

ABOUT G -BUNDLES OVER ELLIPTIC CURVES

by Yves LASZLO(*)

1. Introduction.

In this note, we study principal bundles over a complex elliptic curve X with reductive structure group G . As in the vector bundle case, we first show that a non semistable bundle has a canonical semistable L -structure with L some Levi subgroup of G reducing the study of G -bundles to the study of semistable bundles (Proposition 3.2). We then look at the coarse moduli space M_G of topologically trivial semistable bundles on X (there is not any stable topologically trivial G -bundle) and prove that it is isomorphic to the quotient $[\Gamma(T) \otimes_{\mathbf{Z}} X]/W$ where $\Gamma(T)$ is the group of one parameter subgroups of a maximal torus T and $W = N(G, T)/T$ is the Weyl group (Theorem 4.16). Suppose that G is simple and simply connected and let θ be the longest root (relative to some basis $(\alpha_1, \dots, \alpha_l)$ of the root system $\Phi(G, T)$). The coroot θ^\vee of θ has a decomposition $\theta^\vee = \sum_i g_i \alpha_i^\vee$ with g_i a positive integer. Using Theorem 4.16 and Looijenga's isomorphism

$$[\Gamma(T) \otimes_{\mathbf{Z}} X]/W \xrightarrow{\sim} \mathbf{P}(1, g_1, \dots, g_l),$$

one gets that M_G is isomorphic to the weighted projective space $\mathbf{P}(1, g_1, \dots, g_l)$ (see 4.17), generalizing the well-known isomorphism $M_{\mathbf{SL}_{l+1}} \xrightarrow{\sim} \mathbf{P}^l$ (see [T] for instance). One recovers for instance the Verlinde formula in this case.

We know that these results are certainly well-known from experts, but we were unable to find any reference in the literature, except of course when G is either \mathbf{SL} or \mathbf{GL} . For another point of view, see [BG].

During the referee process of this paper, an independent paper of Friedman, Morgan and Witten has appeared in Duke's eprints (see [FMW]), where the link between Looijenga's result and bundles on elliptic curves is studied from another point of view.

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Notations. — By scheme, we implicitly mean a complex scheme. Let G be a reductive group with a Borel subgroup (resp. a maximal torus) B (resp. $T \subset B$). The corresponding Lie algebras will be denoted by \mathfrak{t} , \mathfrak{b} and \mathfrak{g} .

We denote by $W = N(G, T)/T$ the Weyl group and by $\Gamma(T)$ the W -module $\text{Hom}(G_m, T)$.

2. Review on the Harder-Narasimhan reduction.

Let X be an algebraic curve which is smooth, projective and connected and E be a G -bundle on X . Recall that E is semistable if and only if the adjoint bundle $\mathcal{E} = \text{Ad}(E)$ is a semistable vector bundle. Following [AB], let me recall how to define the Harder-Narasimhan reduction of E . Pick a non degenerate invariant quadratic form q on the Lie algebra of G . Then q defines a non degenerate quadratic form on \mathcal{E} .

LEMMA 2.1. — *The length $r - 1$ of the Harder-Narasimhan filtration*

$$0 = \mathcal{E}_0 \subset \dots \subset \mathcal{E}_i \subset \dots \subset \mathcal{E}_r = \mathcal{E}$$

of \mathcal{E} is even.

Proof. — Let

$$0 = \mathcal{E}_0 \subset \dots \subset \mathcal{E}_i \subset \dots \subset \mathcal{E}_r = \mathcal{E}$$

be the Harder-Narasimhan filtration of \mathcal{E} . The Harder-Narasimhan filtration of \mathcal{E}^* is

$$0 \subset \dots \subset (\mathcal{E}/\mathcal{E}_{r-i})^* \subset \dots \subset \mathcal{E}^*.$$

