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SYMPLECTIC SUBVARIETIES OF PROJECTIVE FIBRATIONS OVER SYMPLECTIC MANIFOLDS

by Roberto PAOLETTI

1. Introduction.

Suppose that (M, ω) is a compact symplectic manifold of dimension $2n$, such that the cohomology class $[\omega] \in H^2(M, \mathbb{R})$ lies in the integral lattice $H^2(M, \mathbb{Z})/\mathrm{Torsion}$; we shall say that (M, ω) is *almost-Hodge.* It has been recently proved by Donaldson that for any sufficiently large integer k there exists a symplectic submanifold $W \subset M$ representing the Poincaré dual of any fixed integral lift of $[k\omega]$, [D].

In this paper, we specialize this result to the case of a symplectic fibration $p : E \to M$ whose fibre is a projective manifold F with a fixed Hodge form σ on it. For instance, E could be the relative projective space, or a relative flag space, associated to a complex vector bundle on M. Then, as follows from well-known symplectic reduction techniques ([W], [GLS]) E has an almost Hodge structure $\widetilde{\omega}$ restricting to σ on each fibre of p, [MS]. We adapt Donaldson's arguments to show that the symplectic divisor guaranteed by his theorem may be chosen compatibly with the vertical holomorphic structure. More precisely,

THEOREM 1.1. — *Let (M, ω) be an almost Hodge manifold. Let $F \subseteq \mathbb{P}^N$ be a connected complex projective manifold and set $L = \mathcal{O}_F(1)$,*

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the restriction to F of the hyperplane bundle on $\mathbb{P}^N$. Denote by σ the restriction to F of the Fubini-Study form on $\mathbb{P}^N$. Suppose that G is a compact group of automorphisms of $\mathbb{P}^N$ preserving F. Let $p : E \to M$ be a fibre bundle with fibre F and structure group G, so that in particular there is a line bundle $L_E \to E$ extending $L \to F$. Then E admits an almost Hodge structure $\widetilde{\omega}$ vertically compatible with σ. Furthermore, perhaps after replacing $\widetilde{\omega}$ by $kp^*(\omega_M) + \widetilde{\omega}$ for $k \gg 0$, any integral lift of $[\widetilde{\omega}]$ is Poincaré dual to a codimension-2 symplectic submanifold $W \subset E$, meeting any fibre $F_m = p^{-1}(m)$ $(m \in M)$ in a complex subvariety.

In general the submanifold W may not be transverse to every fibre. For example, if $\mathcal{E}$ is a rank-2 complex vector bundle on M and $E = \mathbb{P}\mathcal{E}^*$ with general fibre $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, then W is the blow-up of M along the zero locus Z of a section of a suitable twist of $\mathcal{E}$, and therefore contains all the fibres over Z.

In practice one may have a fibre bundle $E \to M$ with fibre a complex projective manifold (F, J_F) and structure group G preserving the complex structure J_F and some fixed Hodge form σ on F, and complexification $\widetilde{G} \subseteq \mathrm{Aut}(F, J_F)$. If L is a line bundle on F such that $c_1(L) = [\sigma]$, then by general principles from geometric invariant theory a lifting to $L^{\otimes k}$ of the action of G exists if $k \gg 0$. Therefore,

COROLLARY 1.1. — *Suppose that (F, σ), M and E are as just described. Then for $r \gg 0$ and $k > k(r)$ any integral lift of $[r\widetilde{\omega} + kp^*(\omega_M)]$ is Poincaré dual to a codimension-2 symplectic submanifold intersecting each fibre F_m in a divisor of the linear series $|L^{\otimes r}|$.*

Again, W is not transversal to every fibre. In the case of a $\mathbb{P}^1$-bundle $E = \mathbb{P}\mathcal{E}^* \to M$, the projection $W \to M$ is a branched cover with non-empty ramification locus.

The theorem also yields that top Chern classes of symplectically very positive vector bundles have symplectic representatives, as already shown by Auroux, [A]:

COROLLARY 1.2. — *Let (M, ω) be a $2n$-dimensional almost Hodge manifold and let $\mathcal{E}$ be a complex vector bundle on M of complex rank $r < n$. Let H be a complex line bundle on M with $c_1(H) = [\omega]$. Then for $k \gg 0$ there is a transverse section s of $\mathcal{E} \otimes H^{\otimes k}$ whose zero locus Z is a connected symplectic submanifold of M; in fact, $H_j(M, Z) = 0$ if $j \leq n - r$.*

As we shall see, these sections are also asymptotically almost holomorphic in the sense of [A].

Notation. — For any integer $r > 0$, we shall denote by $\omega_0^{(r)} = (i/2) \sum_{\alpha=1}^{r} dz_\alpha \wedge d\bar{z}_\alpha$ the standard symplectic structure on $\mathbb{C}^r$. Furthermore, by C we shall often indicate an appropriate constant, appearing in various estimates, which is allowed to vary from line to line.

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2. Proof of the theorem and corollaries.

