ANNALES DE L'INSTITUT FOURIER

J. ECALLE

D. SCHLOMIUK

The nilpotent part and distinguished form of resonant vector fields or diffeomorphisms

Annales de l'institut Fourier, tome 43, nº 5 (1993), p. 1407-1483

<http://www.numdam.org/item?id=AIF_1993__43_5_1407_0>

Ann. Inst. Fourier, Grenoble
43, 5 (1993), 1407-1483

THE NILPOTENT PART AND DISTINGUISHED FORM OF RESONANT VECTOR FIELDS OR DIFFEOMORPHISMS

by J. ÉCALLE and D. SCHLOMIUK

In honour of Bernard Malgrange

Contents.

Key words: Analytic singularities – Normal forms – Resurgent functions.
A.M.S. Classification: 30B99 – 32S65 – 30E15 – 32B99 – 32G99 – 32H02 – 32S05.

1. Introduction and overview.

The present paper investigates two notions — the classical notion of *nilpotent part* and the novel concept of *distinguished form* — that arise naturally in the parallel study of (local, analytic) *resonant vector fields* and *resonant diffeomorphisms.* For simplicity, however, we forget about diffeos in this introduction, and discuss only vector fields. Throughout, *localness* and *analyticity* are tacitly assumed.

The *nilpotent part* is intrinsical, *i.e.* chart-invariant. Indeed, any resonant vector field decomposes canonically into a diagonalizable part X^{dia} and nilpotent part X^{nil}, each having a simple geometric characterization. The *distinguished form* X^{dist}, on the other hand, is a special *prenormal form*, *i.e.* a formal vector field conjugate to X and with nothing but resonant terms in it. In its own way, X^{dist}, too, is undisputably "canonical", and this is even the whole point of introducing it, since the existence of *merely prenormal* forms is a triviality. Like X^{nil}, it is also generically divergent and resurgent. But unlike X^{nil}, the *distinguished form* X^{dist} is chart-dependent. Above all, it results from an *analytical construction* (see (1.2) infra) and doesn't appear to be capable of any simple geometric characterization.

We investigate X^{nil} and X^{dist} successively under three viewpoints:

(i) *the analytical viewpoint*, which is concerned with deriving the Taylor expansions of X^{nil} and X^{dist} from that of X.

(ii) *the analytic viewpoint*, which aims at understanding the divergence/resurgence properties of X^{nil} and X^{dist}.

(iii) *the algebraic viewpoint*, which focuses on the case of algebraic data (*e.g.* polynomial vector fields X) and attempts to use the analytical expressions for X^{nil} and X^{dist} to make some headway in certain long-standing problems, like the center-focus problem (see below).

The *analytical study* (§§2,3,4,5,6) culminates in the following expressions of X^{nil} and X^{dist}:

$$(1.1)\quad X^{\text{nil}} = \sum \$^{\bullet}\,\mathbb{B}_{\bullet} = \sum_{1\leq r}\sum_{n_i} \$^{\omega_1,\ldots,\omega_r}\,\mathbb{B}_{n_r}\cdots\mathbb{B}_{n_1}$$

$$(1.2)\quad X^{\text{dist}} = X^{\text{lin}} + \sum \$\!\!\$^{\bullet}\,\mathbb{B}_{\bullet} = X^{\text{lin}} + \sum_{1\leq r}\sum_{n_i} \$\!\!\$^{\omega_1,\ldots,\omega_r}\,\mathbb{B}_{n_r}\cdots\mathbb{B}_{n_1}$$

in terms of the *homogeneous components* $\mathbb{B}_n$ of the vector field X:

$$(1.3)\qquad X = X^{\text{lin}} + \sum_{n} \mathbb{B}_n, \qquad (n \in \mathbb{N}^{\nu}_{*})$$

$$(1.3^*)\quad X^{\mathrm{lin}} = \lambda_1 x_1 \partial_{x_1} + \cdots + \lambda_\nu x_\nu \partial_{x_\nu};\ \mathbb{B}_n x^m \equiv \beta_{n,m} x^{n+m}, \qquad (n, m \in \mathbb{N}_*^\nu;\ \beta_{m,n} \in \mathbb{C})$$

and of some well-defined *universal coefficients* $\$^{\boldsymbol{\omega}}$ and $\$\!\!\$^{\boldsymbol{\omega}}$ indexed by finite sequences $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ with $\omega_i = \langle n_i, \lambda \rangle \in \mathbb{C}$. Functions of such sequences $\boldsymbol{\omega}$ are known as *moulds*. Moulds constitute a non commutative algebra, with a rich structure and numerous derivations. Above all, they facilitate the construction and study of "useful universal coefficients". In the present instance, the relevant moulds, namely $\$^\bullet$ and $\$\!\!\$^\bullet$, are related to the moulds $S^\bullet$ and $\not{S}^\bullet$ (useful in the linearization of non-resonant vector fields and the study of *diophantine* small denominators) and even more so to the "compensators" $S^\bullet_{\mathrm{co}}(t)$ and $\not{S}^\bullet_{\mathrm{co}}(t)$ (useful in the study of quasiresonance, *i.e.* of *liouvillian* small denominators). In fact, the moulds $\$^\bullet$ and $\$\!\!\$^\bullet$ come up rather naturally in the study of "degenerate compensators". Or, to put it another way, they shed light on the passage from *quasiresonance* to *resonance*. It should be noted, however, that $\$^\bullet$ is definitely more elementary that $\$\!\!\$^\bullet$: the construction of $\$\!\!\$^\bullet$ is rather painstaking, to say nothing of the study of certain *generating functions* (the so-called *amplification* and *coamplification*) attached to $\$\!\!\$^\bullet$. But no matter how technical these developments, they are indispensible to an in-depth understanding of X^{dist}.

After the analytical spadework, we are in a position to tackle the *analytic study* (§§7,8,9). It turns out that both X^{nil} and X^{dist} are generically divergent *and* resurgent, though each in its own way. The *resurgence equations* which govern the divergence of X^{nil} and describe its resurgence pattern, are merely a variant (but a rather interesting one) of the so-called *Bridge Equation.* Like the usual Bridge Equation, they yield, as a byproduct, a complete system of *holomorphic invariants* for X. The *distinguished form* X^{dist}, on the other hand, satisfies resurgence equations which *do not* involve the holomorphic invariants, but the original field X itself, and are of "rigid" or "universal" type. The resurgence "lattice" Ω^{dist} also is different, and the singularities much "worse".

These features are often met with in "man-made" divergent series, *i.e.* divergent series which are not *obtained* as formal solutions of natural (meaning *analytic*) equations or systems, but are rather *defined* by analytic*al* means, to meet certain demands — such as finding canonical representatives in analytic congugacy classes. Summing up, one would like to say that the resurgence of X^{nil} and X^{dist} illustrates the prevalence of resurgence not only among the divergent series that one *encounters*, but also among those that one *constructs*.

The *algebraic part* (§10) is more than sketchy: it outlines a program of investigations without really tackling it. It originated, *as indeed the whole paper*, in a question by one of us (see [S1], [S2]) about the center-focus problem for polynomial vector fields of degree d in $\mathbb{R}^2$:

$$X = x\partial_y - y\partial_x + (\cdots) \tag{1.4}$$

One natural question which comes to mind about such fields (and which can be rephrased so as to make sense for all resonant vector fields, in any dimension) is this: what is the minimal number $\mathrm{nil}(d)$ of polynomial identities between the Taylor coefficients of X, that guarantee the existence of a center-focus at the origin? The whole thing, of course, boils down to the study of certain *finitely generated ideals*, but the moulds $\$^{\bullet}$ and $\$\!\!\!\mid^{\bullet}$ make it possible to replace *commutative ideals* by more tightly structured *Lie ideals*; to produce explicit generators for those ideals; and even to suggest an approach, based on the *splitting properties* of the Lie elements $\mathbb{B}^{\$}_{\underline{\mathbf{n}}}$ and $\mathbb{B}^{\$\!\!\!\mid}_{\underline{\mathbf{n}}}$ constructed by "contraction" with $\$^{\bullet}$ and $\$\!\!\!\mid^{\bullet}$.

We are keenly aware that the present paper, such as it stands, is somewhat lopsided, with more than half its length being devoted to the analy*tical* prerequisites, *i.e.* $\$^{\bullet}$, $\$\!\!\!\mid^{\bullet}$ and the whole *mould apparatus* that surrounds them. But the analy*tic* study (§§7,8,9) already shows to what use these tools can be put, and we cherish the hope that the algebraic program outlined in §10, when implemented, will further reinforce their claim to "usefulness".

2. The alternal moulds $\$^{\bullet}$ and $\$\!\!\!\mid^{\bullet}$ in the context of symmetral compensation.

Reminder about moulds.

As usual, a *mould* $M^{\bullet}$ denotes a family of elements $M^{\boldsymbol{\omega}}$ of a given commutative ring or algebra, with upper indexation by sequences $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$. These sequences have arbitrary length $r = r(\boldsymbol{\omega}) \geq 0$ and their components ω_i range over a set Ω that may be any abelian group or semigroup. Moulds *multiply* (non-commutatively) according to:

$$C^{\bullet} = A^{\bullet} \times B^{\bullet} \Longrightarrow C^{\omega_1,\ldots,\omega_r} = \sum_{0 \leq i \leq r} A^{\omega_1,\ldots,\omega_i} B^{\omega_{i+1},\ldots,\omega_r} \tag{2.1}$$

with a sum beginning with $A^{\emptyset} B^{\omega_1,\ldots,\omega_r}$ and ending with $A^{\omega_1,\ldots,\omega_r} B^{\emptyset}$. The symbol $\emptyset$ denotes of course the *empty sequence*, to which we assign zero length ($r(\emptyset) = 0$).

Useful moulds tend to display certain symmetries. Thus, a mould $A^\bullet$ is said to be *symmetral* (resp. *alternal*) if it verifies $A^\emptyset = 1$ (resp. $= 0$) and:

$$\sum_{\boldsymbol{\omega}} A^{\boldsymbol{\omega}} \equiv A^{\boldsymbol{\omega}^1} A^{\boldsymbol{\omega}^2} \quad (\text{resp. } \equiv 0), \ (\forall \boldsymbol{\omega}^1, \forall \boldsymbol{\omega}^2) \tag{2.2}$$

with a sum extending to all $(r_1 + r_2)!/(r_1!r_2!)$ sequences $\boldsymbol{\omega}$ obtainable by *shuffling* two given, non-empty sequences $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$ of length r_1 and r_2, *i.e.* by intermixing their components under preservation of the internal order of each sequence. (N.B.: throughout, we shall use boldface with upper indexation for sequences $\boldsymbol{\omega}$ or $\boldsymbol{\omega}^j$, and plain print with lower indexation for their components ω_i or ω_i^j).

Similarly, a mould $A^\bullet$ is said to be *symmetrel* (resp. *alternel*) if it verifies (2.2), but relatively to the "*contracting shuffling*" of $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$, under which one or several pairs of consecutive elements (ω_i^1, ω_j^2) from $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$ may contract to $\omega_i^1 + \omega_j^2$. As a consequence, for a symmetrel (or alternal) mould, the left-hand side of identity (2.2) involves exactly Q^{r_1,r_2} terms, with:

$$Q^{r_1,r_2} = \sum_r Q_r^{r_1,r_2} \quad (\sup(r_1,r_2) \le r \le r_1 + r_2) \tag{2.3}$$

$$Q_r^{r_1,r_2} \overset{\text{def}}{=} r!((r-r_1)!(r-r_2)!(r_1+r_2-r)!)^{-1} \tag{2.3 bis}$$

where $Q_r^{r_1,r_2}$ denotes the number of sequences $\boldsymbol{\omega}$ of length $r(\boldsymbol{\omega}) = r$.

Thus, whereas any *symmetral* mould $A^\bullet$ verifies identities like:

$$A^{\omega_1} A^{\omega_2,\omega_3} = A^{\omega_1,\omega_2,\omega_3} + A^{\omega_2,\omega_1,\omega_3} + A^{\omega_2,\omega_3,\omega_1} \tag{2.4}$$

$$\begin{aligned} A^{\omega_1,\omega_2} A^{\omega_3,\omega_4} = {} & A^{\omega_1,\omega_2,\omega_3,\omega_4} + A^{\omega_1,\omega_3,\omega_2,\omega_4} + A^{\omega_3,\omega_1,\omega_2,\omega_4} \\ & + A^{\omega_1,\omega_3,\omega_4,\omega_2} + A^{\omega_3,\omega_1,\omega_4,\omega_2} + A^{\omega_3,\omega_4,\omega_1,\omega_2} \end{aligned} \tag{2.5}$$

etc., any *symmetrel* mould $A^\bullet$ verifies identities like:

$$A^{\omega_1} A^{\omega_2,\omega_3} = \text{as above } + A^{\omega_1+\omega_2,\omega_3} + A^{\omega_2,\omega_1+\omega_3} \tag{2.4*}$$

$$\begin{aligned} A^{\omega_1,\omega_2} A^{\omega_3,\omega_4} = {} & \text{as above } + A^{\omega_1,\omega_2+\omega_3,\omega_4} + A^{\omega_3,\omega_1+\omega_4,\omega_2} \\ & + A^{\omega_1+\omega_3,\omega_2,\omega_4} + A^{\omega_1+\omega_3,\omega_4,\omega_2} + A^{\omega_1,\omega_3,\omega_2+\omega_4} \\ & + A^{\omega_3,\omega_1,\omega_2+\omega_4} + A^{\omega_1+\omega_3,\omega_2+\omega_4} \end{aligned} \tag{2.5*}$$

etc.

