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THE BERGMAN KERNEL OF THE MINIMAL BALL AND APPLICATIONS

by K. OELJEKLAUS, P. PFLUG & E.H. YOUSSEFI

1. Introduction.

Let \mathbb{B}_* be the domain in \mathbb{C}^n , $n \geq 2$, defined by

$$\mathbb{B}_* := \{z \in \mathbb{C}^n : |z|^2 + |z \bullet z| < 1\},$$

where $z \bullet z := \sum_{j=1}^n z_j^2$. This is the ball of radius $\frac{\sqrt{2}}{2}$ with respect to the norm

$$N_*(z) := \sqrt{\frac{|z|^2 + |z \bullet z|}{2}}, \quad z \in \mathbb{C}^n.$$

The norm N_* was introduced by Hahn and Pflug [HP], and was shown to be the smallest norm in \mathbb{C}^n that extends the euclidean norm in \mathbb{R}^n under certain restrictions. The automorphism group of \mathbb{B}_* is compact and its identity component is $\text{Aut}_{\mathcal{O}}^0(\mathbb{B}_*) = S^1 \cdot SO(n, \mathbb{R})$, where the S^1 -action is diagonal and the $SO(n, \mathbb{R})$ -action is the matrix multiplication, see [K] or [OY]. This shows that for $n \geq 3$, the ball \mathbb{B}_* is not biholomorphic to any Reinhardt domain. For $n = 2$, \mathbb{B}_* is linearly biholomorphic to the Reinhardt triangle $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$.

The main purpose of this note is to establish the following

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THEOREM. — *The Bergman kernel of \mathbb{B}_* is given by the formula*

$$K_{B_*}(z, w) = \frac{1}{n(n+1)V(B_*)} \frac{\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} X^{n-1-2j} Y^j (2nX - (n-2j)[X^2 - Y])}{(X^2 - Y)^{n+1}},$$

where

$$X = 1 - \langle z, w \rangle, \text{ and } Y = (z \bullet z)\overline{w \bullet w},$$

and $V(\mathbb{B}_*)$ is the Lebesgue volume of \mathbb{B}_* .

In particular, when $n = 2$, the Bergman kernel of \mathbb{B}_* is

$$K_{\mathbb{B}_*}(z, w) = \frac{2}{\pi^2} \frac{3(1 - \langle z, w \rangle)^2(1 + \langle z, w \rangle) + (z \bullet z)\overline{w \bullet w}(5 - 3\langle z, w \rangle)}{((1 - \langle z, w \rangle)^2 - (z \bullet z)\overline{w \bullet w})^3}.$$

It should be noted that for $n = 2$ this formula can be obtained from the Bergman kernel of the above mentioned Reinhardt triangle whose Bergman kernel can be found in ([JP], p. 176).

Remark. — To the best of our knowledge, the domain \mathbb{B}_* is the first bounded domain in \mathbb{C}^n which is neither Reinhardt nor homogeneous, and for which we have an explicit formula for its Bergman kernel.

2. Preparatory results.

Let G be a semi-simple complex Lie group and K a maximal compact subgroup of G . Suppose that G acts irreducibly on a finite dimensional complex vector space E_Λ via a representation (Π_Λ, E_Λ) with dominant weight Λ and dominant vector v_Λ . Assume further that E_Λ is furnished with a K -invariant hermitian scalar product $[\cdot, \cdot]$. If $G = KAN$ is the Iwasawa decomposition of G , let ϱ denote the sum of the roots associated with the complex decomposition in the Lie algebra \mathfrak{g} of G . If \mathfrak{a} is the Lie algebra of A , then we have the following orthogonal decomposition with respect to the Cartan-Killing form

$$\mathfrak{a} = a_\Lambda \oplus a_\Lambda^\perp,$$

where a_Λ is the annihilator of Λ . If H_0 is the unique vector in a_Λ^\perp such that $\Lambda(H_0) = 1$, we set

$$(2.1) \quad \sigma := 2\varrho(H_0).$$

Let \mathbb{M}^* be the intersection of the G -orbit of v_Λ and the unit ball in E_Λ .

