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DISTANCE FORMULAE AND INVARIANT SUBSPACES, WITH AN APPLICATION TO LOCALIZATION OF ZEROS OF THE RIEMANN ζ-FUNCTION

by Nikolai NIKOLSKI

We consider two distance functions which can be used to describe and to explore z-invariant subspaces of Banach spaces of analytic functions. In the case of Hilbert spaces a unicity theorem is proved and some remarks are made about the localization of zeros of Beurling inner functions in terms of the distance functions. As an example (and an improvement of a theorem by Beurling and Nyman from the 1950's) we consider a series of z-invariant subspaces related to the Riemann ζ-function. The following theorem is proved.

0.1. Theorem. — *Let $s \in \mathbb{C}$, $\operatorname{Re} s > 0$ and $\gamma > 0$. Let further*

$$E_{\alpha,\gamma}(x) = x^\gamma\Big(\Big[\frac{\alpha}{x}\Big] - \alpha\Big[\frac{1}{x}\Big]\Big), \qquad 0 < x < 1,$$

where $0 \le \alpha \le 1$, and

$$d_\gamma^2(s) = \inf \int_0^1 \Big|x^s - \sum_\alpha a_\alpha E_{\alpha,\gamma}\Big|^2 \frac{dx}{x},$$

the inf *being taken over all finite linear combinations of $E_{\alpha,\gamma}$ for $0 \le \alpha \le 1$. Then the disc*

$$D_{s,\gamma} = \gamma + D_s = \gamma + \Big\{z : \Big|\frac{z-s}{z-s_*}\Big|^2 < 1 - 2d_\gamma(s)^2 \operatorname{Re} s\Big\}$$

is free of zeros of the Riemann ζ-function

$$\zeta(z) = \sum_{n\ge 1} \frac{1}{n^z}\cdot$$

Here s_* stands for the point symmetric to s with respect to the imaginary axis.

For commentary see Sections 3 and 4 below.

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1. Introduction. Problem of z-invariant subspaces.

Let X be a Banach space of holomorphic functions continuously imbedded into the space $\mathrm{Hol}(\mathbb{D})$ of all such functions in the unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

and stable with respect to multiplication by the independent variable z:

$$f \in X \implies zf \in X.$$

The problem (very far from being solved) is to describe all (closed) z-invariant subspaces of X:

$$E \subset X, \quad E = \overline{E}, \quad zE \subset E. \tag{1.1}$$

Solutions are known for very rare cases related to the Hardy H^p spaces; they are always based on the techniques of the canonical Riesz–Nevanlinna–Smirnov factorization. The case of the Hardy spaces H^p, where $1 \le p \le \infty$,

$$H^p = H^p(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \sup_{0 \le r < 1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) = \|f\|_p^p < \infty \right\}$$

is classical with the following Beurling description of z-invariant subspaces $E \subset H^p$:

$$E = \Theta H^p, \tag{1.2}$$

where Θ stands for an *inner* function in the disc $\mathbb{D}$ (this means that Θ is analytic and bounded in $\mathbb{D}$ (i.e., $\Theta \in H^\infty$) and unimodular on the unit circle $\mathbb{T} = \partial\mathbb{D}$:

$$|\Theta(t)| = 1 \quad \text{a.e. on } \mathbb{T}$$

(boundary values)). Similar but more complicated descriptions hold true for some spaces of analytic functions smooth up to the boundary (like the disc algebra C_A, the spaces

$$C_A^{(n)} = \{f \in \mathrm{Hol}(\mathbb{D}) : f^{(n)} \in C_A\} \text{ and } H_n^p = \{f \in \mathrm{Hol}(\mathbb{D}) : f^{(n)} \in H^p\})$$

and sometimes for their dual spaces (usually in a Fréchet, not Banach space setting). For these results see [Ho], [K], [Ko], [Sh].

Unfortunately, factorization theory is not yet available for the majority of Banach spaces of analytic functions, even for such popular ones as the Bergman and weighted Bergman spaces (despite the recent breakthrough due to H. Hedenmalm, [H]). Moreover, even though a description of type (1.2) is known, it is not a remedy for all purposes related to invariant subspaces. For instance, it does not always help in the study of cyclic vectors of the adjoint operator z^*; another difficulty is to find an explicit formula for the canonical function Θ if, say, an invariant subspace E is given in terms of its Fourier transform. In this last case, it is usually a problem to localize zeros and singular masses of Θ; the standard formula, see [Ho], [N], saying « *Θ is the greatest common inner divisor of all functions f from E* » is not always efficient. We hope the distance functions defined in Section 2 below can be useful for these purposes.

The paper is organized as follows. In Section 2 we introduce distance functions for handling invariant subspaces and, in particular, establish a one-to-one correspondence between subspaces and distance functions. In Section 3 we show how it is possible to use distance functions to localize zeros of a subspace of the Hardy space H^p. As an illustration, a theorem on zeros of the Riemann ζ-function (generalizing a result due to B. Nyman, [Ny]) is proved in Section 4. We finish with some unsolved problems on invariant subspaces, Section 5.

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2. Distance functions.