Trivial moulds, *i.e.* moulds $M^\bullet$ such that $M^{\boldsymbol{\omega}}$ depends solely on the length r of the sequence $\boldsymbol{\omega}$, are of no direct interest, but they keep cropping up in equations that serve to define important moulds. Foremost among trivial moulds is of course the unit mould $1^\bullet$:

$$1^\emptyset = 1 \ \text{ and } \ 1^{\omega_1,\dots,\omega_r} = 0 \quad (\forall r \ge 1) \tag{2.6}$$

and the four moulds:

$$I^\bullet = \text{ alternal} \; ; \;\; I^\bullet_{\text{ex}}(t) = \text{ symmetral} \tag{2.7}$$

$$J^\bullet = \text{ alternel} \; ; \;\; J^\bullet_{\text{ex}}(t) = \text{ symmetrel} \tag{2.8}$$

which are defined as follows:

$$I^\emptyset = 0 \; ; \;\; I^{\omega_1} \equiv 1 \; ; \;\; I^{\omega_1,\ldots,\omega_r} \equiv 0, \qquad (\forall r \geq 2) \tag{2.9}$$

$$J^\emptyset = 0 \; ; \;\; J^{\omega_1,\ldots,\omega_r} \equiv (-1)^{r+1}/r, \qquad (\forall r \geq 1) \tag{2.10}$$

$$I^\emptyset_{\text{ex}}(t) \equiv 1 \; ; \;\; I^{\omega_1,\ldots,\omega_r}_{\text{ex}}(t) \equiv (1/r!)t^r, \qquad (\forall r \geq 1) \tag{2.11}$$

$$J^\emptyset_{\text{ex}}(t) \equiv 1 \; ; \;\; J^{\omega_1,\ldots,\omega_r}_{\text{ex}}(t) \equiv (1/r!)t(t-1)(t-2)\cdots(t-r+1), \qquad (\forall r \geq 1). \tag{2.12}$$

The *mould exponential* of any alternal (resp. alternel) mould is a symmetral (resp. symmetrel) mould, and the above examples are a case in point, since:

$$I^\bullet_{\text{ex}}(t) \equiv \exp(tI^\bullet) \;\text{ and }\; J^\bullet_{\text{ex}}(t) \equiv \exp(tJ^\bullet) \tag{2.13}$$

with $\exp(\cdots)$ denoting the *mould exponential*:

$$\exp(M^\bullet) \overset{\text{def}}{=} 1^\bullet + M^\bullet + (1/2!)(M^\bullet \times M^\bullet) + (1/3!)(M^\bullet \times M^\bullet \times M^\bullet) + \cdots \tag{2.13*}$$

Two useful operators on the mould algebras, which we shall constantly require, are the *derivation* ∇ and the *automorphism* t^∇, which operate as follows:

$$(B^\bullet = \nabla A^\bullet) \Longrightarrow (B^{\boldsymbol{\omega}} = \|\boldsymbol{\omega}\| A^{\boldsymbol{\omega}}) \tag{2.14}$$

$$(C^\bullet = t^\nabla A^\bullet) \Longrightarrow (C^{\boldsymbol{\omega}} = t^{\|\boldsymbol{\omega}\|} A^{\boldsymbol{\omega}}) \tag{2.14*}$$

with t on $\mathbb{C}_\bullet$ (the Riemann surface of the logarithm) and:

$$\|\boldsymbol{\omega}\| \overset{\text{def}}{=} \omega_1 + \cdots + \omega_r \;\text{ if }\; \boldsymbol{\omega} = (\omega_1, \ldots, \omega_r). \tag{2.15}$$

We shall now construct three alternal moulds $T^\bullet$, $\$^\bullet$, $\not{\$}^\bullet$ and eight symmetral, pairwise inverse moulds:

$$1^\bullet = S^\bullet \times \not{S}^\bullet = S^\bullet_{\text{ext}} \times \not{S}^\bullet_{\text{ext}} = S^\bullet_{\text{co}}(t) \times \not{S}^\bullet_{\text{co}}(t) = S^\bullet_{\text{aco}}(t) \times \not{S}^\bullet_{\text{aco}}(t). \tag{2.16}$$

Some of these will exhibit discontinuities or singularities for certain "degenerate" sequences $\boldsymbol{\omega}$, which have to be singled out. If a sequence $\boldsymbol{\sigma} = (\sigma_i)$ contains exactly n elements, but these assume only n^* distinct values, the difference $n - n^*$ is said to be the *repetitiveness* of $\boldsymbol{\sigma}$. Similarly, we define the degeneracy $\text{dgn}(\boldsymbol{\omega})$ of a sequence $\boldsymbol{\omega} = (\omega_i)$ as being equal to the repetitiveness of the sequence:

$$0, \check{\omega}_1, \check{\omega}_2, \ldots, \check{\omega}_r \;\text{ with }\; \check{\omega}_i \overset{\text{def}}{=} \omega_1 + \omega_2 + \cdots + \omega_i \tag{2.17}$$

or, equivalently, of the sequence:

$$0, \hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_r \quad \text{with} \quad \hat{\omega}_i \overset{\text{def}}{=} \omega_i + \omega_{i+1} + \cdots + \omega_r. \tag{2.18}$$

Lastly, the *vanishing order* $\mathrm{van}(\boldsymbol{\omega})$ of $\boldsymbol{\omega} = (\omega_i)$ is taken to be 0 if $\|\boldsymbol{\omega}\| \neq 0$ and, if $\|\boldsymbol{\omega}\| = 0$, $\mathrm{van}(\boldsymbol{\omega})$ is equal to the number of zeros in either of the sequences $(\hat{\omega}_i)$ or $(\check{\omega}_i)$.

The elementary moulds $S^\bullet$, $\mathcal{S}^\bullet$, $T^\bullet$.

They are defined for almost all sequences $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ by the relations:

$$S^{\boldsymbol{\omega}} \overset{\text{def}}{=} (-1)^r (\check{\omega}_1 \check{\omega}_2 \cdots \check{\omega}_r)^{-1} \quad \text{with } \check{\omega}_i \text{ as in (2.17)} \tag{2.19}$$

$$\mathcal{S}^{\boldsymbol{\omega}} \overset{\text{def}}{=} (\hat{\omega}_1 \hat{\omega}_2 \cdots \hat{\omega}_r)^{-1} \quad \text{with } \hat{\omega}_i \text{ as in (2.18)} \tag{2.20}$$

$$T^{\boldsymbol{\omega}} \overset{\text{def}}{=} 0 \qquad \text{if } \|\boldsymbol{\omega}\| \neq 0 \tag{2.21}$$

$$T^{\boldsymbol{\omega}} \overset{\text{def}}{=} (\hat{\omega}_2 \hat{\omega}_3 \cdots \hat{\omega}_r)^{-1} = (-1)^{r-1} (\check{\omega}_1 \check{\omega}_2 \cdots \check{\omega}_{r-1})^{-1} \quad \text{if } \|\boldsymbol{\omega}\| = 0 \tag{2.21*}$$

and of course:

$$S^{\emptyset} \overset{\text{def}}{=} 1 \;; \quad \mathcal{S}^{\emptyset} \overset{\text{def}}{=} 1 \;; \quad T^{\emptyset} \overset{\text{def}}{=} 0. \tag{2.22}$$

The alternality of $T^\bullet$ or symmetrality of $S^\bullet$ and $\mathcal{S}^\bullet$ is easily checked by induction on r, but can also be inferred from the equations:

$$\nabla S^\bullet = -S^\bullet \times I^\bullet \tag{2.23}$$

$$\nabla \mathcal{S}^\bullet = I^\bullet \times \mathcal{S}^\bullet \tag{2.24}$$

$$\left\{ \begin{array}{l} (I^\bullet \text{ as in (2.9)}) \\ (\nabla \text{ as in (2.14)}). \end{array} \right.$$

From the two scalar-valued moulds $S^\bullet$ and $\mathcal{S}^\bullet$ we shall now derive two others, the so-called *symmetral compensators* $S^\bullet_{\text{co}}(t)$ and $\mathcal{S}^\bullet_{\text{co}}(t)$, which depend on a variable t in $\mathbb{C}_\bullet$, but have the advantage of being defined for *all* sequences $\boldsymbol{\omega}$. Then, by investigating the behaviour of the compensators close to *degenerate sequences* $\boldsymbol{\omega}$, we shall stumble upon the moulds $\$^\bullet$ and $\$\!\!\!\$^\bullet$, which are central to our purpose.

The compensators and compensation-related moulds.

DEFINITION 2.1 (Symmetral compensators). — *For t in $\mathbb{C}_\bullet$ and t^∇ as in (2.14*), we put:*

$$S^\bullet_{\text{co}}(t) \overset{\text{def}}{=} (t^\nabla S^\bullet) \times (\mathcal{S}^\bullet) \tag{2.25}$$

$$\mathcal{S}^\bullet_{\text{co}}(t) \overset{\text{def}}{=} (S^\bullet) \times (t^\nabla \mathcal{S}^\bullet). \tag{2.26}$$

Clearly, $S^\bullet_{\text{co}}(t)$ and $\mathcal{S}^\bullet_{\text{co}}(t)$ are mutually inverse and, as products of symmetral moulds, they are symmetral themselves. They also satisfy equations analogous to (2.23) and (2.24):

$$(\nabla - t\partial_t)S^{\bullet}_{\mathrm{co}}(t) = -S^{\bullet}_{\mathrm{co}}(t) \times I^{\bullet} \tag{2.27}$$

$$(\nabla - t\partial_t)\,\mathcal{S}^{\bullet}_{\mathrm{co}}(t) = +I^{\bullet} \times \mathcal{S}^{\bullet}_{\mathrm{co}}(t). \tag{2.28}$$

Furthermore:

PROPOSITION 2.2 (Continuity of the compensator moulds). — *For sequences $\boldsymbol{\omega}$ of a given length r, both $S^{\boldsymbol{\omega}}_{\mathrm{co}}(t)$ and $\mathcal{S}^{\boldsymbol{\omega}}_{\mathrm{co}}(t)$ are continuous functions of t in $\mathbb{C}_\bullet$ and $\boldsymbol{\omega}$ in $\mathbb{C}^r$. Moreover, for a fixed $\boldsymbol{\omega}$ of degeneracy s, $S^{\boldsymbol{\omega}}_{\mathrm{co}}(t)$ and $\mathcal{S}^{\boldsymbol{\omega}}_{\mathrm{co}}(t)$ are polynomials of degree s in $\log t$ (apart from involving various powers of the form $t^{\omega_i+\cdots+\omega_j}$).*

Proof. — There are three steps. *First*, we introduce the so-called *symmetric compensators $t^{\boldsymbol{\sigma}}$*, which for non-repetitive sequences $\boldsymbol{\sigma}$ are given by:

$$t^{\sigma_0,\sigma_1,\ldots,\sigma_r} \overset{\mathrm{def}}{=} \sum_{0\leq i\leq r} t^{\sigma_i} \prod_{j\neq i}(\sigma_i - \sigma_j)^{-1}, \quad (t \in \mathbb{C}_\bullet, \sigma_i \in \mathbb{C}, \sigma_i \neq \sigma_j) \tag{2.29}$$

with unambiguously defined powers t^{σ_i} (since t is in $\mathbb{C}_\bullet$).

Second, we observe that the compensator $t^{\boldsymbol{\sigma}}$ extends to a continuous function of $(t, \boldsymbol{\sigma})$ defined on the whole of $\mathbb{C}_\bullet \times \mathbb{C}^{1+r}$, with the following expression in case of a repetitive $\boldsymbol{\sigma}$:

$$t^{\sigma_0^{(1+s_0)},\sigma_1^{(1+s_1)},\ldots,\sigma_r^{(1+s_r)}} \equiv (\partial^{s_0}_{\sigma_0}/s_0!)(\partial^{s_1}_{\sigma_1}/s_1!)\cdots(\partial^{s_r}_{\sigma_r}/s_r!)t^{\sigma_0,\sigma_1,\ldots,\sigma_r} \tag{2.30}$$

where of course $\sigma_i^{(1+s_i)}$ means that σ_i is repeated $(1+s_i)$ times.

Third, we check (recursively on r) the following relations between *symmetric* and *symmetral* compensators, under which the *repetitiveness* of $\boldsymbol{\sigma}$ translates into the *degeneracy* of $\boldsymbol{\omega}$:

$$S^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(t) \equiv t^{0,\breve{\omega}_1,\breve{\omega}_2,\ldots,\breve{\omega}_r} \tag{2.31}$$

$$\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(t) \equiv (-1)^r t^{0,\hat{\omega}_1,\hat{\omega}_2,\ldots,\hat{\omega}_r}. \tag{2.32}$$

□

Non-degenerate compensators are quite useful in so-called "small denominator problems", in particular for the study of *quasiresonant local objects* (see [E4], [E8], [E10] and also §7 infra). Here, however, we are concerned with *resonance* rather than quasiresonance, and so what we require is above all a closer analysis of *degenerate compensators*. The requested information will be provided, on the one hand, by the *lateral decomposition* of degenerate compensators (*Proposition 2.2*), which is easily derivable and uniquely defined, but somehow “less than complete”, and on the other hand, by the *central decomposition* (*Proposition 2.3*), which is much more thoroughgoing, but correspondingly more costly.