In his work [Lo], Loeb proved that the Bergman kernel of the manifold \mathbb{M}^* with an invariant form on \mathbb{M}^* is given for $\zeta = \Pi_\Lambda(g_1)v_\Lambda, \eta = \Pi_\Lambda(g_2)v_\Lambda, g_1, g_2 \in G$, by

$$(2.2) \quad K_{\mathbb{M}^*}(\zeta, \eta) = \sum_{j=0}^{\infty} (2j + \sigma) T_\Lambda(j) [\zeta, \eta]^j,$$

where $T_\Lambda(j)$ is the dimension of the representation with dominant weight $j\Lambda$.

Here we consider the special case $G = SO(n + 1, \mathbb{C})$ with its natural linear representation on the complex hermitian space $(\mathbb{C}^{n+1}, \langle \cdot, \cdot \rangle)$, where Λ is the dominant weight associated to this representation and $v_\Lambda = \frac{\sqrt{2}}{2}(1, i, 0, \dots, 0)$. The intersection of the G -orbit of v_Λ and the unit ball in \mathbb{C}^{n+1} is $\mathbb{M}^* = \mathbb{M} \setminus \{0\}$, where

$$\mathbb{M} := \{z = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : |z| < 1, z \bullet z = 0\}.$$

Then by formula (2.2), we see that the Bergman kernel of \mathbb{M}^* with respect to an $SO(n+1, \mathbb{C})$ -invariant form $\alpha(z) \wedge \overline{\alpha(z)}$ is given (up to a multiplicative constant) by

$$(2.3) \quad K_{\mathbb{M}^*}(\zeta, \eta) = \sum_{j=0}^{\infty} (2j + \sigma) T_\Lambda(j) \langle \zeta, \eta \rangle^j,$$

for $\zeta, \eta \in \mathbb{M}^*$.

LEMMA 2.1. — *If $\alpha(z)$ is an $SO(n + 1, \mathbb{C})$ -invariant nonzero n -form on \mathbb{M}^* (the invariant n -form α is unique up to a constant), then the Bergman kernel of \mathbb{M}^* with respect to the invariant form $\alpha(z) \wedge \overline{\alpha(z)}$ is given (up to a multiplicative constant) by*

$$\begin{aligned} K_{\mathbb{M}^*}(\zeta, \eta) &= \frac{2(n + 1) \langle \zeta, \eta \rangle + 2n - 2}{(1 - \langle \zeta, \eta \rangle)^{n+1}} \\ &= \frac{4n}{(1 - \langle \zeta, \eta \rangle)^{n+1}} - \frac{2n + 2}{(1 - \langle \zeta, \eta \rangle)^n}, \end{aligned}$$

for $\zeta, \eta \in \mathbb{M}^*$.

Proof. — Using the notations above, a calculation involving the Weyl character formula implies that

$$T_\Lambda(j) = \frac{n + 2j - 1}{n - 1} \binom{n - 2 + j}{j}, \text{ for all positive integers } j.$$

See ([FH], pp. 267-315). In addition, some computing shows that $\sigma = 2n - 2$. See ([FH], pp. 399-414). The lemma now follows from (2.3). \square

LEMMA 2.2. — *The n -form on $(\mathbb{C} \setminus \{0\})^{n+1}$*

$$\tilde{\alpha}(z) := \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1},$$

induces by restriction an $SO(n + 1, \mathbb{C})$ -invariant and holomorphic n -form α on \mathbb{M}^ .*

