MASAKI KASHIWARA

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Annales de l'institut Fourier, tome 43, nº 5 (1993), p. 1597-1618

<http://www.numdam.org/item?id=AIF_1993__43_5_1597_0>

Ann. Inst. Fourier, Grenoble
43, 5 (1994), 1597-1618

D-MODULES AND REPRESENTATION THEORY OF LIE GROUPS

by Masaki KASHIWARA

Dedicated to Professor B. Malgrange on his 65-th birthday

Introduction.

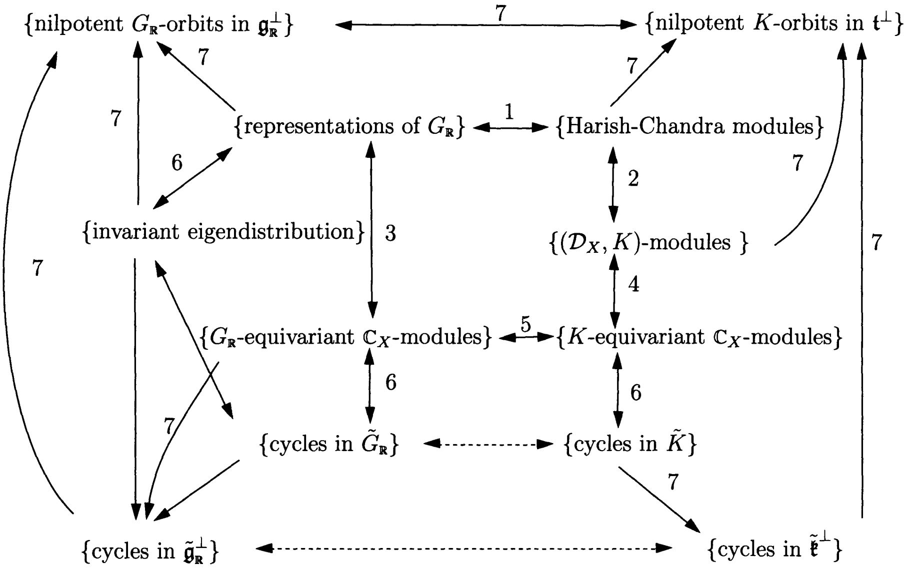
The theory of linear differential equations and the representation theory have been intimately related since the dawn of the representation theory of Lie groups. In fact one of the motivations of the representation theory was to analyze invariant linear differential equations, as seen in the Fourier transform for the constant coefficient equations or the spherical functions for the Laplace equations.

In this article, we shall not discuss this direction but the opposite direction, i.e. the application of D-modules to the representation theory. Here I mean by the representation theory the one for real semisimple Lie group.

The main ingredient in this paper is to understand the representation theory by the geometry of the flag manifold. This is illustrated as the diagram below.

We shall explain the arrows in each section with assigned number. In this diagram, the right columns are (complex) algebraic but the left columns are purely analytic (or real).

Key words : Representation – Real semisimple group – Flag manifold.
A.M.S. Classification : 22E45 – 22E46.



Notations

$G_{\mathbb{R}}$: a real semisimple Lie group with finite center

$K_{\mathbb{R}}$: a maximal compact subgroup of $G_{\mathbb{R}}$

$\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{k}_{\mathbb{R}}$: the Lie algebras of $G_{\mathbb{R}}$, $K_{\mathbb{R}}$

$\mathfrak{g}$, $\mathfrak{k}$: the complexification of $\mathfrak{g}$, $\mathfrak{k}$

G, K : complex reductive group with $\mathfrak{g}$, $\mathfrak{k}$ as Lie algebra with

$$\begin{array}{ccc} G & \leftarrow & G_{\mathbb{R}} \\ \uparrow & & \cup \\ K & \hookleftarrow & K_{\mathbb{R}} \end{array}, \qquad \begin{array}{ccc} \mathfrak{g} & \hookleftarrow & \mathfrak{g}_{\mathbb{R}} \\ \cup & & \cup \\ \mathfrak{k} & \hookleftarrow & \mathfrak{k}_{\mathbb{R}} \end{array}$$

X : the flag manifold of G

$B(x)$, $\mathfrak{b}(x)$: the isotropy group (Lie algebra) at $x \in X$.

$\mathfrak{n}(x) = [\mathfrak{b}(x), \mathfrak{b}(x)]$: the nilpotent radical of $\mathfrak{b}(x)$

$\tilde{G} := \{(g, x) \in G \times X; gx = x\}$

$\tilde{\mathfrak{g}}^* := \{(\xi, x) \in \mathfrak{g}^* \times X; \xi \in \mathfrak{n}(x)^{\perp}\}$

B_0 : a Borel subgroup of G

U_0 : the unipotent radical of B_0

T : the Cartan subgroup of G (contained in B_0)

$\gamma : \tilde{G} \to T$ the projection

$\mathfrak{t}$: the Lie algebra of T

$\Delta \subset \mathfrak{t}^*$: the root system of G

$\check{\alpha}$: the coroot $\in \mathfrak{t}$ corresponding to $\alpha \in \Delta$

Δ^+ : the positive root system (corresponding to U_0)

W : the Weyl group

G_{reg} : the set of regular semisimple elements of G

$\mathrm{Mod}(\mathcal{A})$: the category of $\mathcal{A}$-modules for a sheaf of rings $\mathcal{A}$.

$E^\perp := \{\xi \in V^*; \langle \xi, E\rangle = 0\}$ for a vector subspace E of a vector space V.

1. Harish-Chandra modules.

Let $G_\mathbb{R}$ be a real semisimple Lie group with finite center. A representation (ρ, E) of $G_\mathbb{R}$ is a complete locally convex Hausdorff topological vector space E with a continuous action $\rho : G_\mathbb{R} \to \mathrm{End}_\mathbb{C}(E)$, i.e. the action map

$$G_\mathbb{R} \times E \to E \quad ((g, v) \leftrightarrow \rho(g)v)$$

is a continuous map.

Since we know well the representations of compact groups, the first step is to regard E as a representation of a maximal compact subgroup $K_\mathbb{R}$ of $G_\mathbb{R}$. Let $HC(E)$ be the space of $K_\mathbb{R}$-finite vectors of E. Recall that an element v of E is called $K_\mathbb{R}$-finite if it is contained in a finite-dimensional $K_\mathbb{R}$-modules. Since any finite-dimensional representation of $K_\mathbb{R}$ can be extended uniquely to a representation of the complexification K of $K_\mathbb{R}$, $HC(E)$ is regarded as a K-module.

This space is extensively studied by Harish-Chandra [HC]. In particular he obtained :

(1.1.1) $HC(E)$ is a dense subspace of E.

(1.1.2) For any A in the Lie algebra $\mathfrak{g}_\mathbb{R}$ of $G_\mathbb{R}$ and $v \in HC(E)$, $\rho(e^{tA})v$ is a differentiable function in t and $\rho(A)(v) \underset{\mathrm{def}}{=} \frac{d}{dt}\rho(e^{tA})v\Big|_{t=0}$ defines a structure of a $\mathfrak{g}_\mathbb{R}$-module on $HC(E)$.

Since the $\mathfrak{g}_\mathbb{R}$-module structure can be extended to a $\mathfrak{g}$-module structure, $HC(E)$ has a structure of $\mathfrak{g}$-module and K-module. They satisfy

(1.1.3) The infinitesimal action of K coincides with the restriction of the $\mathfrak{g}$-module structure to $\mathfrak{k}$.

(1.1.4) The action homomorphism $\mathfrak{g}\otimes HC(E) \to HC(E)$ is K-linear, i.e. for $A \in \mathfrak{g}$, $k \in K$ and $v \in HC(E)$ we have $k(Av) = (Ad(k)A)(kv)$.

We refer this as $(\mathfrak{g}, K)$-module structure.