PROPOSITION 2.2 (The $\$^\bullet$ mould and the lateral decomposition of compensators). — *There exists a uniquely defined, alternal, scalar-valued mould $\$^\bullet$ such that:*

$$\begin{aligned} S^\bullet_{\mathrm{co}}(t) &\equiv \exp((\log t)t^\nabla\, \$^\bullet) \times S^\bullet_{\mathrm{aco}}(t) \\ &\equiv S^\bullet_{\mathrm{aco}}(t) \times \exp((\log t)\, \$^\bullet) \end{aligned} \tag{2.33}$$

$$\begin{aligned} \mathcal{S}^\bullet_{\mathrm{co}}(t) &\equiv \exp(-(\log t)\, \$^\bullet) \times \mathcal{S}^\bullet_{\mathrm{aco}}(t) \\ &\equiv \mathcal{S}^\bullet_{\mathrm{aco}}(t) \times \exp(-(\log t)t^\nabla\, \$^\bullet) \end{aligned} \tag{2.34}$$

*where the symmetral moulds $S^\bullet_{\mathrm{aco}}(t)$ and $\mathcal{S}^\bullet_{\mathrm{aco}}(t)$ denote the logarithm-free part of $S^\bullet_{\mathrm{co}}(t)$ and $\mathcal{S}^\bullet_{\mathrm{co}}(t)$ (*__a__ for alogarithmic; __co__ for compensated*) and where* exp *should be construed, as usual, as the* mould exponential *(see (2.13*)). For any non-degenerate sequence $\boldsymbol{\omega}$, $\$^{\boldsymbol{\omega}}$ vanishes and, for any fixed degeneracy type, $\$^{\boldsymbol{\omega}}$ is a homogeneous function of $\boldsymbol{\omega}$ of degree $1-r(\boldsymbol{\omega})$ and, more precisely, a polynomial in some of the variables $(\omega_i+\cdots+\omega_j)^{-1}$.*

Proof. — See after Proposition 2.3.

PROPOSITION 2.3 (The $\$\!\!\$^\bullet$ mould and the central decomposition of compensators). — *There exist scalar-valued moulds $S^\bullet_{\mathrm{ext}}$, $\mathcal{S}^\bullet_{\mathrm{ext}}$ (symmetral) and $\$\!\!\$^\bullet$ (alternal), which remain defined for all sequences $\boldsymbol{\omega}$, no matter how degenerate, and verify:*

$$S^\bullet_{\mathrm{co}}(t) \equiv (t^\nabla\, \mathcal{S}^\bullet_{\mathrm{ext}}) \times \exp((\log t)\, \$\!\!\$^\bullet) \times (S^\bullet_{\mathrm{ext}}) \tag{2.35}$$

$$\mathcal{S}^\bullet_{\mathrm{co}}(t) \equiv (\mathcal{S}^\bullet_{\mathrm{ext}}) \times \exp(-(\log t)\, \$\!\!\$^\bullet) \times (t^\nabla S^\bullet_{\mathrm{ext}}). \tag{2.36}$$

*For non-degenerate sequences $\boldsymbol{\omega}$, the moulds $S^\bullet_{\mathrm{ext}}$ and $\mathcal{S}^\bullet_{\mathrm{ext}}$ (*__ext__ for extended*) coincide with $S^\bullet$ and $\mathcal{S}^\bullet$ but, unlike the latter, they remain defined for all $\boldsymbol{\omega}$. They also provide a factorization of the logarithm-free part of compensators:*

$$S^\bullet_{\mathrm{aco}}(t) = (t^\nabla\, \mathcal{S}^\bullet_{\mathrm{ext}}) \times (S^\bullet_{\mathrm{ext}}) \tag{2.37}$$

$$\mathcal{S}^\bullet_{\mathrm{aco}}(t) = (\mathcal{S}^\bullet_{\mathrm{ext}}) \times (t^\nabla S^\bullet_{\mathrm{ext}}) \tag{2.38}$$

which, unlike (2.25) (2.26), is valid for all $\boldsymbol{\omega}$.

As for the mould $\$\!\!\$^\bullet$, it is conjugate to $\$^\bullet$ under $S^\bullet_{\mathrm{ext}}$:

$$S^\bullet_{\mathrm{ext}} \times \$^\bullet = \$\!\!\$^\bullet \times S^\bullet_{\mathrm{ext}} \tag{2.39}$$

but it is much "slimmer" than $\$^\bullet$*, since* $\$\!\!\$^{\boldsymbol{\omega}}$ *vanishes unless* $\boldsymbol{\omega}$ *be of zero sum (i.e.* $\|\boldsymbol{\omega}\| = 0$*), whereas* $\$^{\boldsymbol{\omega}}$ *vanishes only for non-degenerate* $\boldsymbol{\omega}$*.*

The triplet $(S^\bullet_{\text{ext}}, \rlap{/}S^\bullet_{\text{ext}}, \$\!\!\$^\bullet)$ *is not uniquely determined by the above equations, but it becomes so if we add the further requirement that, for any sequence* $\boldsymbol{\omega}$ *of a fixed vanishing pattern:*

(2.40) $S^{\boldsymbol{\omega}}_{\text{ext}}$ *be a polynomial of degree* r *in the acceptable variables* $(1/\check{\omega}_i)$

(2.41) $\rlap{/}S^{\boldsymbol{\omega}}_{\text{ext}}$ *be a polynomial of degree* r *in the acceptable variables* $(1/\hat{\omega}_i)$

(2.42) $\$\!\!\$^{\boldsymbol{\omega}}$ *be a polynomial of degree* $(r-1)$ *in the acceptable variables* $(1/\check{\omega}_i)$ *or* $(1/\hat{\omega}_i)$*.*

("Acceptable" means of course that we must discard those $\check{\omega}_i$ *or* $\hat{\omega}_i$ *which vanish. For* $\$\!\!\$^\bullet$*, the two sets of variables clearly coincide, since* $\$\!\!\$^{\boldsymbol{\omega}} = 0$ *unless* $\|\boldsymbol{\omega}\| = 0$*.)*

From now on, unless stated otherwise, the symbols $S^\bullet_{\text{ext}}$, $\rlap{/}S^\bullet_{\text{ext}}$, $\$\!\!\$^\bullet$ shall refer to those three unique and perfectly canonical moulds.

Remark. — Were it not for the constraints (2.40), (2.41), (2.42), we might replace the canonical triplet:

$$(S^\bullet_{\text{ext}},\ \rlap{/}S^\bullet_{\text{ext}},\ \$\!\!\$^\bullet) \tag{2.43}$$

by the triplet:

$$(A^\bullet \times S^\bullet_{\text{ext}},\ \rlap{/}S^\bullet_{\text{ext}} \times B^\bullet,\ A^\bullet \times \$\!\!\$^\bullet \times B^\bullet) \tag{2.44}$$

for any pair $(A^\bullet, B^\bullet)$ of scalar-valued, symmetral, mutually inverse moulds such that:

$$A^{\boldsymbol{\omega}} = B^{\boldsymbol{\omega}} = 0 \ \text{ whenever } \ \|\boldsymbol{\omega}\| = 0. \tag{2.45}$$

But, as we shall show in section 4, the imposition of conditions (2.40), (2.41), (2.42), or even any one of the three, suffices to remove the indeterminacy. For the time being, however, we must be content with proving Proposition 2.2 and the "existence part" of Proposition 2.3.

Proof of Proposition 2.2 and the first part of Proposition 2.3. — The argument will rely on *mould-comould contractions*, i.e. on formal sums of type:

$$\sum M^\bullet \mathbb{B}_\bullet = \sum_{\boldsymbol{\omega}} M^{\boldsymbol{\omega}} \mathbb{B}_{\boldsymbol{\omega}} = \sum_{0 \le r} \sum_{\omega_i \in \Omega} M^{\omega_1,\ldots,\omega_r} \mathbb{B}_{\omega_1,\ldots,\omega_r} \tag{2.46}$$

relative to a given *mould* $M^\bullet$, a given *comould* $\mathbb{B}_\bullet$ (see below) and a given subset Ω of $\mathbb{C}$.

But first we observe that the symmetral *moulds* $P^\bullet(t)$ and $Q^\bullet(t)$ characterized by:

$$S^\bullet_{\mathrm{co}}(t) = S^\bullet_{\mathrm{aco}}(t) \times P^\bullet(t) = (t^{\nabla} Q^\bullet(t)) \times S^\bullet_{\mathrm{aco}}(t) \tag{2.47}$$

satisfy:

$$Q^\bullet(t) \times P^\bullet(t^{-1}) = 1^\bullet. \tag{2.48}$$

Indeed, we have on the one hand:

$$P^\bullet(t) = S^\bullet_{\mathrm{aco}}(t) \times S^\bullet_{\mathrm{co}}(t) \tag{2.49}$$

and on the other hand:

$$Q^\bullet(t) = t^{-\nabla}(S^\bullet_{\mathrm{co}}(t) \times S^\bullet_{\mathrm{aco}}(t)) = (t^{-\nabla} S^\bullet_{\mathrm{co}}(t)) \times (t^{-\nabla}\, S^\bullet_{\mathrm{aco}}(t)) \tag{2.50}$$

which in view of (2.25), (2.26) reads:

$$Q^\bullet(t) = S^\bullet_{\mathrm{co}}(t^{-1}) \times S^\bullet_{\mathrm{aco}}(t^{-1}). \tag{2.51}$$

Pairing (2.49) and (2.50), we find precisely (2.48).

We now fix some (enumerable) additive semigroup Ω in $\mathbb{C}$, and we introduce the *free associative algebra* $\mathcal{A}$ and the *free Lie algebra* $\mathcal{L}$ generated by the same set of symbols $\mathbb{B}_{\omega_i}$ ($\omega_i \in \Omega$). Both $\mathcal{A}$ and $\mathcal{L}$ possess a natural *coproduct* induced by:

$$\mathrm{cop}(\mathbb{B}_{\boldsymbol{\omega}}) = \sum \mathbb{B}_{\boldsymbol{\omega}^1} \otimes \mathbb{B}_{\boldsymbol{\omega}^2} \qquad (\boldsymbol{\omega} \in \text{ shuffle } (\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)). \tag{2.54}$$

By setting :

$$\mathrm{grad}(\mathbb{B}_{\boldsymbol{\omega}}) \overset{\mathrm{def}}{=} \|\boldsymbol{\omega}\| = \omega_1 + \cdots + \omega_r \tag{2.55}$$

we turn $\mathcal{A}$ and $\mathcal{L}$ into graded algebras, and we then enlarge them into $\overline{\mathcal{A}}$ and $\overline{\mathcal{L}}$ by allowing enumerable (rather than finite) sums of base elements $\mathbb{B}_{\boldsymbol{\omega}}$, and also by introducing one additional Lie element X^{lin}, of gradation 0, along with the bracket rules:

$$[X^{\mathrm{lin}}, \mathbb{B}_{\boldsymbol{\omega}}] \overset{\mathrm{def}}{=} \|\boldsymbol{\omega}\| B_{\boldsymbol{\omega}} \qquad (\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)). \tag{2.56}$$

We first assume that 0 is *not* in Ω, and consider the following mould-comould contractions relative to Ω:

$$X \overset{\mathrm{def}}{=} X^{\mathrm{lin}} + \sum I^\bullet \mathbb{B}_\bullet = X^{\mathrm{lin}} + \sum_{\omega_i \in \Omega} \mathbb{B}_{\omega_i} \in \overline{\mathcal{L}} \tag{2.57}$$

$$\Theta \overset{\mathrm{def}}{=} \sum S^\bullet \mathbb{B}_\bullet \in \overline{\mathcal{A}} \tag{2.58}$$

$$\Theta^{-1} \overset{\mathrm{def}}{=} \sum \not{S}^\bullet\, \mathbb{B}_\bullet \in \overline{\mathcal{A}} \tag{2.59}$$

(The sums in (2.58) and (2.59) extend to *all* sequences $\boldsymbol{\omega}$, including $\boldsymbol{\omega} = \emptyset$). The moulds $S^\bullet$ and $\not{S}^\bullet$ being symmetral and mutually inverse, it is plain that Θ and Θ^{-1} are two mutually inverse, formal automorphisms:

$$\operatorname{cop}(\Theta^{\pm 1}) = \Theta^{\pm 1} \otimes \Theta^{\pm 1} \tag{2.60}$$

and that we have the conjugacy equation in $\overline{\mathcal{L}}$:

$$X = \Theta X^{\mathrm{lin}} \Theta^{-1} \tag{2.61}$$

which readily follows from (2.56) combined with (2.17) and (2.18).

However, the operators Θ and Θ^{-1}, involving as they do the moulds $S^\bullet$ and $\not{S}^\bullet$, are defined *only* if, as we assumed, $0 \notin \Omega$. To get rid of this restriction, we introduce the compensators $S^\bullet_{\mathrm{co}}(t)$ and $\not{S}^\bullet_{\mathrm{co}}(t)$ relative to an auxiliary variable t in $\mathbb{C}_\bullet$, and we construct two new formal automorphisms:

$$\Theta_{\mathrm{co}} = \sum S^\bullet_{\mathrm{co}}(t) \mathbb{B}_\bullet \in \overline{\mathcal{A}} \tag{2.62}$$

$$\Theta^{-1}_{\mathrm{co}} = \sum \not{S}^\bullet_{\mathrm{co}}(t) \mathbb{B}_\bullet \in \overline{\mathcal{A}}. \tag{2.63}$$

Still assuming (provisionally) that $0 \notin \Omega$, we deduce from (2.61) or (2.27), (2.28) the new conjugacy:

$$X - t\partial_t = \Theta_{\mathrm{co}} (X^{\mathrm{lin}} - t\partial_t) \Theta^{-1}_{\mathrm{co}} \quad (\partial_t \equiv \partial/\partial t) \tag{2.64}$$

which (unlike (2.61) and due to the continuity of compensators: *see Proposition 2.1*), retains both its meaning and validity even when $0 \in \Omega$.