Let $\pi : P \to M$ be the principal G-bundle associated with the fibration. 'Given a connection for π, the existence of a compatible almost Hodge form on E follows from well-known symplectic reduction arguments, [MS]. In fact, minimal coupling produces a compatible closed 2-form $\vartheta = \vartheta_{\min}$ on E, [GS]. Explicitly, let the induced connection be given by the horizontal distribution $\mathcal{H}(E/M) \subset TE$ and denote by $V(E/M) \subset TE$ the vertical tangent space. Let $\mathbf{g}$ be the Lie algebra of G and view the curvature F as a $\mathbf{g}$-valued 2-form on M. Let $\mu : F \to \mathbf{g}^*$ be the moment map for the action. If $e \in E$ and $x = p(e)$, let $U \subseteq M$ be an open subset over which P trivializes and let $\gamma : U \times F \to p^{-1}(U)$ be the corresponding trivialization. Then $\mathcal{H}(E/M)$ and $V(E/M)$ are mutually orthogonal for σ. Furthermore, with abuse of language, $\vartheta|_{V(P/M)} = \sigma$, while if $X, Y \in T_xM$ and $X^\sharp, Y^\sharp$ are their horizontal lifts at $e = \gamma(x, f)$, then $\vartheta_e(X^\sharp, Y^\sharp) = \langle \mu(f), F_x(X, Y)\rangle$. Therefore $\widetilde{\omega}_{(k)} = \vartheta + kp^*(\omega)$ is a compatible symplectic structure on E if $k \gg 0$. However, in order to adapt Donaldson's construction we shall need to describe $-2\pi i\vartheta$ as the curvature of a connection on a suitable line bundle on E.

Clearly, the action of G lifts to L and preserves the unit circle bundle $S_L \subset L$. Let ∇_L be the unique covariant derivative on L compatible with the complex and hermitian structures, that is, the restriction to F of the connection on $\mathcal{O}_{\mathbb{P}^N}(1)$. Let $\mathcal{H}(S_L/F) \subset TS_L$ be the corresponding S^1-invariant horizontal distribution, which by uniqueness is also G-invariant. The line bundle $L_E := P \times_G L$ over E restricts to L on every fibre of p and has an hermitian metric extending that of L. Then the unit circle bundle $S_{L_E} = P \times_G S_L \subset L_E$ has a connection over E, as follows. Let $p' : S_{L_E} \to M$ be the projection, a fibre bundle over M with general fibre S_L. Given $s \in S_{L_E}$ mapping to $e \in E$, set $x = p(e)$ and choose as above a trivialization of P in a neighbourhood U of x, with induced trivializations $\gamma : U \times F \to p^{-1}(U)$ and $\gamma' : U \times S_L \to p'^{-1}(U)$. If $e = \gamma(x, f)$ and $s = \gamma'(x, \ell)$ ($\ell \in S_L$ lies over $f \in F$), then the horizontal space of S_{L_E} at s is $\mathcal{H}(S_{L_E}/E) = \mathcal{H}(S_{L_E}/M) \oplus d\gamma'_{(x,\ell)}\Big(\mathcal{H}_\ell(S_L/F)\Big)$. This gives a well-defined connection ∇_{L_E} on L_E, and we leave it to the reader to check that $\vartheta_{\min}$ may also be obtained as the normalized curvature of ∇_{L_E}:

LEMMA 2.1. — *Let ϑ be the normalized curvature form on E of the connection $\mathcal{H}(S_E/E)$. Then for $k \gg 0$ the 2-form $\widetilde{\omega}_{(k)} = \vartheta + kp^*(\omega)$ is a compatible symplectic structure, and $\mathcal{H}(E/M)$ is the symplectic complement of $V(E/M)$ for $\widetilde{\omega}$. In particular, the subbundle $\mathcal{H}(E/M) \subset TE$ is symplectic with respect to $\widetilde{\omega}$.*