Let $K^\wedge$ be the set of isomorphic classes of irreducible K-modules and for $\xi \in K^\wedge$, let $(HC(E))_\xi$ denote the isotypic component of $HC(E)$ of type ξ. Then

$$HC(E) = \bigoplus_{\xi\in K^\wedge} (HC(E))_\xi.$$

If $\dim_{\mathbb{C}}(HC(E))_\xi < \infty$ for any $\xi \in K^\wedge$, the representation (ρ, E) is called *admissible*. The admissible representations form a good class and we shall only treat those representations. Similarly, we call a $(\mathfrak{g}, K)$-module M *Harish-Chandra module* if it satisfies the following equivalent conditions :

(i) $\dim M_\xi < \infty$ for any $\xi \in K^\wedge$.

(ii) M has finite length.

(iii) M is finitely generated over $U(\mathfrak{g})$ and $\dim_{\mathbb{C}} \mathfrak{z}(\mathfrak{g})u < \infty$ for any $u \in M$.

Here $\mathfrak{z}(\mathfrak{g})$ denotes the center of the universal enveloping algebra $U(\mathfrak{g})$. For a given Harish-Chandra module M, we can construct a representation E of $G_{\mathbb{R}}$ with $HC(E) \cong M$ as follows. Let $M^* = \bigoplus_{\xi\in K^\wedge} (M_\xi)^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})_{K-\mathrm{fini}}$. Then M^* is also a Harish-Chandra module. We call M^* the dual of M. This satisfies : $(M^*)^* \cong M$. Let $C^\infty(G_{\mathbb{R}})$ be the space of C^∞ functions on $G_{\mathbb{R}}$ with the structure of $\mathfrak{g}$-modules and representation of $K_{\mathbb{R}}$ via the left translation on $G_{\mathbb{R}}$. Set

$$M_{\max} = \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})}(M^*, C^\infty(G_{\mathbb{R}}))$$

with the induced topology of $C^\infty(G_{\mathbb{R}})$. Since $\mathcal{D}_G \otimes_{U(\mathfrak{g})} M^*$ is an elliptic D-module on $G_{\mathbb{R}}$ (i.e. the characteristic variety of $\mathcal{D}_G \otimes_{U(\mathfrak{g})} M^*$ does not intersect $T^*_{G_{\mathbb{R}}}G$ outside the zero section), the regularity theorem for elliptic $\mathcal{D}$-modules shows that we can replace $C^\infty(G_{\mathbb{R}})$ with other function spaces, such as real analytic functions, distributions, hyperfunctions. Then $M_{\max}$ has a structure of Fréchet space induced from the topology of $C^\infty(G_{\mathbb{R}})$ and the right translation by $G_{\mathbb{R}}$ on $G_{\mathbb{R}}$ endows $M_{\max}$ with the structure of representation of $G_{\mathbb{R}}$ on $M_{\max}$. By the evaluation at the origin we obtain a linear map $C^\infty(G_{\mathbb{R}}) \to \mathbb{C}$ and it induces a homomorphism

$$M_{\max} \to \mathrm{Hom}_{\mathbb{C}}(M^*, \mathbb{C}). \tag{1.1.5}$$

By a result of Malgrange [M],

$$\mathrm{Hom}_{\mathbb{C}}(M^*, \mathbb{C}) \cong \mathrm{Hom}_{\mathfrak{g}}(M^*, \hat{\mathcal{O}}_{G,e}).$$

Here $\hat{\mathcal{O}}_{G,e}$ is the ring of formal power series at $e \in G$. Hence (1.1.5) is injective. We can show that (1.1.5) induces a $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$-linear-isomorphism

$$M_{\max} \supset HC(M_{\max}) \xrightarrow{\sim} M \cong M^{**} \subset \mathrm{Hom}_{\mathbb{C}}(M^*, \mathbb{C}).$$

This representation is studied extensively by W. Schmid ([Sd],[SW]) and he called it the *maximal globalization* of M because this has the following universal property :

(1.1.6) $$\mathrm{Hom}(E, M_{\max}) \cong \mathrm{Hom}_{(\mathfrak{g},K)}(HC(E), M)$$
for any admissible representation E.

Here the Hom in the left hand side of (1.1.6) is the set of continuous linear map from E to $M_{\max}$ commuting with the action of $G_{\mathbb{R}}$. He showed also that the functor $M \mapsto M_{\max}$ is an exact functor.

Briefly, the study of representation of $G_{\mathbb{R}}$ is almost equivalent to the study of Harish-Chandra modules.

Thus we have obtained the correspondence :

$$\{\text{representations of } G_{\mathbb{R}}\} \longleftrightarrow \{\text{Harish-Chandra modules}\}.$$

2. Beilinson-Bernstein correspondence.

2.1. Infinitesimal character.

Let $\mathfrak{z}(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. If M is an irreducible Harish-Chandra module, then by a generalization of Schur's lemma $\mathfrak{z}(\mathfrak{g})$ acts on M by a scalar, i.e. $\mathfrak{z}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(M)$ splits to $\mathfrak{z}(\mathfrak{g}) \xrightarrow{\chi} \mathbb{C} \hookrightarrow \mathrm{End}_{\mathbb{C}}(M)$ through a ring homomorphism $\chi : \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$. In general, a $U(\mathfrak{g})$-module M is called with *infinitesimal character* χ if $Pu = \chi(P)u$ for any $u \in M$ and $P \in \mathfrak{z}(\mathfrak{g})$. In the sequel, we consider the Harish-Chandra modules with infinitesimal character. By Chevalley, $\mathfrak{z}(\mathfrak{g})$ is isomorphic to $S[\mathfrak{t}]^W$. Here $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group. The isomorphism is given by

$$\mathfrak{z}(\mathfrak{g}) \ni P \mapsto f \in S(\mathfrak{t}) = U(\mathfrak{t})$$

where $P \equiv f \bmod \mathfrak{n}U(\mathfrak{g})$. Here $\mathfrak{n}$ is a maximal nilpotent subalgebra stable by $\mathfrak{t}$. (Note that this is different from a usual normalization.) Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$ and choose a set of positive roots by $\Delta^+ = \{\alpha \in \Delta; \mathfrak{g}_\alpha \subset \mathfrak{n}\}$. Let ρ be the half sum of the element of Δ^+. Then $f \in S(\mathfrak{t}) = \mathbb{C}[\mathfrak{t}^*]$ is stable by the following action W on $\mathfrak{t}^*$:

$$w \circ \lambda = w(\lambda - \rho) + \rho.$$

Hence a character $\mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$ corresponds to an element of $\mathfrak{t}^*/W$. For $\lambda \in \mathfrak{t}^*$, let $\chi_\lambda : \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$ denote the corresponding ring homomorphism.

Beilinson-Bernstein discovered that the category of $\mathfrak{g}$-modules with infinitesimal character is equivalent to the category of twisted D-modules on the flag manifold ([BB]). This may be compared with the fact that coherent sheaves on an affine variety corresponds to modules on the ring of functions on the affine variety.

2.2. The flag manifold.

Let X be the flag manifold of G. This is characterized as a maximal compact homogeneous space of G. We set $\tilde{G} = \{(g, x) \in G \times X; gx = x\}$ and $\tau : \tilde{G} \to X$ and $p : \tilde{G} \to G$ denote the projection. Note that G acts on $\tilde{G}$ by $G \ni g_0 : (g, x) \mapsto (g_0 g g_0^{-1}, g_0 x)$. For $x \in X$, $p\tau^{-1}(x)$ is the isotropy subgroup G_x at x. Then by $x \leftrightarrow G_x$, X may be regarded as the set of Borel subgroups of G. Let us take $x_0 \in X$ and let B_0 be the corresponding Borel subgroup of G. Then $G/B_0 \xrightarrow{\sim} X$ by $gB_0 \mapsto gx_0$. Let us take a Cartan subgroup T of B_0 and let U_0 be the unipotent part of B_0. Then U_0 is an invariant subgroup of B_0 and $T \xrightarrow{\sim} B_0/U_0$. There exists a unique map

$$\gamma : \tilde{G} \to T \tag{2.2.1}$$

that satisfies $\gamma(g\tilde{g}) = \gamma(\tilde{g})$ for $\tilde{g} \in \tilde{G}$ and $g \in G$, and $\gamma|_{\tau^{-1}(x_0)}$ coincides with the map $B_0 \to B_0/U_0 \cong T$. The action of G on X induces the Lie algebra homomorphism

$$\mathfrak{g} \to \Gamma(X; \Theta_X). \tag{2.2.2}$$

Here Θ_X denotes the sheaf of tangent vectors on X. For $x \in X$, let $\mathfrak{b}(x)$ be the kernel of the map $\mathfrak{g} \to T_x X$ induced by (2.2.2), i.e. the isotropy subalgebra at $x \in X$. Hence $\mathfrak{b}(x)$ is the Lie algebra of G_x. By the correspondence $x \leftrightarrow \mathfrak{b}(x)$, X is also identified with the set of Borel subalgebras of $\mathfrak{g}$.