Let X be a Banach space of analytic functions continuously imbedded into the space Hol($\mathbb{D}$). And let φ_λ, $\lambda \in \mathbb{D}$ be the point evaluation functional on X:

$$\varphi_\lambda(f) = f(\lambda), \qquad f \in X.$$

For a subspace $E \subset X$ we set

$$\Theta_E(\lambda) = \|\varphi_{\lambda|E}\|, \qquad \lambda \in \mathbb{D},$$

and call Θ_E a *distance function* of E. This wording is justified by the following property of Θ_E.

2.1. We have

$$\Theta_E = \operatorname{dist}(\varphi_\lambda, E^\perp),$$

where «dist» stands for the distance in the dual Banach space X^* and

$$E^\perp = \{\varphi \in X^* : \varphi_{|E} \equiv 0\}$$

for the annihilator of E. This is a well-known corollary of the Hahn–Banach theorem.

2.2. The function Θ_E is logarithmically subharmonic: in fact,

$$\log \Theta_E(\lambda) = \sup\{\log |f(\lambda)| : f \in E, \ \|f\| \le 1\}$$

and so $\log \Theta_E$ is subharmonic as the least upper bound of a family of subharmonic functions.

2.3. Let H be a Hilbert space of analytic functions in $\mathbb{D}$, and let $\lambda \mapsto k_\lambda$ be the reproducing kernel of H, i.e., the unique H-valued function in $\mathbb{D}$ such that

$$\varphi_\lambda(f) = f(\lambda) = (f, k_\lambda); \qquad f \in H, \ \lambda \in \mathbb{D}. \tag{2.1}$$

Clearly, we have

$$\Theta_E(\lambda) = \|P_E k_\lambda\| = \operatorname{dist}(k_\lambda, E^\perp),$$

where P_E stands for the orthogonal projection on the subspace E, and $E^\perp$ is the orthogonal complement of E. Moreover,

$$\|P_E k_\lambda\|^2 + \|P_{E^\perp} k_\lambda\|^2 = \|k_\lambda\|^2, \qquad \lambda \in \mathbb{D},$$

and so the distance functions Θ_E and $\Theta_{E^\perp}(\lambda) = \operatorname{dist}(k_\lambda, E)$ completely determine each other. In what follows, when using $\Theta_{E^\perp}$ instead of Θ_E, we put

$$d_E(\lambda) = \Theta_{E^\perp}(\lambda) = \operatorname{dist}(k_\lambda, E), \qquad \lambda \in \mathbb{D}. \tag{2.2}$$

The following uniqueness theorem is an important property of Θ_E (and so, of d_E).

2.4. THEOREM. — *Let E_1, E_2 be two (as always, closed) subspaces of H, and let $\Theta_{E_1}(\lambda) = \Theta_{E_2}(\lambda)$, $\lambda \in \mathbb{D}$. Then $E_1 = E_2$.*

The theorem is an almost immediate corollary of the following lemma.

2.5. LEMMA. — *Let F and G be analytic Hilbert space valued functions on a connected domain $\Omega \subset \mathbb{C}$. If*

$$\|F(\lambda)\| = \|G(\lambda)\|, \qquad \lambda \in \Omega, \tag{2.3}$$

there exists an isometry V from the closed linear hull $\operatorname{span}(F(\lambda) : \lambda \in \Omega)$ *onto* $\operatorname{span}(G(\lambda) : \lambda \in \Omega)$ *such that*

$$VF(\lambda) = G(\lambda), \qquad \lambda \in \Omega.$$

Proof. — Fix a point $\mu \in \Omega$ and consider the following scalar valued functions f and g,

$$f(\lambda) = \big(F(\lambda), F(\mu)\big), \qquad \lambda \in \Omega;$$

$$g(\lambda) = \big(G(\lambda), G(\mu)\big), \qquad \lambda \in \Omega.$$

For all $n \geq 0$, we have

$$f^{(n)}(\mu) = \big(F^{(n)}(\mu), F(\mu)\big) \quad \text{and}$$

$$g^{(n)}(\mu) = \big(G^{(n)}(\mu), G(\mu)\big).$$

On the other hand, let

$$\partial = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \bar{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

be the usual differential operators of complex analysis; then

$$\begin{aligned} \partial\|F(\lambda)\|^2 &= \partial\big(F(\lambda), F(\lambda)\big) \\ &= \big(\partial F(\lambda), F(\lambda)\big) + \big(F(\lambda), \bar{\partial} F(\lambda)\big) \\ &= \big(\partial F(\lambda), F(\lambda)\big), \end{aligned}$$

and hence, for $n \geq 0$,

$$\partial^n\|F(\lambda)\|^2 = \big(\partial^n F(\lambda), F(\lambda)\big) = \big(F^{(n)}(\lambda), F(\lambda)\big).$$

The same is true for G:

$$\partial^n\|G(\lambda)\|^2 = \big(G^{(n)}(\lambda), G(\lambda)\big), \qquad n \geq 0.$$

Using (2.3) we obtain that $f^{(n)}(\mu) = g^{(n)}(\mu)$, $n \geq 0$, and therefore $f(\lambda) \equiv g(\lambda)$, $\lambda \in \Omega$. So, we get the identity

$$\big(F(\lambda), F(\mu)\big) = \big(G(\lambda), G(\mu)\big), \qquad \lambda, \mu \in \Omega.$$

It implies that

$$\Big\|\sum a_\lambda F(\lambda)\Big\|^2 = \Big\|\sum a_\lambda G(\lambda)\Big\|^2 \tag{2.4}$$

whatever a finite family $\{a_\lambda\}$ of complex numbers is. Identity (2.4) shows that the operator V,

$$V\Big(\sum a_\lambda F(\lambda)\Big) = \sum a_\lambda G(\lambda),$$

is well defined, linear and isometric. The lemma follows.