Proof. — Let $A \in SO(n + 1, \mathbb{C})$, $z \in \mathbb{M}^*$ and set $w = Az$. Denote by A_{jk} the $n \times n$ matrix obtained from A by deleting the j th row and the k th column. Since $A \in SO(n + 1, \mathbb{C})$, Cramer’s rule gives that

$$(2.4) \quad a_{jk} = (-1)^{k+j} \det A_{jk}.$$

Note also that for $z \in \mathbb{M}^*$ and $z_j \neq 0$, then

$$(2.5) \quad dz_j = - \sum_{l \neq j} \frac{z_l}{z_j} dz_l \quad \text{on } T_z \mathbb{M}^*,$$

where $T_z \mathbb{M}^*$ denotes the tangent space of \mathbb{M}^* at the point z . Denote by $A^* \alpha$ the pull-back of α . Then

$$\begin{aligned} (A^* \alpha)(z) &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{n+1} \\ &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} \sum_{k=1}^{n+1} \det A_{jk} dz_1 \wedge \cdots \wedge \widehat{dz_k} \cdots \wedge dz_{n+1} \\ &= \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{j=1}^{n+1} \frac{(-1)^{k+j}}{w_j} \det A_{jk} dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1} \end{aligned}$$

$$\begin{aligned}
 \text{by (2.4)} &= \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{j=1}^{n+1} \frac{a_{jk}}{w_j} dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1} \\
 \text{by (2.5)} &= \sum_{k=1}^{n+1} (-1)^k \sum_{j=1}^{n+1} \frac{a_{jk}}{w_j} dz_1 \wedge \cdots \wedge dz_{j-1} \wedge \widehat{dz_j} \wedge \left(\sum_{l \neq j} \frac{z_l}{z_j} dz_l \right) \\
 &\quad \wedge dz_{j+1} \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1} \\
 &= \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{j=1}^{n+1} (-1)^{j-k} \frac{a_{jk}}{z_j w_j} z_k dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1} \\
 &= \sum_{j=1}^{n+1} \left(\frac{(-1)^{j-1}}{w_j z_j} \sum_{k=1}^{n+1} a_{jk} z_k \right) dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1} \\
 &= \sum_{j=1}^{n+1} (-1)^{j-1} \frac{1}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1} = \alpha(z).
 \end{aligned}$$

That the restriction of α to \mathbb{M}^* is holomorphic can be seen by evaluating the form α on the n -fold exterior power of the tangent space. \square

3. Proper holomorphic mappings from \mathbb{M} into \mathbb{C}^n .

Consider the projection $pr : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ defined by

$$pr(z_1, \dots, z_{n+1}) := (z_1, \dots, z_n).$$

The restriction $F := pr|_{\mathbb{M}}$ of pr to \mathbb{M} gives a proper holomorphic mapping of degree 2 from \mathbb{M} onto \mathbb{B}_* . Let W be the branching locus of F and V the image of W under F . Denote by φ and ψ the two local inverses of F defined for $z = (z_1, \dots, z_n) \in \mathbb{B}_* \setminus V$ by

$$\begin{aligned}
 \varphi(z) &= (z, i\sqrt{z \bullet z}) \\
 \psi(z) &= (z, -i\sqrt{z \bullet z}).
 \end{aligned}$$

LEMMA 3.1. — *If $\varphi := (\varphi_1, \dots, \varphi_{n+1})$ and $\psi := (\psi_1, \dots, \psi_{n+1})$ are the local inverses of F defined on $\mathbb{B}_* \setminus V$, then*

$$(3.1) \quad \varphi^*(\alpha) = \frac{1+n}{i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n$$

$$(3.2) \quad \psi^*(\alpha) = \frac{1+n}{-i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n.$$

Proof. — The pull back of α under φ is

$$\begin{aligned} \varphi^*(\alpha) &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{n+1} \\ &= \sum_{j=1}^n \frac{(-1)^{j-1}}{w_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge dz_n \wedge dw_{n+1} + \frac{(-1)^n}{w_{n+1}} dz_1 \wedge \cdots \wedge dz_n. \end{aligned}$$