Because \mathcal{E} is self-dual, the Harder-Narasimhan filtration is self-dual and one has an isomorphism

$$(2.1) \quad \mathcal{E}_i \xrightarrow{\sim} (\mathcal{E}/\mathcal{E}_{r-i})^*$$

inducing isomorphisms

$$(2.2) \quad gr_i \xrightarrow{\sim} (gr_{r+1-i})^*, i = 1, \dots, r$$

where $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$. Assume that the length $r - 1$ is odd. Let us consider the morphisms

$$m_k : \mathcal{E}_{r/2} \otimes \mathcal{E}_k \rightarrow \mathcal{E}/\mathcal{E}_{k-1}, 0 \leq k \leq r - 1$$

deduced from the Lie bracket of \mathcal{E} . The equality (2.2) gives the inequalities

$$(2.3) \quad \mu_1 > \dots > \mu_{r/2} > -\mu_{r/2} > \dots > -\mu_1$$

where μ_i is the slope of the semistable vector bundle $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$. In particular, the slopes $\mu_i + \mu_j$ of the subquotients $gr_i \otimes gr_j, i \leq r/2$ and $j \leq k$ which appear in $\mathcal{E}_{r/2} \otimes \mathcal{E}_k$ are not less than $\mu_{r/2} + \mu_k > \mu_k$ though the slopes of the subquotients $gr_i, i \geq k$ which appear $\mathcal{E}/\mathcal{E}_{k-1}$ are $\leq \mu_k$. This shows that m_k is zero for all k and that all elements of $\mathcal{E}_{r/2}$ are nilpotent. By (2.1), this algebra is also lagrangian. Suppose that the center of G is of positive dimension. Then \mathcal{E} contains a trivial sub-bundle (of positive rank) as a direct summand which implies that some of the μ_i 's is zero, contradicting (2.3). The Lie algebra bundle \mathcal{E} is therefore semisimple and therefore has no non trivial lagrangian sub-Lie algebra (with respect of the Killing form form) consisting of nilpotent elements, just by a dimension argument. \square

It follows that one can index the Harder-Narasimhan filtration of \mathcal{E} such that

$$(2.4) \quad 0 = \mathcal{E}_{-r} \subset \mathcal{E}_{-r+1} \subset \dots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \dots \subset \mathcal{E}_{r-1} = \mathcal{E}$$

where \mathcal{E}_{-j} is the orthogonal of \mathcal{E}_{j-1} . One checks that \mathcal{E}_0 is a subalgebra of \mathcal{E} . Notice that $\mathcal{E}_0/\mathcal{E}_{-1}$ is self-dual and therefore has slope zero. In particular, the slope of $gr_{-j}, j > 0$ is > 0 . As in the proof of the preceding lemma, this immediately implies the sequence of inclusions

$$(2.5) \quad [\mathcal{E}_{-j}, \mathcal{E}_{-1}] \subset \mathcal{E}_{-j-1} \text{ for all } j.$$

In particular, all elements of \mathcal{E}_{-1} are nilpotent. For sake of completeness, let me prove this easy lemma (cf. Th VIII.10.1 of [Bo]).

LEMMA 2.2. — *Let \mathfrak{g} be a reductive algebra endowed with an invariant non degenerate bilinear form. Let \mathfrak{n}' be a subalgebra of \mathfrak{g} whose elements are nilpotent. Then, if the orthogonal of \mathfrak{n}' is a sub-Lie algebra of \mathfrak{g} , it is parabolic.*

Proof. — The Lie algebra \mathfrak{n}' is nilpotent. Let \mathfrak{b} be a maximal solvable Lie subalgebra of \mathfrak{g} containing \mathfrak{n}' and \mathfrak{n} its nilpotent ideal. By [Bo], Definition VIII.3.3.1, \mathfrak{b} is a Borel subalgebra of \mathfrak{g} . Because all elements of \mathfrak{n}' are nilpotent, \mathfrak{n}' is contained in \mathfrak{n} . By [Bo], proposition VII.1.3.10 (iii), the orthogonal of \mathfrak{n} is \mathfrak{b} and the lemma follows. \square

Because the orthogonal of \mathcal{E}_{-1} is \mathcal{E}_0 , the Lie subalgebra \mathcal{E}_0 is therefore parabolic with radical \mathcal{E}_{-1} (see [AB], p. 589). Let P be the unique standard

parabolic subgroup defined by \mathcal{E}_0 . If F is the bundle of local trivialization $G_S \xrightarrow{\sim} E_S$ ($S \rightarrow X$ étale) whose differential sends $\text{Lie}(P)_S$ to \mathcal{E}_0 , then F is a P -structure of E . Let us denote by U the unipotent radical of P and by \bar{P} (resp. \bar{F}) the quotient P/U (resp. F/U). By construction, \bar{F} is semistable (because $\text{Ad}(\bar{F}) = \mathcal{E}_0/\mathcal{E}_{-1}$).