2.6. *Proof of Theorem 2.4.* — Let

$$F(\lambda) = P_{E_1} k_\lambda, \quad G(\lambda) = P_{E_2} k_\lambda, \qquad \lambda \in \mathbb{D}.$$

Then, F and G are H-valued conjugate analytic functions (the weak co-analyticity is immediate from the definition (see (2.1)) and — as is well-known — implies the strong co-analyticity).

Since $\|F(\lambda)\| = \|G(\lambda)\|$, $\lambda \in \mathbb{D}$, one can apply Lemma 2.5 which produces an isometry V from the subspace $\operatorname{span}(F(\lambda) : \lambda \in \mathbb{D}) = E_1$ onto $\operatorname{span}(G(\lambda) : \lambda \in \mathbb{D}) = E_2$ such that

$$V P_{E_1} k_\lambda = P_{E_2} k_\lambda, \qquad \lambda \in \mathbb{D}. \tag{2.5}$$

Due to remarks from Section 2.3 the same arguments are valid for the functions $F_*(\lambda) = P_{E_1^\perp} k_\lambda$, $G_*(\lambda) = P_{E_2^\perp} k_\lambda$, $\lambda \in \mathbb{D}$. So, we get another isometry U,

$$U : E_1^\perp \longrightarrow E_2^\perp$$

such that

$$U P_{E_1^\perp} k_\lambda = P_{E_2^\perp} k_\lambda, \qquad \lambda \in \mathbb{D}. \tag{2.6}$$

Hence, $V \oplus U$ is a unitary operator on H and

$$(V \oplus U) k_\lambda = V P_{E_1} k_\lambda \oplus U P_{E_1^\perp} k_\lambda = P_{E_2} k_\lambda \oplus P_{E_2^\perp} k_\lambda = k_\lambda$$

for all $\lambda \in \mathbb{D}$. This means that $V \oplus U = I$, and therefore $E_1 = E_2$. The proof is finished.

2.7. *Example.* — Now, we shall show that for non-Hilbert spaces X, $X \subset \mathrm{Hol}(\mathbb{D})$ the above unicity property is, in general, no longer true. Let X be the disc algebra,

$$X = C_A = \mathrm{Hol}(\mathbb{D}) \cap C(\operatorname{clos} \mathbb{D})$$

endowed with the usual sup-norm and $\sigma \neq \emptyset$ be a closed subset of $\mathbb{T}$ of Lebesgue measure zero (say, σ is a singleton $\sigma = \{\lambda\}$, $|\lambda| = 1$). Let $E = E_\sigma$ be a subspace $E_\sigma \subset C_A$,

$$E_\sigma = \{f \in C_A : f_{|\sigma} \equiv 0\}.$$

Then E_σ are proper subspaces of C_A and $E_\sigma \neq E_{\sigma'}$ if $\sigma \neq \sigma'$. Let us show that we always have $\Theta_{E_\sigma}(\lambda) \equiv 1, \lambda \in \mathbb{D}$. Indeed, there exists a function $f \in E_\sigma$ such that $\|f\|_{C_A} \leq 1$ and $f(z) \neq 0$ for all $z \in \mathbb{D}$ (for instance, see [Ho] for a construction; for a singleton $\sigma = \{\lambda\}$ one can simply take $f(z) = \frac{1}{2}(1 - \bar{\lambda} z)$). Hence, f^α is well defined in $\mathbb{D}$, belongs to E and $\|f^\alpha\|_{C_A} \leq 1$ whenever $\alpha > 0$. It implies that

$$1 = \lim_{\alpha \to 0} |f^\alpha(\lambda)| \leq \Theta_E(\lambda) \leq 1$$

and hence $\Theta_E(\lambda) = 1$ for all $\lambda \in \mathbb{D}$.

Remark. — Lemma 2.5 also turns out to be false for C-type spaces: the simplest (counter) example is an exponential function $F : \lambda \mapsto \mathrm{e}^{-\lambda x}$ taking values in the space $C[0,1]$ for $\lambda \in \Omega = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$.

On the other hand, the lemma is probably still true for uniformly convex Banach spaces X (or at least for L^p-spaces with $1 < p < \infty$).

3. Example: Hardy spaces H^p and localization of zeros.

Now, in the classical setting of the Hardy spaces H^p, we show how one can use distance functions to localize the zeros of an invariant subspace E, $E \subset H^p$. To this end we compare the canonical Beurling description of z-invariant subspaces (see (1.2) above) with the distance functions.

