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SOME REMARKS ON JAEGER'S DUAL-HAMILTONIAN CONJECTURE

by Bill JACKSON and C.A. WHITEHEAD

We consider finite simple graphs without loops or multiple edges. Let G be a connected graph. A *cocircuit* of G is a minimal set of edges of G whose removal disconnects G. Thus X is a cocircuit of G if and only if $G - X$ has exactly two components and every edge of X is incident with both components. It follows that if X is a cocircuit then $|X| \leq |E|-|V|+2$, with equality if and only if both components of $G - X$ are trees. We shall say that X is a *Hamilton cocircuit* of G if $|X| = |E|-|V|+2$ and, following Jaeger [3], that G is *dual hamiltonian* if G has a Hamilton cocircuit. These definitions are motivated by matroid duality, from which it follows that a plane graph is hamiltonian if and only if its dual graph is dual hamiltonian. Thus conditions implying hamiltonicity in plane graphs can be readily translated to conditions implying dual hamiltonicity in plane graphs. We will be concerned with a conjecture of Jaeger [3] which suggests that one such condition for dual hamiltonicity generalises to non-planar graphs. To state his conjecture we need one further definition. Suppose that G contains two edge disjoint circuits. Then G is said to be *cyclically k-connected* if whenever we partition G into two subgraphs H_1 and H_2 both of which contain circuits, we have $|V(H_1) \cap V(H_2)| \geq k$. Thus, if G is cyclically k-connected, then each circuit of G must have length at least k. The concept of cyclic connectivity is the matroid dual to the 'standard' definition of graph connectivity, see [4]. In particular, a plane graph is k-connected if and only if its dual is cyclically k-connected. It follows that the classical result of

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Whitney [7] that every 4-connected plane triangulation is hamiltonian is equivalent to:

THEOREM 1. — *Every cyclically 4-connected planar cubic graph is dual hamiltonian.*

The above mentioned conjecture of Jaeger is that this result remains valid for non-planar graphs.

CONJECTURE 1. — *Every cyclically 4-connected cubic graph is dual hamiltonian.*

Using Tutte's generalization [6] of Whitney's Theorem, we may deduce that every cyclically 4-connected planar graph is dual hamiltonian. It is not true, however, that this more general result remains valid for non-planar graphs. To see this, consider the graph G whose vertex set is the union of five disjoint independent sets of size three, $S_1, S_2, \ldots, S_5$, in which each vertex of S_i is adjacent to every vertex of S_{i+1} for $1 \leq i \leq 5$, where subscripts are to be read modulo five. Then G is cyclically 4-connected since G is 4-connected and has girth 4. Suppose G is dual hamiltonian. Then $V(G)$ can be partitioned into two subsets T_1, T_2, each of which induces a tree in G. Without loss of generality we may assume $|T_1 \cap S_1| \geq 2$. Then $|T_2 \cap S_2| \geq 2$, otherwise $T_1 \cap (S_1 \cup S_2)$ will induce a 4-circuit in G. Continuing this reasoning we eventually deduce that $|T_1 \cap S_5| \geq 2$. But then $T_1 \cap (S_1 \cup S_5)$ will induce a 4-circuit in G.

For the remainder of this note, we shall restrict our attention to cubic graphs. For such graphs the concept of cyclic 4-connectivity can be simplified as follows: a cubic graph $G \neq K_4$ is cyclically 4-connected if and only if the only edge cuts in G with fewer than four edges are obtained by taking three edges incident with some vertex. Jaeger's main concern in [3] involved the parameter $s(G)$, which he defined to be the size of a largest induced forest in G. He showed that, if G is a cubic graph, then $s(G) \leq \lfloor (3|V| - 2)/4 \rfloor$, and furthermore that if G is dual hamiltonian then equality holds. He was led to make Conjecture 1 by the result of Payan and Sakarovitch [5] that $s(G) = \lfloor (3|V| - 2)/4 \rfloor$ for all cyclically 4-connected cubic graphs G. (There are many conjectures which would imply that the family of cyclically 4-connected cubic graphs has nicer properties than the entire family of 3-connected cubic graphs. The above mentioned result of Payan and Sakarovitch is the only solid evidence we know of in support of this statement.) In the remainder of this note we shall make several remarks on Conjecture 1.