But for $1 \leq j \leq n$

$$\begin{aligned} &\frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge d\varphi_{n+1} \\ &= \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge \left(- \sum_{k=1}^n \frac{z_k}{w_{n+1}} dz_k \right) \\ &= \frac{(-1)^j}{w_{n+1}} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge dz_j \\ &= \frac{(-1)^{j+n-j}}{w_{n+1}} dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \\ &= (-1)^n \frac{dz_1 \wedge \cdots \wedge dz_n}{w_{n+1}}. \end{aligned}$$

Thus

$$\begin{aligned} \varphi^*(\alpha) &= \left(\frac{(-1)^n}{w_{n+1}} + (-1)^n \frac{n}{w_{n+1}} \right) dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1+n}{w_{n+1}} (-1)^n dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1+n}{i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n. \end{aligned}$$

Similarly one has that

$$\psi^*(\alpha) = \frac{1+n}{-i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n.$$

□

If $P_{\mathbb{M}^*}$ denotes the Bergman projection of \mathbb{M}^* with respect to the volume form $\alpha(z) \wedge \overline{\alpha(z)}$, and if $P_{\mathbb{B}_*}$ denotes the Bergman projection of \mathbb{B}_* , then we have the following transformation rule

LEMMA 3.2. — *If $\varphi := (\varphi_1, \dots, \varphi_{n+1})$ and $\psi := (\psi_1, \dots, \psi_{n+1})$ are the local inverses of F defined locally on $\mathbb{B}_* \setminus V$, then*

$$P_{\mathbb{M}^*}(z_{n+1}(h \circ F))(z) = z_{n+1}((P_{\mathbb{B}_*} h) \circ F)(z)$$

for all $h \in L^2(\mathbb{B}_*)$, where V is the image of the branching locus of F .

Proof. — First observe that the lemma holds for all holomorphic functions $h \in L^2(\mathbb{B}_*)$. Indeed, by virtue of Lemma 3.1 we have that

$$\begin{aligned} \int_{\mathbb{M}^*} |z_{n+1}(h \circ F)(z)|^2 \alpha(z) \wedge \overline{\alpha(z)} &= \int_{\mathbb{M}^* \setminus W} |(z_{n+1}(h \circ F))(z)|^2 \alpha(z) \wedge \overline{\alpha(z)} \\ &= \int_{\mathbb{B}_* \setminus V} |\varphi_{n+1}(w)h(w)|^2 \varphi^*(\alpha)(w) \wedge \varphi^*(\overline{\alpha})(w) \\ &\quad + \int_{\mathbb{B}_* \setminus V} |\psi_{n+1}(w)h(w)|^2 \psi^*(\alpha)(w) \wedge \psi^*(\overline{\alpha})(w) \\ &= 2(n+1)^2 \int_{\mathbb{B}_* \setminus V} |h(w)|^2 dv(w) < +\infty. \end{aligned}$$

Thus $z_{n+1}(h \circ F)(z) \in L^2(\mathbb{M}^*, \alpha(z) \wedge \overline{\alpha(z)})$.

Next let $f \in L^2(\mathbb{M}^*, \alpha(z) \wedge \overline{\alpha(z)})$ be a holomorphic function, and let g be an element of the space $C_0^\infty(\mathbb{B}_* \setminus V)$ of all C^∞ -function with compact support in $\mathbb{B}_* \setminus V$. Then

$$\begin{aligned} \int_{\mathbb{M}^*} f(z) z_{n+1} \overline{\left(\frac{\partial g}{\partial w_j} \circ F \right)}(z) \alpha(z) \wedge \overline{\alpha(z)} \\ = (n+1)^2 \left[\int_{\mathbb{B}_*} \frac{(f \circ \varphi)(w)}{\varphi_{n+1}(w)} \overline{\frac{\partial g}{\partial w_j}}(w) dv(w) + \int_{\mathbb{B}_*} \frac{(f \circ \psi)(w)}{\psi_{n+1}(w)} \overline{\frac{\partial g}{\partial w_j}}(w) dv(w) \right] \end{aligned}$$

so that by integration by parts we obtain that

$$P_{\mathbb{M}} \left(z_{n+1} \left(\frac{\partial g}{\partial w_j} \circ F \right) \right) = 0, \text{ for all } j = 1, \dots, n.$$