3.1. Let $E \subset H^p$ be an invariant subspace and $E = \Theta H^p$ its canonical representation, Θ being an inner function. Then

$$\Theta_E(\lambda) = |\Theta(\lambda)| \left(1 - |\lambda|^2\right)^{-1/p}, \qquad \lambda \in \mathbb{D}.$$

Indeed, Θ is an isometric multiplier of H^p and

$$\|\varphi_\lambda\|_{H^p \to \mathbb{C}} = \left(1 - |\lambda|^2\right)^{-1/p}, \qquad \lambda \in \mathbb{D}.$$

3.2. Let $E \subset H^2$ be an invariant subspace, $E = \Theta H^2$ and d_E its distance function (see (2.2) for definition). Then

$$d_E^2(\lambda) = \frac{1 - |\Theta(\lambda)|^2}{1 - |\lambda|^2}, \qquad \lambda \in \mathbb{D}.$$

In fact,

$$\begin{aligned} d_E^2(\lambda) &= \left\|P_{E^\perp} k_\lambda\right\|_2^2 \\ &= \left\|(1 - \overline{\Theta(\lambda)}\,\Theta)\,(1 - \bar{\lambda} z)^{-1}\right\|_2^2 \\ &= \left(1 - |\Theta(\lambda)|^2\right)(1 - |\lambda|^2)^{-1}. \end{aligned}$$

3.3. Let F be a subspace of H^2 (not necessarily invariant), $\lambda \in \mathbb{D}$ and

$$\epsilon_F(\lambda) = \frac{\Theta_F(\lambda)}{\Theta_{H^2}(\lambda)} = \Theta_F(\lambda)(1 - |\lambda|^2)^{1/2}.$$

Let further $\mu, \mu \in \mathbb{D}$ be a zero of F (i.e. $f(\mu) = 0$ whenever $f \in F$). Then

$$\epsilon_F(\lambda) \le \left|b_\mu(\lambda)\right|,$$

where $b_\mu(z) = (\mu - z)(1 - \bar{\mu} z)^{-1}$ stands for a Blaschke factor. For $\lambda \ne \mu$ the equality holds if and only if $b_\mu (1 - \bar{\lambda} z)^{-1} \in F$, and — supposing F to be z-invariant — if and only if $F = b_\mu H^2$.

Indeed, if $f \in F$ we have $f = b_\mu g$, where $\|g\|_2 = \|f\|_2$, and hence

$$\left|f(\lambda)\right| \le \left|b_\mu(\lambda)\right| \cdot \left|g(\lambda)\right| \le \left|b_\mu(\lambda)\right| \cdot \|g\|_2 \cdot \left(1 - |\lambda|^2\right)^{-1/2}$$

for all $\lambda \in \mathbb{D}$, and the inequality follows.

The equality means that there exists a function $f = b_\mu g \in F$ such that

$$f(\lambda) = \Theta_F(\lambda) = \left|b_\mu(\lambda)\right| \left(1 - |\lambda|^2\right)^{-1/2}, \quad \|f\|_2 = 1$$

and so

$$g(\lambda) = \left(1 - |\lambda|^2\right)^{-1/2}, \quad \|g\|_2 = 1.$$

Hence, $g = \alpha(1 - \bar{\lambda} z)^{-1}$ with $|\alpha| = (1 - |\lambda|^2)^{1/2}$.

Moreover, an invariant subspace containing a reproducing kernel $g = (1 - \bar{\lambda} z)^{-1}$ coincides with the whole space H^2 (obviously, g is an outer function).

3.4. Corollary. — *Let F be a subspace of H^2 and $\lambda \in \mathbb{D}$. Then the disc*

$$\{z \in \mathbb{D} : |b_\lambda(z)| < \epsilon_F(\lambda)\}, \tag{3.1}$$

where $\epsilon_F^2(\lambda) = 1 - d_F^2(\lambda)(1 - |\lambda|^2)$, *is free of zeros of the subspace F.*

This is an immediate consequence of 3.3.

3.5. *Remark.* — It is clear that we always have $\epsilon_F(\lambda) \geq 0$ and

$$d_F(\lambda) = \operatorname{dist}(k_\lambda, F) \leq \|k_\lambda\|_2 = \left(1 - |\lambda|^2\right)^{-1/2}.$$

The equality holds (and the disc (3.1) is therefore empty) if and only if λ is a zero of the subspace F. So, given a subspace $F \subset H^2$, $F \neq \{0\}$, the set $\{\lambda \in \mathbb{D} : \epsilon_F(\lambda) = 0\}$ is at most a sequence $\{\lambda_n\}_{n\geq 1}$ satisfying the Blaschke condition

$$\sum_{n\geq 1} (1 - |\lambda_n|) < \infty.$$

4. Invariant subspaces related to the Riemann ζ-function and localization of zeros.

First, we state two corollaries of Theorem 0.1 (see Introduction); the first one is an improvement of B. Nyman's theorem [Ny] (see also [B]).

