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ZETA FUNCTIONS OF JORDAN ALGEBRAS REPRESENTATIONS

by Dehbia ACHAB

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0. Introduction.

Riemann zeta function has been generalized by Epstein as follows : let $\mathfrak{S}$ be a symmetric positive matrix of order k, Epstein zeta function is defined by

$$\zeta_1(\mathfrak{S}, s) = \sum_{g \in \mathbf{Z}^k - \{0\}} \frac{1}{(g'\mathfrak{S}g)^s}, \quad \mathrm{Re}(s) > \frac{k}{2}$$

where g' is the adjoint of g.

In [15], Kœcher has generalized Epstein zeta function as follows :

$$\zeta_m(\mathfrak{S}, s) = \sum_{\mathfrak{U} \in [\mathbf{Z}^{k \times m}/GL(m,\mathbf{Z}), \mathrm{rank}(\mathfrak{U}) = m]} \mathrm{Det}(\mathfrak{U}'\mathfrak{S}\mathfrak{U})^{-s}, \quad (k \geqslant m).$$

Kœcher zeta series converges absolutely and is analytic in the half-plane $\mathrm{Re}(s) > \frac{k}{2}$, it admits an analytic continuation as a meromorphic function on $\mathbf{C}$ and satisfies to the functional equation

$$\mathcal{R}_m(\mathfrak{S}, s) = |\mathfrak{S}|^{-\frac{m}{2}} \mathcal{R}_m\left(\mathfrak{S}^{-1}, \frac{k}{2} - s\right)$$

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where $\mathcal{R}_m(\mathfrak{S}, s)$ is a product of $\zeta_m(\mathfrak{S}, s)$ and some gamma factor, more precisely

$$\mathcal{R}_m(\mathfrak{S}, s) = \pi^{m(\frac{m-1}{4}-s)}\Gamma(s)\Gamma\left(s-\frac{1}{2}\right)\dots\Gamma\left(s-\frac{m-1}{2}\right)\zeta_m(\mathfrak{S}, s)$$

$\Gamma(s)$ being the usual Euler gamma function.

Later, in [15], A. Krieg studied the Kœcher zeta function for the hermitian matrices with quaternionic coefficients defined by

$$\zeta(s) = \sum_{A \in GL(m,\mathcal{O})\backslash[M(m,k,\mathcal{O})|\text{rank}(A)=m]} \text{Det}(A\bar{A}^t)^{-s}$$

where $\mathcal{O}$ is the ring of Hurwitz integers.

This work is situated in a more general context. In fact, we define the Kœcher zeta series associated to a self-adjoint Euclidean Jordan algebra representation and we obtain the above zeta series as particular cases of it. More precisely, let V be an Euclidean simple Jordan algebra of dimension n and rank m, E an Euclidean space of dimension N, ϕ a regular self-adjoint representation of V in the space $\text{Sym}(E)$ of symmetric morphisms of E. Let Q be the quadratic form associated to ϕ, Ω the symmetric cone associated to V and $G(\Omega)$ its automorphism group

$$G(\Omega) = \{g \in GL(V) \mid g(\Omega) = \Omega\}.$$

(H_1) We assume that V and E have $\mathbf{Q}$-structures $V_{\mathbf{Q}}$ and $E_{\mathbf{Q}}$ respectively and that ϕ is defined over $\mathbf{Q}$.

Let L be a lattice in $E_{\mathbf{Q}}$.

We define the zeta series associated to ϕ and L by the following :

$$\zeta_L(s) = \sum_{l \in \Gamma_\circ \backslash L'} [\det(Q(l))]^{-s}, \forall s \in \mathbf{C}$$

where $L' = \{l \in L \mid \det(Q(l)) \neq 0\}$ and $\Gamma_\circ$ is some arithmetic subgroup of $GL(E)$ which we will precise.

Recall that the primitive rank of a Jordan algebra is the cardinality of a maximal complete system of primitive orthogonal idempotents. A Jordan algebra is said to be split if its rank equals its primitive rank.

(H_2) We assume that $V_{\mathbf{Q}}$ is split.

The fundamental results in this work are :

THEOREM 1. — *Under the assumptions* (H_1) *and* (H_2)*, the zeta series converges absolutely for* $\mathrm{Re}(s) > \dfrac{N}{2m}$.

THEOREM 2. — *If the arithmetic subgroup* $\Gamma_\circ$ *is self-adjoint, then the zeta function* ζ_L *admits an analytic continuation as a meromorphic function on the whole plane* $\mathbf{C}$ *and satisfies to the functional equation*

$$\zeta_L\left(\frac{N}{2m}-s\right) = \mathrm{vol}(L)\pi^{\frac{N}{2}-2ms}\frac{\Gamma_\Omega(s)}{\Gamma_\Omega\left(\frac{N}{2m}-s\right)}\zeta_{L^\star}(s)$$

where $\Gamma_\Omega(s)$ *is the Kœcher-Gindikin gamma function of the symmetric cone* Ω *and* $L^\star$ *is the dual lattice of* L.

This article is composed of three parts; the first consisting in the proof of Theorem 1 by using reduction theory, the second is an adaptation of the classical method to prove Theorem 2 and the last one gives some examples.

