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ON BOUNDARY SLOPES OF IMMERSED INCOMPRESSIBLE SURFACES

by Mark D. BAKER

1. Introduction.

Let M be a compact, orientable, irreducible 3-manifold with ∂M a torus. Hatcher [H] showed that there are only finitely many slopes on ∂M realized by boundary curves of embedded incompressible, ∂-incompressible surfaces in M.

In this paper we show that there can be infinitely many slopes on ∂M realized by the boundary curves of immersed, incompressible, ∂-incompressible surfaces in M which are embedded in a neighborhood of ∂M.

2. Notation and statement of results.

Let T_0 be the torus with an open disk removed, pictured in Figure 1. Let D_x (resp. D_y) denote the right handed Dehn twist about the loop x (resp. the loop y).

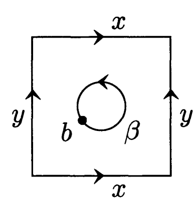

Figure 1

Consider the manifold M, a once punctured torus bundle over S^1, given by

$$M \cong T_0 \times [0,1]/\big(h(s),0\big) \sim (s,1)$$

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where $h = D_y^{-4} \circ D_x^4$. Fix a basepoint $b \in \partial T_0$. The loops $\alpha = b \times [0,1]/\sim$, $\beta = \partial T_0$ give a coordinate system for ∂M; the loop $\alpha^\mu \beta^\lambda$ in ∂M is represented by the pair $(\mu, \lambda) \in \mathbb{Z}^2$ and is said to have *slope* μ/λ.

Now, if $S \subset M$ is an immersed surface, properly embedded in a neighborhood of ∂M, then ∂S consists of parallel simple closed curves in ∂M parametrized by a coprime pair (μ, λ).

THEOREM. — *For M as above, the coprime pair (μ, λ) is realized by the boundary curves of an immersed, incompressible, ∂-incompressible surface provided $\mu \geq 1$ and $|\lambda| > \mu$.*

Remarks.

1) It suffices to prove the theorem for (μ, λ) with $\mu \geq 1$ and $\lambda > \mu$ since M admits an involution sending the curves α, β to $\alpha, -\beta$. Indeed, let $k : T_0 \to T_0$ be a reflection in the diagonal (see Figure 1) followed by D_y^{-4}. Then k^2 is isotopic to $D_y^{-4} \circ D_x^4$ and the map $(x, t) \mapsto (k(x), t + \frac{1}{2})$ induces the desired involution on M.

2) The immersed surfaces of the theorem are virtually embedded in M: they lift to embedded surfaces in finite covers of M (see §3) and one obtains virtually Haken manifolds by Dehn filling on M with respect to these boundary slopes.

3) There exist manifolds with torus boundary (N Seifert fibered for example) for which only finitely many slopes on ∂N are realized as the boundary curves of essential immersed surfaces.

3. Proof of theorem.

3.1. — We prove our result by constructing, for each (μ, λ), a finite covering space $\widetilde{M} \to M$ such that:

(i) The loop $\alpha^\mu \beta^\lambda$ lifts to loops in each of the four components of $\partial \widetilde{M}$.

(ii) In $H_1(\widetilde{M}; \mathbb{Z})$ there exists a relation of the form $\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0$ where γ_i is a lift of $\alpha^\mu \beta^\lambda$ to the i-th component of $\partial \widetilde{M}$.

Property (ii) implies that $\widetilde{M}$ contains an incompressible, ∂-incompressible surface S' whose boundary consists of the loops $\gamma_1, \ldots, \gamma_4$. Indeed, if we consider a triangulation for $\widetilde{M}$ and simplicial homology, the fact that $\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4$ is a primitive element in $H_1(\partial \widetilde{M}; \mathbb{Z})$ that is zero in $H_1(\widetilde{M};\mathbb{Z})$ implies that these loops bound an oriented 2-complex, K, in $\widetilde{M}$ with property that an even number of triangles meet at each interior edge, with oriented sum equal to zero. Thus by cutting and pasting along the edges of K, and then pulling apart at vertices, we obtain an embedded surface which can then be compressed. Now by property (i) this surface projects to an immersed incompressible surface S in M with boundary consisting of four parallel copies of $\alpha^\mu\beta^\lambda$.