4.1. Corollary. — *The Riemann ζ-function has no zeros in the half-plane $\{\operatorname{Re} z > \gamma > 0\}$ if and only if there exists a point s, $\operatorname{Re} s > 0$ such that $d_\gamma(s) = 0$ (where, as well as in the theorem, $d_\gamma(s) = \operatorname{dist}(x^s, K_\gamma)$ and*

$$K_\gamma = \operatorname{span}_{L^2(0,1;\mathrm{d}x/x)}(E_{\alpha,\gamma} : 0 < \alpha < 1); \tag{4.1}$$

the «dist» stands for the distance in the space $L^2(0,1;\mathrm{d}x/x)$).

4.2. Corollary. — *Let F be any subspace of K_γ. Then the disc*

$$\gamma + \left\{z : \left|\frac{z-s}{z-s_*}\right|^2 < 1 - 2\operatorname{Re} s \cdot d_F^2(s)\right\}$$

is free of zeros of the ζ-function; here $d_F(s) = \operatorname{dist}(x^s, F)$.

The last corollary is an immediate consequence of the theorem and remarks from Section 3; Corollary 4.1 will be justified a little bit later. For other remarks see Section 4.5 below.

4.3. *Proof of theorem 0.1.* — Let us consider the standard imbedding of the space $L^2(0,1;\mathrm{d}x/x)$ into $L^2(0,\infty;\mathrm{d}x/x)$ putting $f(x)=0$ for $x>1$ for a function $f\in L^2(0,1;\mathrm{d}x/x)$, and let K_γ be the subspace defined in (4.1).

The Mellin transform $\mathcal{F}_*$ (i.e., the Fourier transform on the multiplicative group $\mathbb{R}_+=(0,\infty)$),

$$\mathcal{F}_*g(z)=\frac{1}{\sqrt{2\pi}}\int_0^1 g(x)x^z\frac{\mathrm{d}x}{x},\qquad \operatorname{Re} z>0,$$

maps isometrically the space $L^2(0,1;\mathrm{d}x/x)$ onto the Hardy space $H^2(\operatorname{Re} z>0)$ endowed with the usual norm $\sup_{\sigma>0}\big(\int_{\mathbb{R}}|f(\sigma+it)|^2\,\mathrm{d}t\big)^{1/2}$. For the $\mathcal{F}_*$-image of the functions $E_{\alpha,\gamma}$ we have

$$\begin{aligned}(4.2)\qquad \mathcal{F}_*E_{\alpha,\gamma}(z)&=\frac{1}{\sqrt{2\pi}}\int_0^1 x^{z+\gamma-1}\Big(\Big[\frac{\alpha}{x}\Big]-\alpha\Big[\frac{1}{x}\Big]\Big)\,\mathrm{d}x\\&=\frac{1}{\sqrt{2\pi}}\,(\alpha^{z+\gamma}-\alpha)\int_1^\infty [t]\,t^{-z-\gamma-1}\,\mathrm{d}t\\&=\frac{1}{\sqrt{2\pi}}\,(\alpha^{z+\gamma}-\alpha)\,\frac{\zeta(z+\gamma)}{z+\gamma}.\end{aligned}$$

The last equality is a well-known integral representation of the Riemann ζ-function: for an integer $n\ge 1$ one has

$$s\int_1^{n+1}[t]\,t^{-s-1}\,\mathrm{d}t=s\sum_{k=1}^{n}k\int_k^{k+1}t^{-s-1}\,\mathrm{d}t=\sum_{k=1}^{n}k^{-s}-n(n+1)^{-s}$$

which tends to $\sum_{k\ge1}k^{-s}=\zeta(s)$ for $\operatorname{Re} s>1$.

Let $E=E_\gamma=\mathcal{F}_*K_\gamma$ be the corresponding subspace of the Hardy space $H^2(\operatorname{Re} s>0)$; our goal is to distinguish zeros of this subspace. By definition, these are common zeros of functions (4.2). As to the latter, they vanish for z with $\operatorname{Re} z>0$, $\zeta(z+\gamma)=0$ and for all z solving the following system of equations:

$$(z+\gamma-1)\log\alpha=2\pi ik,\qquad k\in\mathbb{Z}\backslash\{0\}.$$

Note that the zero z corresponding to $k = 0$, i.e. $z + \gamma = 1$, is killed by the pole of $\zeta(z + \gamma)$ which is easily visible from another classical identity,

$$\frac{\zeta(s)}{s} = \int_1^\infty ([x] - x)x^{-s-1}\,\mathrm{d}x + \frac{1}{s-1},$$

which can be verified in a similar way. Thus, the common zeros of the family $\mathcal{F}_* E_{\alpha,\gamma}$, $0 < \alpha < 1$ are $\{z : \operatorname{Re} z > 0,\ \zeta(z + \gamma) = 0\}$.