1. Construction and convergence of the zeta series.

Let V be a simple Euclidean Jordan algebra with unity e, of dimension n and rank m, E an Euclidean space of dimension N, ϕ a representation of V in the space $\mathrm{Sym}(E)$ of self-adjoint endomorphisms of E such that

$$\forall x, y \in V, \quad \phi(xy) = \frac{1}{2}(\phi(x)\phi(y) + \phi(y)\phi(x)),$$

and $Q : E \to V$ the quadratic form associated to ϕ determined by

$$(Q(\xi) \mid x)_V = (\phi(x)\xi \mid \xi)_E \quad \forall x \in V, \quad \forall \xi \in E.$$

For $x \in V$, we denote by $L(x)$ the multiplication endomorphism, $L(x) : V \to V, y \mapsto xy$ and by $P(x)$ the quadratic representation of V, i.e $P(x) = 2L(x)^2 - L(x^2)$. Let Ω be the symmetric cone associated to V and $G(\Omega)$ the automorphism group of Ω,

$$G(\Omega) = \{g \in GL(V) \mid g(\Omega) = \Omega\}.$$

In the sequel, we assume that ϕ is regular, that is $\exists\xi \in E$ such that $\det(Q(\xi)) \neq 0$; then $Q(E) = \bar{\Omega}$ where $\bar{\Omega}$ is the closure of Ω. We assume too that $\phi(e) = \mathrm{id}_E$.

(H_1) Assume that V, E and ϕ are defined over $\mathbf{Q}$, that is there exists a $\mathbf{Q}$-Jordan subalgebra $V_{\mathbf{Q}}$ of V, and a $\mathbf{Q}$-subspace $E_{\mathbf{Q}}$ of E such that

$$V = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}, \quad E = E_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R},$$

and for each $x \in V_{\mathbf{Q}}$ we have $\phi(x) \in \mathrm{Sym}(E)_{\mathbf{Q}}$.

Let L be a lattice in the space $E_{\mathbf{Q}}$. In all the sequel, we denote by *det* the determinant in the Jordan algebra and by *Det* the usual determinant of matrices.

1.1. Arithmetic subgroups associated to ϕ and L.

Let $H = \{(\tilde{h}, h) \in GL(V) \times GL(E) \mid Q(h.\xi) = \tilde{h}.Q(\xi), \forall \xi \in E\}$.

H is non empty because, for each invertible $x \in V$,

$$Q(\phi(x)\xi) = P(x)Q(\xi).$$

It is clear that H is an algebraic subgroup of $GL(V) \times GL(E)$. As ϕ is defined over $\mathbf{Q}$ then it is the same for H. If π_1 and π_2 are the projections of H, then the groups $\pi_1(H)$ and $\pi_2(H)$ are algebraic, defined over $\mathbf{Q}$, we denote them by $G(\phi)$ and $F(\phi)$ respectively.

Notice that $G(\phi) \subseteq G(\Omega)$ and that π_2 is injective. Denote by F and G the identity connected components of $F(\phi)$ and $G(\Omega)$ for the ordinary topologies respectively. Consider the map

$$\begin{aligned} \rho : & F(\phi) \to G(\phi) \\ & f \mapsto \tilde{f} \end{aligned}$$

ρ is well defined because π_2 is injective and we have :

Proposition 1.1.1. — *ρ satisfies to the following :*

(1) *$\rho(F) = G$.*

(2) *ρ is a surjective $\mathbf{Q}$-morphism of algebraic groups.*

(3) *The groups $F(\phi)$ and $G(\phi)$ are self-adjoint $\rho(h^\star) = \rho(h)^\star$. So they are reductive. Moreover $|\mathrm{Det}\rho(h)| = |\mathrm{Det}(h)|^{\frac{2n}{N}}$.*

Proof. — (1) Denote by $\mathfrak{f}$ and $\mathfrak{g}$ the Lie algebras of F and G respectively. It suffices to show that the differential $d\rho : \mathfrak{f} \to \mathfrak{g}$ is surjective. We start by showing the following lemma :

LEMMA 1.1.2. — *For each $x \in V$ we have $\phi(x) \in \mathfrak{f}$ and $d\rho(\phi(x)) = 2L(x)$.*

Proof of the lemma. — For $x \in V, \xi \in E, t \in \mathbf{R}$,

$$\begin{aligned} Q(\exp(\phi(tx)).\xi) &= Q(\phi(\exp(tx)).\xi) \\ &= P(\exp(tx)).Q(\xi) \\ &= \exp(2tL(x)).Q(\xi) \end{aligned}$$

then $\rho(\exp(t\phi(x))) = \exp(2tL(x))$, and,

$$d\rho(\phi(x)) = \frac{d}{dt}|_{t=0}[\rho(\exp(t\phi(x)))] = \frac{d}{dt}|_{t=0}[\exp(2tL(x))] = 2L(x). \quad \square$$

As $\mathfrak{g}$ is generated by the $L(x), x \in V$, then the lemma shows that $d\rho$ is surjective.

(2) As $\rho = \pi_1 \circ i_2$ where i_2 is the injection $F(\phi) \to H, f \mapsto (\tilde{f}, f)$, then ρ is clearly a morphism of algebraic groups. Moreover, as

$$Q(f\xi) = \rho(f)Q(\xi) \forall \xi \in E \Leftrightarrow f^\star \phi(x) f = \phi(\rho(f)^\star x) \quad \forall x \in V,$$

then, if $(e_i)_{1 \leqslant i \leqslant n}$ is a basis of $V_{\mathbf{Q}}$, and $(\epsilon_\alpha)_{1 \leqslant \alpha \leqslant N}$ a basis of E, we have

$$\sum_{\beta,\gamma=1}^{N} f_{\beta\alpha}\phi(e_i)_{\beta\gamma} f_{\gamma\delta} = \phi\left(\sum \rho(f)_{ij} e_i\right) = \sum_{j=1}^{n} \rho(f)_{ij}\phi(e_i)_{\alpha\delta}.$$

The above formula shows that the coefficients of $\rho(f)$ are polynomials of degree 2, with rational coefficients, in the coefficients of f.