We shall need an auxiliary non-degenerate 2-form ω_{aux} on E. The vertical tangent bundle $V(E/M)$ has an obvious symplectic structure, the restriction of $\widetilde{\omega}$, that we shall also indicate by σ, and an obvious complex structure J_{vert}, inherited by that of TF. The horizontal distribution $\mathcal{H}(E/M)$, on the other hand, carries the symplectic structure $p^*\omega$. Then $\omega_{\mathrm{aux}} \in \Omega^2(E)$ will denote the orthogonal direct sum of σ and $p^*\omega$. In general ω_{aux} will not be closed, and in view of the minimal coupling horizontal component of ϑ we see that $\omega_{\mathrm{aux}} \neq \widetilde{\omega}_{(1)}$ when P is not flat. Let us pick some $J_M \in \mathcal{J}(M, \omega)$ and view it in a natural manner as a complex structure on $\mathcal{H}(E/M)$; then $J_{\mathrm{aux}} := J_M \oplus J_{\mathrm{vert}} \in \mathcal{J}(E, \omega_{\mathrm{aux}})$. Thus $g_{\mathrm{aux}}(\cdot,\cdot) = \omega_{\mathrm{aux}}(\cdot, J_{\mathrm{aux}}\cdot)$ is a riemannian metric on E. On the other hand, we have $\widetilde{\omega}_{(k)} = \widetilde{\omega}^h_{(k)} \oplus \widetilde{\omega}^v_{(k)}$, where $\widetilde{\omega}^h_{(k)}$ and $\widetilde{\omega}^v_{(k)} = \sigma$ denote, respectively, the horizontal and vertical components. Now $\alpha_k := (1/k)\widetilde{\omega}^h_{(k)}$ is a sequence of symplectic structures on the vector bundle $\mathcal{H}(E/M)$, converging to $p^*\omega$ in the $\mathcal{C}^1$-topology, namely $\|\alpha_k - p^*\omega\| < C/k$ and $\|\nabla(\alpha_k - p^*\omega)\| < C/k$. Given a vector bundle $\mathcal{F}$ on a manifold and any symplectic structure η on $\mathcal{F}$, there is a retraction $r_\eta : \mathcal{M}et(\mathcal{F}) \to \mathcal{J}(\mathcal{F}, \eta)$ depending pointwise analytically on η, where $\mathcal{M}et(\mathcal{F})$ is the space of all riemannnian metrics on $\mathcal{F}$, and $\mathcal{J}(F, \eta)$ denotes the space of all complex structures on $\mathcal{F}$ compatible with η ([MS], ch. 2). Denote by g^h_{aux} the restriction of g_{aux} to $H(E/M)$, and let $J^h_k := r_{\alpha_k}(g^h_{\mathrm{aux}}) \in \mathcal{J}(H(E/M), \alpha_k)$ for each k; then $\|J^h_k - J_M\| < C/k$, $\|\nabla(J^h_k - J_M)\| < C/k$. Therefore $J_k := J^h_k \oplus J_{\mathrm{vert}} \in \mathcal{J}(E, \widetilde{\omega}_k)$ and

$\|J_k - J_{\mathrm{aux}}\| < C/k$, $\|\nabla(J_k - J_{\mathrm{aux}})\| < C/k$. Let $\overset{(1,0)}{\underset{J_{\mathrm{aux}}}{\bigwedge}} T_E^*$ and $\overset{(0,1)}{\underset{J_{\mathrm{aux}}}{\bigwedge}} T_E^*$ denote, respectively, the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear complex functionals on (T_E, J_{aux}), and let $\mu_k : \overset{(1,0)}{\underset{J_{\mathrm{aux}}}{\bigwedge}} T_E^* \to \overset{(0,1)}{\underset{J_{\mathrm{aux}}}{\bigwedge}} T_E^*$ be the morphism of vector bundles relating J_k to J_{aux}, [D]. Then $\|\mu_k\| < C/k$ and $\|\nabla\mu_k\| < C/k$.

The riemannian metric $g_M = \omega(\cdot, J_M\cdot)$ on M induces a distance function d; for k a positive integer, let d_k denote the distance function associated to the pair $(k\omega, J_M)$, that is to the metric kg_M. Similarly, let d_F be the distance function on F associated to the pair (σ, J_F). Furthermore, on M there is an hermitian line bundle H together with a unitary connection on it having curvature form $-2\pi i\omega$. Replacing $\widetilde{\omega}$ by $\widetilde{\omega}_{(k)}$ amounts to replacing L_E by $B = p^*(H^{\otimes k}) \otimes L_E$ with the tensor product connection. Thus we are looking for a section s of B for some $k \gg 0$ whose zero locus is a symplectic submanifold $Z \subset E$ with respect to $\widetilde{\omega}$, meeting each fibre F_x in a complex subvariety.

Let ∇_B be the covariant derivative on B. Given the almost complex structure J_E, we have a decomposition $\nabla_B = \partial_B + \overline{\partial}_B$. The zero locus $Z = Z(s)$ of a smooth section s of B will be symplectic if $|\overline{\partial}_{J_k,B}s| < |\partial_{J_k,B}s|$ at every point of Z ([D]; Lemma 4.30 of [MS]); the two latter terms represent, respectively, the $(0,1)$ and $(1,0)$ components of $\nabla_B s$ with respect to the almost complex structure J_k. Following the path of Donaldson's construction, we shall produce such a section as a linear combination of certain "concentrated" building blocks. In order for $Z \cap F_x$ to be a complex subvariety of F_x for every $x \in M$, these basic pieces must be chosen in an appropriate way.

DEFINITION 2.1. — If $U \subset E$ is an open set, a smooth function $f : U \to \mathbb{C}$ will be called *vertically holomorphic* (in short, v-holomorphic) if its restriction to $U \cap F_x$ is holomorphic, whenever the latter set is non-empty. Let A be any complex line bundle on E. A v-holomorphic structure on A is the datum of an open cover $\mathcal{U} = \{U_\alpha\}$ of A, together with v-holomorphic transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$. With such an assignment, H will be called a v-holomorphic line bundle. There is a natural notion of equivalence of v-holomorphic structures. Clearly, the restriction of A to any fibre F_x is a holomorphic line bundle A_x. A local section of A on $U \subset E$ is called v-holomorphic if it restricts to a holomorphic local section of A_x for every $x \in M$ for which $U \cap F_x \neq \emptyset$. Let $\mathcal{O}_E^v$ denote the sheaf of rings of v-holomorphic functions on E; the sheaf of v-holomorphic sections of A, denoted $\mathcal{O}_E^v(A)$, is a sheaf of $\mathcal{O}_E^v$-modules.