On the other hand, the reproducing kernel of the space $H^2(\operatorname{Im} z > 0)$ is $k_\lambda = (2\pi i)^{-1}(\bar{\lambda} - z)^{-1}$ (for a moment, we change the half-plane $\{\operatorname{Re} s > 0\}$ to $\{\operatorname{Im} z > 0\}$ to deal with the commonly used Hardy spaces of the upper half-plane related to the usual Fourier transform, see [Ho], [N], [Ni]). The Fourier transform $\mathcal{F}$ takes k_λ to

$$\mathcal{F}k_\lambda = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-i\bar{\lambda}t}\chi_{(0,\infty)}(t) = \frac{1}{\sqrt{2\pi}}\,x^s\chi_{(0,1)}(x)$$

with $x = \mathrm{e}^{-t}$, $s = i\bar{\lambda}$. Hence,

$$\begin{aligned}
d_\gamma(s) &= \operatorname{dist}_{L^2(0,1;\mathrm{d}x/x)}(x^s, K_\gamma) = \sqrt{2\pi}\operatorname{dist}_{H^2}(k_\lambda, \mathcal{F}_* K_\gamma) \\
&= \sqrt{2\pi}\, d_{\mathcal{F}_* K_\gamma}(\lambda), \\
\|k_\lambda\| &= (4\pi \operatorname{Im}\lambda)^{-1/2} = (4\pi \operatorname{Re} s)^{-1/2}, \\
\epsilon_\gamma(s)^2 &= \epsilon^2_{\mathcal{F}_* K_\gamma}(\lambda) = 1 - d^2_{\mathcal{F}_* K_\gamma}(\lambda)\|k_\lambda\|^{-2} = 1 - 2\operatorname{Re} s \cdot d^2_\gamma(s).
\end{aligned}$$

The theorem now follows from proposition 3.4.

4.4. *Verifying Corollary 4.1.*

We observe first that if $d_\gamma(s) = 0$ the theorem implies $\zeta(z) \neq 0$ for $\operatorname{Re} z > \gamma$.

The «only if» part of the Corollary depends on the Beurling invariant subspace theorem. Defining the multiplicative shifts τ_β,

$$(\tau_\beta f)(x) = f\Big(\frac{x}{\beta}\Big), \qquad 0 < \beta < 1$$

we get $\tau_\beta E_{\alpha,\gamma} = \beta^{-\alpha}(E_{\alpha\beta,\gamma} - \alpha E_{\beta,\gamma})$ and hence

$$\tau_\beta K_\gamma \subset K_\gamma, \qquad 0 < \beta < 1.$$

It means the Mellin transform $E = \mathcal{F}_* K_\gamma$ is an invariant subspace of the semi-group $\mathcal{F}_* \tau_\beta \mathcal{F}_*^{-1}$, $0 < \beta < 1$ which coincides with the multiplication semi-group

$$f(z) \longmapsto e^{-\lambda z} f(z), \quad \lambda = \log\Big(\frac{1}{\beta}\Big),$$

$f \in H^2(\operatorname{Re} z > 0)$. Hence, due to Beurling's theorem, there exists an inner function Θ such that $E = \Theta H^2(\operatorname{Re} z > 0)$. Moreover, $\Theta = BS$ where B stands for the Blaschke product for the zero set of E (already identified with zeros of $\zeta(z+\gamma)$, $\operatorname{Re} z > 0$) and S for a singular inner function. It is clear that S is an inner divisor of all the functions from E and, in particular, of all the functions $\mathcal{F}_* E_{\alpha,\gamma}$ generating E.

Since $-1 \le [\alpha/x] - \alpha[1/x] \le \alpha$ for $0 < \alpha < 1, 0 < x \le 1$ formula (4.2) shows the functions $\mathcal{F}_* E_{\alpha,\gamma}$ to be analytic and bounded for $\operatorname{Re} z \ge 0$. It follows then (see, for instance, [N], [Ni]) that the function S is analytic at all points of $i\mathbb{R}$, and hence has to be of the form $S = e^{-az}$, $a \ge 0$.

On the other hand, having $\zeta(\sigma) = \sum_{n\ge 1} n^{-\sigma}$ for $\sigma > 1$, we get

$$\limsup_{\sigma \to \infty} \frac{1}{\sigma} \log \Big| \frac{\zeta(\sigma+\gamma)}{\sigma+\gamma} \Big| = 0$$

and hence

$$\limsup_{\sigma \to \infty} \frac{1}{\sigma} \log \big| \mathcal{F}_* E_{\alpha,\gamma}(\sigma) \big| = 0.$$

Since S is an inner divisor of $\mathcal{F}_* E_{\alpha,\gamma}$ we obtain $a = 0$.

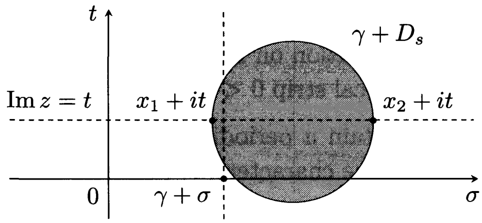
Now, we finish the proof simply mentioning that if $B = 1$ one has $E = H^2(\operatorname{Re} z > 0)$ and $K_\gamma = L^2(0,1;dx/x)$, and consequently $d_\gamma(s) = 0$ for every $s, \operatorname{Re} s > 0$.