Similarly to $\tilde{G}$, we define

$$\tilde{\mathfrak{g}} = \{(A, x) \in \mathfrak{g} \times X; A \in \mathfrak{b}(x)\}. \tag{2.2.3}$$

Then G acts on $\tilde{\mathfrak{g}}$ so that the projection map $\tilde{\mathfrak{g}} \to X$ is G-equivariant. Let $\mathfrak{t}$ be the Lie algebra of T. Then similarly to (2.2.1) we have

$$\gamma : \tilde{\mathfrak{g}} \to \mathfrak{t}. \tag{2.2.4}$$

Since $T_xX \cong \mathfrak{g}/\mathfrak{b}(x)$ for any $x \in X$, we obtain an exact sequence of vector bundles on X :

$$0 \to \tilde{\mathfrak{g}} \to \mathfrak{g} \times X \to TX \to 0. \tag{2.2.5}$$

Similarly to (2.2.4), we set

$$\tilde{\mathfrak{g}}^* = \{(\xi, x) \in \mathfrak{g}^* \times X; \xi \in \mathfrak{n}(x)^{\perp}\}. \tag{2.2.6}$$

Here $\mathfrak{n}(x) = [\mathfrak{b}(x), \mathfrak{b}(x)]$ is the nilpotent radical of $\mathfrak{b}(x)$. By a G-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, $\tilde{\mathfrak{g}}^*$ and $\tilde{\mathfrak{g}}$ are isomorphic. Taking the dual of an exact sequence

$$0 \to \mathfrak{n}(x) \to \mathfrak{b}(x) \to \mathfrak{t} \to 0,$$

we have an exact sequence

$$0 \to \mathfrak{b}(x)^{\perp} \to \mathfrak{n}(x)^{\perp} \to \mathfrak{t}^* \to 0.$$

This induces the exact sequence of vector bundles on X

$$0 \to T^*X \to \tilde{\mathfrak{g}}^* \to \mathfrak{t}^* \times X \to 0. \tag{2.2.7}$$

The moment map $\rho : T^*X \to \mathfrak{g}^*$ (the dual of $\mathfrak{g} \to T_xX$) coincides with the composition $T^*X \to \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$.

2.3. Twisted equivariance.

Let Z be a complex manifold with an action of a complex group H. Let $\mathfrak{h}$ be the Lie algebra of H and λ an H-invariant element of $\mathfrak{h}^*$ with respect to the coadjoint action. Let us denote by L_λ the subsheaf of $\mathcal{O}_H$ consisting of the sections u satisfying $R(A)u = \langle A, \lambda\rangle u$ for $A \in \mathfrak{h}$. Here $R(A)u(h) = \frac{d}{dt}u(he^{tA})|_{t=0}$ is the left invariant tangent vector corresponding to A. This is given (locally at the origin) $u(e^A) = \text{const.}e^{\langle A,\lambda\rangle}$ $A \in \mathfrak{h}$. Hence L_λ

is a locally constant sheaf of rank 1. The multiplication of $\mathcal{O}_H$ gives an isomorphism :

$$m : (L_\lambda)_h \otimes (L_\lambda)_{h'} \xrightarrow{\sim} (L_\lambda)_{hh'} \quad \text{for} \quad h, h' \in H.$$

Let $\mu : H \times Z \to Z$ be the action map. A sheaf F is called *twisted H-equivariant* sheaf with twist λ if F is endowed with an isomorphism $\beta : \mu^{-1}F \xrightarrow{\sim} L_\lambda \boxtimes F$ satisfying the chain condition :

$$(2.3.1) \qquad \begin{array}{ccc} F_{hh'z} & \xrightarrow{\beta} & (L_\lambda)_h \otimes F_{h'z} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ (L_\lambda)_{hh'} \otimes F_z & \xleftarrow{m} & (L_\lambda)_h \otimes (L_\lambda)_{h'} \otimes F_z \end{array}$$

is commutative for $h, h' \in H$ and $z \in Z$. If $\lambda = 0$, F is simply called *H-equivariant.* Let us denote by $\mathrm{Mod}_{H,\lambda}(\mathbb{C}_Z)$ the category of twisted H-equivariant sheaves with twist λ.

Assume moreover the action of H to be free. Set $Z_0 = H \setminus Z$ and let $\pi : Z \to Z_0$ be the projection. Then $Z_0 \supset U \mapsto \mathrm{Mod}_{H,\lambda}(\mathbb{C}_{\pi^{-1}U})$ is a stack (a sheaf of category) over Z_0 locally isomorphic to $U \mapsto \mathrm{Mod}(\mathbb{C}_U)$. Hence we regard $\mathrm{Mod}_{H,\lambda}(\mathbb{C}_{\pi^{-1}U})$ living on U and denote it by $\mathrm{Mod}_\lambda(\mathbb{C}_U)$. We call an object of this a *twisted sheaf* with twist λ on U (cf. [K3]).

2.4. Twisted sheaves on the flag manifold.

Now we shall come back to our situation of the semisimple group with the notations in §2.2..

Let $X' = G/U_0$ and let $\pi : X' \to X = G/B_0$ be the projection. Then T acts on X' by $t(gU_0) = gt^{-1}U_0$ for $g \in G$ and $t \in T$. With these structures, X' becomes a principal T-bundle on X. Let $\mu : T \times X' \to X'$ be the action map, and $\mathrm{pr} : T \times X' \to X'$ be the projection. For $\lambda \in \mathfrak{t}^*$, let $e^\lambda(t) = t^\lambda = e^{\langle A,\lambda\rangle}$ for $t = e^A, A \in \mathfrak{t}$. Then e^λ is a multivalued holomorphic function on T. The sheaf L_λ introduced in the preceding paragraph coincides with the subsheaf $\mathbb{C}e^\lambda$ of $\mathcal{O}_T$.

A twisted sheaf F on X with twist λ is a sheaf F on X' with an isomorphism $\beta : \mu^{-1}F \cong L_\lambda \boxtimes F$ satisfying the chain condition (2.3.1). We denote by $\mathrm{Mod}_\lambda(\mathbb{C}_X)$ the category of twisted sheaves with twist λ. Then for $F, F' \in \mathrm{Mod}_\lambda(\mathbb{C}_X)$, $\mathcal{H}om(F, F')$ is a sheaf on X and $\mathrm{Hom}_{\mathrm{Mod}_\lambda(\mathbb{C}_X)}(F, F') \cong \Gamma(X; \mathcal{H}om(F, F'))$.

Let $\mathcal{O}(\lambda)$ be the sheaf on X' satisfying $\{\varphi \in \mathcal{O}_{X'}; \varphi(tx) = t^\lambda\varphi(x)$ for t sufficiently near $1\}$. Then $\mathcal{O}(\lambda)$ is a twisted sheaf on X with twist λ. We denote the subsheaf $\mathcal{D}_\lambda$ of $\mathcal{E}nd(\mathcal{O}(\lambda))$ consisting of sections expressed by a differential operator locally on X'. Then $\mathcal{D}_\lambda$ is a sheaf of rings on X locally isomorphic to $\mathcal{D}_X$. We know that

$$\Gamma(X;\mathcal{D}_\lambda) = U(\mathfrak{g})/U(\mathfrak{g})\ker(\chi_\lambda : \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}).$$

We denote this ring by $U_\lambda(\mathfrak{g})$. Hence for any $\mathcal{D}_\lambda$-module $\mathcal{M}$, $\Gamma(X;\mathcal{M})$ is a $U(\mathfrak{g})$-module with infinitesimal character χ_λ.