DEFINITION 2.3. — *With the notation above, the P -bundle F is the Harder-Narasimhan reduction of E .*

Remark 2.4. — It is easy to check that the filtration and therefore the corresponding reduction does not depend on the particular choice of the invariant non degenerate quadratic form on $\text{Lie}(G)$.

Example 2.5. — Suppose that E is the \mathbf{GL}_n -bundle of local frames of a vector bundle \mathcal{E} on X with Harder-Narasimhan filtration $0 = \mathcal{E}_0 \dots \subset \mathcal{E}_i \dots \subset \mathcal{E}_k = \mathcal{E}$. Let P be the quasi-triangular subgroup of \mathbf{GL}_n -defined by the partition $[r_i = \text{rk}(\mathcal{E}_{i+1}) - \text{rk}(\mathcal{E}_i)]_{0 \leq i < k}$ of $\text{rk}(\mathcal{E})$. Then F is the P -bundle of local frames compatible with the filtration and \bar{F} is the $\times_i \mathbf{GL}_{r_i}$ -bundle of local frames of $\bigoplus_i \mathcal{E}_{i+1}/\mathcal{E}_i$.

We suppose once for all that X is an elliptic curve.

3. Non semistable G -bundles.

Let E be a G -bundle on X and let F be the Harder-Narasimhan reduction of E . Let us consider a Levi factor of P thought as a section $\sigma : \bar{P} \rightarrow P$ of the canonical projection $P \twoheadrightarrow \bar{P} = P/U$.

Remark 3.1. — Following Humphreys (see [Hu], 30.2) a Levi factor is a factor of the unipotent radical and not of the radical itself (as in Bourbaki for instance).

PROPOSITION 3.2. — *With the notations above, the P -bundle $\sigma_*(\bar{F})$ is isomorphic to F .*

Remark 3.3. — This is the generalization of the well-known (and easy) fact that any vector bundle on X is a direct sum of semistable vector bundles.

Proof. — Let us denote by

$$1 \rightarrow \mathcal{U} \rightarrow \mathcal{P} \rightarrow \bar{\mathcal{P}} \rightarrow 1$$

be the twist of

$$1 \rightarrow U \rightarrow P \rightarrow \bar{P} \rightarrow 1$$

by F (see [S], chap. I & 5). Geometrically, \mathcal{P} (resp. $\bar{\mathcal{P}}$) is the group scheme $\mathcal{A}ut_P(F)$ (resp. $\mathcal{A}ut_{\bar{P}}(\bar{F})$). The twisted group \mathcal{U} is the unipotent radical of \mathcal{P} and is isomorphic $F \times_P U$ (P acts on the normal subgroup U by conjugation). As usual, the map

$$\left\{ \begin{array}{ccc} H^1(X, P) & \rightarrow & H^1(X, \mathcal{P}) \\ F'' & \mapsto & \mathcal{I}som_P(F, F'') \end{array} \right. \text{ resp. } \left\{ \begin{array}{ccc} H^1(X, \bar{P}) & \rightarrow & H^1(X, \bar{\mathcal{P}}) \\ \bar{F}'' & \mapsto & \mathcal{I}som_{\bar{P}}(\bar{F}, \bar{F}'') \end{array} \right.$$

are bijective. The image of $\mathcal{I}som_P(F, \sigma_*\bar{F})$ in $H^1(X, \bar{\mathcal{P}})$ is the trivial torsor $\mathcal{I}som_{\bar{P}}(\bar{F}, \bar{F})$ and it is enough to show the equality $H^1(X, \mathcal{U}) = \{[\mathcal{U}]\}$ to prove the isomorphism $F \xrightarrow{\sim} \sigma_*(\bar{F})$. With the notations of (2.4), the Lie algebra of \mathcal{U} is \mathcal{E}_{-1} . By (2.5), the Lie algebra $\mathcal{E}_{-j}/\mathcal{E}_{-j-1}$ is abelian for any $j \geq 1$. This induces a filtration

$$1 = \mathcal{U}_{-r} \subset \mathcal{U}_{-r+1} \subset \dots \subset \mathcal{U}_{-2} \subset \mathcal{U}_{-1} = \mathcal{U}$$