The cover $\widetilde{M}$ is obtained by constructing a cover $F \to T_0$ to which $h = D_y^{-4} \circ D_x^4$ lifts to a homeomorphism $\tilde{h}: F \to F$. Then $\widetilde{M}$ is the mapping torus of the pair $(F, \tilde{h})$.

3.2. — We construct $F \to T_0$ by cutting and pasting together copies of the following two covers of T_0:

a) The four-fold cover $X' \to T_0$ corresponding to the kernel of the map $\theta: \pi_1(T_0) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ defined by $\theta([x]) = (1,0)$ and $\theta([y]) = (0,1)$.

b) The two-fold cover $Y' \to T_0$ corresponding to the kernel of the map $\theta: \pi_1(T_0) \to \mathbb{Z}/2$ defined by $\theta([x]) = 0$ and $\theta([y]) = 1$.

Note that X' (resp. Y') is a torus with four (resp. two) boundary circles.

Now alter X' (resp. Y') by making two (resp. one) vertical cuts τ_1, τ_2 (resp. τ) between boundary circles as shown in Figure 2a (resp. Figure 2b). Denote the cut surfaces by X and Y.

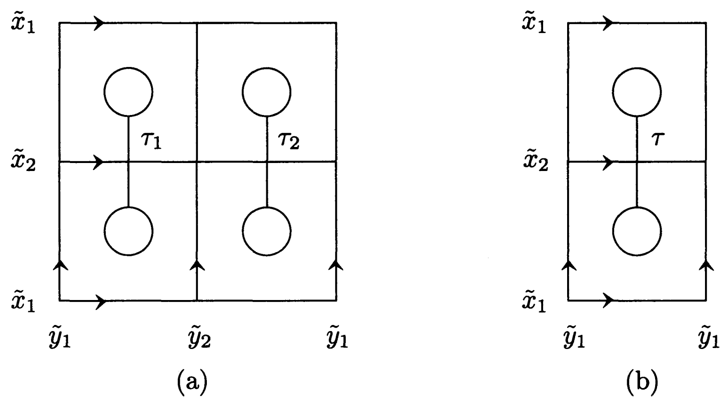

Figure 2

3.3. — Since $\lambda > \mu$, we have that $\lambda - \mu = 2r + s$ for $0 \leq s \leq 1$. Then $F \to T_0$ is obtained by gluing together $\mu + 2r$ copies of X and $2s$ copies of Y as indicated in steps i) – v) below. The cover F for $(\mu, \lambda) = (2, 7)$, with edges glued as numbered, is illustrated in Figure 3.

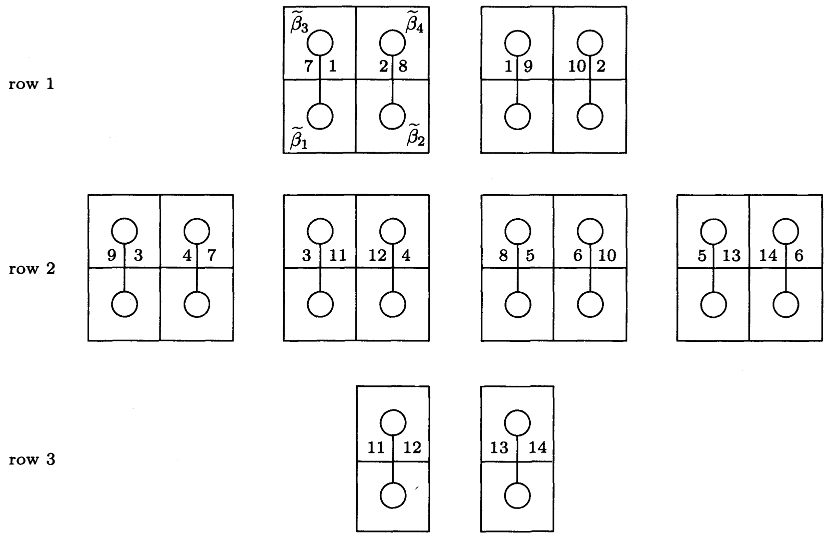

Figure 3

i) First arrange μ copies of X, indexed $X_1, \ldots, X_\mu$, in a row, followed by the remaining $2r$ copies of X, indexed $X_1^2, \ldots, X_{2r}^2$ in a second row. The $2s$ copies of Y, indexed $Y_1, \ldots, Y_{2s}$ form the third row.