THEOREM 2.4.2 ([BB]).

(i) *Assume* $\langle\check{\alpha}, \lambda-\rho\rangle \neq 1,2,3,\cdots$ *for any* $\alpha \in \Delta^+$. *For any coherent* $\mathcal{D}_\lambda$*-module* $\mathcal{M}$, $H^n(X;\mathcal{M}) = 0$ *for* $n \neq 0$.

(ii) *Assume* $\langle\check{\alpha}, \lambda-\rho\rangle \neq 0,1,2,\cdots$ *for any* $\alpha \in \Delta^+$. *Then the category* $\mathrm{Mod}^{\mathrm{coh}}(\mathcal{D}_\lambda)$ *of coherent* $\mathcal{D}_\lambda$*-modules is equivalent to the category* $\mathrm{Mod}^{\mathrm{fin}}(\mathfrak{g},\chi_\lambda)$ *of finitely generated* $U(\mathfrak{g})$*- modules with infinitesimal character* χ_λ *by the functors quasi-inverse to each other :* $\mathrm{Mod}^{\mathrm{coh}}(\mathcal{D}_\lambda) \ni \mathcal{M} \mapsto \Gamma(X;\mathcal{M}) \in \mathrm{Mod}^{\mathrm{fin}}(\mathfrak{g},\chi_\lambda)$ *and* $\mathrm{Mod}^{\mathrm{fin}}(\mathfrak{g},\chi_\lambda) \ni M \mapsto \mathcal{D}_\lambda \otimes_{U(\mathfrak{g})} M \in \mathrm{Mod}^{\mathrm{coh}}(\mathcal{D}_\lambda)$.

Moreover we know that the derived category $D^b(\mathrm{Mod}^{\mathrm{coh}}(\mathcal{D}_\lambda))$ is equivalent to the derived category $D^b(\mathrm{Mod}^{\mathrm{fin}}(\mathfrak{g},\chi_\lambda))$ if $\langle\check{\alpha}, \lambda-\rho\rangle \neq 0$ for any $\alpha \in \Delta$. If M is a $(\mathfrak{g}, K)$-module then the corresponding $\mathcal{D}_\lambda$-module $\mathcal{D}_\lambda \otimes M$ is a $(\mathcal{D}_\lambda, K)$-module. i.e. $\mathcal{D}_\lambda$-module with K-actions (cf. [K3]). Thus we obtain the correspondence :

$$\{\text{Harish-Chandra modules}\} \leftrightarrow \{(\mathcal{D}_\lambda, K)-\text{modules}\}.$$

3. Construction of representation.

One of the method to construct a representation of $G_\mathbb{R}$ is as follows. We consider a real analytic manifold Z with an action of $G_\mathbb{R}$ and a $G_\mathbb{R}$-equivariant vector bundle $\mathcal{V}$ on Z. Then the C^∞ sections of $\mathcal{V}$ over Z form a Fréchet $G_\mathbb{R}$-module $C^\infty(\mathcal{V})$. Here we can take many other function spaces instead of C^∞-functions, e.g. locally L^2-functions, Sobolev spaces, real analytic functions, hyperfunctions (if Z is compact), etc. With appropriate conditions, they give the same $K_\mathbb{R}$-finite vectors.

If we take an arbitrary Z, then $C^\infty(\mathcal{V})$ may be too big that it is not admissible, e.g. if we take $G_\mathbb{R}/K_\mathbb{R}$ as Z, then $C^\infty(\mathcal{V})$ is not admissible in general. To obtain admissible representation, there are two methods :

(i) We take a "small" Z.

(ii) Taking a linear differential equation invariant by $G_{\mathbb{R}}$ and consider the solution space.

For a Harish-Chandra module M, $\mathrm{Hom}_{(\mathfrak{g},K_{\mathbb{R}})}(M, C^{\infty}(G_{\mathbb{R}}))$ may be regarded as the case (ii) with $Z = G_{\mathbb{R}}/K_{\mathbb{R}}$.

We shall employ the method (i) by taking the flag manifold as Z.

Let us explain our method more precisely by taking $SL_2(\mathbb{R})$ as an example.

Example 3.1. — Take the case $G_{\mathbb{R}} = SL_2(\mathbb{R}) \subset G = SL_2(\mathbb{C})$. We shall treat only the trivial infinitesimal character case. Let us take the flag manifold $X = \mathbb{P}^1_{\mathbb{C}}$. The action of G on X is given by $G \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $z \mapsto \dfrac{az+b}{cz+d}$. Then X has three $G_{\mathbb{R}}$-orbits :

$$X_{\mathbb{R}} = \mathbb{R} \cup \{\infty\},$$
$$X^{\pm} = \{z \in \mathbb{C}; \mathrm{Im}\, z \gtrless 0\}.$$

On $X_{\mathbb{R}}$ there are two $G_{\mathbb{R}}$-equivariant locally constant sheaves of rank 1, the trivial one and the other F_-. Thus we obtain the Fréchet representations of $G_{\mathbb{R}}$

$$\Gamma(X_{\mathbb{R}}; \mathcal{B}_{G_{\mathbb{R}}}) \quad \text{and} \quad \Gamma(X_{\mathbb{R}}; F_- \otimes \mathcal{B}_{G_{\mathbb{R}}}).$$

Here $\mathcal{B}_{G_{\mathbb{R}}}$ is the sheaf of hyperfunctions. On the other hand $\mathcal{O}_X(X_{\pm})$ is also a representation of $G_{\mathbb{R}}$ and there is a canonical exact sequence

$$0 \to \mathbb{C} \to \mathcal{O}_X(X_+) \oplus \mathcal{O}_X(X_-) \to \Gamma(X_{\mathbb{R}}; \mathcal{B}_{G_{\mathbb{R}}}) \to 0.$$

In fact there are four irreducible representations of $G_{\mathbb{R}}$ with trivial infinitesimal character :

$$\mathbb{C}, \;\; \mathcal{O}_X(X_+)/\mathbb{C}, \;\; \mathcal{O}_X(X_-)/\mathbb{C}, \;\; \Gamma(X_{\mathbb{R}}; F_- \otimes \mathcal{B}_{G_{\mathbb{R}}}).$$

The representation space $\mathbb{C}$ may be regarded as $\Gamma(X; \mathcal{O}_X) = \mathrm{Hom}(\mathbb{C}_X, \mathcal{O}_X)$. We have $\mathcal{O}_X(X_{\pm}) = \mathrm{Hom}(\mathbb{C}_{X_{\pm}}, \mathcal{O}_X)$ and the exact sequence

$$0 \to \mathbb{C}_{X_{\pm}} \to \mathbb{C}_X \to \mathbb{C}_{\overline{X_{\mp}}} \to 0$$

induces the exact sequence

$$\begin{array}{ccccccc}\mathrm{Hom}(\mathbb{C}_X,\mathcal{O}_X) & \to & \mathrm{Hom}(\mathbb{C}_{X_\pm},\mathcal{O}_X) & \to & \mathrm{Ext}^1(\mathbb{C}_{\overline{X_\mp}},\mathcal{O}_X) & \to & \mathrm{Ext}^1(\mathbb{C}_X,\mathcal{O}_X) \\ \| & & & & & & \| \\ \mathbb{C} & & & & & & 0.\end{array}$$

Hence we obtain

$$\mathcal{O}(X_\pm)/\mathbb{C} = \mathrm{Ext}^1(\mathbb{C}_{\overline{X_\mp}},\mathcal{O}_X).$$

Moreover $\mathcal{B}_{G_\mathbb{R}} = \mathcal{E}xt^1(\mathbb{C}_{X_\mathbb{R}},\mathcal{O}_X)$ implies $\Gamma(X_\mathbb{R}; F_-\otimes\mathcal{B}_{X_\mathbb{R}}) = \mathrm{Ext}^1(F_-,\mathcal{O}_X)$. Thus four irreducible representations are

$$\mathrm{Ext}^0(\mathbb{C}_X,\mathcal{O}_X),\quad \mathrm{Ext}^1(\mathbb{C}_{\overline{X_-}},\mathcal{O}_X),\quad \mathrm{Ext}^1(\mathbb{C}_{\overline{X_-}},\mathcal{O}_x),\quad \mathrm{Ext}^1(F_-,\mathcal{O}_X). \quad \square$$

In general, for an arbitrary $G_\mathbb{R}$, let us take a $G_\mathbb{R}$-equivariant (constructible) twisted sheaf F (or in general in the derived category) of twist λ. Then $\mathrm{Ext}^i(F,\mathcal{O}_X(\lambda))$ has a structure of a Fréchet $G_\mathbb{R}$-module.