Now, in view of the lateral decomposition (2.33) and of the obvious inversion rule, valid for any two scalar moulds $(M^\bullet, N^\bullet)$:

$$\sum (M^\bullet \times N^\bullet) \mathbb{B}_\bullet = \left(\sum N^\bullet \mathbb{B}_\bullet\right) \times \left(\sum M^\bullet \mathbb{B}_\bullet\right) \tag{2.65}$$

the conjugacy relation (2.64) becomes:

$$X - t\partial_t = (\Theta_{\mathrm{aco}})({}^t\Theta_{\mathrm{log}})(X^{\mathrm{lin}} - t\partial_t)({}^t\Theta_{\mathrm{log}})^{-1}(\Theta_{\mathrm{aco}})^{-1} \tag{2.66}$$

with a neat separation into two *logarithm-free* and two *logarithm-ridden* factors:

$$\Theta_{\mathrm{aco}} = \sum S^\bullet_{\mathrm{aco}}(t) \mathbb{B}_\bullet \tag{2.67}$$

$$\Theta^{-1}_{\mathrm{aco}} = \sum \not{S}^\bullet_{\mathrm{aco}}(t) \mathbb{B}_\bullet \tag{2.68}$$

$$\Theta_{\mathrm{log}} = \sum Q^\bullet(t) \mathbb{B}_\bullet \tag{2.69}$$

$${}^t\Theta_{\mathrm{log}} = \sum (t^\nabla Q^\bullet(t)) \mathbb{B}_\bullet \tag{2.70}$$

with $Q^\bullet(t)$ as in (2.47).

If we now introduce the Lie element ${}^tX^{\mathrm{nil}}$ defined by:

$$ {}^tX^{\mathrm{nil}} = [{}^t\Theta_{\log}, X^{\mathrm{lin}} - t\partial_t] \tag{2.71} $$

(beware of mixing up *nil* and *lin*), the conjugacy relation (2.66) becomes:

$$ X - t\partial_t = \Theta_{\mathrm{aco}}(X^{\mathrm{lin}} - t\partial_t)\Theta_{\mathrm{aco}}^{-1} + \Theta_{\mathrm{aco}}({}^tX^{\mathrm{nil}})\Theta_{\mathrm{aco}}^{-1}. \tag{2.72} $$

But (2.72) involves three summands, two of which, namely $X - t\partial_t$ and $\Theta_{\mathrm{aco}}(X^{\mathrm{lin}} - t\partial_t)\Theta_{\mathrm{aco}}^{-1}$, are patently logarithm-free, meaning that they have no $\log t$ in them. So the third summand $\Theta_{\mathrm{aco}}({}^tX^{\mathrm{nil}})\Theta_{\mathrm{aco}}^{-1}$ must also be logarithm-free, and since Θ_{aco} is logarithm free, the commutator ${}^tX^{\mathrm{nil}}$ introduced in (2.71)must itself be logarithm-free. This clearly compels ${}^t\Theta_{\log}$ to be of the form:

$$ {}^t\Theta_{\log} = \exp((\log t)({}^tX^{\mathrm{nil}})) \tag{2.73} $$

with a Lie element ${}^tX^{\mathrm{nil}}$ of the form:

$$ {}^tX^{\mathrm{nil}} = \sum (t^{\nabla}\, \$^{\bullet})\mathbb{B}_{\bullet} \tag{2.74} $$

relative to some alternal, t-independent mould $\$^{\bullet}$. If we now recall (2.47) and (2.48), this implies:

$$ P^{\bullet} = Q^{\bullet} = \$^{\bullet} \tag{2.75} $$

which establishes Proposition 2.2 along with the relation:

$$ (t^{\nabla}\, \$^{\bullet}) \times S^{\bullet}_{\mathrm{aco}}(t) = S^{\bullet}_{\mathrm{aco}}(t) \times \$^{\bullet}\,. \tag{2.76} $$

The above identity in turn shows that:

$$ \Theta_{\mathrm{aco}}{}^tX^{\mathrm{nil}} = X^{\mathrm{nil}}\Theta_{\mathrm{aco}} \tag{2.77} $$

with ${}^tX^{\mathrm{nil}}$ as in (2.71) and:

$$ X^{\mathrm{nil}} = \sum \$^{\bullet}\,\mathbb{B}_{\bullet}. \tag{2.78} $$

Thus, equation (2.72) becomes:

$$ X - t\partial_t = \Theta_{\mathrm{aco}}(X^{\mathrm{lin}} - t\partial_t)\Theta_{\mathrm{aco}}^{-1} + X^{\mathrm{nil}} \tag{2.79} $$

with two t-independent Lie elements X and X^{nil}. Therefore the difference $X - X^{\mathrm{nil}}$ itself has to be t-independent, which is patently impossible unless Θ_{aco} be of the form:

$$ \Theta_{\mathrm{aco}} = \Theta_{\mathrm{ext}}({}^t\Theta_{\mathrm{ext}})^{-1} \tag{2.80} $$

with

$$ \Theta_{\mathrm{ext}} = \sum S^{\bullet}_{\mathrm{ext}}\mathbb{B}_{\bullet}\ ;\quad \Theta_{\mathrm{ext}}^{-1} = \sum \rlap{-}S^{\bullet}_{\mathrm{ext}}\,\mathbb{B}_{\bullet} \tag{2.81} $$

$$(2.82) \qquad {}^{t}\Theta_{\text{ext}} = \sum (t^{\nabla} S^{\bullet}_{\text{ext}})\mathbb{B}_{\bullet} \; ; \quad {}^{t}\Theta^{-1}_{\text{ext}} = \sum (t^{\nabla} \not{S}^{\bullet}_{\text{ext}})\mathbb{B}_{\bullet}$$

relative to two symmetral, mutually inverse and t-independent moulds $S^{\bullet}_{\text{ext}}$ and $\not{S}^{\bullet}_{\text{ext}}$ that verify (2.37) and (2.38). If we now *define* a (necessarily alternal) mould $\not{\$}^{\bullet}$ by the relation:

$$(2.83) \qquad S^{\bullet}_{\text{co}}(t) = (t^{\nabla} \not{S}^{\bullet}_{\text{ext}}) \times \exp((\log t)\, \not{\$}^{\bullet}) \times (S^{\bullet}_{\text{ext}})$$

we see, in view of (2.33), that $\$^{\bullet}$ and $\not{\$}^{\bullet}$ are mutually conjugate under $S^{\bullet}_{\text{ext}}$, as in (2.39), which implies that $\not{\$}^{\bullet}$, like $\$^{\bullet}$ and $S^{\bullet}_{\text{ext}}$, is t-independent. Moreover, again by comparing (2.83) with (2.25) and (2.33), we infer that $t^{\nabla} \not{\$}^{\bullet} = \not{\$}^{\bullet}$, which means that $\nabla \not{\$}^{\bullet} = \not{\$}^{\bullet}$. Thus, $\not{\$}^{\omega}$ necessarily vanishes when $\|\omega\| \neq 0$. This establishes (2.35), (2.36) and completes the proof of the "existence part" of Proposition 2.3. □

3. Construction and properties of the $\$^{\bullet}$ mould.

The compensation-related moulds introduced thus far fall into two quite distinct classes.

On the one hand, we have the $\$^{\bullet}$ mould and all the "soft" moulds involved in the lateral decomposition of Proposition 2.2. They are rather elementary and fairly easy to calculate, because they are entirely determined by the equations (2.33) or (2.34).

On the other hand, we have the $\not{\$}^{\bullet}$ mould and the other two "tough" moulds $S^{\bullet}_{\text{ext}}$ and $\not{S}^{\bullet}_{\text{ext}}$ involved in the central decomposition of Proposition 2.4. These more elusive moulds, as we observed, are not unambiguously characterized by equation (2.35) or (2.36), unless we add the rationality requirements (2.40), (2.41), (2.42). These latter conditions, however, aren't too easy to translate analytically, and this considerably complicates the study of the three "tough" moulds.

The corresponding construction will be postponed to the next section. In this section, we shall deal with the properties of the "soft" moulds, and indicate several ways of calculating them.

Direct calculation of the "soft" moulds.

The logarithm-free parts $S^{\bullet}_{\text{aco}}(t)$ and $\not{S}^{\bullet}_{\text{aco}}(t)$ of $S^{\bullet}_{\text{co}}(t)$ and $\not{S}^{\bullet}_{\text{co}}(t)$ may be obtained directly, by translating the symmetral compensators into symmetric ones according to (2.31), (2.32) and then applying (2.30), but letting the differential operators ∂_{σ_i} act only on the variables σ_i sitting in the denominators $\prod(\sigma_i - \sigma_j)^{-1}$, *not* in the powers t^{σ_i}. Similarly, $\$^\bullet$ may be calculated by letting the ∂_{σ_i} act *once* on the powers, and all the other times on the denominators. In fact, from (2.33), (2.34) we derive:

$$S^\bullet_{\mathrm{co}}(t) = S^\bullet_{\mathrm{aco}}(t) + (\log t)\, \$^\bullet_*(t) + o(\log t) \tag{3.1}$$

$$\not{S}^\bullet_{\mathrm{co}}(t) = \not{S}^\bullet_{\mathrm{aco}}(t) - (\log t)\, \$^\bullet_{**}(t) + o(\log t) \tag{3.2}$$

with:

$$\$^\bullet_*(t) \equiv S^\bullet_{\mathrm{aco}}(t) \times \$^\bullet \;;\;\; \$^\bullet_{**}(t) = \$^\bullet \times \not{S}^\bullet_{\mathrm{aco}}(t) \tag{3.4}$$

$$S^\bullet_{\mathrm{co}}(1) = \not{S}^\bullet_{\mathrm{co}}(1) = S^\bullet_{\mathrm{aco}}(1) = \not{S}^\bullet_{\mathrm{aco}}(1) = 1^\bullet \tag{3.5}$$

and therefore:

$$\$^\bullet = \$^\bullet_*(1) = \$^\bullet_{**}(1). \tag{3.6}$$

Inductive calculation of the "soft" moulds.

There is also a more convenient, induction-based alternative for calculating our moulds — which moreover is intimately related to their *geometric meaning* (see the proof towards the end of the section). But in order to spell out that induction, we require moulds $I^\bullet_{\omega_0}$ similar to $I^\bullet$, and mould operators ∇_{ω_0} similar to ∇. For any simple index ω_0, the alternal mould $I^\bullet_{\omega_0}$ is defined by:

$$I^{\omega_1}_{\omega_0} = 1 \text{ if } \omega_1 = \omega_0 \;;\;\; I^{\omega_1}_{\omega_0} = 0 \text{ if } \omega_1 \neq \omega_0 \tag{3.7}$$

$$I^{\omega_1,\dots,\omega_r}_{\omega_0} = 0 \text{ if } r \neq 1. \tag{3.7 bis}$$

Again, for any simple index ω_0, the operator ∇_{ω_0} acts on *any* mould $M^\bullet$ according to the rule:

$$(\nabla_{\omega_0} M)^{\omega_1,\dots,\omega_r} \overset{\mathrm{def}}{=} \sum_{\omega_i=\omega_0} \{\omega_i M^{\omega_1,\dots,\omega_r} + M^{\omega_1,\dots,\omega_i+\omega_{i+1},\dots,\omega_r} - M^{\omega_1,\dots,\omega_{i-1}+\omega_i,\dots,\omega_r}\} \tag{3.8}$$

the term $M^{\dots,\omega_{i-1}+\omega_i,\dots}$ (resp. $M^{\dots,\omega_i+\omega_{i+1},\dots}$) being systematically omitted if $i = 1$ (resp. $i = r$).

Like ∇, the operator ∇_{ω_0} is a *derivation* of the mould algebra:

$$\nabla_{\omega_0}(A^\bullet \times B^\bullet) \equiv (\nabla_{\omega_0} A^\bullet) \times B^\bullet + A^\bullet \times (\nabla_{\omega_0} B^\bullet) \tag{3.9}$$

and the notation parallelism between $(I^\bullet, \Delta)$ and $(I^\bullet_{\omega_0}, \Delta_{\omega_0})$ is justified not only by the obvious relation:

$$I^\bullet = \sum_{\omega_0} I^\bullet_{\omega_0} \;;\;\; \nabla = \sum_{\omega_0} \nabla_{\omega_0}$$

but, more pointedly, by the fact that, for many important moulds, equations involving ∇ and $I^\bullet$ tend to specialize to similar-looking equations with ∇_{ω_0} and $I^\bullet_{\omega_0}$. Such indeed is the case with our "soft" moulds (but, significantly, *not* with the "tough" moulds).

Induction rules for $S^\bullet$ and $\mathcal{S}^\bullet$.

We have

$$\nabla S^\bullet = -S^\bullet \times I^\bullet \tag{3.10}$$

$$\nabla_{\omega_0} S^\bullet = -S^\bullet \times I^\bullet_{\omega_0} \tag{3.10*}$$

$$\nabla \mathcal{S}^\bullet = I^\bullet \times \mathcal{S}^\bullet \tag{3.11}$$

$$\nabla_{\omega_0} \mathcal{S}^\bullet = I^\bullet_{\omega_0} \times \mathcal{S}^\bullet \tag{3.11*}$$

with the induction-starting conditions $S^\emptyset = \mathcal{S}^\emptyset = 1$.

