

**A SPLITTING THEOREM  
FOR THE KUPKA COMPONENT  
OF A FOLIATION OF  $\mathbb{C}\mathbb{P}^n$ ,  $n \geq 6$ .  
Addendum to an addendum to a paper  
by Calvo-Andrade and Soares**

by Edoardo BALLICO

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A codimension one singular holomorphic foliation  $F$  of  $\mathbb{C}\mathbb{P}^n$  is given by  $\omega \in H^0(\mathbb{C}\mathbb{P}^n, \Omega(k))$  (for some  $k$ ) with  $\omega \neq 0$ ,  $\omega$  not vanishing on a hypersurface. The Kupka subset  $K(F) := \{P \in \mathbb{C}\mathbb{P}^n : \omega(P) = 0, d\omega(P) \neq 0\}$  of the singular set  $S(F) := \{P \in \mathbb{C}\mathbb{P}^n : \omega(P) = 0\}$  of  $F$  has remarkable properties (e.g. if not empty it is a smooth submanifold of pure codimension 2 with strong stability properties with respect to deformations of  $F$ ). For much more on this topic, see [GL] and [GS]. Let  $K \neq \emptyset$  be a Kupka component of  $F$ , i.e. ([CS]) a connected component of  $K(F)$ . It was proved in [CL] that if  $K$  is a complete intersection, then  $F$  has a meromorphic first integral. Motivated by this result in [CS] it was conjectured and proved in some cases that every Kupka component,  $K$ , is a complete intersection. This conjecture was proved in [B] under the assumption that  $\deg(K)$  is not a square. Here we will remove this restrictive assumption and prove the following result.

**THEOREM.** — *Let  $F$  be a codimension 1 singular holomorphic foliation of  $\mathbb{C}\mathbb{P}^n$ ,  $n \geq 6$ , induced by  $\omega \in H^0(\mathbb{C}\mathbb{P}^n, \Omega^1(k))$  and such that the*

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*Keywords:* Singular foliations – Codimension 1 foliations – Kupka component – Complete intersection – Unstable vector bundle – Rank 2 vector bundle – Splitting of a vector bundle – Meromorphic first integral – Barth-Lefschetz theorems.

*Math. classification:* 58F18 – 14F05 – 14M07.

*codimension 2 component of the singular set of  $F$  contains a Kupka component  $K$ . Then  $K$  is a complete intersection and  $F$  has a meromorphic first integral.*

Let  $N_K$  be the normal bundle of  $K$  in  $\mathbb{C}\mathbb{P}^n$ . By [CS], Corollary 3.5,  $N_K$  is the restriction  $E|_K$  to  $K$  of a rank 2 vector bundle  $E$  on  $\mathbb{C}\mathbb{P}^n$ .  $K$  is a complete intersection if and only if  $E$  is the direct sum of two line bundles ([OSS]). If  $n \geq 6$  every line bundle on  $K$  is the restriction of a line bundle on  $\mathbb{C}\mathbb{P}^n$  (see [FL]). Hence, by a very nice result of Faltings ([F]) if  $n \geq 6$  and  $N_K$  is the direct sum of two line bundles,  $K$  is a complete intersection. Now we use the notations of [GL], page 320. There are two complex numbers (not both zero) such that the linear part of the foliation near each point of  $K$  depends from these two complex numbers. Up to a normalization we may assume that one of these numbers is 1; call  $\mu$  the other one. If  $\mu \neq \pm 1$ , then  $N_K$  splits into the direct sum of two line bundles ([GL]), Remark 2 at p. 320). If  $\mu = -1$ , then there is a two-to-one unramified covering  $\pi : Z \rightarrow K$  such that  $\pi^*(N_K)$  is the direct sum of two isomorphic line bundles (see [GL], the two lines before eq. (2.6)). Since  $K$  is simply connected ([FL]), Cor. 6.3),  $N_K$  splits into the direct sum of two isomorphic line bundles. Now assume  $\mu = 1$ . This is the only difference with respect to [B]. By [GL], eq. (2.12) and the two following lines,  $N_K$  is an extension of a line bundle  $L_1$  by itself. Since  $n \geq 6$  we have  $H^1(K, \mathbb{C}) \cong \mathbb{C}$  by a theorem of Barth's (see e.g. [FL], eq. (\*\*)) at p. 83). Hence by standard Hodge theory ([GH], bottom of p. 105) we have  $H^1(K, \mathcal{O}_K) = 0$ . Thus any such extension of  $L_1$ , by itself splits and hence  $N_K \cong L_1 \oplus L_1$ , concluding the proof of the theorem.

The author was partially supported by MURST and GNSAGA of CNR (Italy).

## BIBLIOGRAPHY

- [B] E. BALLICO, A splitting theorem for the Kupka component of a foliation of  $\mathbb{C}\mathbb{P}^n$ ,  $n \geq 6$ . Addendum to a paper by Calvo-Andrade and Soares, Ann. Inst. Fourier, 45-4 (1995), 1119-1121.
- [CS] O. CALVO-ANDRADE, M. SOARES, Chern numbers of a Kupka component, Ann. Inst. Fourier, 44-4 (1994), 1219-1236.
- [CL] D. CERVEAU, A. LINS, Codimension one foliations in  $\mathbb{C}\mathbb{P}^n$   $n \geq 3$ , with Kupka components, in Complex analytic methods in dynamical systems, Astérisque, (1994), 93-133.
- [F] G. FALTINGS, Ein Kriterium für vollständige Durchschnitte, Invent. Math., 62 (1981), 393-401.

- [FL] W. FULTON, R. LAZARSEFELD, Connectivity in algebraic geometry, in Algebraic Geometry, Proceedings Chicago 1980, Lect. Notes in Math. 862, Springer-Verlag (1981), 26–92.
- [GN] X. GOMEZ-MONT, A. LINS-NETO, A structural stability of foliations with a meromorphic first integral, *Topology*, 30 (1990), 315–334.
- [GH] Ph. GRIFFITHS, J. HARRIS, Principles of Algebraic Geometry, John Wiley & Sons, 1978.
- [OSS] Ch. OKONEK, M. SCHNEIDER, H. SPINDLER, Vector Bundles on Complex Projective spaces, Progress in Math., 3, Birkhäuser, Basel, 1978.

Manuscrit reçu le 1er octobre 1998,  
accepté le 12 février 1999.

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