(3) Let $h \in F(\phi)$. We know that

$$Q(h\xi) = \rho(h)Q(\xi) \quad \forall \xi \in E \Leftrightarrow h^\star \phi(x) h = \phi(\rho(h)^\star x) \quad \forall x \in V$$

then, for each invertible $x \in V$, we have

$$h^{-1}\phi(x^{-1})h^{\star -1} = \phi((\rho(h)^\star x)^{-1}).$$

As $(\rho(h)^\star x)^{-1} = \rho(h)^{-1} x^{-1}$, we find

$$h^{-1}\phi(x^{-1})h^{\star -1} = \phi(\rho(h)^{-1}x^{-1}) \quad \forall x \in \mathcal{I}(V)$$

where $\mathcal{I}(V)$ is the set of all invertible elements of V. It follows that

$$h^{-1}\phi(x)h^{\star-1} = \phi(\rho(h)^{-1}x) \quad \forall x \in V, \text{ i.e } \rho(h^\star)^{-1} = (\rho(h)^\star)^{-1}.$$

The last assertion is a direct consequence of the properties
$$\operatorname{Det}(\phi(x)) = \det(x)^{\frac{N}{m}} \text{ and } \det(gx) = \operatorname{Det}(g)^{\frac{m}{n}}\det(x)$$
for $x \in V$ and $g \in G(\Omega)$. □

Now consider the arithmetic subgroup $\Gamma_\circ$ of $F(\phi)$ defined by

$$\Gamma_\circ = \{f \in F(\phi) \mid f(L) = L\}.$$

As ρ is a surjective $\mathbf{Q}$-morphism of algebraic groups, then $\Gamma = \rho(\Gamma_\circ)$ is an arithmetic subgroup of $G(\phi)$. Moreover,

$$\forall \gamma \in \Gamma_\circ, \det(Q(\gamma.\xi)) = \det(Q(\xi)), \quad \forall \xi \in E.$$

1.2. Reduction theory.

The following hypothesis is essential to use in this context reduction theory and to obtain some Minkowski inequality.

(H_2) In all the sequel, we assume that $V_{\mathbf{Q}}$ is a split Jordan algebra, that is, its primitive rank (which is the cardinality of maximal system of primitive orthogonal idempotents), equals its rank.

As $\operatorname{rank}(V) = m$, then the assumption (H_2) implies that there exists in $V_{\mathbf{Q}}$ a complete system of orthogonal primitive idempotents $\{c_1, \ldots, c_m\}$ which we will fix along this paper.

The corresponding Peirce decomposition $V = \underset{i\leqslant j}{\oplus} V_{ij}$ is defined over $\mathbf{Q}$, that is

$$V_{\mathbf{Q}} = \bigoplus_{i\leqslant j} V_{ij\,\mathbf{Q}}.$$

Let P be the subgroup of $G(\phi)$ defined by

$$P = \{g \in G(\phi) \mid (gx_{ij})_{ij} = \lambda_{ij}x_{ij} \quad \forall i,j \quad (gx_{ij})_{kl} = 0 \quad \forall (k,l) < (i,j)\}$$

where the λ_{ij} are reals, and for each y in V, the y_{ij} are the Peirce components of y, with respect to the Jordan frame $\{c_1, \ldots, c_m\}$. The order on the pairs (i,j) is the lexicographic one.

PROPOSITION 1.2.1. — *P is a Borel subgroup of $G(\phi)$ defined over $\mathbf{Q}$.*

Proof — As the Peirce decomposition is defined over $\mathbf{Q}$, then there exists some basis of $V_{\mathbf{Q}}$ whose each element lies in some $V_{ij\mathbf{Q}}$. An element of $G(\phi)$ lies in P iff its matrix in such a basis is upper triangular. □

Now consider the subgroup A of G defined by

$$A = \left\{ P(a) \mid a = \sum_{i=1}^{m} a_i.c_i, \quad a_i > 0 \quad \forall i, 1 \leqslant i \leqslant m \right\}.$$

PROPOSITION 1.2.2. — *A is a maximal $\mathbf{Q}$-split algebraic torus of P (cf. [15], chapter 2, proposition 3.5).*

We denote by N the unipotent radical of P,

$$N = \{t \in P \mid \lambda_{ij} = 1 \quad \forall i,j\}.$$

We know (cf.[15]), that

$$N = \left\{ n(z) \mid z \in \bigoplus_{j<k} V_{jk} \right\}$$

where

$$n(z) = \tau(z^{(1)}) \ldots \tau(z^{(m-1)})$$

$$z^{(j)} = \sum_{k=j+1}^{m} z_{jk}, \quad \tau(z^{(j)}) = \exp(2z^{(j)} \,\square\, c_j)$$

the operation $\square$ being defined by

$$x \,\square\, y = L(xy) + [L(x), L(y)]$$

(cf [15], Chapter 6, Theorem 6.3.6).

Let K be the maximal compact subgroup of $G(\phi)$ defined by

$$K = \{g \in G(\phi) \mid g.e = e\},$$

where e is the unity of V. Then we have the Iwasawa decomposition of $G(\phi)$,

$$G(\phi) = N.A.K.$$

DEFINITION 1.2.3. — *A Siegel set of* $G(\phi)$ *(with respect to* K, N, A*) is the cartesian product* $\mathfrak{S}_{t,u} = N_u.A_t.K$ *with*

$$A_t = \left\{ P(a) \in A \mid a_i \leqslant t a_{i+1}, \forall 1 \leqslant i \leqslant m-1, \quad a = \sum_{i=1}^{m} a_i c_i \right\}$$

$$N_u = \{ n(z) \in N \mid \|z_{kj}\| \leqslant u \},$$

where t, u *are two positive constants.*

PROPOSITION 1.2.4. — *There exist positive constants* t *and* u *and some finite subset* $\mathcal{B}$ *of* $G(\phi)_{\mathbf{Q}}$ *such that*

$$G(\phi) = \Gamma.\mathcal{B}.\mathfrak{S}_{t,u},$$

moreover, as $\Omega = G(\phi)/K$*, then*

$$\Omega = \Gamma\mathcal{B}.N_u.A_t.e.$$

Proof. — It is a direct consequence of Theorem 13.1 of [15], page 90, applicated to the $\mathbf{Q}$-reductive group $G(\phi)$ under the action of the arithmetic subgroup Γ. □