Note that row 1 is never empty whereas either row 2 or row 3 (but not both) can be.

ii) In the first row, glue the right edge of τ_1 (resp. left edge of τ_2) in X_i to the left edge of τ_1 (resp. right edge of τ_2) in X_{i+1} for $i = 1, \ldots, \mu - 1$. In the second row, perform the same gluing as in the first row, first on the $X_1^2, \ldots, X_r^2$ for $i = 1, \ldots, r-1$ and then on $X_{r+1}^2, \ldots, X_{2r}^2$ for $i = r+1, \ldots, 2r-1$.

The gluing pattern for the remaining edges depends on the values of r and s. The three distinct cases are treated in iii) – v) below.

iii) *The case* $r > 0$ *and* $s = 1$. There are sixteen remaining edges to glue in this case. Glue the left edge of τ_1 in X_1 to the right edge of τ_2 in X_1^2; the right edge of τ_2 in X_1 to the left edge of τ_1 in X_{r+1}^2; the right edge of τ_1

in X_μ to the left edge of τ_1 in X_1^2; the left edge of τ_2 in X_μ to the right edge of τ_2 in X_{r+1}^2.

Finally, attach Y_1 to X_r and Y_2 to X_{2r} by, in each case, gluing the left edge (resp. right edge) of τ to the right edge of τ_1 (resp. left edge of τ_2).

iv) *The case* $r > 0$ *and* $s = 0$. Glue the copies of X as in iii). Finish by gluing the right edge of τ_1 to the left edge of τ_2, first in X_1^2 and then in X_{2r}^2.

v) *The case* $r = 0$ *and* $s = 1$. There are eight edges to glue. Glue the left edge of τ_1 (resp. right edge of τ_2) in X_1 to the right edge of τ in Y_1 (resp. left edge of τ in Y_2).

Finally, pair the right edge of τ_1 (resp. left edge of τ_2) in X_μ with the left edge of τ in Y_1 (resp. right edge of τ in Y_2).

Note that the surface F is indeed a cover of T_0; some of its properties are given in:

LEMMA 1. — *The surface F is a 4λ-fold cover of T_0. It is of genus $(2\lambda - 1)$ with four boundary components, each of which projects λ to 1 onto $\beta = \partial T_0$.*

3.4. — Now the loop x (resp. y) in T_0 is covered by loops in F which project d to 1 onto x for $d \in 1, 2, 4$ (resp. 2 to 1 onto y). Thus D_x^4 and D_y^{-4} lift to appropriate powers of Dehn twist homeomorphisms in F, whence $h = D_y^{-4} \circ D_x^4$ lifts to a homeomorphism $\tilde{h} : F \to F$, which fixes ∂F pointwise. Denote by $\widetilde{M}$ the mapping torus of $(F, \tilde{h})$:

$$\widetilde{M} = F \times [0,1]/\big(\tilde{h}(s), 0\big) \sim (s, 1).$$

From the construction of $\widetilde{M}$, it is clear that we can choose on each boundary torus, T_i, of $\widetilde{M}$ a pair of loops $\widetilde{\alpha}_i, \widetilde{\beta}_i$ which cover α, β in ∂M. The loop $\widetilde{\alpha}_i$ (resp. $\widetilde{\beta}_i$) projects 1 to 1 onto α (resp. λ to 1 onto β). We will index the four tori in all cases so that the labelling matches that of X_1 in Figure 3.

3.5. — It is clear from the preceding paragraph that the loops $\widetilde{\alpha}_i^\mu \widetilde{\beta}_i$ in $\partial\widetilde{M}$ project homeomorphically to $\alpha^\mu \beta^\lambda$ in ∂M. Thus property (i) of §3.1 is verified and all that remains to show is that property (ii) holds.

3.6. LEMMA 2. — *Let γ_i denote the loop $\widetilde{\alpha}_i^{\mu}\widetilde{\beta}_i$. Then, in $H_1(\widetilde{M};\mathbb{Z})$,*

$$\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0.$$

Hence there is a properly embedded incompressible surface $S' \subset \widetilde{M}$ with $\partial S'$ the collection of boundary curves $\widetilde{\alpha}_i^{\mu}\widetilde{\beta}_i$ which projects to an immersed incompressible surface S in M with boundary consisting of four parallel copies of $\alpha^{\mu}\beta^{\lambda}$.