Since the space

$$\mathcal{H} := \left\{ \frac{\partial g}{\partial w_j} : g \in C_0^\infty(\mathbb{B}_* \setminus V) \right\}$$

is dense in the orthogonal complement in $L^2(\mathbb{B}_*)$ of the subspace $L^2_h(\mathbb{B}_*)$ of all square integrable holomorphic functions on \mathbb{B}_* , the lemma follows. \square

LEMMA 3.3. — *If φ and ψ are as before, then*

$$z_{n+1} K_{\mathbb{B}_*}(F(z), w) = (n+1)^2 \left[\frac{K_{\mathbb{M}^*}(z, \varphi(w))}{\varphi_{n+1}(w)} + \frac{K_{\mathbb{M}^*}(z, \psi(w))}{\psi_{n+1}(w)} \right],$$

$z \in \mathbb{M}^*, w \in \mathbb{B}_*.$

Proof. — Let $w \in \mathbb{B}_* \setminus V$ and let $r > 0$ be chosen so small that $w + r\Delta^n \subset \mathbb{B}_* \setminus V$, where Δ is the unit disc in \mathbb{C} . By Remark 6.1.4 in [JP],

there is a C^∞ -function $u : \mathbb{C}^n \rightarrow [0, +\infty)$ with compact support in $w + r\Delta^n$ such that

$$P_{\mathbb{B}_*} u = K_{\mathbb{B}_*}(\cdot, w).$$

By virtue of Lemma 3.2 we see that for $z \in \mathbb{M}^*$,

$$\begin{aligned} z_{n+1}K_{\mathbb{B}_*}(F(z), w) &= z_{n+1}(P_{\mathbb{B}_*} u)(F(z)) = P_{\mathbb{M}^*}(z_{n+1}(u \circ F)(z)) \\ &= \int_{\mathbb{M}^*} \zeta_{n+1}(u \circ F)(\zeta) K_{\mathbb{M}^*}(z, \zeta) \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \\ &= (n+1)^2 \int_{\mathbb{B}_*} u(\eta) \left[\frac{K_{\mathbb{M}^*}(z, \varphi(\eta))}{\varphi_{n+1}(\eta)} + \frac{K_{\mathbb{M}^*}(z, \psi(\eta))}{\psi_{n+1}(\eta)} \right] dv(\eta) \\ &= (n+1)^2 \left(\frac{K_{\mathbb{M}^*}(z, \varphi(w))}{\varphi_{n+1}(w)} + \frac{K_{\mathbb{M}^*}(z, \psi(w))}{\psi_{n+1}(w)} \right), \end{aligned}$$

and the lemma is proved. □

4. Proof of the main result.

Proof of the theorem. — For $z, w \in \mathbb{B}_* \setminus V$, set

$$\begin{cases} s := 1 - \langle z, w \rangle, & t := \varphi_{n+1}(z) \overline{\varphi_{n+1}(w)} \\ x := \langle z, w \rangle + t & \text{and } y := \langle z, w \rangle - t. \end{cases}$$

Then using the notations in the main theorem we have $X = s$ and $Y = t^2$. By Lemma 2.1 we see that for some positive constant C we have

$$\begin{aligned} K_{\mathbb{M}^*}(\varphi(z), \varphi(w)) &= C \left(\frac{4n}{(1-x)^{n+1}} - \frac{2n+2}{(1-x)^n} \right) \\ K_{\mathbb{M}^*}(\varphi(z), \psi(w)) &= C \left(\frac{4n}{(1-y)^{n+1}} - \frac{2n+2}{(1-y)^n} \right), \end{aligned}$$

so that by Lemma 3.3 we obtain that

$$\begin{aligned} K_{\mathbb{B}_*}(z, w) &= 4C(n+1)^2 \frac{f(x) - f(y)}{x - y}, \quad \text{where} \\ f(u) &= \frac{2n}{(1-u)^{n+1}} - \frac{n+1}{(1-u)^n}. \end{aligned}$$

