\infty$ as $t \uparrow T$: the wave breaks in finite time exactly at zero and nowhere else.

Let $y := u - u_{xx}$ be the potential associated to the solution $u(t,x)$. Using the fact that $y(t,x)$ is odd in x, we obtain

$$\begin{aligned} u(t,x) &= (p * y)(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-\xi|}y(t,\xi)\,d\xi \\ &= \sinh(x)\int_x^{\infty} e^{-\xi}y(t,\xi)\,d\xi \\ &\quad + e^{-x}\int_0^x \sinh(\xi)y(t,\xi)\,d\xi, \quad (t,x) \in [0,T) \times \mathbb{R}_+ \end{aligned}$$

and

$$\begin{aligned} u_x(t,x) &= \partial_x\Big[\frac{1}{2}\int_{\mathbb{R}} e^{-|x-\xi|}y(t,\xi)\,d\xi\Big] \\ &= \cosh(x)\int_x^{\infty} e^{-\xi}y(t,\xi)\,d\xi \\ &\quad - e^{-x}\int_0^x \sinh(\xi)y(t,\xi)\,d\xi, \quad (t,x) \in [0,T) \times \mathbb{R}_+. \end{aligned}$$

But $y(t,\xi) \leqslant 0$ for $\xi \geqslant 0$ and $u(t,x)$ is uniformly bounded on $[0,T)\times\mathbb{R}$. Therefore, there exists a constant $K>0$ such that

$$\left|\sinh(x)\int_x^\infty e^{-\xi}y(t,\xi)\,d\xi\right| \leqslant K, \quad (t,x)\in[0,T)\times\mathbb{R}_+,$$

and

$$\left|e^{-x}\int_0^x \sinh(\xi)y(t,\xi)\,d\xi\right| \leqslant K, \quad (t,x)\in[0,T)\times\mathbb{R}_+.$$

Using these estimates in the above formula for $u_x(t,x)$ one obtains

$$|u_x(t,x)| \leqslant K + K\,\frac{\cosh(x)}{\sinh(x)}, \quad t\in[0,T),\ x>0.$$

This shows that $|u_x(t,x)| = |u_x(t,-x)|$ is uniformly bounded on $0\leqslant t<T$, $x\geqslant\epsilon$ for $\epsilon>0$ arbitrarily small and completes the proof of the theorem. □

6. Geodesic flow on the diffeomorphism group.

In this last section we will use the qualitative aspects of the shallow water flow analyzed in the previous sections to obtain information about the geodesics on $\mathcal{D}^3(\mathbb{R})$.

THEOREM 6.1. — *On $\mathcal{D}^3(\mathbb{R})$ there are geodesics with infinite life span.*

Proof. — Let $u_0\in\mathcal{M}_1^3\cap H^4(\mathbb{R})$, $u_0\not\equiv 0$, be such that the associated potential $y_0 := u_0 - u_{0,xx}$ does not change sign on $\mathbb{R}$ or changes properly sign exactly once by passing from nonpositive to nonnegative. Further, assume that

$$\sup_{x\in\mathbb{R}}\left(\Big[|u_0(x)| + |\partial_x u_0(x)| + |\partial_x^2 u_0(x)| + |\partial_x^3 u_0(x)|\Big]\,|x|^5\right) < \infty.$$

From Theorem 4.4 or Theorem 4.1 we know that the shallow water equation (2.1) has a unique global solution $u\in C(\mathbb{R}_+;H^4(\mathbb{R}))\cap C^1(\mathbb{R}_+;H^3(\mathbb{R}))$ with initial profile u_0. Moreover, if we take into account Lemma 3.2, we can easily show that the assumed decay property for u_0 ensures that $u(t,\cdot)\in\mathcal{M}_1^3$ for every $t\geqslant 0$.

Then the geodesic $t\mapsto q(t)$ on the diffeomorphism group $\mathcal{D}^3(\mathbb{R})$ starting at the identity in direction u_0 can be continued indefinitely in time. This statement follows by combining the above information with the local existence result for geodesics proved in Theorem 2.2. □

Remark 6.2. — Note that by right multiplication, a similar statement holds for the geodesic starting at a diffeomorphism $q_0 \in \mathcal{D}^3(\mathbb{R})$ in the tangent direction u_0. □

We show now that the formation of singularities of certain solutions to the shallow water equation in finite time yields the breakdown of the geodesic flow on $\mathcal{D}^3(\mathbb{R})$.

THEOREM 6.3. — *If $u_0 \in \mathcal{M}_1^3$ satisfies the hypotheses of Theorem 5.2, then the geodesic starting at the identity with initial velocity u_0 breaks down in finite time.*

Proof. — Assuming that the geodesic could be continued indefinitely in time, we would obtain from the results in Section 2 a global solution of the shallow water equation (2.1) with initial profile $u_0 \in \mathcal{M}_1^3 \subset H^3(\mathbb{R})$, which contradicts the statement of Theorem 5.2. □

One might think that the breakdown of the geodesic flow is caused by the smoothness assumptions on the diffeomorphisms in $\mathcal{D}^3(\mathbb{R})$ or by an unfortunate choice of the weighted space. The next result shows that this is not the case, as sometimes the breakdown occurs due to the flattening out of the diffeomorphisms.

