LIOUVILLE FORMS IN A NEIGHBORHOOD OF AN ISOTROPIC EMBEDDING(*)

by Frank LOOSE

1. Introduction.

Consider a symplectic manifold \((X, \omega)\). If the symplectic 2-form \(\omega\) is exact, a choice of a potential, i.e., a 1-form \(\beta\) satisfying \(-d\beta = \omega\), is called a Liouville form on \(X\). Since \(\omega\) is non-degenerate there is a unique vector field \(\eta\) on \(X\) given by \(i_\eta \omega = \beta\). Here \(i_\eta \omega\) denotes the contraction of the form \(\omega\) by \(\eta\), i.e., \(\langle i_\eta \omega, \xi \rangle = \langle \omega, \eta \wedge \xi \rangle\) for all \(\xi \in TX\). The vector field \(\eta\) is called the associated contracting vector field.

The importance of Liouville forms and contracting vector fields for symplectic geometry has been pointed out among others by Eliashberg and Gromov [EG]. The aim of the present paper is to investigate the flexibility of Liouville forms in a special case, i.e.: When is it possible to transform one Liouville form into another by a symplectomorphism, at least locally?

The center of a Liouville form \(\beta\) is the set \(M = \{x \in X \mid \beta(x) = 0\}\). Equivalently, it is the fixed point locus of the associated contracting vector field.

As a basic example consider the standard symplectic vector space of dimension \(2l\) with its canonical 2-form \(\omega = \sum_{\lambda=1}^{l} dx^\lambda \wedge dy^\lambda\), \((x, y)\) being a coordinate for \(\mathbb{R}^{2l}\). A natural choice for a Liouville form is

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\[ \alpha = \sum_{\lambda=1}^{t} \left(y^\lambda dx^\lambda - x^\lambda dy^\lambda\right) \] (e.g., \(\alpha\) is the unique \(\text{Sp}_{2l}(\mathbb{R})\)-invariant potential of \(\omega\)). The center is \(M = \{0\}\) and the associated contracting vector field is up to the factor \(-\frac{1}{2}\) the Euler vector field, \(\eta = -\frac{1}{2} \sum(x^\lambda \partial / \partial x^\lambda + y^\lambda \partial / \partial y^\lambda)\).

Another basic example is given by the cotangent bundle \(C = T^*M\) of a differentiable manifold (of dimension \(n\)). In fact, \(C\) carries a canonical 1-form \(\alpha\) given by \(\sum_{\nu=1}^{n} v^\nu du^\nu\), where \(u\) is a coordinate on \(M\) and \((u, v)\) is the corresponding bundle chart of \(\pi: C \to M\). The canonical symplectic structure on \(C\) is given by \(\omega = -d\alpha\); thus \(\alpha\) is a Liouville form. Obviously the center of \(\alpha\) is given by the zero section \(\sigma: M \hookrightarrow C\) of \(\pi\) and the associated contracting vector field is up to the factor \(-1\) the Euler field of the vector bundle \(\pi\), i.e., \(\eta = -\sum_{\nu=1}^{n} v^\nu \partial / \partial v^\nu\).

The submanifolds \(M = \{0\} \subseteq \mathbb{R}^{2l}\) and \(\sigma: M \hookrightarrow T^*M\) are both extreme cases of the notion of an isotropic embedding. Recall that a submanifold \(\iota: M \hookrightarrow X\) of a symplectic manifold \((X, \omega)\) is isotropic, if \(\iota^* \omega = 0\). Its dimension \(n = \dim M\) can vary between \(n = 0\) (i.e., \(M\) is a point) and half of the dimension of \(X\) (i.e., \(M\) is Lagrangean). Thus we let \(\dim X = 2(n+l)\) with \(l \in \mathbb{Z}_+\).

Now, let \(X\) be a manifold, let \(\beta\) be a 1-form on \(X\) so that \(\omega := -d\beta\) is non-degenerate, and let \(M := \beta^{-1}(0)\) be smooth (and non-empty). It is easy to see that \(M\) must be necessarily isotropic. If one wants to find a normal form for \(\beta\) in a neighborhood of its center (as we do), it is natural to look first for a normal form for \(\omega\) around \(M\).

Weinstein's isotropic embedding theorem (see [Wel]) gives the appropriate answer. To formulate that recall that \(\iota: M \hookrightarrow X\) isotropic means that \(TM_m \subseteq TM_m^{\perp}\). Thus \(E_m := TM_m^{\perp} / TM_m\) is a symplectic vector space and \(E = \{E_m\}_{m \in M}\) a symplectic vector bundle over \(M\). \(N(\iota) := E\) is called the symplectic normal bundle of \(\iota: M \hookrightarrow X\). If \(\iota_1: M \hookrightarrow (X_1, \omega_1)\) and \(\iota_2: M \hookrightarrow (X_2, \omega_2)\) are isotropic embeddings and \(N(\iota_1) \cong N(\iota_2)\) as symplectic vector bundles, then there exists neighborhoods \(U_1 \subseteq X_1, U_2 \subseteq X_2\) of \(M\) and a diffeomorphism \(f: U_1 \to U_2\) with \(f |M = \text{id}_M\) and \(f^* \omega_2 = \omega_1\). Moreover Weinstein proved the following existence result. Given a symplectic vector bundle \(E\) over a manifold \(M\), then there exists a canonical symplectic manifold \(C = C(E)\), together with an isotropic embedding \(\sigma: M \hookrightarrow C\), so that \(N(\sigma) = E\).