Let $f : U \to \mathbb{C}$ be a smooth function on an open subset $U \subset E$, and let $(df)_{\text{vert}} \in V(E/M)^* \otimes \mathbb{C}$ be the restriction of its differential to the vertical tangent bundle. Let j denote the complex structure of $\mathbb{C}$. Then f is v-holomorphic if and only if $\overline{\partial}_{\text{vert}} f := (df)_{\text{vert}} + j \circ (df)_{\text{vert}} \circ J_{\text{vert}} = 0$; the left hand side is the $\mathbb{C}$-antilinear component of $(df)_{\text{vert}}$. Now the line bundle L_E is naturally v-holomorphic, and restricts to L on each fibre. Thus Theorem 1.1 is a consequence of the following:

Proposition 2.1. — *For $k \gg 0$ there is a v-holomorphic section s of B such that $|\overline{\partial}_{J_k,B} s| < |\partial_{J_k,B} s|$ at all points of the zero locus of s.*

To prove the proposition, we shall first produce a suitable choice of compactly supported v-holomorphic sections, peaked at points of E in an appropriate sense, to be used as the basic buiding blocks in Donaldson's construction. Next we shall give an appropriate open cover of E on which to perform the inductive part of his argument.

Fix $e_0 \in E$ and let $U_0 \subseteq M$ be an open neighbourhood of $x_0 = p(e_0)$ over which P is trivial; perhaps after replacing ω by some multiple, there is a Darboux cooordinate chart $\chi : B^{2n} \to U_0 \subseteq M$ centred at x_0 for ω, which is $\mathbb{C}$-linear at the origin. Let η be a unitary section of H over U_0 such that the connection matrix θ_M of H on U_0 with respect to η satisfies $\chi^*\theta_M = A$, where $A =: (1/4) \sum_{\alpha=1}^{n} (\overline{z}_\alpha dz_\alpha - z_\alpha d\overline{z}_\alpha)$, [D]. We have an induced trivialization $\gamma : U_0 \times F \to p^{-1}(E|_{U_0})$, under which $\gamma^*(L_E) \cong q_2^*(L)$, where q_2 is the projection on the second factor; suppose $e_0 = \gamma(x_0, f_0)$. We may assume that $\forall\ f \in F$ the local section $\gamma_f(y) = \gamma(y, f)$ defined over U satisfies $d_{x_0}\gamma_f(T_{x_0}M) = H_e$, where $e = \gamma_f(x_0)$. The product map $\phi = \gamma \circ (\chi, \mathrm{id}_F) : B^{2n} \times F \to E$ is holomorphic along F_{x_0} with respect to J_{aux}, *i.e.* $d_{(0,f)}\phi : \mathbb{C}^n \times T_f F \to T_{\gamma(x,f)}E$ is $\mathbb{C}$-linear for all $f \in F$.

The picture may be rescaled on the base. If $\delta_k(z) = z/\sqrt{k}$ for $z \in \mathbb{C}^n$, define $\widetilde{\chi}_k = \chi \circ \delta_k : \sqrt{k}B^{2n} \to U_0$, [D]. There are product maps

$$\widetilde{\phi}_k : \sqrt{k}B^{2n} \times F \overset{(\widetilde{\chi}_k, \mathrm{id}_F)}{\longrightarrow} U_0 \times F \overset{\gamma}{\hookrightarrow} E.$$

The function $\widetilde{\phi}_k$ maps diffeomorphically onto $p^{-1}(U_0)$, and is holomorphic along F_{x_0} and on $B^{2n} \times F$ we have $\widetilde{\phi}_k^* \widetilde{\omega}_{(k)} = \omega_0 + \sigma + O(1/k)$. One can check arguing as in [D] that it is approximately holomorphic, in the following sense.

LEMMA 2.2. — *Let J_{pr} denote the product complex structure $J_0 \times J_F$ on $\sqrt{k}B^{2n} \times F$, and let $\mu'_k(z,f) : \bigwedge^{1,0}_{J_{\mathrm{pr}}} (\mathbb{C}^n \times T_fF) \to \bigwedge^{0,1}_{J_{\mathrm{pr}}} (\mathbb{C}^n \times T_fF)$, $(z,f) \in \sqrt{k}B^{2n} \times F$, be the bundle morphism relating $\widetilde{\phi}^*_k(J_k)$ to J_{pr}. Then $|\mu'_k| \le C|z|/\sqrt{k}$, $|\nabla\mu'_k| \le C/\sqrt{k}$.*

If $\nu \in H^0(F,L)$, the product $\eta^{\otimes k} \otimes \nu$ may be regarded as a v-holomorphic section of B on $p^{-1}(U_0)$. We may choose $\nu_0 \in H^0(F,L)$ and an open neighbourhood $V_0 \ni f_0$ so that $1/2 \le |\nu_0| \le 1$ on V, $|\nu_0| \le 1/2$ on $F \setminus V_0$ and $|\nu_0(f)| = 1 \Leftrightarrow f = f_0$. The connection matrix θ of ∇_L with respect to the trivialization ν_0 satisfies $\theta(f_0) = 0$.