Induction rules for $S^\bullet_{\text{co}}$ and $\mathcal{S}^\bullet_{\text{co}}$.

$$\nabla S^\bullet_{\text{co}}(t) = (t^\nabla I^\bullet) \times S^\bullet_{\text{co}}(t) - S^\bullet_{\text{co}}(t) \times I^\bullet \tag{3.12}$$

$$\nabla_{\omega_0} S^\bullet_{\text{co}}(t) = (t^\nabla I^\bullet_{\omega_0}) \times S^\bullet_{\text{co}}(t) - S^\bullet_{\text{co}}(t) \times I^\bullet_{\omega_0} \tag{3.12*}$$

$$\nabla \mathcal{S}^\bullet_{\text{co}}(t) = I^\bullet \times \mathcal{S}^\bullet_{\text{co}}(t) - \mathcal{S}^\bullet_{\text{co}}(t) \times (t^\nabla I^\bullet) \tag{3.13}$$

$$\nabla_{\omega_0} \mathcal{S}^\bullet_{\text{co}}(t) = I^\bullet_{\omega_0} \times \mathcal{S}^\bullet_{\text{co}}(t) - \mathcal{S}^\bullet_{\text{co}}(t) \times (t^\nabla I^\bullet_{\omega_0}) \tag{3.13*}$$

with the induction-starting conditions:

$$S^{\boldsymbol{\omega}}_{\text{co}}(t) \equiv (\log t)^r/r! \ \text{ if } \ \boldsymbol{\omega} = (0,\dots,0) \ (r \text{ times}) \tag{3.14}$$

$$\mathcal{S}^{\boldsymbol{\omega}}_{\text{co}}(t) \equiv (-\log t)^r/r! \ \text{ if } \ \boldsymbol{\omega} = (0,\dots,0) \ (r \text{ times}). \tag{3.14*}$$

Induction rules for $S^\bullet_{\text{aco}}$ and $\mathcal{S}^\bullet_{\text{aco}}$.

The logarithm-free parts $S^\bullet_{\text{aco}}(t)$ and $\mathcal{S}^\bullet_{\text{aco}}(t)$ satisfy exactly the same induction as $S^\bullet_{\text{co}}(t)$ and $\mathcal{S}^\bullet_{\text{co}}(t)$, but with different induction-starting conditions:

$$S^\emptyset_{\text{aco}}(t) \equiv \mathcal{S}^\emptyset_{\text{aco}}(t) \equiv 1 \tag{3.15}$$

$$S^{\boldsymbol{\omega}}_{\text{aco}}(t) \equiv \mathcal{S}^{\boldsymbol{\omega}}_{\text{aco}}(t) \equiv 0 \qquad \text{if } \boldsymbol{\omega} = (0,\dots,0). \tag{3.15*}$$

Induction rules for $\$^\bullet$.

$$\nabla \$^\bullet = I^\bullet \times \$^\bullet - \$^\bullet \times I^\bullet \tag{3.16}$$

$$\nabla_{\omega_0} \$^\bullet = I^\bullet_{\omega_0} \times \$^\bullet - \$^\bullet \times I^\bullet_{\omega_0} \tag{3.16*}$$

with the induction-starting conditions:

$$\$^{\emptyset} = 0\ ;\quad \$^{0} = 1\ ;\quad \$^{0,0} = \$^{0,0,0} = \$^{0,0,0,0} = \cdots = 0 \tag{3.17}$$

(beware that $\emptyset \neq (0)$). In view of the importance of $\$^{\bullet}$, let us explicit the compact formalism of (3.16*). For $\omega_0 = \omega_1$ and $\omega_0 = \omega_r$ we get:

$$\omega_1\,\$^{\omega_1,\dots,\omega_r} + \$^{\omega_1+\omega_2,\omega_3,\dots,\omega_r} = \$^{\omega_2,\omega_3,\dots,\omega_r} \tag{3.18}$$

$$\omega_r\,\$^{\omega_1,\dots,\omega_r} - \$^{\omega_1,\dots,\omega_{r-2},\omega_{r-1}+\omega_r} = -\,\$^{\omega_1,\dots,\omega_{r-2},\omega_{r-1}} \tag{3.19}$$

and for $\omega_0 = \omega_i$ with $1 < i < r$ we get:

$$\omega_i\,\$^{\omega_1,\dots,\omega_i,\dots,\omega_r} - \$^{\omega_1,\dots,\omega_{i-1}+\omega_i,\dots,\omega_r} + \$^{\omega_1,\dots,\omega_i+\omega_{i+1},\dots,\omega_r} = 0. \tag{3.20}$$

Proof of the induction rules for the "soft" moulds. — Let us first recall the main decomposition rules established in §2 for the graded Lie algebra $\overline{\mathcal{L}}$. In the special case when $0 \notin \Omega$, we found the conjugacy relation:

$$X = \Theta X^{\mathrm{lin}} \Theta^{-1} \tag{3.21}$$

with

$$X = X^{\mathrm{lin}} + \sum \mathbb{B}_{\omega_i}, \qquad (\omega_i \in \Omega) \tag{3.22}$$

$$\Theta = \sum S^{\bullet}\mathbb{B}_{\bullet}\ ;\quad \Theta^{-1} = \sum S^{\bullet}\,\mathbb{B}_{\bullet}. \tag{3.23}$$

In the general case, *i.e.* when the semi-group Ω may contain 0, we introduced an auxiliary variable t, which led to a more stable conjugacy relation:

$$X - t\partial_t = \Theta_{\mathrm{co}}(X^{\mathrm{lin}} - t\partial_t)\Theta_{\mathrm{co}}^{-1} \tag{3.24}$$

$$X = X^{\mathrm{dia}} + X^{\mathrm{nil}} \tag{3.25}$$

with

$$\Theta_{\mathrm{co}} = \sum S^{\bullet}_{\mathrm{co}}(t)\mathbb{B}_{\bullet}\ ;\quad \Theta_{\mathrm{co}}^{-1} = \sum S^{\bullet}_{\mathrm{co}}(t)\mathbb{B}_{\bullet} \tag{3.26}$$

$$X^{\mathrm{dia}} = X^{\mathrm{lin}} + \sum (I^{\bullet} - \$^{\bullet})\mathbb{B}_{\bullet} \tag{3.27}$$

$$X^{\mathrm{nil}} = \sum \$^{\bullet}\,\mathbb{B}_{\bullet}. \tag{3.28}$$

However, due to the uniqueness of the decomposition (3.21), valid in the special case when $0 \in \Omega$, if we subject X to an automorphism U_ε of $\overline{\mathcal{L}}$ of the form:

$$X \longmapsto \underline{X} = U_\varepsilon X U_\varepsilon^{-1} \tag{3.29}$$

$$U_\varepsilon = \exp(\varepsilon \mathbb{B}_{\omega_0}), \qquad (\varepsilon \in \mathbb{C},\ \omega_0 \in \Omega) \tag{3.30}$$

the conjugacy equation(3.21) still holds, provided we effect the simultaneous change:

$$\Theta \longmapsto \underline{\Theta} = U_\varepsilon \Theta. \tag{3.31}$$

By an easy continuity argument, we see that, in the general case also (when Ω may contain 0), the decompositions (3.24), (3.25) retain their validity after the simultaneous changes:

$$\Theta_{\text{co}} \longmapsto \underline{\Theta}_{\text{co}} = U_\varepsilon \Theta_{\text{co}} \tag{3.32}$$

$$X^{\text{dia}} \longmapsto \underline{X}^{\text{dia}} = U_\varepsilon X^{\text{dia}} U_\varepsilon^{-1} \tag{3.33}$$

$$X^{\text{nil}} \longmapsto \underline{X}^{\text{nil}} = U_\varepsilon X^{\text{nil}} U_\varepsilon^{-1}. \tag{3.34}$$

However, it is plain, from the construction at the end of §2, that the conjugacies (3.21), (3.24) and the decomposition (3.25) hold not just for a Lie element X of the form (3.22), but for any Lie element $\underline{X}$ of the form:

$$\underline{X} = X^{\text{lin}} + \sum \underline{\mathbb{B}}_{\omega_i} \text{ with } \underline{\mathbb{B}}_{\omega_i} \in \mathcal{L} \text{ and } \operatorname{grad}(\underline{\mathbb{B}}_{\omega_i}) = \omega_i. \tag{3.35}$$

Now, the particular Lie element $\underline{X}$ introduced in (3.29) admits an expansion of type (3.35) with:

$$\mathbb{B}_{\omega_i} \longmapsto \underline{\mathbb{B}}_{\omega_i} \overset{\text{def}}{=} \mathbb{B}_{\omega_i} + \varepsilon[\mathbb{B}_{\omega_0}, \mathbb{B}_{\omega_i-\omega_0}] + o(\varepsilon) \tag{3.36}$$

and this affords us with a second means of calculating $\underline{\Theta}$, $\underline{\Theta}_{\text{co}}$, $\underline{X}^{\text{dia}}$, $\underline{X}^{\text{nil}}$, namely by applying the formulae (3.26), (3.27), (3.28) with $\underline{\mathbb{B}}_{\omega_i}$ instead of $\mathbb{B}_{\omega_i}$, but with the same universal moulds $S^\bullet$, $S^\bullet_{\text{co}}(t)$, $\$^\bullet$. Now, comparing the result of these calculations with the direct formulae (3.31), (3.32), (3.33), (3.34), and equating, in each instance, the coefficients in front of ε (viewed as an infinitesimal parameter) we obtain all the rules (from (3.16) to (3.20)) that govern the ∇_{ω_0}-derivation of the "soft" moulds — which is what we had set out to prove. (As we shall see in the next section, the "tough" moulds $\$\!\!\$^\bullet$, $S^\bullet_{\text{ext}}$, $\cancel{S}^\bullet_{\text{ext}}$ do not possess such simple ∇_{ω_0}-derivatives.)

4. Construction and properties of the $\$\!\!\$^\bullet$ mould.

General scheme.

Just after (2.18) we defined the *vanishing order* $\text{van}(\boldsymbol{\omega})$ of a sequence $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$. When $\|\boldsymbol{\omega}\| \neq 0$, $\text{van}(\boldsymbol{\omega})$ is automatically 0, but we still have a *forward* (resp. *backward*) *vanishing order*, defined by:

$$\overrightarrow{\text{van}}\,(\boldsymbol{\omega}) = \#\{\check{\omega}_i = 0\} \qquad (\text{resp. } \overleftarrow{\text{van}}\,(\boldsymbol{\omega}) = \#\{\hat{\omega}_i = 0\}). \tag{4.1}$$

We shall require all three notions for the construction of the three "tough" moulds, and shall proceed as follows:

$$(S^\bullet, \cancel{S}^\bullet, T^\bullet) \overset{\text{rest}}{\longmapsto} (S^\bullet_{\text{rest}}, \cancel{S}^\bullet_{\text{rest}}, T^\bullet_{\text{rest}}) \overset{\text{diff}}{\longmapsto} (S^\bullet_{\text{ext}}, \cancel{S}^\bullet_{\text{ext}}, \text{\$}^\bullet). \tag{4.2}$$

The step *rest* ("restriction") will rid us of the vanishing denominators $\check{\omega}_i$ or $\hat{\omega}_i$. It will also decrease the homogeneous degree by an integer s equal, respectively, to $\overrightarrow{\text{van}}\,(\boldsymbol{\omega})$, $\overleftarrow{\text{van}}\,(\boldsymbol{\omega})$, $\text{van}(\boldsymbol{\omega})$. But at the next step *diff* ("differentiation") we shall apply to the "restrictions" suitable differential operators:

$$\text{Rad}^{\boldsymbol{\omega}}, \quad \text{Ra}\text{đ}^{\boldsymbol{\omega}}, \quad \text{Ral}^{\boldsymbol{\omega}} \tag{4.3}$$

of order s in the variables ω_i, so that the right degree will be restored. This will also take care of the rationality conditions (2.40), (2.41), (2.42). The main point, however, is to ensure the symmetrality (resp. alternality) of the resulting moulds $S^\bullet_{\text{ext}}$ and $\cancel{S}^\bullet_{\text{ext}}$ (resp. $\text{\$}^\bullet$) and of course to check that they relate to one another in the same way as in Proposition 2.3. Those requirements happen to totally determine the shape of the operators (4.3), but in order to construct these, we shall need three auxiliary moulds:

$$\text{rad}^{\mathbf{w}}, \ \text{ra}\text{đ}^{\mathbf{w}}, \ \text{ral}^{\mathbf{w}} \quad \left(\mathbf{w} = (w_1, \dots, w_r), w_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, u_i \in \mathbb{C}, v_i \in \mathbb{N}\right) \tag{4.4}$$

which, though rather elementary, are interesting in their own right.

The restrictions $S^\bullet_{\text{rest}}$, $\cancel{S}^\bullet_{\text{rest}}$, $T^\bullet_{\text{rest}}$.

DEFINITION 4.1. — *For any sequence* $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$, *we put:*

$$S^{\boldsymbol{\omega}}_{\text{rest}} \overset{\text{def}}{=} \prod_{\check{\omega}_i \neq 0} (-\check{\omega}_i)^{-1} \qquad (\textit{with } \check{\omega}_i = \omega_1 + \dots + \omega_i) \tag{4.5}$$

$$\cancel{S}^{\boldsymbol{\omega}}_{\text{rest}} \overset{\text{def}}{=} \prod_{\hat{\omega}_i \neq 0} (+\hat{\omega}_i)^{-1} \qquad (\textit{with } \hat{\omega}_i = \omega_i + \dots + \omega_r). \tag{4.6}$$

For $\|\boldsymbol{\omega}\| \neq 0$, *we put* $T^{\boldsymbol{\omega}}_{\text{rest}} \overset{\text{def}}{=} 0$ *and for* $\|\boldsymbol{\omega}\| = 0$, *we adopt either of the alternative definitions:*

$$T^{\boldsymbol{\omega}}_{\text{rest}} \overset{\text{def}}{=} \prod_{\check{\omega}_i \neq 0} (-\check{\omega}_i)^{-1} \qquad (\textit{with } \|\boldsymbol{\omega}\| = 0) \tag{4.7}$$

$$T^{\boldsymbol{\omega}}_{\text{rest}} \overset{\text{def}}{=} \prod_{\hat{\omega}_i \neq 0} (+\hat{\omega}_i)^{-1} \qquad (\textit{with } \|\boldsymbol{\omega}\| = 0). \tag{4.8}$$

Remark. — Although $\check{\omega}_i$ is removed from the product (4.5), (4.7) if $\check{\omega}_i = 0$, the variables $\omega_1, \omega_2, \dots, \omega_i$ constitutive of $\check{\omega}_i$ *remain* inside

$\check{\omega}_{i+1}, \check{\omega}_{i+2}, \ldots$. Similarly, $\hat{\omega}_i$ is removed from the products (4.6), (4.8) if $\hat{\omega}_i = 0$, but the variables $\omega_i, \omega_{i+1}, \ldots$ must be kept *inside* $\hat{\omega}_{i-1}, \hat{\omega}_{i-2}, \ldots$ (see example (4.35)–(4.38)).