On the other hand,

$$\frac{f(x) - f(y)}{x - y} = n \frac{(s+t)^{n+1} - (s-t)^{n+1}}{t(s^2 - t^2)^{n+1}} - \frac{n+1}{2} \frac{(s+t)^n - (s-t)^n}{t(s^2 - t^2)^n},$$

and

$$\begin{aligned} \frac{(s+t)^{n+1} - (s-t)^{n+1}}{t} &= \frac{(s+t)^{n+1} - s^{n+1}}{t} + \frac{(s-t)^{n+1} - s^{n+1}}{-t} \\ &= \sum_{j=1}^{n+1} \binom{n+1}{j} s^{n+1-j} t^{j-1} \\ &\quad + \sum_{j=1}^{n+1} \binom{n+1}{j} s^{n+1-j} (-t)^{j-1} \\ &= \sum_{j=1}^{n+1} \binom{n+1}{j} s^{n+1-j} [t^{j-1} + (-t)^{j-1}] \\ &= \sum_{k=0}^n \binom{n+1}{k+1} s^{n-k} [t^k + (-t)^k] \\ &= 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} s^{n-2k} t^{2k}. \end{aligned}$$

Similarly we have that

$$\frac{(s+t)^n - (s-t)^n}{t} = 2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} s^{n-1-2k} t^{2k}.$$

Therefore,

$$\begin{aligned} \frac{f(x) - f(y)}{x - y} &= \frac{2n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} s^{n-2k} t^{2k}}{(s^2 - t^2)^{n+1}} \\ &\quad - (n+1) \frac{\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} s^{n-1-2k} t^{2k}}{(s^2 - t^2)^n}. \end{aligned}$$

But $\binom{n}{2k+1} = \frac{n-2k}{n+1} \binom{n+1}{2k+1}$. Thus

$$\begin{aligned} \frac{f(x) - f(y)}{x - y} &= \frac{2n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} s^{n-2k} t^{2k}}{(s^2 - t^2)^{n+1}} \\ &\quad - \frac{\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-2k) \binom{n+1}{2k+1} s^{n-1-2k} t^{2k} (s^2 - t^2)}{(s^2 - t^2)^{n+1}} \\ &= \frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} s^{n-1-2k} t^{2k} [2ns - (n-2k)(s^2 - t^2)]}{(s^2 - t^2)^{n+1}}. \end{aligned}$$

It follows that

$$K_{\mathbb{B}_*}(z, w) = 4C(n + 1)^2 \frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n + 1}{2k + 1} X^{n-1-2k} Y^k [2nX - (n - 2k)(X^2 - Y)]}{(X^2 - Y)^{n+1}},$$

where X and Y are as in the statement of the theorem. To compute the constant we use the formula

$$1 = \int_{\mathbb{B}_*} K_{\mathbb{B}_*}(0, w) dV(w).$$

□

5. Applications.

THEOREM 5.1. — *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain with C^2 -boundary and let $f : D \rightarrow \mathbb{B}_*$ be a proper holomorphic mapping. Then $A_f \subset V(f)$ where*

$$A = A_f := \{z \in D : f(z) \bullet f(z) = 0\}, V(f) := \{z \in D : \det f'(z) = 0\}.$$

Proof. — Observe that A is an analytic subset of D . Assume that there exists a point $a \in A$ with $\det f'(a) \neq 0$. To get a contradiction it suffices to show that:

$$\text{if } z^\nu \in A, z^\nu \rightarrow z^0 \in \partial D, \text{ then } \det f'(z^\nu) \rightarrow 0.$$

We choose a ball $B(z^0, s)$, $0 < \eta < 1$ so that $\eta(n + 2) > n + 1$, and a defining function r of $D \cap B(z^0, s)$ such that $\tilde{r} := -(-r)^\eta$ is plurisubharmonic on $D \cap B(z^0, s)$; this can be achieved using a result of Diederich-Fornaess [DF]. Moreover, we may assume that $f(z^\nu) \rightarrow w^0 \in \mathbb{H} \cap \partial\mathbb{B}_*$, where $\mathbb{H} := \{\zeta \in \mathbb{C}^n : \zeta \bullet \zeta = 0\}$.