PROPOSITION 1.2.5 (Minkowski inequality). — *Let* $x = \sum_{i=1}^{m} x_i c_i + \sum_{i<j} x_{ij}$ *be the Peirce decomposition of* $x \in V$*. For positive reals* t, u*, there exists a positive constant* $C_{t,u}$ *such that, for each* $x \in \mathfrak{S}_{t,u}.e$*,*

$$\prod_{i=1}^{m} x_i \leqslant C_{t,u} \cdot \det(x).$$

Proof. — Let $x = n \cdot P(a)(e)$ be an element of the Siegel set $\mathfrak{S}_{t,u}.e$ of the symmetric cone Ω, i.e.

$$n = \tau(z^{(1)}) \dots \tau(z^{(m-1)})$$

with

$$z^{(j)} = \sum_{k=j+1}^{m} z_{jk}, \quad \|z_{jk}\| \leqslant u$$

and $a = \sum_{i=1}^{m} a_i c_i$ such that

$$a_i \leqslant t a_{i+1} \quad \forall i, 1 \leqslant i \leqslant m-1.$$

The Peirce componants of x are as follows :

$$x_j = a_j^2 + \frac{1}{2}\sum_{k=1}^{j-1} a_k^2 \|z_{kj}\|^2$$

$$x_{jk} = a_j^2 z_{jk} + 2\sum_{l=1}^{j-1} a_l^2 z_{lj} z_{lk}.$$

So we find the following inequality :

$$x_j \leqslant a_j^2 + \frac{1}{2} a_j^2 \sum_{k=1}^{j-1} t^{2(j-k)} \|z_{kj}\|^2$$

$$\leqslant a_j^2 \left(1 + \frac{1}{2}\sum_{k=1}^{j-1} t^{2(j-k)} \|z_{kj}\|^2 \leqslant a_j^2 \left(1 + \frac{1}{2} u^2 \sum_{k=1}^{j-1} t^{2(j-k)}\right)\right).$$

Otherwise, as $\det(x) = \prod_{j=1}^{m} a_j^2$, we find

$$\prod_{j=1}^{m} x_j \leqslant C_{t,u} \det(x)$$

where

$$C_{t,u} = \left(1 + \frac{1}{2} u^2 \sum_{k=1}^{m-1} t^{2(j-k)}\right)^m . \qquad \square$$

1.3. Convergence of the zeta series ζ_L.

The zeta series associated to the representation ϕ and the lattice L is defined by

$$\zeta_L(s) = \sum_{l \in \Gamma_\circ \backslash L'} \det(Q(l))^{-s}, \quad s \in \mathbf{C}$$

where L' is the set $L' = \{l \in L \mid \det(Q(l)) \neq 0\}$.

THEOREM 1.3.1. — *Under the assumptions* (H_1) *(section 1.1) and* (H_2) *(section 1.2), the zeta series* $\zeta_L(s)$ *converges absolutely for* $\mathrm{Re}(s) > \dfrac{N}{2m}$.

Proof. — For $a \in \Omega$, we set

$$\nu(a) = \#\{l \in L' \mid Q(l) = a\}$$

$$\epsilon(a) = \#\{\gamma \in \Gamma \mid \gamma(a) = a\}$$

$$\mu(a) = \frac{\nu(a)}{\epsilon(a)}.$$

Assume s real. By Proposition 1.2.4, there exist a Siegel set $\mathfrak{S}_{t,u}$ and a finite subset $\mathcal{B}$ of $G(\phi)_{\mathbf{Q}}$ such that $\Omega = \Gamma\mathcal{B}.\mathfrak{S}_{t,u}.e$ and then

$$\zeta_L(s) = \sum_{a\in\Gamma\backslash Q(L')} \mu(a)\det(a)^{-s} \leqslant \sum_{a\in Q(L')\cap\mathcal{B}.\mathfrak{S}_{t,u}.e} \nu(a).\det(a)^{-s}.$$

Before getting to the proof of the theorem, we will show the following lemma :

LEMMA 1.3.2. — *The series* $S = \sum_{a\in Q(L')} \nu(a)\left(\prod_{i=1}^{m} a_i^{-s}\right)$ *converges for* $s > \dfrac{N}{2m}$.

Proof of the lemma. — Let $E_i = \phi(c_i)E$. We have $E = \overset{m}{\underset{i=1}{\oplus}} E_i$ and this decomposition is defined over $\mathbf{Q}$. Then we can find lattices $R_i \subseteq (E_i)_{\mathbf{Q}}$ such that

$$L \subseteq R = \bigoplus_{i=1}^{m} R_i.$$

For $\xi \in E$, denote by $\xi_i = \phi(c_i)\xi \in E_i$. The series S becomes

$$S = \sum_{l\in L'}\left(\prod_{i=1}^{m} \|l_i\|^{-2s}\right)$$

then

$$S \leqslant \sum_{i=1}^{m} \sum_{l_i\in R_i-\{0\}} \prod_{i=1}^{m} \|l_i\|^{-2s} = \prod_{i=1}^{m}\left(\sum_{l_i\in R_i-\{0\}} \|l_i\|^{-2s}\right).$$

Each one of these series is an Epstein zeta series which converges for $s > \frac{1}{2}\dim(E_i) = \dfrac{N}{2m}$. □

Let's now return to the proof of the theorem. For each equivalence class of $Q(L')$ modulo Γ, we choose a representant of the form $a = b\alpha$, $b \in \mathcal{B}$, $\alpha \in \mathfrak{S}_{t,u}.e$.