by unipotent group schemes where the exponential defines isomorphisms

$$\mathcal{U}_{-j}/\mathcal{U}_{-j-1} \xrightarrow{\sim} \mathcal{E}_{-j}/\mathcal{E}_{-j-1} \quad j \geq 1$$

of abelian group schemes. By construction, $\mathcal{E}_{-j}/\mathcal{E}_{-j-1}, j \geq 1$ is semistable of positive slope and therefore

$$H^1(X, \mathcal{E}_{-j}/\mathcal{E}_{-j-1}) = 0, j \geq 1$$

because $g(X) = 1$. This implies the equality

$$H^1(X, \mathcal{U}) = \{[\mathcal{U}]\}.$$

□

4. The coarse moduli space M_G .

Let M_G be the coarse moduli space of semistable G -bundle of trivial topological type (what is the same, the component containing the trivial torsor G_X). Recall that the (closed) points of M_G are S -equivalence classes of semistable G -bundles. The only thing which will be needed about this equivalence relation is the following (cf. [Ra1], Corollary 3.12.1):

4.1. Every class ξ defines a Levi subgroup L such that there exists a stable L -bundle F with $F(G) \in \xi$. Moreover, $F(G)$ is well defined up to isomorphism.

Remark 4.2. — Ramanathan’s construction of M_G is written for a curve of genus ≥ 2 , but the construction can be made in general (see for instance [LeP] in the case of $G = GL_n$ from which the general case follows).

4.3. We denote by $a \otimes b$ the product of two T -bundles a and b (for the natural structure of abelian group of $H^1(X, T)$). Let $\underline{\psi} = (\psi)_{i \in I}$ be a finite family of one parameter subgroups and $\underline{L} = (L_i \in I)$ a family of line bundles of degree 0 on X (thought as G_m -torsors). Then, $\otimes_{i \in I} L_i(\psi_i)$ is a T -structure of a G -bundle $\underline{L}_{\underline{\psi}}$ on X which is semistable. This defines a morphism of abelian groups

$$p : \Gamma(T) \otimes_{\mathbf{Z}} X \rightarrow H^1(X, T).$$

Chose a (closed) point x of X which defines an isomorphism $\text{Pic}^0(X) \xrightarrow{\sim} X$ and a Poincaré line bundle \mathcal{P} on $X \times \text{Pic}^0(X)$. This allows to construct a universal semistable T -bundle \mathbf{L} on $X \times \Gamma(T) \otimes_{\mathbf{Z}} X$.

Remark 4.4. — The theta line bundle Θ on $X = \text{Pic}^1(X)$ becomes through the isomorphism $X \xrightarrow{\sim} \text{Pic}^0(X)$ the determinant bundle $\det(R\Gamma\mathcal{P})^*$.

The family of semistable bundles $\mathbf{L}(G)$ defines a morphism of (reduced) schemes

$$\Gamma(T) \otimes_{\mathbf{Z}} X \rightarrow M_G.$$

The action of the Weyl group W on $\Gamma(T)$ defines an action $\Gamma(T) \otimes_{\mathbf{Z}} X$ such that $w.L_{\psi} \xrightarrow{\sim} L_{\psi}$ for all $w \in W$. Let

$$\pi : [\Gamma(T) \otimes_{\mathbf{Z}} X] / W \rightarrow M_G$$

be the induced morphism. We want to prove that π is an isomorphism.

4.5. Let us prove that π is finite. Let $G \rightarrow \mathbf{SL}_N$ be a faithful representation of G inducing a morphism $M_G \rightarrow M_{\mathbf{SL}_N}$. Let L be the inverse of the determinant bundle on $M_{\mathbf{SL}_N}$.

Remark 4.6. — Notice that in this case, $M_{\mathbf{SL}_N} = \mathbf{P}^{N-1}$ and that the determinant bundle is just $\mathcal{O}(1)$ (see [Tu], Theorem 7 for instance).