Construction of the auxiliary moulds $\mathrm{rad}^{\mathbf{w}}$, $\text{rađ}^{\mathbf{w}}$, $\mathrm{ral}^{\mathbf{w}}$.

In this subsection, we will have to do with moulds indexed by sequences $\mathbf{w} = (w_1, \ldots, w_r)$ with $w_i = \binom{u_i}{v_i}$, $u_i \in \mathbb{C}$, $v_i \in \mathbb{R}^+$. On such moulds, there act the operators $\square_{w_0}$, which are defined as the ∇_{ω_0} in (3.8), but with differentiation by $v_0 \partial_{u_0}$ in place of multiplication by w_0. Thus we have:

$$(4.9)\quad (\square_{w_i} M)^{w_1,\ldots,w_r} = v_i \partial_{u_i} M^{w_1,\ldots,w_r} + M^{w_1,\ldots,w_i+w_{i+1},\ldots,w_r} - M^{w_1,\ldots,w_{i-1}+w_i,\ldots,w_r}$$

and the term with the contraction $w_{i-1} + w_i$ (resp. $w_i + w_{i+1}$) should of course be omitted if $i = 1$ (resp. $i = r$). Each operator $\square_{w_0}$ is a *derivation*, relative to the non-commutative mould product.

We also require moulds $I^{\bullet}_{w_0}$, which we define exactly as in (3.7), (3.7 bis), but with w_0 in place of ω_0. We may note (for future use) that the straightforward application of (3.8) to $I^{\bullet}$ yields:

$$(4.10)\qquad \square_{w_0} I^{\bullet} = I^{\bullet}_{w_0} \times I^{\bullet} - I^{\bullet} \times I^{\bullet}_{w_0}, \quad (\forall w_0).$$

PROPOSITION 4.1 (Characterization of the moulds $\mathrm{rad}^{\bullet}$, $\text{rađ}^{\bullet}$, $\mathrm{ral}^{\bullet}$). *The mould equations:*

$$(4.11)\qquad \square_{w_i} \mathrm{rad}^{\bullet} = -\,\mathrm{rad}^{\bullet} \times I^{\bullet}_{w_i}, \qquad (\forall i)$$

$$(4.12)\qquad \square_{w_i} \text{rađ}^{\bullet} = +I^{\bullet}_{w_i} \times \text{rađ}^{\bullet}, \qquad (\forall i)$$

$$(4.13)\qquad \square_{w_i} \mathrm{ral}^{\bullet} = 0, \qquad (\forall i)$$

along with the initial conditions:

$$(4.14)\qquad \mathrm{rad}^{\emptyset} = \text{rađ}^{\emptyset} = 1\ ;\quad \mathrm{ral}^{\emptyset} = 0$$

$$(4.15)\quad \mathrm{rad}^{\mathbf{w}} = \text{rađ}^{\mathbf{w}} = \mathrm{ral}^{\mathbf{w}} = 0 \ \textit{if}\ \mathbf{w} = (w_1, \ldots, w_r) \ \textit{with}\ 0 = u_1 = u_2 = \cdots = u_r, \ (\forall v_i)$$

admit, as their unique solution, two symmetral moulds $\mathrm{rad}^{\bullet}$ *and* $\text{rađ}^{\bullet}$, *and an alternal mould* $\mathrm{ral}^{\bullet}$, *which are related as follows:*

$$(4.16)\qquad 1^{\bullet} = \mathrm{rad}^{\bullet} \times \text{rađ}^{\bullet}$$

$$(4.17)\qquad \mathrm{ral}^{\bullet} = \mathrm{rad}^{\bullet} \times I^{\bullet} \times \text{rađ}^{\bullet}.$$

Short proof. — To clarify the convenient but all too concise formalism of Proposition 4.1, let us first write out in full the equations (4.11) for $r(\mathbf{w}) = 3$ and $i = 1,2,3$. We find

$$v_1 \partial_{u_1} \mathrm{rad}^{w_1,w_2,w_3} + \mathrm{rad}^{w_1+w_2,w_3} = 0$$

$$v_2 \partial_{u_2} \mathrm{rad}^{w_1,w_2,w_3} + \mathrm{rad}^{w_1,w_2+w_3} - \mathrm{rad}^{w_1+w_2,w_3} = 0$$

$$v_3 \partial_{u_3} \mathrm{rad}^{w_1,w_2,w_3} - \mathrm{rad}^{w_1,w_2+w_3} = - \mathrm{rad}^{w_1,w_2} \,.$$

Clearly, the equations (4.11), (4.12), (4.13), along with the initial conditions (4.14), (4.15), amount to an overdetermined differential system. So we must first check its consistency, by establishing the relations:

$$\partial_{u_j} \partial_{u_i} M^{\mathbf{w}} \equiv \partial_{u_i} \partial_{u_j} M^{\mathbf{w}} \quad \text{for} \quad M^{\mathbf{w}} = \mathrm{rad}^{\mathbf{w}},\ \mathrm{rađ}^{\mathbf{w}},\ \mathrm{ral}^{\mathbf{w}} \,. \tag{4.18}$$

This is easily done, through applying the rules (4.11), (4.12), (4.13) twice in succession, for w_i and then w_j (resp. for w_j and then w_i), but we have to distinguish the case when $|i-j| \geq 2$ from the case $|i-j| = 1$.

Then we must check the alternality of $\mathrm{ral}^\bullet$ (resp. symmetrality of $\mathrm{rad}^\bullet$, $\mathrm{rađ}^\bullet$). As earlier with the "soft" moulds, this is a matter of straightforward induction on r, but we may note that the conclusion (*i.e.* alternality and symmetrality) follows from the very *shape* of the system (4.11), (4.12), (4.13), not from its ingredients: *it would remain in force even if we replaced* $\square_{w_i}$ *by some other mould derivation, and* $I^\bullet_{w_i}$ *by some other alternal mould* (provided, of course, the self-consistency of the system is preserved).

Lastly, the relations (4.16), (4.17) follow from the uniqueness of the solution of the system (4.11), (4.12), (4.13). Indeed, if we take $\mathrm{rad}^\bullet$ to be the solution of (4.11), and then *define* $\mathrm{rađ}^\bullet_*$ as the mould-inverse of $\mathrm{rad}^\bullet$ (as in (4.16)) and $\mathrm{ral}^\bullet_*$ as the conjugate of $I^\bullet$ under $\mathrm{rad}^\bullet$ (as in (4.17)), it is a easy matter to check that $\mathrm{rađ}^\bullet_*$ and $\mathrm{ral}^\bullet_*$ automatically verify the systems (4.12) and (4.13) along with the corresponding initial conditions. Therefore $\mathrm{rađ}^\bullet_* = \mathrm{rađ}^\bullet$ and $\mathrm{ral}^\bullet_* = \mathrm{ral}^\bullet$. □

Remark. — It is plain that for sequences $\mathbf{w}$ of fixed length $r(\mathbf{w}) = r$, the functions $\mathrm{rad}^{\mathbf{w}}$ and $\mathrm{rađ}^{\mathbf{w}}$ (resp. $\mathrm{ral}^{\mathbf{w}}$) are homogeneous polynomials of degree r (resp. $r-1$) not only in the variables u_i $(1 \leq i \leq r)$ but also in the variables:

$$(v_i + v_{i+1} + \cdots + v_j)^{-1} \qquad (1 \leq i \leq j \leq r). \tag{4.19}$$

Thus, if we introduce the short-hand notations:

$$u_{ij} = u_i + u_j \ ; \quad u_{ijk} = u_i + u_j + u_k \ ; \quad \text{etc.} \ ; \quad v_{ij} = v_i + v_j \ ; \ \text{etc.} \tag{4.20}$$

we find as first values of rad$^\bullet$ and rađ$^\bullet$:

$$\text{rađ}^{w_1} = -\,\text{rad}^{w_1} = +\frac{u_1}{v_1}$$

$$\text{rađ}^{w_1,w_2} = +\,\text{rad}^{w_2,w_1} = +\frac{1}{2}\frac{(u_{12})^2}{v_2v_{12}} - \frac{1}{2}\frac{(u_1)^2}{v_1v_2}$$

$$\begin{aligned}\text{rađ}^{w_1,w_2,w_3} = -\,\text{rad}^{w_3,w_2,w_1} &= +\frac{1}{6}\frac{u_{123}^3}{v_3v_{23}v_{123}} - \frac{1}{6}\frac{u_{12}^3}{v_2v_3v_{12}} - \frac{1}{3}\frac{u_1^3}{v_1v_2v_{23}}\\ &\quad - \frac{1}{2}\frac{u_{123}u_1^2}{v_1v_3v_{23}} + \frac{1}{2}\frac{u_{12}u_1^2}{v_1v_2v_3}\end{aligned}$$

etc.,

and as first values of ral$^\bullet$:

$$\text{ral}^{w_1} = 1;\ \ \text{ral}^{w_1,w_2} = -\frac{u_1}{v_1} + \frac{u_2}{v_2}\ ;$$

$$\text{ral}^{w_1,w_2,w_3} = +\frac{1}{2}\frac{u_{12}^2}{v_1v_{12}} + \frac{1}{2}\frac{u_{23}^2}{v_3v_{23}} - \frac{1}{2}\frac{u_2^2}{v_1v_2} - \frac{1}{2}\frac{u_2^2}{v_2v_3} - \frac{u_1u_3}{v_1v_3}.$$

Construction of the operators Rad$^{\boldsymbol{\omega}}$, Rađ$^{\boldsymbol{\omega}}$, Ral$^{\boldsymbol{\omega}}$.

We shall have to factorize each sequence $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ into *unbreakable*, *zero-sum* factors $\boldsymbol{\omega}^i$, *i.e.* into factor sequences:

$$\boldsymbol{\omega}^i = (\omega_j, \omega_{j+1}, \dots, \omega_{k-1}, \omega_k), \quad (\omega_q \in \mathbb{C}) \tag{4.21}$$

such that:

$$\|\boldsymbol{\omega}^i\| \overset{\text{def}}{=} \omega_j + \cdots + \omega_k = 0, \quad \text{but } \omega_j + \cdots + \omega_q \neq 0 \text{ if } j < q < k. \tag{4.22}$$

To any such factor $\boldsymbol{\omega}^i$ we associate an ordinary differential operator u_i and an integer v_i:

$$u_i \overset{\text{def}}{=} D_{\boldsymbol{\omega}^i} \overset{\text{def}}{=} \partial_{\omega_j} + \partial_{\omega_{j+1}} + \cdots + \partial_{\omega_k}, \quad (\partial_{\omega_q} \overset{\text{def}}{=} \partial/\partial\omega_q) \tag{4.23}$$

$$v_i \overset{\text{def}}{=} r(\boldsymbol{\omega}^i) = k - j + 1 = \text{ length of } \boldsymbol{\omega}^i. \tag{4.24}$$

For a general sequence $\boldsymbol{\omega}$, we must distinguish the *forward* and *backward* factorizations:

$$\boldsymbol{\omega} = \boldsymbol{\omega}^1\boldsymbol{\omega}^2 \cdots \boldsymbol{\omega}^s\boldsymbol{\omega}^* \quad \text{(forward)} \tag{4.25}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}^*\boldsymbol{\omega}^1\boldsymbol{\omega}^2 \cdots \boldsymbol{\omega}^s \quad \text{(backward)} \tag{4.26}$$

with unbreakable, zero-sum factors $\boldsymbol{\omega}^i$ and either $\boldsymbol{\omega}^* = \emptyset$ or $\|\boldsymbol{\omega}^*\| \neq 0$. Both factorizations coincide *iff* $\|\boldsymbol{\omega}\| = 0$, since in that case $\boldsymbol{\omega}^* = \emptyset$ and:

$$\boldsymbol{\omega} = \boldsymbol{\omega}^1\boldsymbol{\omega}^2 \cdots \boldsymbol{\omega}^s. \tag{4.27}$$

Definition 4.2 (Operators $\mathrm{Rad}^{\boldsymbol{\omega}}$, $\text{Rađ}^{\boldsymbol{\omega}}$, $\mathrm{Ral}^{\boldsymbol{\omega}}$). — *To each sequence* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ *we associate three differential operators by putting:*

$$\mathrm{Rad}^{\boldsymbol{\omega}} \overset{\mathrm{def}}{=} \mathrm{rad}^{w_1,\ldots,w_s} \tag{4.28}$$

with $\boldsymbol{\omega} = \boldsymbol{\omega}^1 \cdots \boldsymbol{\omega}^s \boldsymbol{\omega}^*$ *as in (4.25) and* $w_i = \binom{u_i}{v_i}$ *as in (4.23), (4.24)*

$$\text{Rađ}^{\boldsymbol{\omega}} \overset{\mathrm{def}}{=} \text{rađ}^{w_1,\ldots,w_s} \tag{4.29}$$

with $\boldsymbol{\omega} = \boldsymbol{\omega}^* \boldsymbol{\omega}^1 \cdots \boldsymbol{\omega}^s$ *as in (4.26) and* $w_i = \binom{u_i}{v_i}$ *as in (4.23), (4.24)*

$$\mathrm{Ral}^{\boldsymbol{\omega}} \overset{\mathrm{def}}{=} \mathrm{ral}^{w_1,\ldots,w_s} \tag{4.30}$$

with $\boldsymbol{\omega} = \boldsymbol{\omega}^1 \cdots \boldsymbol{\omega}^s$ *as in (4.27) and* $w_i = \binom{u_i}{v_i}$ *as in (4.23), (4.24).*

Remark 1. — Since the moulds $\mathrm{rad}^{\mathbf{w}}$, $\text{rađ}^{\mathbf{w}}$, $\mathrm{ral}^{\mathbf{w}}$ depend polynomially on the variables u_i, and since the operators $D_{\boldsymbol{\omega}^i}$ commute pairwise, the substitution $u_i \mapsto D_{\boldsymbol{\omega}^i}$ offers no difficulty.