4.5. *Concluding remarks.*

4.5.1. *Picture of* $\gamma + D_s$,

$$D_s = \Big\{ z : \Big| \frac{z-s}{z-s_*} \Big| < \epsilon(s) \Big\}, \quad \epsilon(s) = \sqrt{1 - 2\operatorname{Re} s \cdot d_\gamma^2(s)}.$$

Let $s = \sigma + it$. Then $[x_1 + it, x_2 + it]$ is the diameter of the disc $\gamma + D_s$, where $x_1 = \gamma + \sigma(1-\epsilon(s))/(1+\epsilon(s))$, $x_2 = \gamma + \sigma(1+\epsilon(s))/(1-\epsilon(s))$.



4.5.2. Estimation of number of zeros. — One can improve a little bit the final conclusion of Section 3 (see 3.4) estimating the number of zeros in a larger disc. Namely, it is easy to see that the same arguments imply that the disc

$$\{z \in \mathbb{D} : |b_\gamma(z)| < \epsilon_E(\lambda)^{1/n}\}$$

contains no more than $(n-1)$ zeros of a subspace E $(n = 1, 2, \ldots)$. And hence, a similar claim is true for the ζ-function: the disc

$$\gamma + \left\{z : \left|\frac{z-s}{z-s_*}\right| < \epsilon(s)^{1/n}\right\}$$

contains fewer than n zeros of the ζ-function $(n = 1, 2, \ldots)$.

4.5.3. Simplifying approximation. — The functions $x^{-\gamma}E_{\alpha,\gamma}$ are step functions taking values in the interval $[-1, \alpha]$, periodic in $1/x$ if $\alpha = p/q$ is rational. To get a rough approximation of $d_\gamma(s)$ one can simplify the integrals to deal with periodic functions only: taking a finite set A of rationals α, let us consider a finite family of rational translates $\tau_\beta E_{\alpha,\gamma}$, $\beta \in B$ and form a subspace

$$K_{A,B,\gamma} = \operatorname{span}(\tau_\beta E_{\alpha,\gamma} : \alpha \in A,\ \beta \in B).$$

Then, defining $d_{A,B,\gamma}(s) = \operatorname{dist}(x^s, K_{A,B,\gamma})$, we conclude that the disc

$$\left\{z : \left|\frac{z-s}{z-s_*}\right|^2 < 1 - 2\operatorname{Re} s \cdot d^2_{A,B,\gamma}(s)\right\}$$

is free of zeros of the function $\zeta(z+\gamma)$, $\operatorname{Re} z > 0$ and of common points of the sequences

$$z = 1 - \gamma + \frac{2\pi i k}{\log \alpha}, \quad k \in \mathbb{Z}\backslash\{0\}, \quad \alpha \in A.$$

The functions $x^{-\gamma}\tau_\beta E_{\alpha,\gamma}$, $\alpha \in A$, $\beta \in B$ will be periodic in $1/x$ with a period not exceeding the least common divisor of denominators of fractions $\alpha\beta$. There are no excessive zeros if the set $\log A$ is rationally independent. Otherwise, to get an information on zeros of the ζ-function one has to put a test point s into the critical strip $0 < \operatorname{Re} s < 1$.

In particular, we obtain a periodic approximation taking $A = \{\frac{1}{2}\}$; then $x^{-\gamma}E_{1/2,\gamma}$ is simply the characteristic function of a union of intervals, namely,

$$\bigcup_{n\geq 1}\Big[\frac{1}{2n}, \frac{1}{2n-1}\Big],$$

and for the Mellin transform we get

$$\mathcal{F}_* E_{1/2,\gamma}(z) = f(z+\gamma),$$

where $f(z) = 2^{-1}(1-2^{1-z})\zeta(z)$.

One can obtain some other step functions approximations with entire steps considering products

$$\Big(\sum_{k\geq 1} a_k k^{-s}\Big)\zeta(s)$$

with $\sum_{k\geq 1} a_k k^{-1} = 1$ as well as corresponding combinations $\sum_{k\geq 1} a_k E_{1/k,\gamma}$.

4.5.4. Numerical experiments. — One can make use of non-invariant finite dimensional approximations to K_γ: $F \subset K_\gamma$ and $d_F(s) = \operatorname{dist}(x^s, F)$ (see Corollary 4.2). For instance, for a one dimensional $F = \mathcal{L}in(E_{\alpha,\gamma})$ we have

$$\epsilon_F^2(s) = 1 - 2\operatorname{Re} s d_F^2(s) = 2\operatorname{Re} s\frac{|(x^s, E_{\alpha,\gamma})|^2}{\|E_{\alpha,\gamma}\|^2},$$

where, say, for $s = 1+it$ and $\alpha = \frac{1}{2}$ we have

$$(x^s, E_{\alpha,\gamma}) = -\frac{1}{2}\sum_{n\geq 0}(\gamma+1+it)^{-1}\Big(\frac{1}{(2n+1)^{\gamma+1+it}} - \frac{1}{(2n+2)^{\gamma+1+it}}\Big),$$

$$\begin{aligned}\|E_{\alpha,\gamma}\|^2 &= \sum_{n\geq 0}\int_{1/2n+2}^{\min(1/2n,1)} x^{2\gamma}\Big|\Big[\frac{1}{2x}\Big] - \frac{1}{2}\Big[\frac{1}{x}\Big]\Big|^2\frac{dx}{x}\\ &= \sum_{n\geq 0}\frac{1}{8\gamma}\Big(\frac{1}{(2n+1)^\gamma} - \frac{1}{(2n+2)^\gamma}\Big)\\ &\approx \text{const}\times\gamma^{-1}.\end{aligned}$$

Some more sophisticated numerical experiments related to the distance formulae are presented in [V].