By the Minkowski inequality, we have

$$\det(a) = \mathrm{Det}(b)^{\frac{m}{n}} \det(\alpha) \geqslant \mathrm{Det}(b)^{\frac{m}{n}} C_{t,u}^{-1} \prod_{i=1}^{m} \alpha_i$$

and if $s > 0$, then

$$\det(a)^{-s} \leqslant M^s \prod_{i=1}^{m} \alpha_i^{-s}, \qquad \text{with} \qquad M = C_{t,u} \sup_{b \in \mathcal{B}} \mathrm{Det}(b)^{-\frac{m}{n}}.$$

Set $\mathcal{B} = \{b_1, \dots, b_r\}, b_j = \rho(f_j), \quad L_j = f_j^{-1}(L)$. If $a = b_j \alpha$, then

$$\nu(a) = \#\{l \in L_j' \mid Q(l) = \alpha\} = \nu_j(\alpha).$$

Otherwise, if $f_1, f_2 \in F(\phi)$, then

$$(*) \qquad \rho(f_1) = \rho(f_2) \Leftrightarrow \forall x \in V, \quad f_1^\star.\phi(x).f_1 = f_2^\star.\phi(x).f_2$$

so, if $b \in G(\phi)$, then

$$x \in b^{-1}(Q(L')) \Leftrightarrow \exists l \in L', \exists f \in F(\phi), \quad x = b^{-1}.Q(l) = Q(fl).$$

Notice that f is not unique, but if $x = \sum_{j=1}^{m} x_j.c_j + \sum_{k<l} x_{kl}$ is the Peirce decomposition of x, then

$$x_j = Q(f.l)_j = (Q(f \cdot l) \mid c_j) = (\phi(c_j)f.l \mid f.l) = (f^\star \cdot \phi(c_j) \cdot f.l \mid l)$$

and $x_j = \|\phi(c_j)f \cdot l\|^2$ does not depend on the choice of the antecedent f of b^{-1} by the map ρ.

Finally, we find

$$\zeta_L(s) \leqslant M^s \sum_{j=1}^{r} \sum_{\alpha \in Q(L_j')} \mu_j(\alpha) \prod_{i=1}^{m} \alpha_i^{-s} = M^s \sum_{j=1}^{r} S_j,$$

and the announced result is just a consequence of the above lemma. □

2. Analytic continuation and the functional equation.

We use the classical method which consists to see the zeta series as the Mellin transform of the theta series associated to the representation ϕ and the lattice L, and the functional equation is a consequence of the transformation formula of the theta series.

First recall some results about zeta integrals.

2.1. Zeta integrals associated to ϕ.

For each function f in the Schwartz space $\mathcal{S}(E)$, the zeta integral associated to the representation ϕ is defined by

$$Z(f,s) = \int_E [\det Q(\xi)]^s f(\xi) d\xi, \quad \forall s \in \mathbf{C}.$$

PROPOSITION 2.1.1. — *The zeta integral $Z(f,s)$ converges absolutely for* $\mathrm{Re}(s) > \frac{d}{2}(m-1) - \frac{N}{2m}$ *(d denotes the dimension of the subspaces V_{ij} for $i \neq j$ in the Peirce decomposition of V). It admits an analytic continuation as a meromorphic function on the whole plane* $\mathbf{C}$, *and satisfies to the functional equation*

$$Z\left(\hat{f}, s - \frac{N}{2m}\right) = \gamma(s) Z(f, -s),$$

where

$$\gamma(s) = \pi^{\frac{N}{2} - 2ms} \frac{\Gamma_\Omega(s)}{\Gamma_\Omega\left(\frac{N}{2m} - s\right)},$$

Γ_Ω being the Kœcher-Gindikin gamma function of the cone Ω, that is

$$\Gamma_\Omega(s) = \int_\Omega e^{-tr(x)} \det(x)^{s - \frac{n}{m}} dx,$$

and

$$\hat{f}(\xi) = \int_E e^{-2\pi i (\xi|\eta)} f(\eta) d\eta$$

is the Fourier transform of f (cf [15], Chapter 16, Theorem 16.4.3).

2.2. Theta series associated to ϕ and L.

For each $f \in \mathcal{S}(E)$, the theta series associated to ϕ and L is defined by

$$\Theta(x, f, L) = \sum_{l \in L} f[\phi(x^{\frac{1}{2}})l], \quad \forall x \in \Omega.$$

It is clear that this series converges absolutely for $x \in \Omega$.

PROPOSITION 2.2.1 (Transformation formula).

$$\Theta(x^{-1}, f, L) = \mathrm{vol}(L)^{-1} \det(x)^{\frac{N}{2m}} \Theta(x, \hat{f}, L^\star),$$

where $\hat{f}$ is the Fourier transform of f and $L^\star$ is the dual lattice of L, that is

$$L^\star = \{b \in E \mid (b \mid a) \in \mathbf{Z}, \quad \forall a \in L\},$$

and $\mathrm{vol}(L) = \mathrm{vol}(E/L)$.

Proof. — It is a consequence of the Poisson summation formula. If $\psi \in \mathcal{S}(E)$, then

$$\sum_{l \in L} \psi(l) = \mathrm{vol}(L)^{-1} \sum_{l \in L^\star} \hat{\psi}(l).$$

If $\psi(\xi) = f[\phi(x^{-\frac{1}{2}})\xi]$, then

$$\hat{\psi}(\eta) = \mathrm{Det}(\phi(x^{\frac{1}{2}}))\hat{f}[\phi(x^{\frac{1}{2}})\eta] = \det(x)^{\frac{N}{2m}} \hat{f}[\phi(x^{\frac{1}{2}})\eta]. \qquad \square$$

2.3. Invariance property of theta series.

LEMMA 2.3.1. — *If F is a K-invariant function defined on $\bar{\Omega}$, then there exists a kernel $F'(x,y)$ defined on $\bar{\Omega} \times \bar{\Omega}$ such that*

$$F'(x,e) = F(x),$$
$$F'(gx,y) = F'(x, g^\star y),$$
$$F'(x,y) = F'(y,x), \quad \forall x, y \in \bar{\Omega}, \forall g \in G(\Omega).$$

Proof. — The function F_1 defined on $\bar{\Omega} \times G(\Omega)$ by $F_1(x,g) = F(g^\star x)$ is right-invariant by K as a function of g.