Let θ_{L_E} and $\widetilde{\theta}$ be the connection matrices of ∇_{L_E} and ∇_B with respect to the trivializations ν_0 and $\eta^{\otimes k} \otimes \nu_0$, respectively. We may assume that $\theta_{L_E}(e_0) = 0$; let ς_0 denote the resulting section of B over U_0. If the t_i's are local coordinates on F centred at f_0 and the $x_1, \cdots, x_{2n}$ are the local coordinates on M centred at x_0 given by the chart χ, in the resulting trivialization on $\widetilde{\chi}_k(B^{2n} \times F)$ we have $\widetilde{\phi}^*_k \theta_B = \theta + A + \beta_k$, where $|\beta_k| = O(1/\sqrt{k})$.

The function $g(z) = \exp(-|z|^2/4)$ is a holomorphic section of the trivial line bundle ξ on $\mathbb{C}^n$ with the connection A, [D]. If β is the standard cut-off function centred at the origin and $\beta_k(z) = \beta(k^{-1/6}|z|)$, then $\varphi_k = \beta_k g$ is the compactly supported, approximately holomorphic section of (ξ, A) constructed in [D]. The following lemma shows that $\vartheta_0(e) = \varphi_k(\widetilde{\chi}_k^{-1}(x))\varsigma_0(e)$, where $e = \gamma(x,f)$, is a good candidate for the seeked concentrated v-holomorphic section of B.

Let us consider, as in [D], the following real function on $M \times M$:

$$\ell_k(x,x') = \begin{cases} e^{-d_k(x,x')^2/5} & \text{if } d_k(x,x') \le k^{1/4} \\ 0 & \text{if } d_k(x,x') > k^{1/4}. \end{cases}$$

LEMMA 2.3. — *If $x = p(e)$ then $|\vartheta_0(e)| \le \ell_k(x,x_0)$. If $d_k(x,x_0) \le k^{1/6}/4$, then $|\vartheta_0(e)| \ge \exp(-d_k(x,x_0)^2/3)|\nu_0(f)|$; in particular, for a fixed $R > 0$ and all $k \gg 0$, if $d_k(x,x_0) \le R$ and $f \in V_0$ then $|\vartheta_0(e)| \ge 1/C$. For all $e \in E$, we have*

$$|\nabla_B \vartheta_0(e)| \le C(1 + d_k(x_0,x))\ell_k(x_0,x),$$

$$|\overline{\partial}_{J_k,B}\vartheta_0(e)| \le Ck^{-1/2}(1 + d_k(x_0,x) + d_k(x_0,x)^2)\ell_k(x_0,x),$$

and

$$|\nabla_B \overline{\partial}_{J_k,B}\vartheta_0(e)| \le Ck^{-1/2}(1 + d_k(x,x_0) + d_k(x_0,x)^2 + d_k(x_0,x)^3)\ell_k(x_0,x).$$

Proof of Lemma 2.3. — We may introduce an additional almost Kähler structure on $E|_U$, as follows. Given the trivialization $\gamma : U \times F \cong E|_U$, for each $e = \gamma(x, f) \in E|_U$ we have $T_eE \cong d_x\gamma_f(T_xE) \oplus V_e$. We define a horizontal distribution $H' \subset TE$ over U by setting $H'_e = d_x\gamma_f(T_xE)$, so that $TE \cong H' \oplus V$. Let us pull back the almost complex structure J_M to an almost complex structure J'_M on H' and then set $J' = J'_M \oplus' J_{\mathrm{vert}}$, where $\oplus'$ is the direct sum with respect to the latter decomposition. By construction $H'_e = H_e$ and so $J_{\mathrm{aux}}(e) = J'(e)\ \forall\ e \in F_{x_0}$. Similarly set $\omega' := \omega \oplus' \sigma$, where ω is implicitly pulled-back to H'. Then ω' is a nondegenerate 2-form on $E|_U$ and $J' \in \mathcal{J}(E|_U, \omega')$. Hence $g' := \omega'(\cdot, J'\cdot)$ is a riemannian metric on $E|_U$ and $g'_k = g_{\mathrm{aux}}$ on F_{x_0}. Let $\mu' = \mu'(x,t) : \bigwedge_{J'}^{1,0} TE \to \bigwedge_{J'}^{0,1} TE$ be the morphism of vector bundles relating J_{aux} to J'. Thus $\mu'(e) = 0$ $\forall\ e \in F_{x_0}$ and so $|\mu'| \leq C|x|$. Let μ'_k be the vector bundle morphism relating $\widetilde{\phi}_k^* J_{\mathrm{aux}}$ to $\widetilde{\phi}_k^* J'$; then $\mu'_k = \delta_k^* \mu_1$, hence $|\mu'_k| \leq C d_k(x, x_0)/\sqrt{k}$ and $|\nabla \mu'_k| < C/\sqrt{k}$. Similarly, replacing ω by $k\omega$ in the above construction but leaving the vertical component σ unchanged, we get non-degenerate 2-forms $\omega_{\mathrm{aux}}^{(k)}$ and $\omega'^{(k)}$, and riemannian metrics $g_{\mathrm{aux}}^{(k)}$ and $g'^{(k)}$; perhaps after restricting U for $k \gg 0$ the corresponding quadratics forms $q_{\mathrm{aux}}^{(k)}$ and $q'^{(k)}$ are equivalent on $E|_U$. In turn, $q_{\mathrm{aux}}^{(k)}$ is equivalent to $q^{(k)}$ (the quadratic form associated to g_k). On the upshot the claimed estimates may be proved using $q'^{(k)}$, by an adaptation of the arguments in [D]. Let us give some detail for ϑ_0 and $\nabla_B\vartheta_0$. As to the former, the claim follows direclty from the definition. As to the latter, the proof is straightforward on the region T where $d_k(x_0, x) \leq k^{1/6}/4$ and $f \in V_0$. Fix $e_1 \notin T$. Let ϑ_1 be a section constructed as above, but with reference point e_1. Then $\vartheta_0 = s\vartheta_1$ near e_1 for a suitable v-holomorphic function s, and therefore $|\nabla_B\vartheta_0(e_1)| = |ds(e_1)|$. The claim easily follows from this.