LEMMA 4.7. — *The line bundle $\pi^*(L)$ is ample.*

Proof. — One can assume that G is semisimple. Let q be the natural morphism

$$q : \Gamma(T) \otimes_{\mathbf{Z}} X \rightarrow M_G.$$

It is enough to prove that $q^*(L)$ is ample. Let us choose a basis of $\Gamma(L)$ identifying $\Gamma(T) \otimes_{\mathbf{Z}} X$ with X^l (l is the rank of G). Let $\gamma : G_m \rightarrow T$ be a non trivial element in $\Gamma(T)$. Let $q_\gamma : X \rightarrow M_G$ be the morphism defined by γ . One can assume that $\gamma(z) = \text{diag}(z^{\gamma_1}, \dots, z^{\gamma_l})$ for $z \in \mathbf{C}^*$ (with $\sum \gamma_i = 0$). Then (see Remark 4.4),

$$q_\gamma^*(L) = \Theta^{\sum_i \gamma_i^2}$$

which is ample because $\sum_i \gamma_i^2 > 0$ (recall that γ is non trivial). The rank- N vector bundle bundle parameterized by (x_1, \dots, x_l) is

$$\bigoplus_i \mathcal{O} \left(\sum_{\gamma \in \gamma} \gamma_i (x_\gamma - x) \right).$$

By additivity of the determinant bundle, $q^*(L)$ is of the form

$$\boxtimes_{1 \leq i \leq l} \Theta^{b_i} \text{ with } b_i > 0$$

and therefore is ample. □

The fibers of π are therefore finite, and the proper morphism π is finite.

4.8. Let $\pi^{-1}(0)$ be the fiber of π at the trivial bundle G_X . Let us first prove that $\pi^{-1}(0)$ is set-theoretically reduced to $[0]$, the class $W.0$. Let us first prove the following general result.

4.9. Let us consider the following situation: let $p : \mathcal{X} \rightarrow S$ be a proper morphism such that $\mathcal{O}_S \rightarrow p_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism. Assume that p has a section $\sigma : S \rightarrow \mathcal{X}$. Let $A \subset B$ be a reductive subgroup of a linear group B .

LEMMA 4.10. — *Let α be an A -bundle trivial along σ . Then, if the associated B -bundle $\beta = \alpha(B)$ is trivial, the A -bundle α is so.*

Proof. — Let s be the section of β/A defined by α . Because β/A is affine over S , the section s factors through p in a section \tilde{s} . Because α is trivial along σ , the section \tilde{s} comes from a section of the restriction to σ of the trivial bundle β and can be lifted to a section s' of β . The section $s' \text{ mod } A$ of β/A is equal to s and defines a trivialization of $\alpha = s^* \beta$. □

4.11. Choose an embedding G in a product $G' = \prod_i \mathbf{GL}_{n_i}$ of linear groups such that $Z_0(G) \subset Z_0(G')$ (Z_0 denotes the neutral component). Let T' be a maximal torus of G' containing T . Let $f : M_G \rightarrow M_{G'}$ be a natural morphism (see [Ra2], Corollary of Theorem 7.1). Let E be a T -bundle such that $E \in \pi^{-1}(0)$ and let E' be the corresponding T' -bundle.

Because $f(E(G)) = [E'(G')]$, the semistable bundle is equivalent to the trivial bundle and is therefore trivial (a direct sum of line bundles of degree 0 is equivalent to the trivial bundle if and only if all summands are trivial). Applying the preceding lemma with $\alpha = E, A = T, B = G'$ and $\mathcal{X} = X$ for instance, one gets that E is trivial.

4.12. It remains to show that π is étale at the origin: this will follow from the fact that the completion of π at the origin can be identified to the completion at the origin of the Chevalley isomorphism $\mathfrak{t}/W \xrightarrow{\sim} \mathfrak{g}/G$.

LEMMA 4.13. — *The morphism $\pi : (\Gamma(T) \otimes \text{Pic}^0(X))/W \rightarrow M_G$ is étale at the origin.*

Proof. — Let's briefly recall how to construct the moduli space M_G (see [Ra1], [BLS]), or better of an affine neighborhood M of the trivial bundle $X \times G$ as a GIT quotient Y/H of a smooth affine scheme Y by some reductive group H (with Lie algebra \mathfrak{h}). One choose first a faithful representation $G \hookrightarrow \mathbf{GL}_n$ inducing an embedding $\Gamma(T) \otimes \text{Pic}^0(X) \hookrightarrow (\text{Pic}^0(X))^n$. For m big enough, one knows that the canonical morphism

$$\iota_P : H^0(P(\mathbf{C}^n) \otimes \mathcal{O}(mx)) \otimes \mathcal{O} \rightarrow P(\mathbf{C}^n) \otimes \mathcal{O}(mx)$$