In the Darboux case, i.e., \(n = 0\) (i.e., \(M = \text{pt}\)), the canonical model
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$E$ comes down to $\mathbb{R}^{2l}$ with its standard structure and the theorem reduces to the classical Darboux theorem. In the Lagrange case, i.e., if $l = 0$ (i.e., $\dim M = \frac{1}{2} \dim X$), the canonical model is just $C = T^*M$. In both cases we already observed that there is a canonical choice of a Liouville form. In the general case, the model $C = C(E)$ is a combination of the extreme cases $n = 0$ and $l = 0$ using the Marsden-Weinstein reduction procedure. As a general fact we will show that the reduction procedure is also valid for Liouville forms, not only for symplectic forms. (For a precise statement see section 2.) In particular, Weinstein's canonical model $C = C(E)$ for isotropic embeddings carries a canonical 1-form, i.e., a Liouville form $\alpha$.

What we have discussed so far, results in the following. If one starts with a manifold $X$, with a 1-form $\beta$ so that $-d\beta$ is non-degenerate, and with $M = \beta^{-1}(0)$ smooth, and one is interested in a normal form for $\beta$ in a neighborhood of $M$, one may assume that $X = C(E)$ (where $E$ is the symplectic normal bundle of $M \hookrightarrow X$), $M \hookrightarrow X$ is the standard isotropic embedding $\sigma$, and $-d\beta = \omega = -d\alpha$, where $\alpha$ is the canonical 1-form on $C$.

We want to characterize those potentials $\beta$ of $\omega$ which we can transform into $\alpha$, i.e., those potentials for which $\alpha$ is a normal form.

The following argument shows that there is a necessary condition coming from considerations on the first order of $\beta$ along $M$. Precisely, if $\beta$ is a Liouville form vanishing along the isotropic submanifold $M \hookrightarrow X$, then so does the associated vector field $\eta$. The derivative of $\eta$ in $m \in M \subseteq X$ is a linear transformation $L_m: TX_m \to TX_m$. Now, from the symplectic nature, one computes easily that $L_m + \frac{1}{2} \text{id} \in \mathfrak{sp}(TX_m)$, the symplectic linear algebra. Moreover, the subspaces $TM_m$ and $TM_m^\perp$ are $L_m$-invariant. Thus $L_m$ induces a linear transformation $\Lambda_m: E_m \to E_m$ which is again conformal symplectic, $\Lambda_m + \frac{1}{2} \text{id} \in \mathfrak{sp}(E_m)$. In conclusion one gets a bundle homomorphism $\Lambda: E \to E$ with $\Lambda + \frac{1}{2} \text{id} \in \mathfrak{sp}(E)$. The canonical Liouville form on $C = C(E)$ fulfills $\Lambda = -\frac{1}{2} \text{id}$, as is easily shown. It is clear that this is invariant under diffeomorphisms $f$ of $C$ with $f|_M = \text{id}_M$, since the induced $f^*(\Lambda)$ coming from $f^*\alpha$ is just a conjugation of $\Lambda$. We call therefore a Liouville form $\beta$ on a manifold $X$ special, if the associated bundle transformation $\Lambda: E \to E$ fulfills $\Lambda = -\frac{1}{2} \text{id}$. We can state now the main result of that paper.

**Theorem (Existence).** Let $E \to M$ be a symplectic vector bundle, let $C$ be the canonical model associated with $E$, and let $\alpha$ be the canonical Liouville form on $C$. If $\beta$ is any potential of $\omega = -d\alpha$, vanishing along $M$, and being special in the above sense, then there exist
neighborhoods $U$ and $V$ of $M$ in $C$ and a diffeomorphism $f: U \to V$ with $f|_{M} = \text{id}_{M}$ satisfying $f^{*}\beta = \alpha$.

The theorem was proved in the Lagrangean case by Kostant-Sternberg [GuSt], chap. 5. However, to the best of our knowledge it is even unknown in the other extreme, i.e., the Darboux case $M = \text{pt}$. Kostant-Sternberg also proved that the diffeomorphism $f$ is unique. In section 3 it is shown that the canonical model $C$ comes along with a certain fibre bundle structure over $M$ and that the standard embedding is given by the zero section of that bundle. Moreover there is a natural sequence of fibre bundles

$$0 \to T^{*}M \to C \to E \to 0$$

over $M$. Using that we are able to state the following uniqueness result.

**THEOREM (Uniqueness).** — Let $\pi: C \to M$ be the fibre bundle projection of the standard model $C$ and assume that there are neighborhoods $U$ and $V$ of the zero-section $\sigma: M \to C$ and a diffeomorphism $f: U \to V$ satisfying $f|_{M} = \text{id}_{M}$ and $f^{*}\alpha = \alpha$, where $\alpha$ is the canonical 1-form on $C$. Then $f$ is already (restriction of) a bundle isomorphism of $C$ which fixes the subbundle $T^{*}M \subseteq C$.

**2. Proof of the existence theorem.**

We start with the observation that the natural reduction procedure for Hamiltonian $K$-spaces is valid for Liouville forms. Recall that a Hamiltonian $K$-space is given by a symplectic manifold $(X, \omega)$, a (connected) Lie group $K$ acting on $X$ by symplectic diffeomorphisms, and a moment map $\Phi: X \to k^{*}$ (here $k$ denotes the Lie algebra of $K$ and $k^{*}$ its dual vector space), i.e., a $K$-equivariant map (with respect to the given $K$-action on $X$ and the coadjoint action on $k^{*}$) satisfying the moment condition

$$d\Phi_{a} = i_{a_{X}}\omega.$$  

Here $\Phi_{a}$ denotes the $a$-th component of $\Phi$, i.e., $\Phi_{a} = \langle \Phi, a \rangle$, and $a_{X}$ the vector field on $X$ associated with $a \in k$.

An important case where a moment map exists is the following. Suppose $(X, \omega)$ is symplectic, suppose $\beta$ is a Liouville form (i.e., $-d\beta = \omega$), and suppose that $K$ acts by diffeomorphisms respecting $\beta$, $k^{*}\beta = \beta$ for all $k \in K$. Then a natural moment map is given by the formula

$$\Phi_{a} = \langle \beta, a_{X} \rangle.$$  

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Consider next the moment level $Z = \Phi^{-1}(0) \subseteq X$. Suppose now that $K$ acts freely and properly on $X$. Then $Z$ is a submanifold and the natural projection $\nu: Z \to Z/K$ gives $Z$ the structure of a $K$-principal bundle over $X_0 := Z/K$. Marsden-Weinstein observed that there exists a unique 2-form $\omega_0$ on $X_0$ so that $\nu^*\omega_0 = i^*\omega$, where $i: Z \hookrightarrow X$ is the inclusion map.