Proof. — It suffices to show that in $H_1(\widetilde{M};\mathbb{Z})$ the following relation holds:

(*) $$(\mu\widetilde{\alpha}_1 + \widetilde{\beta}_1) - (\mu\widetilde{\alpha}_2 + \widetilde{\beta}_2) + (\mu\widetilde{\alpha}_3 + \widetilde{\beta}_3) - (\mu\widetilde{\alpha}_4 + \widetilde{\beta}_4) = 0.$$

Note that

(a) $\widetilde{\beta}_1 + \widetilde{\beta}_2 + \widetilde{\beta}_3 + \widetilde{\beta}_4 = 0$ the relation being given by the fiber surface F, and

(b) $\mu(\widetilde{\alpha}_1 - \widetilde{\alpha}_2 + \widetilde{\alpha}_3 - \widetilde{\alpha}_4) + 2(\widetilde{\beta}_1 + \widetilde{\beta}_3) = 0.$

Relations (a) and (b) imply (*).

One sees (b) as follows: Consider the loops $\tilde{y}_1$, $\tilde{y}_2$ on X, pictured in Figure 2, and let $\tilde{y}_{1,i}$ and $\tilde{y}_{2,i}$ denote the corresponding loops in $X_i \subset F$, $i = 1, \dots, \mu$ (see §3.3). Now on each of these X_i, let $\gamma_i \subset X_i$ (resp. $\delta_i \subset X_i$) denote a horizontal, properly embedded arc between $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ crossing $\tilde{y}_{2,i}$ (resp. between $\widetilde{\beta}_3$ and $\widetilde{\beta}_4$ crossing $\tilde{y}_{1,i}$). Then the disks $\gamma_i \times I \subset F \times I$ and $\delta_i \times I \subset F \times I$ provide the relations

$$\widetilde{\alpha}_1 - \widetilde{\alpha}_2 + \left(\tilde{h}(\gamma_i) * \gamma_i^{-1}\right) = 0,$$
$$\widetilde{\alpha}_3 - \widetilde{\alpha}_4 + \left(\tilde{h}(\delta_i) * \delta_i^{-1}\right) = 0,$$

where $\tilde{h}(\gamma_i) * \gamma_i^{-1}$ (resp. $\tilde{h}(\delta_i) * \delta_i^{-1}$) denotes path composition and is equal to $2\tilde{y}_{2,i}$ (resp. $-2\tilde{y}_{2,i}$).

Combining the 2μ relations thus obtained gives:

$$\mu(\widetilde{\alpha}_1 - \widetilde{\alpha}_2 + \widetilde{\alpha}_3 - \widetilde{\alpha}_4) + 2(\tilde{y}_2, 1 + \cdots + \tilde{y}_2, \mu) - 2(\tilde{y}_{1,1} + \cdots + \tilde{y}_{1,\mu}) = 0$$

and hence

$$\mu(\widetilde{\alpha}_1 - \widetilde{\alpha}_2 + \widetilde{\alpha}_3 - \widetilde{\alpha}_4) + 2(\widetilde{\beta}_1 + \widetilde{\beta}_3) = 0$$

since the sum of loops about the boundary circles contributing to $\widetilde{\beta}_1$ and $\widetilde{\beta}_3$ in the X_i^2 and Y_i are all homologous to zero. □

4. Concluding remarks.

1) A once-punctured torus bundle will have infinitely many slopes realized by immersed incompressible surfaces if its characteristic homeomorphism is of the form $D_x^{r_1} \circ D_y^{s_1} \circ \cdots \circ D_x^{r_k} \circ D_y^{s_k}$ where $2 \mid s_i$ and $4 \mid r_i$, $i = 1, \ldots, k$, provided that $s_1 + \cdots + s_k \neq 0$. Hence the same will be true for any once-punctured torus bundle whose monodromy (in $\mathrm{SL}_2(\mathbb{Z})$) has a power that is conjugate to the monodromy of one of the above bundles.

2) The cut and paste techniques in §3 can also be used to produce families of once-punctured surface bundles over S^1 of any given genus $g > 1$ having infinitely many slopes realized by immersed incompressible surfaces.

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