is surjective for all semistable bundles P and that $H^0(\iota_P)$ is bijective. Let χ be the Euler characteristic of some $P(\mathbf{C}^n) \otimes \mathcal{O}(mx)$. By the theory of Hilbert schemes, the pairs (P, ι) where P is a semistable G -bundle and ι an isomorphism

$$H^0(P(\mathbf{C}^n) \otimes \mathcal{O}(mx)) \xrightarrow{\sim} \mathbf{C}^\chi$$

are parameterized by a smooth scheme Y and M_G is a GIT quotient of this scheme by $H = \mathbf{GL}_\chi$ (see [BLS]). Notice that the stabilizer of the “trivial pair” is G itself.

Let \mathcal{U} be the universal T -bundle on $X \times (\Gamma(T) \otimes \text{Pic}^0(X))$. Let us chose a trivialization of the vector bundle $R\Gamma(\mathcal{L} \boxtimes \mathcal{O}(mx))$ on some symmetric affine neighborhood S^0 of 0 in $\text{Pic}^0(X)$. Therefore, the direct image of $\mathcal{U}(\mathbf{C}^n) \boxtimes \mathcal{O}(m)$ is trivial on $S = (S^0)^n \cap \Gamma(T) \otimes \text{Pic}^0(X)$ and the trivialization is W -equivariant. The induced morphism $\pi : S \rightarrow M_G = Y/H$ is therefore induced by a W -equivariant morphism $S \rightarrow Y$ mapping 0 to y . Notice that the orbit $H.y$ is closed. By considering some H -invariant affine open neighborhood of y , one can assume that Y is affine (the quotient Y/H is now a neighborhood of $H.y$ in M_G).

Let's consider the following commutative diagram:

$$(4.1) \quad \begin{array}{ccc} \mathbf{C}[Y]_+ & \longrightarrow & \mathbf{C}[S]_+ \\ \downarrow & & \downarrow \\ V = (T_y^*Y)/\mathfrak{h} & \xrightarrow{k} & T_0^*S \end{array}$$

where $\mathbf{C}[Y]_+$ (resp. $\mathbf{C}[S]_+$) denotes the maximal ideal of y (resp. 0). The transpose of k is the tangent map of $S \rightarrow \mathcal{M}_G$ from S to the stack of G -bundles on X , namely the Kodaira-Spencer map

$$k : \mathfrak{t} = \mathfrak{t} \otimes H^1(X, \mathcal{O}_X) = T_0S \rightarrow (T_yY)/\mathfrak{h} = \mathfrak{g} \otimes H^1(X, \mathcal{O}) = \mathfrak{g}.$$

LEMMA 4.14. — *The Kodaira-Spencer map k is the canonical inclusion $\mathfrak{t} \hookrightarrow \mathfrak{g}$.*

Proof. — By functoriality, one is reduced to the case where $G = \mathbf{GL}_n$ and T is the torus of invertible diagonal matrices. Consider the one parameter subgroup of differential $aE_{i,i}$ for some integer a ($E_{i,i}$ is the standard diagonal rank 1 matrix). If $(\lambda_{\alpha,\beta})$ is a Cech-cocycle representing $\lambda \in H^1(\mathcal{O})$, the derivative

$$\frac{\partial \pi}{\partial(\gamma \otimes \lambda)}(0)$$

is defined by the vector bundle on $X[\epsilon]/(\epsilon^2 = 0)$ with cocycle $1 + a\epsilon\lambda_{\alpha,\beta}E_{i,i}$. In other words,

$$\frac{\partial \pi}{\partial(\gamma \otimes \lambda)}(0) = \lambda d\gamma,$$

which proves the lemma. □

Notice that k is $N(G, T)$ -equivariant. By Luna's results ([Lu]), one obtains an étale slice of $Y \rightarrow Y/H$ as follows.