Assuming that f is a mapping with multiplicity m , we know by Pinchuk [Pi2] that

$$mK_D(z, z) \geq |\det f'(z)|^2 K_{\mathbb{B}_*}(f(z), f(z)), z \in D.$$

It is well known that $K_D(z, z) \leq C_1 \text{dist}(z, \partial D)^{-(n+1)}$, $z \in D$. Hence we get

$$|\det f'(z)|^2 \leq C_2 (K_{\mathbb{B}_*}(f(z), f(z)))^{-1} \text{dist}(z, \partial D)^{-(n+1)}.$$

Now we apply the theorem to obtain that

$$|\det f'(z^\nu)|^2 \leq C_3(1 - |f(z^\nu)|^2)^{n+2} / \text{dist}(z^\nu, \partial D)^{n+1}, \nu \gg 1.$$

Fix $s' < s$ and define on D the following function:

$$v(z) := \begin{cases} \max\{\tilde{r}(z), |z - z^0|^2 - s'^2\} & \text{if } z \in D \cap \overline{B(z^0, s')}, \\ |z - z^0|^2 - s'^2, & \text{if } z \in D \setminus B(z^0, s'). \end{cases}$$

It is clear that v is plurisubharmonic on D and that $v(z) = \tilde{r}(z)$ for $z \in D \cap B(z^0, s'')$, $0 < s'' < s'$ sufficiently small.

For $w \in \mathbb{B}_*$ we put $\rho(w) := \max\{v(z) : z \in D, f(z) = w\}$. Obviously, ρ is plurisubharmonic on \mathbb{B}_* . In particular, for $\nu \gg 1$ we have $\rho(f(z^\nu)) \geq v(z^\nu) = \tilde{r}(z^\nu)$.

Exploiting that \mathbb{B}_* is balanced and the Hopf-Lemma on $\mathbb{H} \cap \mathbb{B}_*$ leads to the following estimate: $\rho(f(z^\nu)) \leq C_4(|f(z^\nu)|^2 - 1)$, $\nu \gg 1$; $C_4 > 0$ independent of z^ν . Therefore

$$|\det f'(z^\nu)|^2 \leq C_5(-r(z^\nu))^{n(n+2)} / \text{dist}(z^\nu, \partial D)^{n+1} \rightarrow 0, \text{ if } \nu \rightarrow \infty,$$

which leads to that contradiction we mentioned at the beginning of the proof. □

COROLLARY 5.2. — *There are no unbranched proper holomorphic mappings from D onto \mathbb{B}_* for any bounded pseudoconvex domain with a C^2 -boundary; in particular, such a D is never biholomorphically equivalent to \mathbb{B}_* .*

Moreover, if D is assumed to be strongly pseudoconvex we get even more:

THEOREM 5.3. — *Let $D \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 -boundary. If $f : D \rightarrow \mathbb{B}_*$ is a proper holomorphic mapping, then $A = V(f)$.*