THEOREM 3.3 [KSd]. — $\mathrm{Ext}^i(F,\mathcal{O}_X(\lambda))$ *is a maximal globalization.*

This is conjectured in [K2] and proved in [KSd]. Thus we can associate the representations of $G_\mathbb{R}$ with $G_\mathbb{R}$-equivariant (constructible twisted) sheaves.

Thus we obtained

$$\{\text{representations of } G_\mathbb{R}\} \longleftrightarrow \{G_\mathbb{R}\text{-equivariant sheaves on } X\}.$$

4. Riemann-Hilbert correspondence.

4.1. Regularity of $(\mathcal{D}_X, K)$-module.

Let $\mathcal{M}$ be a $(\mathcal{D}_\lambda, K)$-module. Then we can easily see that the characteristic variety $\mathrm{Ch}(\mathcal{M})$ of $\mathcal{M}$ is contained in $\tilde{\mathfrak{k}}^\perp = \rho^{-1}(\mathfrak{k}^\perp)$. Here $\rho : T^*X \to \mathfrak{g}^*$ is the moment map and $\mathfrak{k}^\perp \subset \mathfrak{g}^*$ is the orthogonal complement of $\mathfrak{k}$. We know that there are only finitely many K-orbits in X (cf. §5) and we have

$$\tilde{\mathfrak{k}}^\perp = \bigcup_S T^*_S X \tag{4.1.1}$$

where S ranges over the set of K-orbits. In particular $\tilde{\mathfrak{k}}^{\perp}$ is a Lagrangian subvariety and $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module. In fact we can say more by the following theorem.

THEOREM 4.1.1. — *Let Z be an algebraic manifold, H an algebraic group acting on Z. If there are only finitely many H-orbits, then any H-equivariant $\mathcal{D}_Z$-module is regular holonomic.*

By this theorem, we can show that $\mathcal{M}$ is a regular holonomic $\mathcal{D}_\lambda$-module. By the Riemann-Hilbert correspondence the category of regular holonomic $\mathcal{D}_X$-module corresponds to so-called the category of perverse sheaves. The correspondence is given by

(4.1.2) $$\mathrm{Sol}_X : \mathcal{M} \longmapsto R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

In the twisted case, by the functor $\mathrm{Sol}_X : \mathcal{M} \mapsto R\mathcal{H}om_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{O}_X(\lambda))$, the $(\mathcal{D}_\lambda, K)$-module corresponds to a K-equivariant perverse twisted sheaf of twist λ.

Let $\mathrm{Mod}_\lambda(\mathbb{C}_X)$ denote the category of twisted sheaves with twist λ and let $D^b_\lambda(\mathbb{C}_X)$ be the derived category of bounded complexes in $\mathrm{Mod}_\lambda(\mathbb{C}_X)$. Then K-equivariant perverse twisted sheaf F is an object of $D^b_\lambda(\mathbb{C}_X)$ with an isomorphism $\beta : \mu_K^{-1} F \simeq \mathrm{pr}^{-1} F$ (where $\mu_K : K \times X \to X$ is the action map and $\mathrm{pr} : K \times X \to X$ is the second projection) satisfying the following conditions :

(4.1.3) (K-equivariance) β satisfies the chain condition,

(4.1.4) (constructibility) $\dim H^k(F_x) < \infty$ for any $x \in X$ and $k \in \mathbb{Z}$,

(4.1.5) (perversity) For any K-orbit S, $H^k(F)|_S = 0$ for $k > \operatorname{codim} S$ and $H^k_S(F)|_S = 0$ for $k < \operatorname{codim} S$.

By the Riemann-Hilbert correspondence , we have thus (in the derived category)

$$\{(\mathcal{D}_\lambda, K)\text{-modules}\} \longleftrightarrow \{K\text{-equivariant twisted } \mathbb{C}_X\text{-modules of twist } \lambda\}.$$

5. Matsuki correspondence.

Matsuki [Mt] showed the following facts :

(5.1) There are finitely many K-orbits on X.

(5.2) There are finitely many $G_{\mathbb{R}}$-orbits on X.

(5.3) There is a one-to-one correspondence between the set of K-orbits and the set of $G_{\mathbb{R}}$-orbits.

A K-orbit S and a $G_{\mathbb{R}}$-orbit S^a correspond (by the correspondence above) if and only if they satisfy

(5.4) $S \cap S^a$ is non empty and compact $\Leftrightarrow S \cap S^a$ is a $K_{\mathbb{R}}$-orbit.

More generally, there is an equivalence of categories between the K-equivariant derived category $D^b_{K,\lambda}(\mathbb{C}_X)$ of twisted sheaves of twist λ and the $G_{\mathbb{R}}$-equivariant derived category $D^b_{G_{\mathbb{R}},\lambda}(\mathbb{C}_X)$ (conjectured in [K2] and proved by Mirković-Uzawa-Vilonen [MUV]) as follows.

Let $F \in D^b_{K,\lambda}(\mathbb{C}_X)$. Let us consider $F \boxtimes \mathbb{C}_G$. This is K-equivariant with respect to the diagonal action and $(X \times G)/K \cong X \times (G/K)$ by the projection $p : X \times G \to X \times (G/K)$ given by $(x, g) \mapsto (g^{-1}x, g^{-1})$. Hence there exits $\tilde{F} \in D^b_\lambda(\mathbb{C}_{X\times(G/K)})$ such that $F \boxtimes \mathbb{C}_G \cong p^{-1}\tilde{F}$, Since $F \boxtimes \mathbb{C}_G$ is G-equivariant, $\tilde{F} \in D^b_{G,\lambda}(\mathbb{C}_{X\times(G/K)})$. Set

$$\Phi(F) = \mathbb{R}p_{1!}(\tilde{F} \otimes \mathbb{C}_{G_{\mathbb{R}}/K_{\mathbb{R}}})[\dim G/K].$$

Here $p_1 : X \times (G/K) \to X$ is the first projection. Then $\Phi(F)$ belongs to $D^b_{G_{\mathbb{R}},\lambda}(\mathbb{C}_X)$ and hence Φ gives a functor from $D^b_{K,\lambda}(\mathbb{C}_X)$ to $D^b_{G_{\mathbb{R}},\lambda}(\mathbb{C}_X)$.

THEOREM 5.1 ([MUV]). — *Φ is an equivalence of categories between $D^b_{K,\lambda}(\mathbb{C}_X)$ and $D^b_{G_{\mathbb{R}},\lambda}(\mathbb{C}_X)$.*

Thus we obtain

$$\{G_{\mathbb{R}}\text{-equivariant sheaves}\} \leftrightarrow \{K\text{-equivariant sheaves}\}.$$

Thus together with the former results, we obtain correspondences :

$$\begin{tikzcd}
\{\text{representations of } G_{\mathbb{R}}\} \arrow[r, leftrightarrow] \arrow[dd, leftrightarrow] & \{\text{Harish-Chandra modules}\} \arrow[d, leftrightarrow] \\
 & \{(\mathcal{D}_\lambda, K)\text{-modules}\} \arrow[d, leftrightarrow] \\
\{G_{\mathbb{R}}\text{-equivariant sheaves}\} \arrow[r, leftrightarrow] & \{K\text{-equivariant sheaves}\}
\end{tikzcd}$$

This diagramm commutes ([KSd]).