**Proposition.** — Let $(X, \omega)$ be symplectic, let $\beta$ be a Liouville form, let $K$ be a Lie group acting freely and properly and respecting $\beta$. Let $\Phi$ be the natural momentum, $i: Z \hookrightarrow X$ the inclusion and $\nu: Z \to X_0$ the natural projection. Then there exists a unique 1-form $\beta_0$ on $X_0$ satisfying $\nu^*\beta_0 = i^*\beta$.

**Proof.** — Since $\beta$ is $K$-invariant it suffices to prove that $\beta(z)$ vanishes in the fibre direction, for every $z \in Z$. A typical vector $\xi$ tangent to the fibre of $\nu: Z \to X_0$ is given by $\xi = a_X(z)$ for some $a \in \mathfrak{k}$. Thus

$$\langle \beta(z), \xi \rangle = \langle \beta(z), a_X(z) \rangle = \Phi_\alpha(z) = 0,$$

by the definition of $Z$. \hfill \Box

**Remark.** — (a) By the uniqueness of the Marsden-Weinstein symplectic structure $\omega_0$ on $X$ it follows that $-d\beta_0 = \omega_0$, i.e., $\beta_0$ is a Liouville form on $X_0$.

(b) Sjamaar-Lerman [SjLe] have proved a much more general statement of a symplectic structure on the quotient space $X_0$, when $K$ is acting not necessarily freely. In particular they proved that $X_0$ is a stratified space where on each stratum (which is a smooth manifold) there exists a unique symplectic structure compatible with the projection map. The above argument shows that in case of a Liouville form $\beta$ on $X$, there exists a unique Liouville form $\beta_0$ on $X_0$ in the appropriate sense, in particular a Liouville form on each of the strata.

An application of the proposition is given by the existence of a canonical 1-form $\alpha$ on the standard symplectic manifold $C$ associated to a symplectic vector bundle $E$ (of rank $2l$) over a manifold (of dimension $n$) due to Weinstein [We2].

We recall the construction. Let $Q$ be the standard symplectic vector space of dimension $2l$. Let $\pi: P \to M$ be the $\text{Sp}_{2l}(\mathbb{R})$-principal bundle of symplectic frames of $E \to M$. Then $Q$ as well as $T^*P$ carry a natural structure of a Hamiltonian $K$-space, $K = \text{Sp}_{2l}(\mathbb{R})$, coming from the natural Liouville forms. Therefore the product space $T^*P \times Q$ is again a Hamiltonian $K$-space and the moment map comes from the Liouville form
(see ([1])). Moreover, since $K$ acts freely and properly on $P$, the Marsden-Weinstein quotient $C := (T^* P \times Q)_0$ exists and the above proposition shows that $C$ carries again a canonical 1-form.

Consider now a manifold $X$ and let $\beta$ be a 1-form on $X$ such that $\omega := -d\beta$ is non-degenerate. Let $\iota: M \hookrightarrow X$ be a submanifold of $X$ sitting in the zero level of $\beta$, i.e., $\beta|M = 0$. Further let $\eta$ be the associated contracting vector field on $X$ given by $i_\eta \omega = \beta$. Then $\eta$ vanishes on $M$, too. For every $m \in M$ we therefore can build the derivative of $\eta$, i.e., $L_m: TX_m \to TX_m$ given by $L_m(\xi) = [\xi, \eta](m)$ (where $\xi$ is an arbitrary vector field on $X$ with $\xi(m) = 0$).

The so-called infinitesimal conformal symplectic transformations of a symplectic vector space $(V, \omega)$ are defined by linear transformations $T: V \to V$ satisfying

$$\langle \omega, Tv_1 \wedge v_2 \rangle + \langle \omega, v_1 \wedge Tv_2 \rangle = \lambda \langle \omega, v_1 \wedge v_2 \rangle$$

for some $\lambda \in \mathbb{R}$. It is not difficult to see (cf. [GuSt], chap. 4) that our derivative $L_m: TX_m \to TX_m$ is conformal symplectic with factor $\lambda = -1$, i.e., $L_m + \frac{1}{2} \text{id} \in \mathfrak{sp}(TX_m)$. We conclude that the characteristic exponents of $\eta$ in $m$ are symmetric with respect to $\nu = -\frac{1}{2}$, i.e., $-\frac{1}{2} + \nu$ is an exponent if and only if $-\frac{1}{2} - \nu$ is an exponent. From

$$\langle \omega_m, L_m \xi_1 \wedge \xi_2 \rangle + \langle \omega_m, \xi_1 \wedge L_m \xi_2 \rangle = -\langle \omega_m, \xi_1 \wedge \xi_2 \rangle,$$

for all $m \in M$ and $\xi_1, \xi_2 \in TX_m$, it follows now easily that $M$ is necessarily isotropic in $(X, -d\beta)$. In fact, since $L_m|TM_m = 0$, we have $\langle \omega_m, \xi_1 \wedge \xi_2 \rangle = 0$ for all $\xi_1, \xi_2 \in TM_m$. Furthermore $TM_m^\perp$ is an $L_m$-invariant subspace of $TX_m$. For this let $\xi_1 \in TM_m^\perp$ and $\xi_2 \in TM_m$ and compute again:

$$-\langle \omega_m, L_m \xi_1 \wedge \xi_2 \rangle = \langle \omega_m, \xi_1 \wedge L_m \xi_2 \rangle + \langle \omega_m, \xi_1 \wedge \xi_2 \rangle = 0 + 0 = 0.$$