The following result shows that Conjecture 1 is equivalent to a conjecture concerning an edge-partition of a cubic graph.

LEMMA 2. — *A cubic graph G is dual hamiltonian if and only if the edge set of G can be partitioned into two trees.*

Proof. — Suppose G is dual hamiltonian. Then $V(G)$ can be partitioned into two induced trees T_1, T_2. Let H be the spanning subgraph of G containing all edges which join T_1 and T_2. Then H is a bipartite graph of maximum degree two and so has a two edge colouring E_1, E_2. Then $E(T_1) \cup E_1$, $E(T_2) \cup E_2$ is the required partition of $E(G)$ into two trees.

Suppose, on the other hand, that $E(G)$ has a partition T_1', T_2' into two trees. Let T_i be the tree obtained from T_i' by deleting all its end vertices. Then T_1, T_2 is the required partition of $V(G)$ into two induced trees. □

It follows that Conjecture 1 is equivalent to:

CONJECTURE 2. — *The edge set of every cyclically 4-connected cubic graph G can be partitioned into two trees.*

Given a cubic graph G and $e \in E(G)$, let G_e be the cubic (multi)-graph obtained from $G - e$ by suppressing its two vertices of degree two. One may try to use the following lemma as a basis for an inductive proof of Conjecture 1.

LEMMA 3. — *Let G be a cubic graph and $e = v_1v_2$ be an edge of G. Let e_i be the edge of G_e 'containing' v_i for $1 \leq i \leq 2$. Suppose G_e has a Hamilton cocircuit containing e_1. Then G has a Hamilton cocircuit containing e.*

Proof. — Let T_1, T_2 be a partition of $V(G_e)$ into two induced trees such that e_1 is incident with both T_1 and T_2. We may suppose without loss of generality that either e_2 is also incident with both T_1 and T_2 or that e_2 is an edge of the tree induced by T_2. In either case $T_1 + v_1$, $T_2 + v_2$ is the required partition of $V(G)$. □

An obvious approach to using this lemma would be to try to use it to prove:

CONJECTURE 3. — *Let G be a cyclically 4-connected cubic graph and $e \in E(G)$. Then G has a Hamilton cocircuit containing e.*

This approach fails if G_e is not cyclically 4-connected. We have tried to get around this problem by working with various more complicated induction hypotheses, in line with either Conjecture 1 or 2. One such attempt is described below.

The problem with using Lemma 3 occurs when G_e no longer belongs to the family of graphs for which the inductive hypothesis applies. To get round this we could try to work with a larger family than the family of cyclically 4-connected cubic graphs. To define such a family we proceed as follows.

Let G be a 3-connected cubic graph. Suppose that G is not cyclically 4-connected and is not isomorphic to K_4. Let X be a 'non-trivial' 3-edge cut in G and H_1, H_2 the two components of $G - X$. For $1 \leq i \leq 2$ let G_i be the 3-connected cubic graph obtained from G by contracting H_{3-i} to a new vertex v. We shall refer to v as a *marker vertex* in G_1 and G_2. We now iterate this procedure for both G_1 and G_2. We continue until we obtain a collection of cubic graphs S each of which is either cyclically 4-connected or else is isomorphic to K_4. We shall refer to these graphs as *pieces* of G. Note that each 'non-trivial' 3-edge cut of G will be represented by a marker vertex in exactly two pieces of G. We define a new graph D whose vertices are the pieces of G, and in which two pieces are joined by an edge if they have a marker vertex in common. It follows from the decomposition theory developed by Cunningham and Edmonds in [1] that D is a tree and that the set of pieces S and the tree D are uniquely defined by G. We shall refer to D as the *decomposition tree* of G and the pieces of G which correspond to end-vertices of D as *end-pieces* of G. We believe that if G is a 3-connected cubic graph whose decomposition tree has maximum degree at most three then G is dual hamiltonian. The bound on the maximum degree of the decomposition tree comes from the construction described below.