The function F' defined by $F'(x, g \cdot e) = F_1(x,g)$ satisfies to the announced properties; in fact, it is clear that $F'(x,e) = F(x)$ and

$$\begin{aligned} F'(g_1 \cdot x, g.e)) &= F_1(g_1 \cdot x, g) = F(g^\star g_1 \cdot x) \\ &= F((g_1^\star g)^\star \cdot x) = F_1(x, g_1^\star g) \\ &= F'(x, g_1^\star g \cdot e). \end{aligned}$$

Moreover, as there exists $k \in K$ such that

$$P(x^{\frac{1}{2}})y = kP(y^{\frac{1}{2}})x$$

(cf.[15], Chapter 14, Lemma 14.1.2), then

$$F'(x,y) = F(P(x^{\frac{1}{2}})y) = F(P(y^{\frac{1}{2}})x) = F'(y,x). \qquad \square$$

Proposition 2.3.2. — *Let $F \in \mathcal{S}(\bar{\Omega})$, a K-invariant function on $\bar{\Omega}$ and let f be defined by $f(\xi) = F(Q(\xi))$. If the arithmetic subgroup $\Gamma_\circ$ is self-adjoint, then the theta series $\Theta(x, f, L)$ is Γ-invariant i.e.*

$$\Theta(\gamma x, f, L) = \Theta(x, f, L) \quad \forall \gamma \in \Gamma.$$

Proof. — Let F' be the kernel of Lemma 2.3.1. We have

$$f(\phi(x^{\frac{1}{2}})\xi) = F(Q(\phi(x^{\frac{1}{2}})\xi)) = F(P(x^{\frac{1}{2}})Q(\xi))$$

$$= F'(P(x^{\frac{1}{2}})Q(\xi), e) = F'(Q(\xi), x),$$

and then

$$\Theta(x, f, L) = \sum_{a \in L} F'(x, Q(a)).$$

Moreover, for h in $F(\phi)$,

$$\begin{aligned} \Theta(\rho(h)x, f, L) &= \sum_{a\in L} F'(\rho(h)x, Q(a)) = \sum_{a \in L} F'(x, \rho(h)^\star Q(a)) \\ &= \sum_{a \in L} F'(x, Q(h^\star a)) = \Theta(x, f, h^\star L), \end{aligned}$$

and, as we assumed that $\Gamma_\circ^\star = \Gamma_\circ$, then $\Theta(x, f, L)$ is Γ-invariant. $\square$

2.4. Mellin transform of theta series.

The Mellin transform of the theta series $\Theta(x, f, L)$ is defined by

$$\Xi(s, f, L) = \int_{\Gamma\backslash\Omega} \Theta(x, f, L)\det(x)^s d^\star x, \quad \forall s \in \mathbf{C}$$

where $d^\star x$ is the G-invariant measure on Ω, $d^\star x = \det(x)^{-\frac{n}{m}} dx$, dx denoting the Euclidean measure on V.

(H_3) In all the sequel, we assume that $N > m(m-1)d$ and then the image of the Euclidean measure on E under the quadratic form $\mathbf{Q}$ has a density with respect to the Euclidean measure of V.

PROPOSITION 2.4.1. — *Let $F \in \mathcal{S}(\bar{\Omega})$, K-invariant, null on $\partial\Omega$. Let f be the function defined by $f(\xi) = F(Q(\xi))$. For $s \in \mathbf{C}$, if* $\mathrm{Re}(s) > \max\left\{\frac{N}{2m}, (m-1)\frac{d}{2}\right\}$, *then the integral $\Xi(s, f, L)$ converges absolutely and satisfies to*

$$\Xi(s, f, L) = \frac{\Gamma_\Omega\left(\frac{N}{2m}\right)}{\pi^{\frac{N}{2}}} \zeta_L(s).Z\left(f, s - \frac{N}{2m}\right).$$

Proof. — Assume f positive and s real, then

$$\Theta(x, f, L) = \sum_{a \in Q(L')} \nu(a) F'(x, a) = \sum_{a \in \Gamma\backslash Q(L')} \mu(a) \sum_{b \in \Gamma\cdot a} F'(x, b),$$

and as

$$\int_{\Gamma/\Omega} \left[\sum_{b \in \Gamma\cdot a} F'(x, b)\right] \det(x)^s d^\star x = \int_\Omega F'(x, a)\det(x)^s d^\star x$$

$$= \det(a)^{-s} \int_\Omega F(x)\det(x)^s d^\star x < \infty,$$

then

$$\begin{aligned}\Xi(s,f,L) &= \int_{\Gamma\backslash\Omega}\left[\sum_{a\in\Gamma\backslash Q(L')}\mu(a)\left(\sum_{b\in\Gamma\cdot a}F'(x,b)\right)\right]\det(x)^s d^\star x\\ &= \sum_{a\in\Gamma\backslash Q(L')}\mu(a)\int_{\Gamma\backslash\Omega}\left[\sum_{b\in\Gamma\cdot a}F'(x,b)\right]\det(x)^s d^\star x\\ &= \sum_{a\in\Gamma\backslash Q(L')}\mu(a)\det(a)^{-s}\int_\Omega F(x)\det(x)^s d^\star x\\ &= \zeta_L(s)\int_\Omega F(x)\det(x)^s d^\star x < +\infty,\end{aligned}$$

and we find

$$\Xi(s,f,L) = \zeta_L(s)\cdot\textstyle\int_\Omega F(x)\det(x)^s d^\star x.$$

Recall that for $N > m(m-1)d$, the image of the measure $d\xi$ under $\mathbf{Q}$ is

$$d\mu(x) = \frac{\pi^{\frac{N}{2}}}{\Gamma_\Omega\left(\frac{N}{2m}\right)}\det(x)^{\frac{N}{2m}-\frac{n}{m}}dx,$$