Moreover, again using that $TM_m \subseteq \ker(L_m)$, $L_m$ induces a transformation $\Lambda_m$ on the symplectic normal vector space $E_m = TM_m^\perp/ TM_m$, $\Lambda_m(\xi + TM_m) := L_m(\xi) + TM_m$, which is again an infinitesimal conformal symplectic transformation with factor $-1$, i.e.,

$$\Lambda_m + \frac{1}{2} \text{id}_{E_m} \in \mathfrak{sp}(E_m).$$

Recall that we have called $\beta$ a special Liouville form with center $M$, if $\Lambda_m = -\frac{1}{2} \text{id}$ for every $m \in M$. What we just required is, that the tangent space $TX_m$ decomposes into the direct sum of eigenspaces of $L_m$ corresponding to the eigenvalues $\nu = 0$, $\nu = -1$ and $\nu = -\frac{1}{2}$,

$$TX_m = \text{Eig}(L_m, 0) + \text{Eig}(L_m, -1) + \text{Eig}(L_m, -\frac{1}{2}).$$
In fact, a similar computation as above using the conformal symplectic property of $L_m$ shows that $\ker(L_m) \subseteq TM_m^\perp$. Thus, since $TM_m \subseteq \ker(L_m)$ and since $\Lambda_m$ is in particular injective, $\text{Eig}(L_m, 0) = \ker(L_m) = TM_m$. Then, using the conformal symplectic property of $L_m$ again, one must have an eigenspace $\text{Eig}(L_m, -1)$ of the same dimension $n = \dim M$. Moreover, for an infinitesimal symplectic transformation $T$ of a symplectic vector space $(V, \omega)$ one similarly observes (cf. [GuSt], chap. 4) that $\omega$ induces a non-degenerate pairing on the pair of eigenspaces $\text{Eig}(T, \nu) \times \text{Eig}(T, -\nu)$ for any eigenvalue $\nu$ of $T$. This gives an identification of $T^*M_m$ with $\text{Eig}(L_m, -1)$. Therefore the eigenspace of $L_m$ corresponding to $\nu = -\frac{1}{2}$ is a realization of the symplectic normal space $E_m$ in $TX_m$ and in particular such a special Liouville form $\beta$ induces a splitting of the vector bundle sequences

$$0 \rightarrow TM \rightarrow TX|M \rightarrow N_{X/M} \rightarrow 0$$

and

$$0 \rightarrow TM \rightarrow TM^\perp \rightarrow E \rightarrow 0$$

over $M$. Here $N_{X/M} = (TX|M)/TM$ denotes the geometric normal bundle of $M$ in $X$.

Existence Theorem. — Let $X_j$ be a manifold and $\beta_j$ a special Liouville form with center $M \subset X_j$ ($j = 0, 1$). Suppose that the symplectic normal bundles are isomorphic. Then there exists neighborhoods $U_j \subset X_j$ of $M$ and a diffeomorphism $f_j: U_j \rightarrow X_j$ with $f_j|_M = \text{id}_M$ and $f_j^*(-d\beta_j) = -d\alpha$, where $\alpha$ is the canonical 1-form on $C$, by Weinstein's isotropic embedding theorem. Moreover, Weinstein's version of the Darboux-Moser-Weinstein theorem (see [Wel]) shows that one can even achieve that not only $f_j|_M = \text{id}_M$ but moreover $f_j^*|(TX|M) = \text{id}_{TX|M}$. Thus the Liouville forms $f_j^*\beta_j$ are again special. We may therefore assume that $X_0 = X_1 = C$, $\beta_0 = \alpha$, and $-d\beta_1 = -d\alpha = \omega$.

Proof of the theorem. — By using an appropriate bundle isomorphism one may assume that $\beta_1 =: \beta$ and $\beta_0 = \alpha$ coincide along $M$ up to the first order (since both are special). Following the usual proof of the
Darboux-Moser-Weinstein theorem, let
\[ \beta_t := \beta_0 + t(\beta_1 - \beta_0), \]
\[ t \in [0, 1], \]
be the straight line curve of 1-forms connecting \( \beta_0 \) with \( \beta_1 \). Since \( \sigma \) induces an isomorphism on the corresponding cohomology groups and \( \sigma^*(\beta_1 - \beta_0) = 0 \) it follows that there exists a smooth function \( H \) on \( C \) so that
\[ dH = \beta_1 - \beta_0. \]

By using explicit integration formulas one can achieve that \( H|M = 0, \)
\[ dH|M = 0 \] and \( \text{Hess}_M(H) = 0, \]
since \( \beta_1 - \beta_0 \) vanishes of the first order along \( M. \)

We look now for a curve \( t \mapsto f_t \) in \( \text{Diff}(C) \) with \( f_0 = \text{id} \) and \( f_t^*\beta_t = \beta_0 \)
(in particular \( f := f_1 \) fulfills \( f^*\beta = \alpha \)). This is equivalent to
\[ 0 = \frac{d}{dt} (f_t^*\beta_t) = f_t^* \left( \mathcal{L}_{\xi_t}\beta_t + \frac{d}{dt} \beta_t \right) \]
by the Leibniz rule. Here \( t \mapsto \xi_t \) denotes the corresponding curve in the
vector fields of \( C \), i.e.,
\[ \xi_t(f_t(x)) = \frac{d}{dt} f_t(x), \]
and \( \mathcal{L}_{\xi_t} \) denotes the Lie derivative in direction of the vector field \( \xi_t \). Since \( \mathcal{L}_{\xi_t} = i_{\xi_t}d + di_{\xi_t} \) and \( \frac{d}{dt} \beta_t = dH \), we end up with
\[ 0 = d(i_{\xi_t}\beta_t) + i_{\xi_t}(d\beta_t) + dH = d(i_{\xi_t}\beta_t) - i_{\xi_t}\omega + dH. \]
So far we have followed the familiar proof.