THEOREM 6.4. — *Let $u_0 \in H^4(\mathbb{R})$, $u_0 \not\equiv 0$, be such that the associated potential $y_0 := u_0 - u_{0,xx}$ is odd and such that $y_0(x) = 0$ for $x \in [-x_0, x_0]$ for some $x_0 > 0$ while $y_0(x) \leqslant 0$ for $x > x_0$. Moreover, assume that*

$$\sup_{x\in\mathbb{R}} \Big(\Big[|u_0(x)| + |\partial_x u_0(x)| + |\partial_x^2 u_0(x)| + |\partial_x^3 u_0(x)|\Big]\, |x|^5\Big) < \infty.$$

Then the geodesic $t \mapsto q(t)$ on the diffeomorphism group $\mathcal{D}^3(\mathbb{R})$, starting at the identity Id *in direction u_0, breaks down in finite time $T < \infty$. At time T, the diffeomorphisms $q(t,x)$ flatten out.*

Proof. — Let $T > 0$ be the maximal existence time of the solution $u(t,x)$ to the shallow water equation (2.1) with initial profile u_0, as given by Theorem A. We know by Theorem 5.2 that $T < \infty$. Moreover, a simple argumentation based on Lemma 3.2 shows that for every $t \in [0,T)$, the wave $u(t,\cdot)$ preserves the stated decay at infinity of the initial profile.

We deduce from the previous observations that $u(t,\cdot) \in \mathcal{M}_1^3$ for all $t \in [0,T)$. Now, from the results in Section 2 we infer that the geodesic $t \mapsto q(t)$ is well-defined at least until the wave-breaking time T.

Our goal is to understand what happens with the diffeomorphisms $q(t) \in \mathcal{D}^3(\mathbb{R})$ as $t \uparrow T$.

As a solution to (2.1) with odd initial profile remains spatially odd[12] on the time-interval $[0,T)$, it follows that $y := u - u_{xx}$ is odd in the space variable on $[0,T)$.

Setting $x = 0$ in (3.1) we see that $q(t,0) = 0$ for $t \in [0,T)$ by the uniqueness theorem for ODE's, cf. Remark 5.6.

Lemma 3.2 implies that $y(t,x)$ remains nonpositive for $x \geqslant 0$ as long as $t \in [0,T)$.

Since $q(t,0) = 0$ for $t \in [0,T)$ and $q(t,\cdot)$ is an increasing diffeomorphism of the line, we have that $q(t,x) > 0$ for $(t,x) \in [0,T) \times (0,\infty)$. On the other hand, $y(t,x) \leqslant 0$ on $[0,T) \times \mathbb{R}_+$ and $u(t,0) = 0$ on $[0,T)$ by the oddness property. We claim that $u(t,x) \leqslant 0$ for $(t,x) \in [0,T) \times \mathbb{R}_+$.

Indeed, observe that $u(t,\cdot) \in H^3(\mathbb{R})$ implies $\lim_{x\to\infty} u(t,x) = 0$ for fixed $t \in [0,T)$, so that the existence of some $x_1(t) > 0$ with $u(t,x_1(t)) > 0$ would mean that the supremum of $u(t,\cdot)$ on $\mathbb{R}_+$ is positive and attained at some $x_2(t) > 0$. But in this case $u_{xx}(t,x_2(t)) \leqslant 0$ and the desired contradiction follows by

$$0 < u(t,x_2(t)) = y(t,x_2(t)) + u_{xx}(t,x_2(t)) \leqslant 0,$$

thus $u(t,x) \leqslant 0$ on $[0,T) \times \mathbb{R}_+$ as claimed.

According to (3.1), $q(t,x)$ satisfies the differential equation

$$\frac{d}{dt}\, q(t,x) = u(t,q(t,x)), \quad t \in (0,T),$$

so that, for every fixed $x \in \mathbb{R}_+$, $q(t,x)$, by the diffeomorphism property and the fact that $q(t,0) = 0$, is nonincreasing by nonnegative values as $t \uparrow T$. Therefore $\lim_{t\uparrow T} q(t,x)$ exists and is nonnegative for every $x \in \mathbb{R}_+$. If $z \in [0,x]$, we have, by the monotonicity of q, that

$$0 \leqslant q(t,z) \leqslant q(t,x) \quad \text{as} \quad t \in [0,T)$$

so that $\lim_{t\uparrow T} q(t,x) = 0$ implies $\lim_{t\uparrow T} q(t,z) = 0$ for all $z \in [0,x]$.

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[12] See Remark 5.6.

The previous observations show that in order to prove that $q(t, x)$ flattens out in the limit $t \uparrow T$, it is enough to prove that for some $x \in \mathbb{R}_+$ we have $\lim_{t\uparrow T} q(t, x) = 0$.

Assume the contrary. Then

$$q(t, x_0) \geqslant \lim_{t\uparrow T} q(t, x_0) = \epsilon > 0, \quad t \in [0, T). \tag{6.1}$$

The relation

$$y(t, q(t, x))\, q_x^2(t, x) = y_0(x), \qquad t \in [0, T),\ x \in \mathbb{R},$$

obtained in Lemma 3.2 shows that $y(t, q(t, x)) = 0$ for all $(t, x) \in [0, T) \times [0, x_0]$, that is, $y(t, z) = 0$ on $[0, q(t, x_0)]$ for every $t \in [0, T)$. Combining this with (6.1), and the oddness of y with respect to the spatial variable, one gets

$$y(t, z) = 0, \qquad (t, z) \in [0, T) \times [-\epsilon, \epsilon]. \tag{6.2}$$

As a consequence we do not have that $u_x(t, 0) \to -\infty$ as $t \uparrow T$(13) contradicting Theorem 5.8.

The obtained contradiction proves that $q(t, x)$ flattens out in the limit $t \uparrow T$ and we have the breakdown of the geodesic flow. □

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