(cf.[15], Chapter.16, Proposition.16.1.1), we find

$$\begin{aligned}\Xi(s,f,L) &= \frac{\Gamma_\Omega\left(\frac{N}{2m}\right)}{\pi^{\frac{N}{2}}}\zeta_L(s)\int_E f(\xi)[\det(Q(\xi))]^{s-\frac{N}{2m}}d\xi\\ &= \frac{\Gamma_\Omega\left(\frac{N}{2m}\right)}{\pi^{\frac{N}{2}}}\zeta_L(s)\cdot Z\left(f,s-\frac{N}{2m}\right).\end{aligned}$$

Recall the following lemma :

LEMMA 2.4.2. — *Let $f\in\mathcal{S}(E)$ be a radial function, namely, there exists a function F defined on $\bar\Omega$, such that $f(\xi) = F(Q(\xi))$, then the Fourier transform $\hat f$ of f is radial (cf.[15], Chapter 16, Proposition 16.2.5).*

Moreover, we can find a radial function f such that f and its Fourier transform $\hat f$ vanish on the set

$$\{\xi\in E \mid \det(Q(\xi)) = 0\}.$$

PROPOSITION 2.4.3. — *Let $F\in\mathcal{S}(\bar\Omega)$, K-invariant, and $f(\xi) = F(Q(\xi))$. If f and $\hat f$ vanish on the set $\{\xi\in E \mid \det(Q(\xi)) = 0\}$, then $\Xi(s,f,L)$is an analytic function on its convergence domain, it admits an analytic continuation as analytic function on the whole $\mathbf{C}$, and it satisfies* *to the functional equation*

$$\Xi\left(\frac{N}{2m} - s, \hat{f}, L^\star\right) = \mathrm{vol}(L)\Xi(s, f, L).$$

Proof. — Set

$$\Omega_+ = \{x \in \Omega \mid \det(x) \geqslant 1\}$$
$$\Omega_- = \{x \in \Omega \mid \det(x) \leqslant 1\}$$

and we define

$$\Xi_+(s, f, L) = \int_{\Gamma/\Omega_+} \Theta(x, f, L)\det(x)^s d^\star x$$
$$\Xi_-(s, f, L) = \int_{\Gamma/\Omega_-} \Theta(x, f, L)\det(x)^s d^\star x.$$

The integral defining $\Xi_+(s, f, L)$ converges for each $s \in \mathbf{C}$ and is analytic on the whole $\mathbf{C}$. Indeed, for each positive constant A, B there exists a positive constant C such that

$$|F(y)| \leqslant C\det(y)^{-A}(1 + \mathrm{tr}(y))^{-B},$$

and

$$|F'(x, a)| \leqslant C \det(x)^{-A}\det(a)^{-A}(1 + (a \mid x))^{-B}.$$

If $\mathrm{Re}(s) \leqslant A$, and $\det(x) \geqslant 1$, then $|\det(x)^s| \leqslant \det(x)^A$,

and

$$|F'(x, a)\det(x)^{s-\frac{n}{m}}| \leqslant C \det(a)^{-A}(1 + (a \mid x))^{-B}.$$

Otherwise,

$$\int_{\Omega_+} (1 + (a \mid x))^{-B} dx \leqslant \det(a)^{-\frac{n}{m}} \int_{\Omega} (1 + \mathrm{tr}(y))^{-B} dy.$$

So, if $A > \dfrac{N}{2m}$, then

$$\sum_{a \in \Gamma\backslash Q(L')} \mu(a)\det(a)^{-A} \int_{\Omega_+} (1 + (a \mid x))^{-B} dx < \infty.$$

On an other side, as

$$\Theta(x^{-1}, f, L) = \mathrm{vol}(L)^{-1}\det(x)^{\frac{n}{2m}}\Theta(x, \hat{f}, L^\star),$$

then if $\Xi_-(s,f,L)$ converges, i.e. if $\Xi(s,f,L)$ converges, then

$$\begin{aligned}\Xi_-(s,f,L) &= \int_{\Gamma/\Omega_+} \Theta(x^{-1},f,L)\det(x)^{-s}d^\star x \\ &= \mathrm{vol}(L)^{-1}\int_{\Gamma/\Omega_+} \Theta(x,\hat{f},L^\star)\det(x)^{\frac{N}{2m}-s}d^\star x \\ &= \mathrm{vol}(L)^{-1}\Xi_+\left(\frac{N}{2m}-s,\hat{f},L^\star\right)\end{aligned}$$

i.e.

$$\Xi_-(s,f,L) = \mathrm{vol}(L)^{-1}\Xi_+\left(\frac{N}{2m}-s,\hat{f},L^\star\right).$$

We deduce that $\Xi_-(s,f,L)$ is analytic on its convergence domain and as $\Xi_+(s,f,L)$ is analytic on $\mathbf{C}$, then the above equation gives the analytic continuation of $\Xi_-(s,f,L)$ as analytic function on $\mathbf{C}$. It is also the same for the Mellin transform $\Xi(s,f,L)$ which is given by

$$\begin{aligned}\Xi(s,f,L) &= \Xi_+(s,f,L)+\Xi_-(s,f,L) \\ &= \Xi_+(s,f,L)+\mathrm{vol}(L)^{-1}\Xi_+\left(\frac{N}{2m}-s,\hat{f},L^\star\right).\end{aligned}$$

Moreover, it satisfies to the functional equation

$$\Xi\left(\frac{N}{2m}-s,\hat{f},L^\star\right) = \mathrm{vol}(L)\Xi(s,f,L). \qquad \square$$