5. Some open problems.

The problems concern the metric approach to invariant subspaces.

5.1. *Characterize in terms of* Θ_E *the property of a subspace* $E \subset X$ *to be invariant.*

Since

$$|f(\lambda)| \le \Theta_E(\lambda)\, \|f\|$$

for every $\lambda \in \mathbb{D}$ and $f \in E$, one can suppose that for z-invariant subspaces the converse is also true. Unfortunately, this is not the case even for $X = H^2$: to see this, let us consider an invariant subspace E in its canonical Beurling form, $E = \Theta H^2$, where Θ stands for an inner function. What we mean by the converse is the following property: the inequality

$$|f(\lambda)| \le |\Theta(\lambda)|\, \left(1 - |\lambda|^2\right)^{-1/2} \|f\|_2$$

should imply $f \in \Theta H^2$. This is surely true for Blaschke products Θ and for

$$\Theta = \prod_{k=1}^{n} \Theta_{\zeta_k}, \quad \Theta_\zeta = \exp\Big(-a_\zeta \frac{\zeta + z}{\zeta - z}\Big),$$

where $|\zeta| = 1$, $a_\zeta > 0$. But it fails to be true in general: for an arbitrarily small ϵ, $\epsilon > 0$, there exist non-constant inner functions Θ with

$$|\Theta(\lambda)| \ge \left(1 - |\lambda|^2\right)^\epsilon$$

for all $\lambda \in \mathbb{D}$, and hence with

$$\inf_{\lambda \in \mathbb{D}} |\Theta(\lambda)| \left(1 - |\lambda|^2\right)^{-1/2} > 0,$$

see [DSS]; if the inverse conjectured above were true, this would imply $z^n \in \Theta H^2$ for n large enough, and hence $\Theta = 1$ (contradiction).

It is curious to note that a description of the mentioned type of a growth limitation was obtained in [KR] for a class of singular inner functions not for subspaces ΘH^2 but for the orthogonal complements $H^2 \ominus \Theta H^2$. To describe z-invariant subspaces one probably needs some deeper relations between E and Θ_E.

5.2. *Characterize finite co-dimensional z-invariant subspaces E in terms of Θ_E.*

As this is the case for $X = H^p$, is it true that the requested characterization is

$$\underline{\lim}_{|\lambda|\to 1} \frac{\Theta_E(\lambda)}{\Theta_X(\lambda)} > 0?$$

5.3. *Is it true, or under what conditions is it true, that the convergence $\Theta_{E_n}(\lambda) \longrightarrow \Theta_E(\lambda)$, $\lambda \in \mathbb{D}$ implies a convergence of subspaces E_n to E (say, in the sense $E = \underline{\lim}_n E_n = \{f \in X : \lim_n \operatorname{dist}(f, E_n) = 0\}$)?*

Is it true that for any z-invariant subspace E there exist finite co-dimensional z-invariant subspaces E_n such that $\Theta_{E_n}(\lambda) \to \Theta_E(\lambda)$, $\lambda \in \mathbb{D}$? For a discussion of such an approximative spectral synthesis property see [Nik].

Addendum. — The referee provided the author with an interesting example showing that there exist spaces X of holomorphic functions, for which at least one of the questions raised in Section 5.3 has a negative answer. Namely, in the Bergman space $L^2_a(\mathbb{D}) = \operatorname{Hol}(\mathbb{D}) \cap L^2(\mathbb{D})$ there exist z-invariant subspaces E which cannot be represented as lower limits $\underline{\lim}\, E_n = E$ of invariant subspaces of finite co-dimension.

The key observation is that every invariant subspace of finite co-dimension meets the property $\dim(E \ominus zE) = 1$ and that this property is stable with respect to lower limits. Indeed, let $\underline{\lim}\, E_n = E$ and $\dim(E_n \ominus zE_n) = 1$, $n \geq 1$; to prove that $\dim(E \ominus zE) = 1$ it is enough to check that any function f, $f \in E$, having an excessive zero at the point 0 (i.e., a zero of multiplicity exceeding the zero multiplicity of E at 0) is of the form $f = zg$, $g \in E$. Taking $f_n \in E_n$ such that $\lim_n \|f_n - f\| = 0$ one can see that f_n have excessive zeros at some points λ_n tending to 0. Hence, due to the hypothesis (see also [He]), we get $(z - \lambda_n)^{-1} f_n \in E_n$. It is clear that there exists a limit $\lim_n (z - \lambda_n)^{-1} f_n = g \in E$ and that $zg = f$.

On the other hand, it is known (see [He]) that there exist invariant subspaces $E \subset L^2_a(\mathbb{D})$ with $\dim(E \ominus zE) > 1$.

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