Now we observe that the desired curve of diffeomorphisms must satisfy
\[ f_t \in \text{Symp}(C) = \{ f \mid f^*\omega = \omega \}; \]
thus \( \xi_t \in \text{symp}(C) = \{ \xi \mid \mathcal{L}_{\xi}\omega = 0 \}. \)
Inside \( \text{symp}(C) \) there are the Hamiltonians \( \xi_g \), i.e., those which are
associated to functions \( g \) on \( C \) by \( i_{\xi_g}\omega = dg \), \( \text{ham}(C) = \{ \xi_g \mid g \in C^\infty(C) \}. \)
Therefore we make the "ansatz"
\[ \xi_t := \xi_{gt} \]
and look for an equation of the desired curve \( t \mapsto g_t \) in \( \text{Diff}(C) \). Obviously
we have \( i_{\xi_t}\omega = dg_t \) and furthermore \( i_{\xi_t}\beta_t = i_{\xi_t}i_{\eta_t}\omega \), where \( \eta_t \) is the
associated contracting vector field with respect to the Liouville form \( \beta_t. \)
But
\[ i_{\eta_t}\omega = \beta_t = \beta_0 + tdH = i_{\eta_0 + t\xi_H}\omega, \]
and therefore \( \eta_t = \eta_0 + t\xi_H, \) since \( \omega \) is non-degenerate and where \( \eta_0 = \eta \) is the
canonical vector field on \( C. \) We come down to
\[ i_{\xi_t}\beta_t = i_{\xi_t}i_{\eta_t}\omega = -i_{\eta_t}i_{\xi_t}\omega = -i_{\eta_t}(dg_t) = -(\eta + t\xi_H)(g_t). \]
Our equation to solve is therefore
\[ 0 = -(\eta + t\xi_H)(g_t) - g_t + H \]
or
\[ (\text{id} + \eta + t\xi_H)(g_t) = H. \]
Observe now that \( \xi_H \) vanishes of the first order along \( M \), because \( H \) vanishes of the second order. Therefore \( T := \text{id} + \eta + t\xi_H \) may be seen as a perturbation of the differential operator \( \text{id} + \eta \). As a consequence of the next lemma, we will prove that there exists a unique solution \( g_t \) which vanishes of the second order along \( M \). This solution also depends differentiably on \( t \) and thus we have found our curve \( t \mapsto g_t \).

Consider now \( M = \mathbb{R}^n \) linearly embedded in \( X = \mathbb{R}^n \times \mathbb{R}^r \) as \( M = \{ (x_1, x_2) \in \mathbb{R}^{n+r} \ | \ x_2 = 0 \} \). Denote by \( \mathcal{E} = \mathcal{E}_{n,r} \) the set of germs of smooth functions around \( M \) in \( X \). Let \( \mathfrak{m} \) be the ideal of (germs of) functions vanishing on \( M \) and more generally for any positive integer \( k \) let \( \mathfrak{m}^k \) denote the functions vanishing on \( M \) up to the \( (k-1) \)-st order.

**Lemma.** — Let \( k \) be a non-negative integer and \( A: M \to \text{Mat}(r, \mathbb{R}) \), \( A(x_1) = (a^\rho_\sigma(x_1))_{1 \leq \rho, \sigma \leq r} \), a matrix-valued smooth function so that \( A(x_1) \) is semi-simple and for any eigenvalue \( \nu(x_1) \) of \( A(x_1) \) let \( \text{Re}(\nu(x_1)) \leq -1 \).

Let \( \xi \) be (a germ of) a vector field along \( M \) which vanishes of the first order. Then the linear partial differential operator \( T: \mathcal{E} \to \mathcal{E} \),

\[ T = k \cdot \text{id} + \sum_{\rho, \sigma = 1}^r a^\rho_\sigma(x_1)x_2^\rho \frac{\partial}{\partial x_2^\rho} + \xi \]

maps \( \mathfrak{m}^{k+1} \) bijectively to itself.

Let \( (\varphi^s) \) be the flow associated with the vector field \( \eta := T - k \cdot \text{id} \). Then the flow exists for all positive time \( s \) and for every \( h \in \mathfrak{m}^{k+1} \) the preimage under \( T \) is given by

\[ g(x) = -\int_0^\infty e^{ks} h(\varphi^s x) \, ds. \]

Before going to the proof, let us first make some remarks concerning the existence of the integral and on its smooth dependence on \( x \). Denoting by

\[ \varphi^s(x) = (\varphi^s_1(x), \varphi^s_2(x)) \in \mathbb{R}^n \times \mathbb{R}^r \]

the components, it follows from standard results in dynamical systems (see [Ha], chap. 9, e.g.) that the flow \( (\varphi^s) \) converges almost as fast to its limit as its linear part does. Precisely,

\[ \varphi^s_2(x) = o(e^{-(1-\varepsilon)s}) \]
for every fixed $x$ and every $\varepsilon > 0$, since the real parts of the eigenvalues of $A$ have real part less or equal to $-1$. Therefore, since $h$ vanishes up to the $k$-th order, we find that $h(\varphi^s x) = o(e^{-(k+1-\varepsilon)s})$ and this gives the uniform convergence of the functions $g_s(x) = -\int_0^s e^{ks} h(\varphi^s x) \, ds$ for $s \to \infty$. Moreover, the limit $g = \lim_{s \to \infty} g_s$ is smooth and its $x$-derivative, denoted in the sequel by $D$, can be carried out under the integral,

$$Dg(x) = -\int_0^\infty e^{ks} D(h(\varphi^s x)) \, ds.$$ 

A similar statement for the $x$-derivative $D\varphi^s(x)$ of the flow and also the higher derivatives implies that $g$ is in fact smooth.