Remark 2. — If the *forward* (resp. *backward*) decomposition of $\boldsymbol{\omega}$ reduces to the one factor $\boldsymbol{\omega}^*$, the above definitions yield $s = 0$ and $\mathrm{Rad}^{\boldsymbol{\omega}} = 1$ (resp. $\text{Rađ}^{\boldsymbol{\omega}} = 1$). Likewise, if $\|\boldsymbol{\omega}\| \neq 0$, we get $s = 0$ in (4.27) and $\mathrm{Ral}^{\boldsymbol{\omega}} = 0$, but if $\|\boldsymbol{\omega}\| = 0$ and $\boldsymbol{\omega}$ is itself unbreakable, we get $s = 1$ and $\mathrm{Ral}^{\boldsymbol{\omega}} = 1$.

Remark 3. — It should be noted that even those components ω_j (inside an unbreakable, zero-sum factor $\boldsymbol{\omega}^i$) that vanish (*i.e.* $\omega_j = 0$) nonetheless contribute a term ∂_{ω_j} to the operator $u_i = D_{\boldsymbol{\omega}^i}$ of (4.23).

Construction of the "tough" moulds.

Proposition 4.2 (Expression of the canonical moulds $S^{\bullet}_{\mathrm{ext}}$, $\not{S}^{\bullet}_{\mathrm{ext}}$, $\$^{\bullet}$). — *The unique, canonical mould triplet* $S^{\bullet}_{\mathrm{ext}}$, $\not{S}^{\bullet}_{\mathrm{ext}}$ *(symmetral) and* $\$^{\bullet}$ *(alternal) of Proposition 2.3 is explicitely given, for any sequence* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ *of any given vanishing pattern, by the relations:*

$$S^{\bullet}_{\mathrm{ext}} = \mathrm{Rad}^{\boldsymbol{\omega}} \cdot S^{\boldsymbol{\omega}}_{\mathrm{rest}} \tag{4.31}$$

$$\not{S}^{\boldsymbol{\omega}}_{\mathrm{ext}} = \text{Rađ}^{\boldsymbol{\omega}} \cdot \not{S}^{\boldsymbol{\omega}}_{\mathrm{rest}} \tag{4.32}$$

$$\$^{\boldsymbol{\omega}} = \mathrm{Ral}^{\boldsymbol{\omega}} \cdot T^{\boldsymbol{\omega}}_{\mathrm{rest}} \tag{4.33}$$

with the same notations as in Definition 4.1 and 4.2.

Remark. — In (4.33), one may take for $T^{\boldsymbol{\omega}}_{\mathrm{rest}}$ either of the alternative definitions (4.7) and (4.8). The choice doesn't affect the end result. Checking this makes for a nice exercise, which we leave to the reader.

Important caveat. — In all three instances (4.31), (4.32), (4.33), one should take the restricted moulds $S^{\boldsymbol{\omega}}_{\text{rest}}$, $\not{S}^{\boldsymbol{\omega}}_{\text{rest}}$, $T^{\boldsymbol{\omega}}_{\text{rest}}$ with *all* their variables ω_i, *without simplifications*; then apply the operators $\mathrm{Rad}^{\boldsymbol{\omega}}$, $\mathrm{Ra}\text{đ}^{\boldsymbol{\omega}}$, $\mathrm{Ral}^{\boldsymbol{\omega}}$; and then only, at the last stage, effect the simplifications that stem from the identities:

$$0 = \|\boldsymbol{\omega}^1\| = \|\boldsymbol{\omega}^2\| = \cdots = \|\boldsymbol{\omega}^s\|. \tag{4.34}$$

Thus, if we consider a sequence $\boldsymbol{\omega}$ with the factorization:

$$\boldsymbol{\omega} = \boldsymbol{\omega}^1\boldsymbol{\omega}^2\boldsymbol{\omega}^* \text{ if } \boldsymbol{\omega}^1 = (\omega_1, \omega_2), \boldsymbol{\omega}^2 = (\omega_3), \boldsymbol{\omega}^* = (\omega_4, \omega_5) \tag{4.35}$$

the rules of Definition 4.1 yield:

$$\mathrm{Rad}^{\boldsymbol{\omega}} = +\frac{1}{2}\frac{(u_{12})^2}{v_1 v_{12}} - \frac{1}{2}\frac{(u_2)^2}{v_1 v_2}, \quad (u_{12} = u_1+u_2; v_{12} = v_1+v_2) \tag{4.36}$$

$$u_1 = \partial_{\omega_1} + \partial_{\omega_2} \ ; \ \ u_2 = \partial_{\omega_3} \ ; \ \ v_1 = 2 \ ; \ \ v_2 = 1. \tag{4.37}$$

We must apply the operator $\mathrm{Rad}^{\boldsymbol{\omega}}$ to the "restriction"

$$\begin{aligned} S^{\boldsymbol{\omega}}_{\text{rest}} &= -(\check{\omega}_1\check{\omega}_4\check{\omega}_5)^{-1} \\ &= -(\omega_1)^{-1}(\omega_1+\omega_2+\omega_3+\omega_4)^{-1}(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5)^{-1} \end{aligned} \tag{4.38}$$

and only then may we simplify by taking into account the fact that $0 = \omega_1 + \omega_2 = \omega_3$. This procedure alone yields the right result, which reads:

$$\begin{aligned} S^{\boldsymbol{\omega}}_{\text{ext}} = -(\check{\omega}_1\check{\omega}_4\check{\omega}_5)^{-1}\Big\{&\frac{1}{6}(\check{\omega}_1)^{-2}+\frac{1}{2}(\check{\omega}_1\check{\omega}_4)^{-1} \\ &+\frac{1}{2}(\check{\omega}_1\check{\omega}_5)^{-1}+(\check{\omega}_4)^{-2} + (\check{\omega}_5)^{-2}+(\check{\omega}_4\check{\omega}_5)^{-1}\Big\}. \end{aligned} \tag{4.38*}$$

Let us now examine what shape $S^{\boldsymbol{\omega}}_{\text{ext}}$ assumes depending on the number s of "unbreakable" factors in the forward factorization (4.25).

If $s = 0$, then of course $S^{\boldsymbol{\omega}}_{\text{ext}} \equiv S^{\boldsymbol{\omega}}_{\text{rest}} \equiv S^{\boldsymbol{\omega}}$.

If $s = 1$, *i.e.* if $\boldsymbol{\omega} = \boldsymbol{\omega}^1\boldsymbol{\omega}^*$ (with $\|\boldsymbol{\omega}^1\| = 0$, $r(\boldsymbol{\omega}^1) = r_1$) we find:

$$S^{\boldsymbol{\omega}}_{\text{ext}} = \Big\{\prod\nolimits^{*}(-\check{\omega}_i)^{-1}\Big\}\Big\{\sum\nolimits^{*} Q_i(\check{\omega}_i)^{-1}\Big\} \tag{4.39}$$

with

$$Q_i \equiv \frac{i}{r_i} \text{ if } \omega_i \in \boldsymbol{\omega}^1 \text{ and } Q_i \equiv 1 \text{ if } \omega_i \in \boldsymbol{\omega}^*. \tag{4.40}$$

Here, the star $*$ atop $\prod$ and $\sum$ signals that we omit the "inacceptable" terms $\check{\omega}_i \equiv 0$.

If $s = 2$, *i.e.* $\boldsymbol{\omega} = \boldsymbol{\omega}^1\boldsymbol{\omega}^2\boldsymbol{\omega}^*$ (with $\|\boldsymbol{\omega}^i\| = 0$, $r(\boldsymbol{\omega}^i) = r_i$) we find:

$$S^{\boldsymbol{\omega}}_{\text{ext}} = \Big\{\prod_{i}{}^{*}(-\check{\omega}_i)^{-1}\Big\}\Big\{\sum_{i\leq j}{}^{*} Q_{ij}(\check{\omega}_i\check{\omega}_j)^{-1}\Big\} \tag{4.41}$$

with the same omission rules as above, and with coefficients Q_{ij} given by:

$$Q_{ij} \equiv \frac{ij}{r_1 r_{12}} \qquad \text{if } (\omega_i,\omega_j) \in (\boldsymbol{\omega}^1,\boldsymbol{\omega}^1) \text{ or } (\boldsymbol{\omega}^1,\boldsymbol{\omega}^2) \tag{4.42}$$

$$Q_{ij} \equiv \frac{ij}{r_1 r_{12}} - \frac{(i-r_1)(j-r_1)}{r_1 r_2} \qquad \text{if } (\omega_i,\omega_j) \in (\boldsymbol{\omega}^2,\boldsymbol{\omega}^2) \tag{4.42*}$$

$$Q_{ij} \equiv \frac{i}{r_1} \qquad \text{if } (\omega_i,\omega_j) \in (\boldsymbol{\omega}^1,\boldsymbol{\omega}^*) \tag{4.42**}$$

$$Q_{ij} \equiv 1 \qquad \text{if } (\omega_i,\omega_j) \in (\boldsymbol{\omega}^2,\boldsymbol{\omega}^*) \text{ or } (\boldsymbol{\omega}^*,\boldsymbol{\omega}^*). \tag{4.42***}$$

In the general case, for $\boldsymbol{\omega} = \boldsymbol{\omega}^1 \boldsymbol{\omega}^2 \cdots \boldsymbol{\omega}^s \boldsymbol{\omega}^*$ (with $\|\boldsymbol{\omega}^i\| = 0$, $r(\boldsymbol{\omega}^i) = r_i$) we get:

$$\mathcal{S}^{\boldsymbol{\omega}}_{\mathrm{ext}} = \Big\{ \prod_i{}^{*} (-\check{\omega}_i)^{-1} \Big\} \Big\{ \sum_{i_1 \le i_2 \le \cdots \le i_s}{}^{*} Q_{i_1,i_2,\ldots,i_s} (\check{\omega}_{i_1} \check{\omega}_{i_2} \cdots \check{\omega}_{i_s})^{-1} \Big\} \tag{4.43}$$

with nearly $s!$ different expressions for $Q_{\bullet}$, such as:

$$Q_{i_1,i_2,\ldots,i_s} \equiv \frac{i_1 i_2 \cdots i_s}{r_1 r_{12} \cdots r_{12\cdots s}} \qquad \text{if } (\omega_{i_1},\omega_{i_2},\ldots,\omega_{i_s}) \in (\boldsymbol{\omega}^1,\boldsymbol{\omega}^1,\ldots,\boldsymbol{\omega}^1) \tag{4.43*}$$

$$Q_{i_1,i_2,\ldots,i_s} \equiv 1 \qquad \text{if } (\omega_{i_1},\omega_{i_2},\ldots,\omega_{i_s}) \in (\boldsymbol{\omega}^*,\boldsymbol{\omega}^*,\ldots,\boldsymbol{\omega}^*). \tag{4.43**}$$

We observe (first in the case $s = 1$, $s = 2$, etc.) the following *continuity property*: although the outward shape of the coefficients $Q_{i,j,k,\ldots}$ *depends on which factors* $\boldsymbol{\omega}^{i'}, \boldsymbol{\omega}^{j'}, \boldsymbol{\omega}^{k'}, \ldots$ *the components* $\omega_i, \omega_j, \omega_k, \ldots$ *are taken from*, these coefficients *coincide in the boundary cases*, *i.e.* when $i, j, k, \ldots$ assume the "prohibited" values r_1, r_{12}, r_{123}, etc. For instance, in the case $s = 2$, we get the same value for Q_{ij}:

- by putting $i = r_1$ in (4.42) or (4.42*)
- by putting $j = r_{12}$ in (4.42*) or (4.42***)
- by putting $i = r_1$ in (4.42**) or (4.42***).