2.5. Analytic continuation and functional equation.

From the above, we deduce

$$\Xi\left(\frac{N}{2m}-s,f,L\right) = \frac{\Gamma_\Omega\left(\frac{N}{2m}\right)}{\pi^{\frac{N}{2}}}\zeta_L\left(\frac{N}{2m}-s\right)Z(f,-s) = \mathrm{vol}(L)\Xi(s,\hat{f},L^\star)$$

$$= \mathrm{vol}(L)\frac{\Gamma_\Omega\left(\frac{N}{2m}\right)}{\pi^{\frac{N}{2}}}\zeta_{L^\star}(s)Z\left(\hat{f},s-\frac{N}{2m}\right)$$

$$= \mathrm{vol}(L)\frac{\Gamma_\Omega\left(\frac{N}{2m}\right)}{\pi^{\frac{N}{2}}}\zeta_{L^\star}(s)\pi^{\frac{N}{2}-2ms}\frac{\Gamma_\Omega(s)}{\Gamma_\Omega\left(\frac{N}{2m}-s\right)}Z(f,-s)$$

and finally, we have the theorem :

THEOREM 2.5.1. — *Under the assumptions :*

(H_3) $N > m(m-1)d$,

(H_4) *the arithmetic subgroup* $\Gamma_\circ$ *is self-adjoint,*

the zeta function $\zeta_L(s)$ *admits an analytic continuation as a meromorphic function on the whole* $\mathbf{C}$ *and satisfies to the functional equation*

$$\zeta_L\left(\frac{N}{2m}-s\right) = \mathrm{vol}(L)\pi^{\frac{N}{2}-2ms}\frac{\Gamma_\Omega(s)}{\Gamma_\Omega\left(\frac{N}{2m}-s\right)}\zeta_{L^*}(s).$$

Remark. — If $\tilde{\Gamma_\circ}$ is a finite-index subgroup of $\Gamma_\circ$, then the zeta series defined by

$$\tilde{\zeta_L}(s) = \sum_{l\in\tilde{\Gamma_\circ}\backslash L'} \det(Q(l))^{-s}$$

has the same properties than $\zeta_L(s)$.

3. Examples.

In this section we look at some examples of zeta functions.

3.1. Case of symmetric real matrices.

Let $V = \mathrm{Sym}(m,\mathbf{R})$ be the Jordan algebra with the product $A\circ B = \frac{1}{2}(AB+BA)$, then the symmetric associated cone is the cone Ω of positive definite symmetric real matrices. Let $E = M(m,n,\mathbf{R})$ (with $n \geqslant m$), and ϕ the representation

$$\phi : V \to \mathrm{Sym}(E), x \mapsto \phi(x) : \xi \mapsto x\xi.$$

The associated quadratic form Q is given by $Q(\xi) = \xi\xi', \forall\xi \in E$. ($\xi'$ is the adjoint of ξ.) Let $V_{\mathbf{Q}} = \mathrm{Sym}(m,\mathbf{Q})$, then $V_{\mathbf{Q}}$ is a split $\mathbf{Q}$-structure of V, and let $E_{\mathbf{Q}} = M(m,n,\mathbf{Q})$ and L the lattice $L = M(m,n,\mathbf{Z})$. It is clear that ϕ is defined over $\mathbf{Q}$, moreover, the arithmetic group $GL(m,\mathbf{Z})$ is a finite-index subgroup of $\Gamma_\circ$, where

$$\Gamma_\circ = \{f \in F(\phi) \mid f(L) = L\},$$

and the zeta series is

$$\zeta_L(s) = \sum_{A \in GL(m,\mathbf{Z})\backslash M(m,n,\mathbf{Z}),\text{rank}(A)=m} \text{Det}(AA')^{-s},$$

which is the classical Kœcher zeta function.

3.2. Case of Hermitian complex matrices.

Let $V = \text{Herm}(m, \mathbf{C})$ be the Jordan algebra with the above product "$\circ$", $E = M(m, n, \mathbf{C})$ (with $n \geqslant m$), and ϕ the representation

$$\phi(x)\xi = x\xi, \forall x \in V, \xi \in E.$$

The associated quadratic form is given by $Q(\xi) = \xi\bar{\xi}'$.

Let K be an imaginary quadratic field and $\mathcal{O}$ its ring of integers. Then $V_{\mathbf{Q}} = \text{Herm}(m, K)$ is a split $\mathbf{Q}$-structure of V and the space $E_{\mathbf{Q}} = M(m, n, K)$ is a $\mathbf{Q}$-structure of E. Let L be the lattice $L = M(m, n, \mathcal{O})$, then it is clear that the representation ϕ is defined over $\mathbf{Q}$, and the group $GL(m, \mathcal{O})$ is of finite index in $\Gamma_\circ$, and we obtain the zeta series

$$\zeta_L(s) = \sum_{A \in GL(m,\mathcal{O})\backslash M(m,n,\mathcal{O}),\text{rank}(A)=m} [\text{Det}(A\bar{A}')]^{-s},$$

and this case gives a new example of zeta function.

3.3. Case of Hermitian quaternionic matrices.

Let $V = \text{Herm}(m, \mathbf{H})$ with the Jordan product, $E = M(m, n, \mathbf{H})$ and ϕ the representation $\phi(x)\xi = x\xi, \forall x \in V, \xi \in E$. If $\mathcal{O}$ denotes the ring of Hurwitz integers, then $L = M(m, n, \mathcal{O})$ is a lattice in E and the group $GL(m, \mathcal{O})$ is of finite-index in the arithmetic subgroup $\Gamma_\circ$ associated to ϕ. The zeta series is the one studied by A. Krieg in [15], and is given by

$$\zeta_L(s) = \sum_{A \in GL(m,\mathcal{O})\backslash M(m,n,\mathcal{O}),\text{rank}(A)=m} \text{Det}(A\bar{A}')^{-s}.$$

The case of zeta functions of representations of rank 2-Jordan algebras gives new examples of zeta functions and constitutes for itself an other article (cf. [1]).