The second remark concerns the smoothness of the solution with respect to an additional parameter in the case where the vector field $\eta$ depends smoothly on some additional parameter. In particular, if

$$\eta_t = \sum_{\rho, \sigma=1}^r a^\rho_\sigma x^\rho \frac{\partial}{\partial x^\sigma} + t\xi$$

for $t \in \mathbb{R}$, then, again by standard results, the flow $(\varphi^s(x, t))$ depends smoothly on $(s, x, t)$ and one can see by similar arguments as above that the solution of $(k \cdot \text{id} + \eta_t)(g_t) = h$, i.e.,

$$g_t(x) = -\int_0^\infty e^{ks} h(\varphi^s(x, t)) \, ds$$

is smooth in $(x, t)$.

As a third remark, there is a version of the lemma in the manifold setting. In fact, by the uniqueness of the solution, one may assume that $M$ is an arbitrary manifold (of dimension $n$) embedded as a submanifold in another manifold $X$ (of dimension $n+r$). Denoting by $E = E_{X,M}$ the germs of functions around $M$ in $X$ and by $m^k$, $k \in \mathbb{Z}_+$, its ideals as above, let $\eta$ be (a germ of) a vector field on $X$ which vanishes on $M$. Then $\eta$ induces a derivative $L: TX|M \to TX|M$ along $M$, and moreover, since $\eta|M = 0$ and therefore $L|TM = 0$, it induces a bundle homomorphism on the normal bundle $N_{X/M}$ of $M$, $\Lambda: N_{X/M} \to N_{X/M}$. Then we make the assumption that $\Lambda$ is semisimple and that the eigenvalues of $\Lambda$ have real part less or equal to $-1$. If $(\varphi^s)$ denotes the flow of $\eta$ on $X$, and if $h \in m^{k+1}$, the lemma asserts that the operator $T = k \cdot \text{id} + \eta$ gives a bijection of $m^{k+1}$ to itself and moreover the preimage for any $h \in m^{k+1}$ is given by the formula

$$g(x) = -\int_0^\infty e^{ks} h(\varphi^s x) \, ds.$$
Since the canonical vector field \( \eta \) on \( C \) has exponents \( \nu = -\frac{1}{2} \) and \( \nu = -1 \) on the geometrical normal bundle of \( M \), applying the lemma in this version with an additional parameter \( t \) and with \( k = 2 \) to the equation

\[
(2i\eta + 2\alpha t \xi_H)(g_t) = 2H,
\]
we find the desired solution curve \( t \mapsto g_t \) for the proof of the theorem.

**Proof of the lemma.** — For the uniqueness let \( f \in m^{k+1} \) and assume that \( Tf = 0 \). We have to show that \( f = 0 \). Let \((\cdot, \cdot)\) denote the standard inner product on \( \mathbb{R}^{n+r} \) and let \( a: \mathbb{R}^{n+r} \to \mathbb{R}^{n+r} \) be the vector-valued function describing the vector field \( \eta \), i.e., \( \eta(f) = (a, \text{grad})(f) \). Now fix \( x \in X = \mathbb{R}^{n+r} \) and consider the function \( \lambda: [0, \infty) \to X, \ s \mapsto e^{ks} f(\varphi^s x) \). Since \( f \in m^{k+1} \) we have \( \lim_{s \to \infty} \lambda(s) = 0 \) and of course \( \lambda(0) = f(x) \). But

\[
\lambda'(s) = e^{ks} (k \cdot f(\varphi^s x) + (a(\varphi^s x), \text{grad}f(\varphi^s x))) = e^{ks} Tf(\varphi^s x) = 0, \]

which implies \( f(x) = 0 \).

For the existence observe that

\[
D(\varphi^s x)a(x) = a(\varphi^s x),
\]

This is true for \( s = 0 \) and both sides give a solution of the non-autonomous linear differential equation

\[
z' = Da(\varphi^s x)z,
\]

where \( ' \) denotes differentiation with respect to \( s \) and \( x \) is fixed. In fact,

\[
\frac{d}{ds} (a(\varphi^s x)) = Da(\varphi^s x)a(\varphi^s x),
\]

since \( (\varphi^s) \) is the flow for the equation \( x' = a(x) \). On the other hand, differentiating the equation \( \frac{d}{ds} \varphi^s (x) = a(\varphi^s x) \) with respect to \( x \) gives

\[
\frac{d}{ds} D(\varphi^s x) = Da(\varphi^s x)D\varphi^s(x).
\]

So

\[
\frac{d}{ds} (D\varphi^s(x)a(x)) = Da(\varphi^s x)(D\varphi^s(x)a(x)).
\]

Computing directly, we have:

\[
h(x) = - \int_0^\infty \frac{d}{ds} (e^{ks} h(\varphi^s x)) \ ds
\]

\[
= - \int_0^\infty e^{ks} (k \cdot h(\varphi^s x) + (\text{grad}h(\varphi^s x), a(\varphi^s x))) \ ds
\]

\[
= - \int_0^\infty e^{ks} (k \cdot h(\varphi^s x) + (\text{grad}h(\varphi^s x), D\varphi^s(x)a(x))) \ ds
\]

\[
= - \int_0^\infty e^{ks} (k \cdot h(\varphi^s x) + (a(x), \text{grad})(h(\varphi^s x))) \ ds
\]

\[
= -(k + (a(x), \text{grad})) \left( \int_0^\infty e^{ks} h(\varphi^s x) \ ds \right)
\]

\( = Tg(x) \). □
Remark. — An inspection of the proof shows that we can also give a normal form for Liouville forms which are not necessarily special. Of course, a necessary condition that $\beta_0$ is a pullback of $\beta_1$ via a diffeomorphism is that the induced transformations $\Lambda_0$ and $\Lambda_1$ of $E$ must coincide up to conjugation of a bundle isomorphism $h: E \to E$. But the proof only works, if the eigenvalues of $\Lambda_j$ are in the interval $[-\frac{2}{3}, -\frac{1}{3}]$ (i.e., the associated contracting vector field has to contract $C$ to $M$ fast enough). Otherwise, the explicit integration formula ([2]) does not necessarily converge.