6. Character.

6.1. Character.

A finite-dimensional representation is controlled by its character. Harish-Chandra generalized this to infinite-dimensional representations ([HC]). The character in this case is not any more real analytic.

Let (E, ρ) be an admissible representation of $G_{\mathbb{R}}$. Then $\mathrm{tr}\rho(g)$ $(g \in G_{\mathbb{R}})$ does not have a sense but for $\varphi(g) \in C_0^\infty(G_{\mathbb{R}})$

$$\int_{G_{\mathbb{R}}} \varphi(g)\rho(g)dg \in \mathrm{End}(E)$$

has a trace and

$$\varphi \mapsto \mathrm{tr} \int_{G_{\mathbb{R}}} \varphi(g)\rho(g)dg \tag{6.1.1}$$

is a continuous functional on $C_0^\infty(G_{\mathbb{R}})$ and hence it may be regarded as a distribution on $G_{\mathbb{R}}$. We call this the character of E and it will be denoted by $\mathrm{ch}(E)$. Conversely two irreducible representations of $G_{\mathbb{R}}$ have the same character if and only if they have the isomorphic Harish-Chandra modules. The character $\mathrm{ch}(E)$ has the following properties :

(6.1.2) (invariance by the adjoint action)

$$\mathrm{ch}(E)(g_0 g g_0^{-1}) = \mathrm{ch}(E)(g) \text{ for } g, g_0 \in G_{\mathbb{R}}.$$

$$P\,\mathrm{ch}(E) = \chi_\lambda(P)\,\mathrm{ch}(E) \quad \text{for} \quad P \in \mathfrak{z}(\mathfrak{g}). \tag{6.1.3}$$

Here χ_λ is the infinitesimal character for E and we regard $\mathfrak{z}(\mathfrak{g})$ as the ring of differential operators on $G_{\mathbb{R}}$ invariant by the left and the right actions by $G_{\mathbb{R}}$. Let $\mathcal{M}_\lambda$ be the $\mathcal{D}_G$-module corresponding to (6.1.2–3). Namely $\mathcal{M}_\lambda$ is a $\mathcal{D}_G$-module generated by u_λ with the defining equations :

$$(L(A) + R(A))u_\lambda = 0 \quad \text{for} \quad A \in \mathfrak{g} \quad \text{and} \tag{6.1.4}$$

$$Pu_\lambda = \chi_\lambda(P)u_\lambda \quad \text{for} \quad P \in \mathfrak{z}(\mathfrak{g}).$$

A distribution solution to $\mathcal{M}_\lambda$ is called an *invariant eigendistribution.*

Harish-Chandra studied the property of this equation and he proved

THEOREM 6.1.1. — (i) *$\mathcal{M}_\lambda$ has no non trivial submodule whose support is nowhere dense in G.*

(ii) *Any invariant eigendistribution is locally integrable.*

(iii) *Any invariant eigendistribution u is real analytic on the set $G_{\mathbb{R},\mathrm{reg}}$ of regular semisimple elements and for any Cartan subgroup H of $G_{\mathbb{R}}$ if we set*

$$v = \Big(\prod_{\alpha\in\Delta^+}(1-e^{\alpha})\Big)u|_{H_{\mathrm{reg}}}$$

then v satisfies the equation $R(P)v = P(\lambda-\rho)v$ for any $P \in S(\mathfrak{t})^W \subset U(\mathfrak{t})$.

Here $H_{\mathrm{reg}} = H \cap G_{\mathbb{R},\mathrm{reg}}$.

In fact, $\mathcal{M}_\lambda$ is a regular holonomic $\mathcal{D}_G$-module (cf. [HK]). Moreover $\mathcal{M}_\lambda$ is locally isomorphic to $\mathcal{O}_G^{\sharp W}$ on the open subset G_{reg} of regular semisimple elements. If $\lambda-\rho$ is regular then by Theorem 6.1.1 on each connected component of H_{reg}, any solution u to $\mathcal{M}_\lambda$ has the form

$$u(e^A) = \sum_{w\in W} a_w \frac{e^{\langle A, w\circ\lambda\rangle}}{\prod_{\alpha\in\Delta_+}(1-e^{\langle A,\alpha\rangle})} \tag{6.1.5}$$

for some constant $a_w \in \mathbb{C}$.

If $\lambda-\rho$ is not regular, then a_w may be a polynomial in A.

A solution to $\mathcal{M}_\lambda$ is called *constant coefficient invariant eigendistribution* if it has the form (6.1.5) with constant a_w on any H_{reg}.

Fomin-Shapavalov [FS] showed that the character is a constant coefficient invariant eigendistribution for arbitrary λ and its converse was shown by K. Nishiyama [N].

6.2. Cycles in $\tilde{G}_{\mathbb{R}}$.

Let us recall $p : \tilde{G} \to G$, $\tau : \tilde{G} \to X$ and $\gamma : \tilde{G} \to T$ in §2.2.1. Let $\mathcal{D}_T e^\lambda$ be the holonomic $\mathcal{D}_T$-module of rank 1 on T defined by $R(A)e^\lambda = \langle A,\lambda\rangle e^\lambda$ for $A \in \mathfrak{t}$. Here $R(A)$ is the infinitesimal right translation. Then $L_\lambda = \mathcal{H}om_{\mathcal{D}_T}(\mathcal{D}_T e^\lambda, \mathcal{O}_T)$. Let us consider the $\mathcal{D}_G$-module

$$\int_p \gamma^*(\mathcal{D}_T e^\lambda).$$

Then $\mathcal{H}^j(\int_p \gamma^*\mathcal{D}_T e^\lambda) = 0$ for $j \neq 0$ and $\int_p \gamma^*\mathcal{D}_T e^\lambda = \mathcal{H}^0(\int_p \gamma^*\mathcal{D}_T e^\lambda)$ is a regular holonomic $\mathcal{D}_G$-module. Moreover it is the minimal extension of

$$\left(\int_p \gamma^*\mathcal{D}_G e^\lambda\right)|_{G_{\mathrm{reg}}}.$$

Namely there is no non-trivial coherent submodule nor quotient of $\int_p \gamma^*\mathcal{D}_G e^\lambda$ whose support is contained in $G \setminus G_{\mathrm{reg}}$. If we fix a biinvariant n-form dg on G $(n = \dim G)$, then

$$\omega = \gamma^*\left(\frac{1}{\prod\limits_{\alpha\in\Delta^+}(1-e^\alpha)}\right)p^*(dg)$$

has no poles and nowhere vanishing on $\tilde{G}$. Hence

$$v_\lambda = p_*(\omega \otimes (dg)^{-1} \otimes \gamma^*(e^\lambda))$$

defines a section of $\int_p \gamma^*\mathcal{D}_T e^\lambda$. (Note that for $f : Y \to Z$ and a coherent $\mathcal{D}_Y$-module $\mathcal{M}$, there is a canonical embedding $f_*(\Omega_Y^{\dim Y} \otimes (\Omega_Z^{\dim Z})^{\otimes -1} \otimes \mathcal{M}) \to \int_f \mathcal{M}$). One can see easily that v_λ does not depend on the choice of dg and it satisfies the differential equation $\mathcal{M}_\lambda$ for invariant differential equations. Thus we obtain a $\mathcal{D}_G$-linear homomorphism

$$\begin{array}{ccc} \mathcal{M}_\lambda & \longrightarrow & \int_p \gamma^*\mathcal{D}_T e^\lambda \\ \Cup & & \Cup \\ u_\lambda & \longmapsto & v_\lambda. \end{array}$$

This is an isomorphism when $\lambda - \rho$ is regular (i.e. $\langle \lambda - \rho, \check{\alpha} \rangle \neq 0$ for any $\alpha \in \Delta$). In general let $\tilde{\mathcal{M}}_\lambda$ be the image of this homomorphism. Then $\mathrm{Hom}_{\mathcal{D}_G}(\tilde{\mathcal{M}}_\lambda, \mathcal{B}_{G_\mathbb{R}})$ is the space of constant coefficient invariant eigendistributions. On the other hand the calculation as in [K1] shows that

$$\mathrm{Hom}_{\mathcal{D}_G}\left(\int_p \gamma^*\mathcal{D}_T e^\lambda, \mathcal{B}_{G_\mathbb{R}}\right) \cong H_n^{\mathrm{inf}}(\tilde{G}_\mathbb{R}, \gamma^{-1}L_\lambda \otimes \mathrm{or}_{G_\mathbb{R}}).$$

Here $\mathrm{or}_{G_\mathbb{R}}$ denotes the orientations sheaf on $G_\mathbb{R}$ and H_n^{inf} is the n-th homology group of locally finite chains with value in the local system $\gamma^{-1}L_\lambda \otimes \mathrm{or}_{G_\mathbb{R}}$.