The estimates on $\overline{\partial}_{J_k,B}\vartheta_0$ and $\nabla_B\overline{\partial}_{J_k,B}\vartheta_0$ also follow by similar arguments, in view of the fact that, up to $(1 - \overline{\mu}'\overline{\mu}'^{-1})$ etc,

$$\overline{\partial}_{J_k,B}\vartheta_0 = \overline{\partial}_{J_{\mathrm{aux}},B}\vartheta_0 - \mu_k(\partial_{J_{\mathrm{aux}},B}\vartheta_0),$$

$$\overline{\partial}_{J_{\mathrm{aux}},B}\vartheta_0 = \overline{\partial}_{J',B}\vartheta_0 - \mu'_k(\partial_{J_{\mathrm{aux}},B}\vartheta_0),$$

$$\partial_{J_{\mathrm{aux}},B}\vartheta_0 = \partial_{J',B}\vartheta_0 - \mu'_k(\overline{\partial}_{J_{\mathrm{aux}},B}\vartheta_0), \qquad \text{[D]}. \qquad \square$$

We now need to describe a suitable open cover of E. This is obtained by locally taking products of open sets in an open cover of M depending on k as in [D] and in a suitable fixed open cover of F. For $k \gg 0$ let $\mathcal{U} = \{U_i\}$ be an open cover of M by a collection of g_k-unit balls U_i, with centres x_i,

$i = 1, \cdots, M_k$, satisfying the properties stated in Lemmas 12 and 16 of *loc. cit.* In particular, for every $e \in E$ and $r = 0, 1, 2, 3$ one has

$$\sum_{i=1}^{M_k} d_k(x_i, x)^r \ell_k(x_i, x) \leq C. \tag{1}$$

For $D > 0$, let $N = CD^{2n}$ and the partition of $I = \bigcup_{\alpha=1}^{N} I_\alpha$, where $I = \{1, \cdots, M_k\}$ be as in the statement of Lemma 16 of *loc. cit.*

For each i fix a trivialization $\gamma_i : U_i \times F \cong E|_{U_i}$. Consider an open cover $\mathcal{V} = \{V_j\}_{j \in J}$ of F, $J = \{1, \cdots, R\}$, by balls of a suitable g_F-radius $\delta > 0$ centred at points $f_j \in V_j$, so that for each j there exists $\nu_j \in H^0(F, L)$ satisfying $1/2 \leq |\nu_j|_{V_j}| \leq 1$ and $|\nu_j(f)| = 1$ if and only if $f = f_j$. We thus obtain an open cover $\mathcal{W} = \{W_{ij}\}$ of E, where $W_{ij} = \gamma_i(U_i \times V_j)$. For each (i, j) there is a v-holomorphic section ϑ_{ij} of B supported near F_{x_i} and peaked at $e_{ij} = \gamma_i((x_i, f_j))$. Partition the index set $I \times J$ as $I \times J = \bigcup_{\alpha, j} I_\alpha \times \{j\}$, which may be rewritten as $I \times J = \bigcup_{\beta=1}^{NR} S_\beta$, where $S_{kN+\alpha} = I_\alpha \times \{k+1\}$, $k = 0, \cdots, R-1$, $1 \leq \alpha \leq N$. Now let us insert the ϑ_{ij}'s in Donaldson's construction. Given any $\vec{w} \in \mathbb{C}^{NR}$, with $|w_\beta| \leq 1$ $\forall \beta$, set $s_{\vec{w}} = \sum_i w_{ij} \vartheta_{ij}$; since $s_{\vec{w}}$ is v-holomorphic, its zero locus $Z_{\vec{w}}$ meets any fibre F_x in a complex subvariety. For any $(i, j) \in I \times J$, the local functions $f_{ij} = s_{\vec{w}} / \vartheta_{ij}$ are defined on W_{ij}, and by Lemma 2.2, when viewed as functions on a suitable multidisc Δ^+ of fixed radius in $\mathbb{C}^{n+d}$, they satisfy properties as in lemmas 18 and 19 of [D]. We may then proceed by adjusting the coefficients w_β's in NR steps to obtain a $\vec{w}_f \in \mathbb{C}^{NR}$, such that $s_{\vec{w}_f}$ satisfies $|\partial_B s_{\vec{w}_f}| > |\bar{\partial}_B s_{\vec{w}_f}|$ on Z_f, so that Z_f is a symplectic submanifold of E. $\square$