One may construct a 3-connected cubic graph H which is not dual hamiltonian by taking any 3-connected cubic graph G on at least six vertices and 'blowing up' each vertex v of G into a triangle C_v. To see this we use Lemma 2. Suppose $E(H)$ can be partitioned into two trees T_1, T_2. Since each triangle C_v in H must contain two edges of one tree and one edge of the other tree we see that $P_1 = T_1 \cap E(G)$, $P_2 = T_2 \cap E(G)$ is a partition of $E(G)$ into two connected spanning subgraphs of maximum degree two. This is clearly impossible since G is cubic and has more than four vertices. If we begin with the triangular prism G_0 and apply this construction recursively we obtain an infinite sequence of 3-connected cubic graphs $G_0, G_1, \ldots$ such that, for $i \geq 1$, G_i is not dual hamiltonian and has a decomposition tree of maximum degree four. (The decomposition tree for G_0 is K_2. The decomposition tree for G_{i+1} can be obtained from that of G_i by attaching three new leaves to each of its end-vertices.)

One can try to apply Lemma 3 inductively to the family of 3-connected cubic graphs whose decomposition tree has maximum degree three by formulating a conjecture as follows:

CONJECTURE 4. — *Let G be 3-connected cubic graph whose decomposition tree D has maximum degree at most three and e be an edge of G which lies in an end piece of G. Then G has a Hamilton cocircuit which contains e.*

Adopting the notation of Lemma 3, it can be seen that G_e will be a 3-connected cubic graph with a decomposition tree D of maximum degree at most three. Furthermore, either e_1 or e_2 will belong to an end piece of G_e unless the end piece G_1 of G which contains e is isomorphic to K_4 (that is to say e belongs to a triangle of G), and the unique neighbour of G_1 in D has degree three. Unfortunately, if this second alternative occurs, then both e_1 and e_2 will belong to a piece of G_e which has degree two in the decomposition tree of G_e and the inductive argument again fails.

Working with Yu [2], the first named author has recently verified Conjecture 4 for planar graphs by proving the dual statement to:

THEOREM 4. — *Let G be a 3-connected planar cubic graph. Suppose that G has a decomposition tree D of maximum degree at most three. Let G_i be a piece of G corresponding to a vertex of D of degree at most two, and e, f be edges both incident with a vertex of G_i which is not a marker vertex. Then G has a Hamilton cocircuit through e and f.*

Although this result gives some evidence in favour of Conjecture 4, its current proof is discouraging since it makes even greater use of planarity than Whitney's original theorem.

BIBLIOGRAPHY

[1] W.H. CUNNINGHAM and J. EDMONDS, A combinatorial decomposition theory, Canadian J. Math., 32 (1980), 734–765.

[2] B. JACKSON and X. YU, Hamilton cycles in plane triangulations, submitted.

[3] F. JAEGER, On vertex induced forests in cubic graphs, Proc. Fifth Southeastern Conf. on Combinatorics, Graph Theory and Computing, Utilitas Mathematica, Winnipeg (1974), 501–512.

[4] J.G. OXLEY, Matroid Theory, Oxford Univ. Press, Oxford, 1992.

[5] C. PAYAN and M. SAKAROVITCH, Ensembles cycliquement stables et graphes cubiques, Cahiers du C.E.R.O., 17 (1975), 319–343.

[6] W. T. TUTTE, A theorem on planar graphs, Trans. Amer. Math. Soc., 82 (1956), 99–116.

[7] H. WHITNEY, A theorem on graphs, Ann. of Math., 32 (1931), 378–390.

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