3. Proof of the uniqueness theorem.

To formulate the uniqueness result we need some additional information about the standard model $C$ associated with a symplectic vector bundle $E$ over $M$. Let $\pi: P \to M$ be the associated $K$-principal bundle of symplectic frames of $E \to M$, $K = \text{Sp}_{2l}(\mathbb{R})$, $2l = \text{rank}(E)$. Denote by $\text{pr}_P: T^*P \to P$ the natural projection, $\text{pr}_1: T^*P \times Q \to T^*P$ the projection onto the first factor, and by $i: Z \hookrightarrow T^*P \times Q$ the inclusion of the moment level $Z = \Phi^{-1}(0)$ of $T^*P \times Q$. Since all these maps are $K$-equivariant (where $K$ acts trivially on $M$), the composition $\pi \circ \text{pr}_P \circ \text{pr}_1 \circ i: Z \to M$ is $K$-invariant. Therefore there exists a unique map $\pi_C: C \to M$ so that $\pi_C \circ \nu = \pi \circ \text{pr}_P \circ \text{pr}_1 \circ i$, where $\nu: Z \to C$ is the natural projection. It is not hard to see (cf. [Lo]) that $\pi_C$ gives $C$ the structure of a fibre bundle over $M$ with fibre $F := \mathbb{R}^n \times Q$. The structure group of $\pi_C$ is described by the following. Let $H$ be $\text{GL}_n(\mathbb{R}) \times K$ and $V$ be the linear space of homogeneous quadratic polynomials from $Q$ to $\mathbb{R}^n$, $V = \text{Sym}^2(Q, \mathbb{R}^n)$. Then $H$ acts on $V$ by the representation

$$(A, C).b(q) = Ab(C^{-1}q),$$

for $q \in Q$, $b \in V$ and $(A, C) \in H$. Thus we can form the semi-direct product $G := H \times V$ corresponding to that representation. Now $G$ acts on $F = \mathbb{R}^n \times Q$ via

$$((A, C), b).(v, q) = (Av + b(Cq), Cq).$$

This is the structure group of $\pi_C: C \to M$. Essentially it results from the fact that the moment map on $Q$ is homogeneous quadratic (the "angular momentum part," so to say) while the moment map on $T^*P$ is linear on the fibres (the "linear momentum part," so to say). Observe further that the $\mathbb{R}_+$-action on $F$ given by

$$t.(v, q) = (t^2 v, tq)$$

is also $H$-equivariant.
commutes with the $G$-action. This induces a vector field on $C$ which turns out to be $-2$ times the contracting vector field $\eta$ associated with the Liouville form $\alpha$ on $C$ (see [Lo]). Moreover $(0,0) \in F$ is a $G$-fixpoint, i.e., we have a zero section $\sigma: M \hookrightarrow C$, which turns out to be the standard isotropic embedding. Finally we observe that the subspace $\mathbb{R}^n \subseteq F$ is $G$-invariant and $G$ acts on $\mathbb{R}^n$ by its projection on $\text{GL}_n(\mathbb{R})$. Similarly $G$ acts on the quotient $F/\mathbb{R}^n \cong Q$ via its projection on $K$. Thus we have an exact sequence of $G$-spaces

$$0 \rightarrow \mathbb{R}^n \rightarrow F \rightarrow Q \rightarrow 0.$$ 

To this corresponds an exact sequence of $G$-fibre bundles over $M$ with fibres $\mathbb{R}^n$, $F$ and $Q$. Again a computation (see [Lo]) shows that this sequence is given by

$$0 \rightarrow T^*M \xrightarrow{g} C \xrightarrow{h} E \rightarrow 0$$

over $M$. Here $g$ and $h$ are defined in a natural way similar to the construction of the map $\pi_C: C \rightarrow M$.

Denote by $\text{Aut}(C)$ the group of the associated bundle isomorphisms of $C$, i.e., $\tau \in \text{Aut}(C)$, if $\tau$ fixes every fibre $C_m := \pi_C^{-1}(m) \cong F$ and every $\tau_m: C_m \rightarrow C_m$ is a transformation of $F$ which is in $G$ (depending differentiably on $m$, of course).

**UNIQUENESS THEOREM.** — Let $M$ be a manifold, $E \rightarrow M$ a symplectic vector bundle, $C$ the standard model associated with $E \rightarrow M$ and $\alpha$ its canonical 1-form. Let $f: C \rightarrow C$ be a diffeomorphism with $f|_M = \text{id}_M$ and $f^*\alpha = \alpha$. Then $f$ is a bundle isomorphism, $f \in \text{Aut}(C)$, which fixes the subbundle $T^*M$, $f|T^*M = \text{id}_{T^*M}$.

**Proof.** — Let us first consider the case $E = 0$, i.e., $C = T^*M$. Since the diffeomorphism $f$ respects $\alpha$, it respects the associated contracting vector field $\eta$. In particular, for any $m \in M$, $f$ respects the stable manifold $S_m = \{ c \in C \mid \lim_{t \to \infty} \phi^t(c) = m \}$, where $(\phi^t)$ is the flow associated with $\eta$. Of course, in our case $\eta$ is just $-1$ times the Euler vector field on the vector bundle $T^*M \rightarrow M$, i.e., $S_m = T^*M_m$.

Next let us look at the derivative $F$ of $f$ along $M$, i.e., $F_m := df_m: TC_m \rightarrow TC_m$. We have $TC_m = TM_m + T^*M_m$ with its natural symplectic structure. Furthermore, due to the fact that $f|_M = \text{id}_M$, $F_m|TM_m = \text{id}_{T^*M_m}$ and $T^*M_m$ is $F_m$-invariant by the preceding remark. It follows immediately from the definition that a symplectic transformation $T$...
of the symplectic vector space $V + V^*$, which is a direct sum, $T = s + t$ for $s: V \to V$ and $t: V^* \to V^*$, must satisfy $t = s^*$. Therefore $F_m = \text{id}_{TC_m}$ for all $m \in M$. Now $f_m := f|T^*M_m$ is a diffeomorphism of $T^*M_m \cong \mathbb{R}^n$ which commutes with the Euler vector field and which is therefore $\mathbb{R}_+$-equivariant with respect to the natural $\mathbb{R}_+$-action. The condition $df_m(0) = \text{id}$ implies that $f_m$ fixes every orbit, which is just a straight line and therefore (using again that $df_m(0) = \text{id}$) $f_m = \text{id}_{T^*M_m}$, i.e., $f = \text{id}_X$.