Let us prove Corollary 1.1. If L is a holomorphic line bundle on F with $c_1(L) = [\sigma]$, there are an hermitian structure on L and a unitary connection on it whose normalized curvature form is σ. For $r \gg 0$, the action of G on F admits a linearization $\widetilde{\nu} : \widetilde{G} \times L^{\otimes r} \to L^{\otimes r}$ ([M], section 1.3). Let s be the section of $B = L^{\otimes r} \otimes H^{\otimes k}$ for $k > k(r)$ provided by the theorem, Z its zero locus. Given a v-holomorphic line bundle A on E we define its *v-holomorphic direct image*, $p^v_*(A)$, as the sheaf of modules over the ring of smooth functions on M given by $p^v_*(A)(U) = \mathcal{O}^v_E(p^{-1}U, A)$ for any open subset $U \subseteq M$. Then $\mathcal{F} := p^v_*(B)$ is a smooth vector bundle on M of rank $r = h^0(F, L^{\otimes r})$ and $\mathcal{O}^v_E(B) \cong \mathcal{A}(M, \mathcal{F})$, the latter being the space of smooth sections of $\mathcal{F}$. Let V be the vector space of v-holomorphic sections of B spanned by the ϑ_i's and let $W \supseteq V$ be a finite dimensional space of $\mathcal{C}^\infty$ sections of $\mathcal{F}$ that globally generates $\mathcal{F}$. Then $s \in W$ has an open neighbourhood Q consisting of v-holomorphic sections of B whose zero locus is a symplectic submanifold of E. On the other hand, except for those in a subset of W of measure zero the elements of W are transversal to the zero section and this is true in particular for some section $s' \in Q$. But for $r \gg 0$ certainly $\operatorname{rank}(\mathcal{F}) = h^0(F, L^{\otimes r}) > \dim(M)$ and therefore s' is nowhere vanishing. $\square$

Finally let us come to Corollary 1.2. Fix an hermitian metric on $\mathcal{E}$ and thus an associated principal $U(r)$-bundle. With $E = \mathbb{P}\mathcal{E}^*$, L_E is the relative hyperplane line bundle and $p_*^v(L_E) = \mathcal{E}$. Let $\mathcal{H}$ be the connection on L_E induced by the compatible connection on $L = \mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes H^{\otimes k}$, L_E changes to $L_E \otimes p^*(H^{\otimes k})$. When $k \gg 0$ the theorem yields a v-holomorphic section σ of $B = L_E \otimes p^*(H^{\otimes k})$ with zero locus D at each point of which $|\overline{\partial}_{J_k,B}\sigma(e)|_k < Ck^{-1/2}|\partial_{J_k,B}\sigma(e)|_k$, where $|\cdot|_k$ is the norm induced by g_k. By perturbing σ slightly, the section $\widetilde{\sigma}$ of $\mathcal{E} \otimes H^{\otimes k}$ corresponding to it may be assumed transverse, with smooth zero locus $Z \subseteq M$. Now J_{aux} and J_k differ by $O(1/k)$ and $q_{\text{aux}}^{(k)}$ is equivalent to $q^{(k)}$. Thus $|\overline{\partial}_{J_{\text{aux}},B}\sigma(e)|_{\text{aux},k} < |\partial_{J_{\text{aux}},B}\sigma(e)|_{\text{aux},k}$ at all $e \in D$, where $|\cdot|_{\text{aux},k}$ denotes the norm associated to $q_{\text{aux}}^{(k)}$, and therefore $\omega_{\text{aux}}^{(k)}$ restricts to an everywhere non-degenerate 2-form on D. I claim that this implies that Z is a symplectic submanifold of M. If not, there exist $x \in Z$ and $v \in T_xZ$ such that $\omega_x(v, w) = 0 \; \forall\, w \in T_xZ$. The restriction $p|_D : D \to X$ is a $\mathbb{P}^{r-2}$-bundle off Z, while $D_Z = p_D^{-1}(Z)$ is $\mathbb{P}\mathcal{E}^*|_Z$. Identify a tubular neighbourhood of Z in M with a neighbourhood of the zero section in $\mathcal{E}|_Z$. If $v^\perp \subset T_xM$ is the symplectic annhilator of v and $W = E(x) \cap v^\perp$, then $\dim W \geq 2r - 1$ and $\dim W \cap (iW) \geq 2r - 2$, where i is the complex structure of $E(x)$. Thus there is a complex hyperplane Λ of $E(x)$ with $\Lambda \subseteq v^\perp$. If $\lambda \in p^{-1}(x)$ is the corresponding point, $T_\lambda D$ is generated by $T_\lambda D_Z$ and $2(r-1)$ vectors $w_1, \cdots, w_{2r-2}$ projecting to a real basis of Λ. Let $v^\sharp \in \mathcal{H}_\lambda$ be the horizontal lift of v; by construction $v^\sharp$ lies in the kernel of $\omega_{\text{aux}}^{(k)}|_{T_\lambda D}$, a contradiction. Now essentially the same argument as in the proof of Proposition 39 of [D] (with $\omega_{(k)}$ in place of $k\omega$) shows that E is obtained topologically from D by attaching cells of dimension $\geq n + r - 1$, so that by Lefschetz duality $H^k(E \setminus D) = 0$ for $k \geq n + r$. Since $E \setminus D$ is a $\mathbb{C}^{r-1}$-bundle over $M \setminus Z$, this implies $H_j(M, Z) = 0$ for $j \leq n - r$ (*cf.* [S] and [L], §1). $\square$