Proof. — Assume the inclusion $V(f) \subset A$ is not correct. Then, by the maximum principle, there is a sequence $z^\nu \in V(f)$, $z^\nu \rightarrow z^0 \in \partial D$ such that $|f(z^\nu) \bullet f(z^\nu)| > C > 0$. Without loss of generality we assume that $f(z^\nu) \rightarrow w^0$. Since $|w^0 \bullet w^0| > 0$ we conclude that w^0 is a strongly pseudoconvex boundary point of \mathbb{B}_* . By Theorem 3 of [Ber] there is a neighborhood $U = U(z^0)$ such that f extends to a continuous mapping on $U \cap \bar{D}$. Then using Theorem 3' of [Pi1] we obtain that $f \in C^1(\bar{D} \cap U)$. Finally using Theorem 1 of [Pi2] we finally get the contradiction to the fact that $z^0 \in \overline{V(f)}$. □

We recall that a bounded domain Ω is said to satisfy condition (Q) if the Bergman projection of Ω maps $C_0^\infty(\Omega)$ into the space $\mathcal{O}(\bar{\Omega})$ of all holomorphic functions on a neighborhood of $\bar{\Omega}$. It was proved recently in [Th] that Ω satisfies condition Q is if and only if that for every compact subset L of Ω , there is an open neighborhood $U = U(L)$ of $\bar{\Omega}$ such that the Bergman kernel $K_\Omega(z, w)$ of Ω extends to be holomorphic on U as a function of z for each $w \in L$, and K_Ω is continuous on $U \times L$.

LEMMA 5.4. — *The ball \mathbb{B}_* satisfies condition (Q).*

Proof. — For $z, w \in \mathbb{B}_*$, we have

$$\begin{aligned} \left| (1 - \langle z, w \rangle)^2 - (z \bullet z)(\overline{w \bullet w}) \right| &\geq |1 - \langle z, w \rangle|^2 - |z \bullet z||w \bullet w| \\ &\geq (1 - |z||w|)^2 - |z \bullet z||w \bullet w| \\ &\geq \left(1 - |z||w| - \sqrt{|z \bullet z|} \sqrt{|w \bullet w|} \right)^2 \\ &\geq \left(1 - \sqrt{|z|^2 + |z \bullet z|} \sqrt{|w|^2 + |w \bullet w|} \right)^2 \end{aligned}$$

where the last inequality holds because of Cauchy-Schwarz’s inequality. Therefore for some positive constant C we have

$$|K_{\mathbb{B}_*}(z, w)| \leq \frac{C}{\left(1 - \sqrt{|z|^2 + |z \bullet z|} \sqrt{|w|^2 + |w \bullet w|} \right)^{2n+4}}, \text{ for all } z, w \in \mathbb{B}_*.$$

This shows that \mathbb{B}_* satisfies condition (Q). □

THEOREM 5.5. — *Let $D \subset \mathbb{C}^n$ be an arbitrary bounded circular domain which contains the origin.*

- (1) *If $f : \mathbb{B}_* \rightarrow D$ is a proper holomorphic mapping, then f extends holomorphically to a neighborhood of $\bar{\mathbb{B}}_*$.*
- (2) *If D is smooth then there is no proper holomorphic mapping from \mathbb{B}_* into D .*

Proof. — Since, by Lemma 5.4, \mathbb{B}_* satisfies condition Q, part (1) of the theorem becomes a consequence of Theorem 2 of [Bel]. To see that part (2) of theorem holds, it is enough to notice that if there is proper holomorphic mapping $f : \mathbb{B}_* \rightarrow D$, and if ϱ is a defining function of D , then $\varrho \circ f$ is a defining function for \mathbb{B}_* , which will imply that \mathbb{B}_* is smooth and thus leads to a contradiction. □

THEOREM 5.6. — *Let L be a compact subset of \mathbb{B}_* and let ζ be a boundary point of \mathbb{B}_* . Then every holomorphic function f in a*

neighborhood of L is the uniform limit of functions in the complex span of the functions

$$\frac{\partial}{\partial \zeta^\beta} K_{\mathbb{B}_*}(\cdot, \zeta), \quad \beta \in \mathbb{N}_0^n.$$

Proof. — Since \mathbb{B}_* is a Runge domain and satisfies condition (Q) the proposition follows from Theorem 2.5 of [Th]. \square

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