We have

$$H_n^{\mathrm{inf}}(\tilde{G}_\mathbb{R}^{\mathrm{reg}}; \gamma^{-1}L_\lambda \otimes \mathrm{or}_{G_\mathbb{R}}) = \Gamma(\tilde{G}_\mathbb{R}^{\mathrm{reg}}; \gamma^{-1}L_\lambda),$$

where $\tilde{G}_\mathbb{R}^{\mathrm{reg}} = p^{-1}(G_\mathbb{R} \cap G_{\mathrm{reg}})$. Note that $\tilde{G}_\mathbb{R}^{\mathrm{reg}}$ is an n-dimensional real manifold and dense in $\tilde{G}_\mathbb{R}$. For $\sigma \in H_n^{\mathrm{inf}}(\tilde{G}_\mathbb{R}^{\mathrm{reg}}; \gamma^{-1}L_\lambda \otimes \mathrm{or}_{G_\mathbb{R}})$ and $\tilde{g} \in \tilde{G}_\mathbb{R}^{\mathrm{reg}}$

let $\sigma(\tilde{g})$ be the value of $\sigma \in \gamma^{-1}L_\lambda \subset \gamma^{-1}\mathcal{O}_T$ at $\gamma(\tilde{g})$. Then the solution of $u(\sigma)$ of $\mathcal{M}_\lambda$ corresponding to σ is given by

$$(6.2.1)\quad u(\sigma)(g) = \sum_{\tilde{g}\in p^{-1}(g)} \sigma(\tilde{g})\left(\frac{\omega}{dg}\right)(\tilde{g}) = \sum_{\tilde{g}\in p^{-1}(g)} \frac{\sigma(\tilde{g})}{\prod_{\alpha\in\Delta^+}(1-e^{\alpha}(\gamma(\tilde{g})))}$$

$$\text{for}\quad g \in G_{\mathbb{R}}^{\mathrm{reg}}.$$

Since $L_\lambda = \mathbb{C}e^\lambda \subset \mathcal{O}_T, \sigma(\tilde{g})$ gives e^λ. Note that for $\sigma \in H_n^{\mathrm{inf}}(\tilde{G}_{\mathbb{R}}^{\mathrm{reg}}; \gamma^{-1}L_\lambda \otimes \mathrm{or}_{G_{\mathbb{R}}})$, $u(\sigma)$ extends to a solution of $\mathcal{M}_\lambda$ if and only if $d\sigma = 0$ on $\tilde{G}_{\mathbb{R}}$. This condition coincides with the connection formula of T. Hirai [Hi].

Remark that for $w \in W$, there is a canonical isomorphism $\int_p \gamma^*\mathcal{D}_T e^\lambda \xrightarrow{\sim} \int_p \gamma^*\mathcal{D}_T e^{w\circ\lambda}$ that sends v_λ to $v_{w\circ\lambda}$ and also an isomorphism $w : H_n^{\mathrm{inf}}(\tilde{G}_{\mathbb{R}};\, \gamma^{-1}L_\lambda \otimes \mathrm{or}_{G_{\mathbb{R}}}) \xrightarrow{\sim} H_n^{\mathrm{inf}}(\tilde{G}_{\mathbb{R}}; \gamma^{-1}L_{w\circ\lambda} \otimes \mathrm{or}_{G_{\mathbb{R}}})$. By this action we have $u(w\sigma) = u(\sigma)$. Hence $W_\lambda = \{w \in W; w\circ\lambda = \lambda\}$ acts on $\int_p \gamma^*\mathcal{D}_T e^\lambda$ and hence also on $H_n^{\mathrm{inf}}(\tilde{G}_{\mathbb{R}}; \gamma^{-1}L_\lambda \otimes \mathrm{or}_{G_{\mathbb{R}}})$. We have also

$$\tilde{\mathcal{M}}_\lambda \cong \left(\int_p \gamma^*\mathcal{D}_T e^\lambda\right)^{W_\lambda}.$$

Thus we obtain

$$\begin{aligned} &\{\text{constant coefficient invariant eigendistributions}\}\\ &\leftrightarrow \{W_\lambda\text{-invariant cycles in } \tilde{G}_{\mathbb{R}} \text{ with values in } \gamma^{-1}L_\lambda \otimes \mathrm{or}_{G_{\mathbb{R}}}\}. \end{aligned}$$

We call it the Hirai correspondence.

6.3. Character cycle.

In [K1], we defined the character cycle $\mathrm{ch}(F) \in H_n^{\mathrm{inf}}(\tilde{G}_{\mathbb{R}}; \mathrm{or}_{G_{\mathbb{R}}})$ for a $G_{\mathbb{R}}$-equivariant sheaf F. The same method can be applied to a $G_{\mathbb{R}}$-equivariant twisted sheaf F of twist λ and we can define its *character cycle* $\mathrm{ch}(F)$ as an element of $H_n^{\mathrm{inf}}(\tilde{G}_{\mathbb{R}}; \gamma^{-1}L_\lambda \otimes \mathrm{or}_{G_{\mathbb{R}}})$.

CONJECTURE 6.3.1. — The diagram

$$\begin{tikzcd}
& \{\text{representations of } G_{\mathbb{R}}\} \arrow[ddl, leftrightarrow] \arrow[d, leftrightarrow] \\
& \{G_{\mathbb{R}}\text{-equivariant } \mathbb{C}_X\text{-modules}\} \arrow[dd, leftrightarrow] \\
\{\text{constant coefficient invariant eigendistributions}\} \arrow[dr, leftrightarrow] & \\
& \{\text{cycles in } \tilde{G}_{\mathbb{R}}\}
\end{tikzcd}$$

commutes.

7. Microlocal geometry.

7.1. Characteristic variety.

For a finitely generated $U(\mathfrak{g})$-module M, we can define its characteristic variety $\mathrm{Ch}(M)$ as a subset of $\mathfrak{g}^*$ as follows : let us take a filtration $F(U(\mathfrak{g}))$ of $U(\mathfrak{g})$ by

$$F_n(U(\mathfrak{g})) = \begin{cases} 0 & n < 0, \\ \mathbb{C} & n = 0, \\ \mathbb{C} \oplus \mathfrak{g} & n = 1, \\ F_{n-1}(U)F_1(U) & n \geq 1. \end{cases}$$

Then $\mathrm{Gr}^F U(\mathfrak{g}) = \bigoplus_n F_n(U(\mathfrak{g}))/F_{n-1}(U(\mathfrak{g})) \cong S(\mathfrak{g})$ by the Poincaré Birkoff-Witt theorem. For a finitely generated $U(\mathfrak{g})$-module M, let us take a filtration $F(M)$ of M such that

(7.1.1) there are finitely many $m_j \in \mathbb{Z}$ and $u_j \in F_{m_j}(M)$ such that $F_m(M) = \Sigma F_{m-m_j}(U(\mathfrak{g}))u_j$.

Then $\mathrm{Gr}^F M$ is a finitely generated module over $\mathrm{Gr}^F U(\mathfrak{g}) \cong S(\mathfrak{g})$. Therefore $\operatorname{supp} \mathrm{Gr}^F M$ is a closed subset of $\mathfrak{g}^* = \mathrm{Spec}(S(\mathfrak{g}))$. This subset does not depend on the choice of filtrations $F(M)$ and called the *characteristic variety* of M. This variety is involutive with respect to the Poisson structure on $\mathfrak{g}^*$.