We now examine the almost complex geometry of the sections of $\mathcal{E} \otimes H^{\otimes k}$ produced in Corollary 1.2. Let us write $\mathcal{F}$ for $\mathcal{E} \otimes H^{\otimes k}$ and, in the notation of the proof, fix $x \in Z$ and a unitary frame $f_1, \cdots, f_r$ for $\mathcal{F}$ in a neighbourhood U of x. Then $\widetilde{\sigma} = \sum_i a_i f_i$, where the a_i's are smooth functions and $Z \cap U = \{a_i = 0 \,\forall i\}$. Therefore $\nabla_{\mathcal{F}} \widetilde{\sigma}(x) = \sum_i d_x a_i \otimes f_i(x)$ and so $\partial_{J,\mathcal{F}} \widetilde{\sigma}(x) = \sum_i \partial_J a_i(x) \otimes f_i(x)$, $\overline{\partial}_{J,\mathcal{F}} \widetilde{\sigma}(x) = \sum_i \overline{\partial}_J a_i(x) \otimes f_i(x)$ whence $\|\partial_{J,\mathcal{F}} \widetilde{\sigma}(x)\|^2 = \sum_i \|\partial_J a_i(x)\|^2$, $\|\overline{\partial}_{J,\mathcal{F}} \widetilde{\sigma}(x)\|^2 = \sum_i \|\overline{\partial}_J a_i(x)\|^2$. Given that $B = \mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1)$, we have on $\mathbb{P}(\mathcal{E}^*) = \mathbb{P}(\mathcal{F}^*)$ the short exact sequence $0 \to \Omega^1{}_{\text{rel}} \otimes B \to \pi^*(\mathcal{F}) \xrightarrow{\alpha} B \to 0$, where $\Omega^1{}_{\text{rel}}$ is the relative cotangent bundle. In loose notation, on $\pi^{-1}(U)$ we have $\sigma = \alpha(\widetilde{\sigma}) = \sum_i a_i F_i$, where $F_i = \alpha(f_i)$. At any $e \in \pi^{-1}(x)$, we have $\nabla_B \sigma(e) = \sum_i d_x a_i \otimes F_i(e)$, and therefore $\partial_{J_{\text{aux}},B} \sigma(e) = \sum_i \partial_{J_{\text{aux}}} a_i(x) \otimes F_i(e)$, $\overline{\partial}_{J_{\text{aux}},B} \sigma(e) = \sum_i \overline{\partial}_{J_{\text{aux}}} a_i(x) \otimes F_i(e)$. Now $\|\overline{\partial}_{J_{\text{aux}},B} \sigma(e)\|_{\text{aux},k} < C k^{-1/2} \|\partial_{J_{\text{aux}},B} \sigma(e)\|_{\text{aux},k}$ at every $e \in \mathbb{P}(\mathcal{F}^*_x)$. For $i = 1, \cdots, r$ let $e_i \in \mathbb{P}(\mathcal{F}^*_x) \cong \mathbb{P}^{r-1}$ be the point where all the F_j's except F_i vanish. Evaluating the latter inequality at e_i, we obtain $\|\overline{\partial}_{J_{\text{aux}}} a_i(x)\|_{\text{aux},k} < C k^{-1/2} \|\partial_{J_{\text{aux}}} a_i(x)\|_{\text{aux},k}$ and thus $\|\overline{\partial}_{J_M} a_i(x)\| < C k^{-1/2} \|\partial_{J_M} a_i(x)\|$ on M for every i, whence $\|\overline{\partial}_{J,\mathcal{F}} \widetilde{\sigma}(x)\| < C k^{-1/2} \|\partial_{J,\mathcal{F}} \widetilde{\sigma}(x)\|$. In fact, we also know that $\|\partial_{J_{\text{aux}},B} \sigma(e)\|_{\text{aux},k} > \eta$ at all $x \in D$ for some $\eta > 0$ independent of k, and the argument just given then shows that $\|\partial_{J,\mathcal{F}} \widetilde{\sigma}(x)\| > \eta$ for all $x \in Z$.

Furthermore, these sections are asymptotically almost holomorphic in the sense of [A]. By construction, $\sigma = \sum_{i,j} w_{ij} e_j \otimes \sigma_i$, where $|w_{ij}| \leq 1$ for all i, j, while the σ_i's are compactly supported sections of $H^{\otimes k}$ as in Proposition 11 of [D], and the e_j's are local sections of $\mathcal{E}$, chosen once for all and thus independent of k. A slight modification of the arguments proving Lemma 14 of [D] then leads to the estimates stated in Definition 1 of [A].

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