The next step is to transform as far as possible the preceding discussion to the more general case. First, the stable manifold of the canonical vector field $\eta$ on $C$ is again the fibre $C_m := \pi_C^{-1}(m) \cong F = \mathbb{R}^n \times Q$. Furthermore, inside the stable manifold $C_m$ sits the "even more" stable manifold corresponding to the eigenvalue $1 - \varepsilon$ for some $0 < \varepsilon < \frac{1}{2}$, i.e.,

$$S_m = \{ c \in S_m \mid \varphi^t(c) = O(e^{-(1-\varepsilon)t}) \}$$

(see [Ha], chap. 9), which is just $T^*M_m \subset C_m$. Thus we see that $T^*M \subset C$ is $f$-invariant. Now the restriction of the canonical 1-form $\alpha$ on $T^*M$ is just the canonical 1-form on $T^*M$, thereby showing that $f|T^*M = \text{id}_{T^*M}$.

Let $f_m := f|C_m: C_m \to C_m$. For each $m \in M$ we now construct an element $\tau_m = (\langle A_m, c_m \rangle, b_m) \in G$ induced by $f_m$. First observe that every diffeomorphism $f$ respects the symplectic form $\omega$ on $C$, i.e., $f^*\omega = \omega$, and fixing the zero-section pointwise, $f|M = \text{id}_M$. Thus $f$ induces a symplectic bundle isomorphism $\gamma$ of $E \to M$. In fact, since the derivative $F_m: (TC)_m \to (TC)_m$ of $f$ is symplectic, i.e.,

$$\langle \omega_m, F_m \xi_1 \wedge F_m \xi_2 \rangle = \langle \omega_m, \xi_1 \wedge \xi_2 \rangle$$

for all $\xi_1, \xi_2 \in (TC)_m$, and $F_m|TM_m = \text{id}_{TM_m}$, the $\omega_m$-orthogonal $TM_m^\perp$ is also $F_m$-invariant: for $\xi_1 \in TM_m^\perp$ and $\xi_2 \in TM_m$ compute

$$\langle \omega_m, F_m \xi_1 \wedge \xi_2 \rangle = \langle \omega_m, F_m \xi_1 \wedge F_m \xi_2 \rangle = \langle \omega_m, \xi_1 \wedge \xi_2 \rangle = 0.$$ 

Therefore $F_m$ induces a linear transformation $\gamma_m$ on $E_m = TM_m^\perp/\{TM_m \}$ which is clearly symplectic with respect to its natural induced structure.

For our diffeomorphism $f$ of $C$, which even respects $\alpha$, we have already seen that $F_m|T^*M_m = \text{id}_{T^*M_m}$. So we set $A_m := \text{id}_{T^*M_m}$ and $c_m := \gamma_m \in \text{Sp}(E_m)$. To find the element $b_m \in \text{Sym}^2(E_m, T^*M_m)$, we consider the second derivative of $f_m: C_m \to C_m$ in the origin which is a symmetric bilinear map $T(C_m)_m \times T(C_m)_m \to T(C_m)_m$. By restricting this map to $E_m \times E_m \subset T(C_m)_m \times T(C_m)_m$ and then projecting from $T(C_m)_m$ to $T^*M_m$, we obtain a symmetric bilinear map $E_m \times E_m \to T^*M_m$. Here we have used the realization of $E_m$ in $T(C_m)_m$ as the $-\frac{1}{2}$-eigenspace of
Let $b_m \in \text{Sym}^2(E_m, T^*M_m)$ be the associated quadratic form. In summary, using the first and second derivatives of $f$, for each $m \in M$ we have $\tau_m = ((A_m, c_m), b_m) \in \text{Aut}(C_m)$. These fit together to form an automorphism $\tau \in \text{Aut}(C)$ with $\tau|T^*M = \text{id}_{T^*M}$.

Now, in order to prove the uniqueness assertion, by composing $f$ with $\tau^{-1}$, we may assume that $b_m = 0$ and $\gamma_m = \text{id}_{E_m}$ for all $m \in M$. We want to show that $f = \text{id}_C$. Since $f_m: C_m \rightarrow C_m$ respects the canonical vector field $\eta$, it is $\mathbb{R}_+^n$-equivariant with respect to the action $t.(v, q) = (t^2 v, tq)$ on $C_m \cong F$. Using the equivariance and the second derivative in 0, the condition $\tau_m = \text{id}$ implies that $f_m$ must stabilize every orbit. We conclude that $f_m = \text{id}_{C_m}$, i.e., $f = \text{id}_C$. This finishes the proof of the uniqueness theorem.

Remark. — (a) Although not explicitely formulated, the theorem was proved by Kostant-Sternberg in [GuSt] in the Lagrangean case, i.e., $E = (0)$. Note that this means that $f$ is simply the identity.

(b) The proof shows that the theorem is true not only for special Liouville forms. More precisely, the proof works for Liouville forms $\beta$ where the associated bundle $\Lambda: E \rightarrow E$ has its eigenvalues in the open interval $(-1, 0)$, since $\eta$ has to be contracting (cf. the remark in section 2).

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Frank LOOSE,
Mathematisches Institut
der der Eberhard-Karls-Universität
Auf der Morgenstelle 10
72076 Tübingen (Allemagne).
frank.loose@uni-tuebingen.de