One choose an $H_y = \text{Aut}_G(X \times G) = G$ -invariant section σ of $\mathbf{C}[Y]_+ \rightarrow (T_y^*Y)/\mathfrak{h}$ and the induced morphism $Y \rightarrow V$ identifies étale locally Y/H and V/H_y . The group $N(G, T)$ being reductive and ${}^t k$ being surjective, one pick an invariant section τ of ${}^t k : \mathfrak{g}^* \rightarrow \mathfrak{t}^*$ which defines a morphism (still denoted by τ)

$$\mathfrak{t}^* \rightarrow \mathbf{C}[Y]_+ \rightarrow \mathbf{C}[S]_+$$

which is W -equivariant. This is a W -equivariant section of $\mathbf{C}[S]_+ \rightarrow T_0^*S$ and therefore defines an étale slice of $S \rightarrow S/W$. Shrinking S and Y if necessary, one obtains from the diagram (4.1) the commutative diagram

$$\begin{array}{ccc}
 \mathbf{C}[Y]_+^H & \xrightarrow{\pi} & \mathbf{C}[S]_+^W \\
 \mathbf{S}(\sigma) \uparrow & & \uparrow \mathbf{S}(\tau) \\
 (\mathbf{S} \mathfrak{g}^*)^G & \xrightarrow{\mathbf{S}(\iota k)} & (\mathbf{S} \mathfrak{t}^*)^W
 \end{array}$$

where $\mathbf{S}(\sigma)$ and $\mathbf{S}(\tau)$ are étale. By Chevalley’s theorem, π is therefore étale at the origin. □

4.15. The morphism π is therefore a finite morphism between normal varieties and is of degree 1. We have proved the

THEOREM 4.16. — *The morphism*

$$\pi : [\Gamma(T) \otimes_{\mathbf{Z}} X]/W \rightarrow M_G$$

is an isomorphism.

4.17. Assume that G is simple and simply connected. Let θ be the longest root and $\alpha_i, i = 1, \dots, l$ the basis of the root system $\Phi(B, G)$. The coroot θ^\vee is a sum

$$\theta^\vee = \sum_i g_i \alpha_i^\vee$$

where α_i^\vee is the coroot of α . Then Looijenga [Lo] has proved that $[\Gamma(T) \otimes_{\mathbf{Z}} X]/W$ is the weighted projective space $\mathbf{P}(1, g_1, \dots, g_l)$.

Remark 4.18. — The proof in [Lo] is not correct. See [BS] for a more general result and hints for a complete proof.

4.19. It is interesting to remark ([D], remarques 1.8) that $\mathcal{O}(l)$ is locally free if and only if l is a multiple of $\text{lcm}(g_i)$ although it is reflexive (Lemme 4.1 of *loc. cit.*). In particular, M_G is locally factorial if and only if $\text{lcm}(g_i) = 1$, condition which is equivalent to G special in the sense of Serre (look at the table of [Bo]). If one notice that $\text{lcm}(g_i)$ is also the minimal Dynkin index of the representations of G (see [LS]), this funny characterization of special groups in terms of M_G is the version in the genus one case of Proposition 13.2 of [BLS] (which deals with the genus > 1). In all the cases, one has the formula

$$(4.2) \qquad \dim H^0(M_G, \mathcal{O}(l)) = \text{card}(P_l)$$

where P_l is the number of dominant weights w such that $\langle \theta^\vee, w \rangle \leq l$, as predicted by the Verlinde formula (see [Be]).

4.20. Let us explain briefly the link between the theorem of Narasimhan and Seshadri and our description of M_G . Suppose that G is semisimple with maximal compact subgroup K . The theorem of Narasimhan and Seshadri says that the complex points of M_G are parameterized by equivalence classes of pairs of elements of K which commutes (K acting on these pairs diagonally through the adjoint action). Suppose further that G is simply connected. Then such a class has a representative in $T_{\mathbf{R}} \times T_{\mathbf{R}}$ (where $T_{\mathbf{R}}$ is the maximal torus of K). Suppose that $X(\mathbf{C})$ is a complex torus $\mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau$ of period τ in the Poincaré upper half plane. The complex structure $(a, b) \rightarrow a - \tau b$ on $\mathbf{R} \times \mathbf{R}$ induces a complex structure on $T_{\mathbf{R}} \times T_{\mathbf{R}}$ which is naturally the maximal torus T of G . We get a diagram

$$\begin{array}{ccc}
 \Gamma(T) \otimes \mathbf{C} & \xrightarrow{\sim} & T_{\mathbf{R}} \times T_{\mathbf{R}} \\
 \downarrow & & \downarrow \\
 \Gamma(T) \otimes X(\mathbf{C}) & \xrightarrow{\pi} & M_G(\mathbf{C})
 \end{array}$$

One checks easily that this diagram commutes.

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