If M is a Harish-Chandra module then $\mathrm{Ch}(M)$ is a union of nilpotent K orbits in $\mathfrak{k}^{+}$. Thus we obtain

$$\{\text{Harish-Chandra module}\} \to \{\text{nilpotent } K\text{-orbits in } \mathfrak{k}^{\perp}\}.$$

Similarly, for a $\mathcal{D}_\lambda$-module $\mathcal{M}$, we can define its characteristic variety $\mathrm{Ch}(\mathcal{M})$ as a subset of T^*X. If $\mathcal{M}$ is a $(\mathcal{D}_\lambda, K)$-module, then $\mathrm{Ch}(\mathcal{M})$ is contained in $\tilde{\mathfrak{k}}^{\perp} = \rho^{-1}(\mathfrak{k}^{\perp}) = \bigcup\limits_{S} T^*_S X$ where S ranges over K-orbits in X. Thus we obtain

$$\{(\mathcal{D}_\lambda, K)\text{-modules}\} \to \{\text{involutives subsets of } \tilde{\mathfrak{k}}^{\perp}\}.$$

If $\langle \check{\alpha}, \lambda - \rho \rangle \neq 0, 1, 2, \cdots$ then we can prove ([BoBr])

$$\mathrm{Ch}\,\Gamma(X; \mathcal{M}) = \rho(\mathrm{Ch}(\mathcal{M})). \tag{7.1.2}$$

7.2. Singular support.

Let E be an admissible irreducible representation of $G_{\mathbb{R}}$. Then E can be embedded into a principal series $\mathcal{B}_{X^{\mathbb{R}}_{\min}}(\mathcal{V})$, the hyperfunction sections of a $G_{\mathbb{R}}$-equivariant bundle $\mathcal{V}$ on $X^{\mathbb{R}}_{\min}$. Here $X^{\mathbb{R}}_{\min} = G_{\mathbb{R}}/P_{\mathbb{R}}$ where $P_{\mathbb{R}}$ is a minimal parabolic subgroup of $G_{\mathbb{R}}$. Let $\varphi : E \to \mathcal{B}_{X^{\mathbb{R}}_{\min}}(\mathcal{V})$ be the embedding. Set

$$SS'(E) = \{SS\varphi(u); u \in E\}.$$

Here SS denotes the singular spectrum of the hyperfunction. This is a closed subset of $\sqrt{-1}T^*X^{\mathbb{R}}_{\min}$. This subset might depend on the embedding φ. Let us consider the moment map

$$\sqrt{-1}T^*X^{\mathbb{R}}_{\min} \xrightarrow{\rho} \sqrt{-1}\mathfrak{g}^*_{\mathbb{R}} = \mathfrak{g}^{\perp}_{\mathbb{R}} \subset \mathfrak{g}^*.$$

Then we conjecture that

CONJECTURE 7.2.1.

$SS(E) = \rho(SS'(E))$ does not depend on the embedding $\varphi : E \to \mathcal{B}_{X^{\mathbb{R}}_{\min}}(\mathcal{V})$.

If Conjecture 7.2.1 is true, $SS(EE)$ is a union $G_{\mathbb{R}}$-invariant nilpotent orbits in $\mathfrak{g}^{\perp}_{\mathbb{R}}$.

7.3. Kostant-Sekiguchi correspondence.

Sekiguchi and Kostant ([Se]) showed a bijection between the set of nilpotent K-orbits in $\mathfrak{k}^{\perp}$ and nilpotent $G_{\mathbb{R}}$-orbits in $\mathfrak{g}_{\mathbb{R}}^{\perp}$.

CONJECTURE 7.3.1

$$\begin{tikzcd}
\{\text{representations of } G_{\mathbb{R}}\} \arrow[r, leftrightarrow] \arrow[d, "SS"] & \{\text{Harish-Chandra modules}\} \arrow[d, "\mathrm{Ch}"] \\
\{\text{nilpotent } G_{\mathbb{R}}\text{-orbits in } \mathfrak{g}_{\mathbb{R}}^{\perp}\} \arrow[r, leftrightarrow, "\text{Kostant-Sekiguchi}"'] & \{\text{nilpotent } K\text{-orbits in } \mathfrak{k}^{\perp}\}
\end{tikzcd}$$

commutes.

Note that a relation between $SS(E)$ and the K-types of E is shown in Kashiwara-Vergne [KV].

7.4. Microsupport.

For a (twisted) sheaf F we can define its microsupport $SS(F)$ as a subset of T^*X ([KSp]). If F is $G_{\mathbb{R}}$-equivariant, $SS(F)$ is a $G_{\mathbb{R}}$-invariant subset of $\tilde{\mathfrak{g}}_{\mathbb{R}}^{\perp} = \rho^{-1}(\mathfrak{g}_{\mathbb{R}}^{\perp})$. Moreover, we can define a cycle in $\tilde{\mathfrak{g}}_{\mathbb{R}}^{\perp}$ called a characteristic cycle of F ([KSp]). This is also obtained by a kind of Fourier transform of the character cycle of F ([K1]). Note that $\rho(SS(F))$ is again a union of nilpotent $G_{\mathbb{R}}$-orbits of $\mathfrak{g}_{\mathbb{R}}^{\perp}$.

CONJECTURE 7.4.1

$$\begin{tikzcd}
\{\text{representations of } G_{\mathbb{R}}\} \arrow[r, "SS"] \arrow[d, leftrightarrow] & \{\text{nilpotent } G_{\mathbb{R}}\text{-orbits of } \mathfrak{g}_{\mathbb{R}}^{\perp}\} \\
\{G_{\mathbb{R}}\text{-equivariant sheaves on } X\} \arrow[ur] &
\end{tikzcd}$$

commutes.

7.5. Singular spectrum of character.

For an irreducible admissible representation E of $G_{\mathbb{R}}$ let $\mathrm{ch}(E)$ be its character. Then its singular spectrum $SS(\mathrm{ch}(E))$ is a subset of $\sqrt{-1}T^*G_{\mathbb{R}}$.

Hence $SS(\mathrm{ch}(E)) \cap \sqrt{-1}T_e^* G_{\mathbb{R}}$ may be regarded as a subspace of $\mathfrak{g}_{\mathbb{R}}^{\perp}$. This is again a union of nilpotent $G_{\mathbb{R}}$-orbits. It seems that the following is proved by Schmid-Vilonen ([SV]).

$$\begin{tikzcd}
\{\text{invariant eigendistributions}\} \arrow[r, leftrightarrow, "\mathrm{ch}"] \arrow[d] & \{\text{representations of } G_{\mathbb{R}}\} \arrow[d, leftrightarrow] \\
\{\text{nilpotent } G_{\mathbb{R}}\text{-orbits in } \mathfrak{g}_{\mathbb{R}}^{\perp}\} & \{G_{\mathbb{R}}\text{-equivariant sheaves on } X\} \arrow[l]
\end{tikzcd}$$

commutes.

7.6. Final remarks.

So far I explained the diagram in the introduction. There are still unknown arrows such as the correspondence between the cycles in $\tilde{\mathfrak{g}}_{\mathbb{R}}^{\perp}$ and the cycles in $\tilde{\mathfrak{k}}^{\perp}$. Many parts of the commutativity of diagrams are still open.

The duality is not either included in this diagram (e.g. $\mathrm{Ch}(M^*) = \mathrm{Ch}(M)^-$ for a Harish-Chandra module M? What is the duality for $(\mathcal{D}_X, K)$-modules?)

Added in Proof : After writing this article, I noticed the existence of the paper : H. Hecht and J.L. Taylor, *Analytic localization of group representations*, Adv. in Math., 79 (1990), 139–212. They proved Theorem 3.3 in the dual form.

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Masaki KASHIWARA,
Research Institute for
Mathematical Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-01 (Japan).