Proof of Proposition 4.3. — The conditions (2.40), (2.41), (2.42) of Proposition 2.3 are obviously fulfilled by construction, and the main point to prove is the symmetrality (resp. alternality) of $S^{\bullet}_{\mathrm{ext}}$ and $\mathcal{S}^{\bullet}_{\mathrm{ext}}$ (resp. $\$^{\bullet}$). We first establish the symmetrality of $\mathcal{S}^{\bullet}_{\mathrm{ext}}$ with the help of the following lemma:

LEMMA 4.4 (Arborification). — *The relation (4.32) still holds after arborification*

$$\overset{<}{\mathcal{S}}{}^{\boldsymbol{\omega}}_{\mathrm{ext}} = \overset{<}{\mathrm{Rad}}{}^{\boldsymbol{\omega}}\, \overset{<}{\mathcal{S}}{}^{\boldsymbol{\omega}}_{\mathrm{rest}} \tag{4.44}$$

i.e. after replacing the fully ordered sequence $\boldsymbol{\omega}$ *by any partially ordered sequence* $\overset{<}{\boldsymbol{\omega}}$ *such that each* ω_i *in* $\overset{<}{\boldsymbol{\omega}}$ *has* at most one direct antecedent*; and after setting:*

$$S_{\text{ext}}^{\overset{<}{\boldsymbol{\omega}}} = \sum_{\boldsymbol{\omega}} S_{\text{ext}}^{\boldsymbol{\omega}} \tag{4.45}$$

with a sum extending to all sequences $\boldsymbol{\omega}$ *whose* full order *is compatible with the* partial order *of* $\overset{<}{\boldsymbol{\omega}}$*. The operator* $\text{Rađ}^{\overset{<}{\boldsymbol{\omega}}}$ *is still defined by (4.29), (4.23), (4.24), but relatively to the backward factorization which is the exact analogue of (4.26) for the arborescent structure of* $\overset{<}{\boldsymbol{\omega}}$*. Lastly,* $S^{\overset{<}{\boldsymbol{\omega}}}$ *is defined as in (4.29), but with sums* $\hat{\omega}_i = \sum \omega_j$ *that are now relative to the partial order of* $\boldsymbol{\omega}$ *(i.e. they extend to all* ω_j *posterior to* ω_i *in* $\overset{<}{\boldsymbol{\omega}}$*, including* ω_i *itself).*

To prove this lemma, we fix some unbreakable, zero-sum sequences $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \boldsymbol{\omega}^3, \ldots$ and denote the product-sequences $(\boldsymbol{\omega}^1\boldsymbol{\omega}^2)$, $(\boldsymbol{\omega}^1\boldsymbol{\omega}^2\boldsymbol{\omega}^3)$, etc. by the short-hand $\boldsymbol{\omega}^{12}, \boldsymbol{\omega}^{123}$, etc. Then, to each $\boldsymbol{\omega}^i$ or $\boldsymbol{\omega}^{ij}$, etc. we associate pairs $w_i = \binom{u_i}{v_i}$ or $w_{ij} = \binom{u_{ij}}{v_{ij}}$, etc. with operators u_i or $u_{ij} \cdots$ as in (4.23) and integers v_i or $v_{ij} \cdots$ as in (4.24). With such components w_i, the polynomials $\text{rađ}^{\mathbf{w}}$ become ordinary differential operators, and the system (4.12) translates into the following Leibniz type rules:

$$\text{rađ}^{w_1}(\|\boldsymbol{\omega}^1\|\varphi_1) \equiv \varphi_1 \tag{4.46}$$

$$\text{rađ}^{w_1,w_2}(\|\boldsymbol{\omega}^2\|\varphi_2) \equiv \text{rađ}^{w_{12}}((\varphi_2) \tag{4.47}$$

$$\text{rađ}^{w_1,w_2}(\|\boldsymbol{\omega}^{12}\|\varphi_2) \equiv \text{rađ}^{w_1}((\varphi_2) \tag{4.48}$$

$$\text{rađ}^{w_1,w_2,w_3}(\|\boldsymbol{\omega}^3\|\varphi_3) \equiv \text{rađ}^{w_1,w_{23}}((\varphi_3) \tag{4.49}$$

$$\text{rađ}^{w_1,w_2,w_3}(\|\boldsymbol{\omega}^{12}\|\varphi_3) \equiv \text{rađ}^{w_{12},w_3}((\varphi_3) \tag{4.50}$$

$$\text{rađ}^{w_1,w_2,w_3}(\|\boldsymbol{\omega}^{123}\|\varphi_3) \equiv \text{rađ}^{w_2,w_3}((\varphi_3) \tag{4.51}$$

etc., and more generally, for $i \geq 2$ (resp. $i = 1$):

$$\text{rađ}^{w_1,\ldots,w_r}(\|\omega^i\omega^{i+1}\cdots\omega^s\|\varphi_s) \equiv \text{rađ}^{w_1,\ldots,w_i+w_{i+1},\ldots,w_s}(\varphi_s) \tag{4.52}$$

$$(\text{resp. } \equiv \text{rađ}^{w_2,w_3,\ldots,w_s}(\varphi_s))$$

with sums $\|\boldsymbol{\omega}^i\|$, $\|\boldsymbol{\omega}^{ij}\| = \|\boldsymbol{\omega}^i\boldsymbol{\omega}^j\| = \|\boldsymbol{\omega}^i\| + \|\boldsymbol{\omega}^j\|$, etc. defined as usual (see (2.15)) and with test functions φ_s that may be any (almost everywhere smooth) function of the variables ω_j appearing in the sequences $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \ldots$

Now, since in the case when all sums $\hat{\omega}_j$ are $\neq 0$, one has the elementary arborification rule:

$$S^{\boldsymbol{\omega}} = \sum{}^{1} (\hat{\omega}_i)^{-1} \longmapsto S^{\overset{<}{\boldsymbol{\omega}}} = \sum{}^{2} (\hat{\omega}_i)^{-1} \tag{4.53}$$

with sums $\hat{\omega}_i = \sum \omega_j$ relative to the *full order* of $\boldsymbol{\omega}$ in $\sum^1$ (resp. to the *partial order* of $\overset{<}{\boldsymbol{\omega}}$ in $\sum^2$), it is sufficient to show, by induction on s, that the identity (4.44) holds for each $\overset{<}{\boldsymbol{\omega}}$ that has s vanishing sums $\hat{\omega}_i$ in it, but such that each of the subordinated sequences $\boldsymbol{\omega}$ in (4.45) (whose *full order* is compatible with the *partial order* of $\overset{<}{\boldsymbol{\omega}}$) has s or $s+1$ vanishing sums $\hat{\omega}_i$ in it.

Let us start the induction with $s = 0$. It is enough to consider an arborified sequence $\overset{<}{\boldsymbol{\omega}} = (\omega_1, \ldots, \omega_r)^<$ beginning with a fully ordered sequence $(\omega_1, \ldots, \omega_\alpha)$ whose last component ω_α has *at least two* immediate successors ω_j. If we now assume that the only set of components ω_j of $\overset{<}{\boldsymbol{\omega}}$ whose sum vanishes is the set of all ω_j *strictly* posterior to ω_α, in other words:

$$(4.54) \qquad \hat{\omega}_\alpha - \omega_\alpha = 0 \quad (\hat{\omega}_\alpha \text{ relative to the partial order of } \overset{<}{\boldsymbol{\omega}})$$

it is plain that $\overset{<}{\boldsymbol{\omega}}$ has only non-vanishing sums $\hat{\omega}_i$, but that each of the subordinated, fully ordered sequences $\boldsymbol{\omega}$ in (4.45) has *exactly* one vanishing sum, namely $\hat{\omega}_{\alpha+1}$. However, applying the Leibniz rule (4.46) with $\boldsymbol{\omega}^1 = (\omega_{\alpha+1}, \ldots, \omega_r)$ and the following test function φ_1:

$$(4.55) \qquad \varphi_1 = S^{\overset{<}{\boldsymbol{\omega}}}_{\text{rest}} \overset{\text{def}}{=} \sum_{\boldsymbol{\omega}} S^{\boldsymbol{\omega}}_{\text{rest}} \quad (\boldsymbol{\omega} \text{ compatible with } \overset{<}{\boldsymbol{\omega}})$$

and using the identity:

$$(4.56) \qquad \varphi_1 \equiv \|\boldsymbol{\omega}^1\| \prod_{i=1}^{r} (\hat{\omega}_i) \quad (\hat{\omega}_i \text{ relative to } \overset{<}{\boldsymbol{\omega}})$$

we find that the *non-arborified* identity (4.32) implies the *arborified* identity (4.44).

Similarly, whenever the de-arborification $\overset{<}{\boldsymbol{\omega}} \mapsto \boldsymbol{\omega}$ entails a jump from 1 to 2 (resp. 2 to 3, or $s-1$ to s) of the *backward vanishing number* (*i.e.* the number of vanishing sums $\hat{\omega}_j$) one resorts to the relevant Leibniz rule (4.47) or (4.48) (resp. (4.49), (4.50), (4.51) or (4.52) in the general case) and verifies, once more, that the non-arborified identity (4.32) implies its arborified counterpart (4.44), without any change of outward form.

To deduce from this the symmetrality of $S^\bullet_{\text{ext}}$, all we have to do is consider the special case of an arborescent sequence $\overset{<}{\boldsymbol{\omega}}$ consisting of the *juxtaposition* (not *succession!*) of two fully ordered sequences $\boldsymbol{\omega}^1 = (\omega_\alpha, \ldots)$ and $\boldsymbol{\omega}^2 = (\omega_\beta, \ldots)$. (In other words, each ω_j in $\overset{<}{\boldsymbol{\omega}}$ has exactly one direct antecedent ω_i, except for the two *minimal elements* ω_α and ω_β.)

Lastly, to establish the uniqueness of the mould $\mathcal{S}^\bullet_{\text{ext}}$ under the rationality requirement (2.41), one must also resort to the Leibniz rules (4.52) and observe that, in order to ensure *stabilility under arborification* (or even *mere symmetrality*) the operators $\mathrm{Rad}^{\boldsymbol{\omega}}$ should be *invariant* under any internal *reordering of the components of any of the unbreakable sequences* $\boldsymbol{\omega}^i$ *in the backward factorization* (4.26).

One deals with the moulds $S^\bullet_{\text{ext}}$ and $\$\!\!\$^\bullet$ in exactly the same way, and winds up by proving all identities from (2.35) to (2.39) by induction on the number s of unbreakable factors $\boldsymbol{\omega}^i$ in (4.25), (4.26), (4.27).

PROPOSITION 4.3 (∇_{ω_0}-derivatives of $\$\!\!\$^\bullet$). — *Under the* ∇_{ω_0}-*derivation (see (3.8)) the "tough" moulds behave as follows:*

$$\nabla_{\omega_0} S^\bullet_{\text{ext}} = -S^\bullet_{\text{ext}} \times I^\bullet_{\omega_0} + \$\!\!\$^\bullet_{\omega_0} \times S^\bullet_{\text{ext}} \qquad (\forall \omega_0) \tag{4.57}$$

$$\nabla_{\omega_0} \mathcal{S}^\bullet_{\text{ext}} = +I^\bullet_{\omega_0} \times \mathcal{S}^\bullet_{\text{ext}} - \mathcal{S}^\bullet_{\text{ext}} \times \$\!\!\$^\bullet_{\omega_0} \qquad (\forall \omega_0) \tag{4.58}$$

$$\nabla_{\omega_0} \$\!\!\$^\bullet = \$\!\!\$^\bullet_{\omega_0} \times \$\!\!\$^\bullet - \$\!\!\$^\bullet \times \$\!\!\$^\bullet_{\omega_0} \qquad (\forall \omega_0) \tag{4.59}$$

where (for each $\omega_0 \in \mathbb{C}$) $\$\!\!\$^\bullet_{\omega_0}$ denotes a well-defined alternal mould such that $\$\!\!\$^{\omega_1,\ldots,\omega_r}_{\omega_0} = 0$ as soon as one of the following three conditions is fulfilled

$$\omega_0 = 0 \tag{4.60}$$

$$\omega_1 + \cdots + \omega_r \neq 0 \tag{4.60*}$$

$$\omega_i \neq \omega_0 \qquad (\forall i). \tag{4.60**}$$

Proof. — For each $\omega_0 \in \mathbb{C}$, we may regard equations (4.57) and (4.58) as *defining* two (a priori) distinct moulds $\$\!\!\$^\bullet_{\omega_0}$. But if we apply the mould derivation to the mould identity:

$$S^\bullet_{\text{ext}} \times \mathcal{S}^\bullet_{\text{ext}} = 1^\bullet \tag{4.61}$$

we see that the two aforementioned moulds $\$\!\!\$^\bullet_{\omega_0}$ do in fact coincide. Likewise, applying ∇_{ω_0} to the mould identity:

$$\$^\bullet = \mathcal{S}^\bullet_{\text{ext}} \times \$\!\!\$^\bullet \times S^\bullet_{\text{ext}} \tag{4.62}$$

and bearing in mind that:

$$\nabla_{\omega_0} \$^\bullet = I^\bullet_{\omega_0} \times \$^\bullet - \$^\bullet \times I^\bullet_{\omega_0} \quad \text{(see §3)} \tag{4.63}$$

we see at once that (4.57) and (4.58) imply (4.59). So we may regard $\$\!\!\$^\bullet_{\omega_0}$ as being defined by, say, equation (4.57) and prove, with the help of the properties of $\$\!\!\$^\bullet$ and recursively on $\overrightarrow{\mathrm{van}}\,(\boldsymbol{\omega})$, that $\$\!\!\$^{\omega_1,\ldots,\omega_r}_{\omega_0}$ does indeed vanish when either (4.60) or (4.60*) or (4.60**) is fulfilled. As for the alternality of $\$\!\!\$^\bullet_{\omega_0}$, it also follows from (4.57), but has nothing to do with the particular *nature* of the mould $S^{\bullet}_{\text{ext}}$, only with its *symmetrality*. Indeed, it is an easy matter to show that for any three moulds $A^{\bullet}$, $B^{\bullet}$, $C^{\bullet}$ the relation:

$$\nabla_{\omega_0} A^{\bullet} = A^{\bullet} \times B^{\bullet} + C^{\bullet} \times A^{\bullet} \tag{4.64}$$

determines $C^{\bullet}$ (resp. $B^{\bullet}$) in terms of $A^{\bullet}$ and $B^{\bullet}$ (resp. $A^{\bullet}$ and $C^{\bullet}$) and automatically guarantees the alternality of $C^{\bullet}$ (resp. $B^{\bullet}$) if $A^{\bullet}$ is symmetral and $B^{\bullet}$ (resp. $C^{\bullet}$) is alternal. □

To conclude this section, let us review some of the differences between $\$^{\bullet}$ and $\$\!\!\!\$^{\bullet}$. $\$\!\!\!\$^{\boldsymbol{\omega}}$ vanishes more often than $\$^{\boldsymbol{\omega}}$, since $\$\!\!\!\$^{\boldsymbol{\omega}} = 0$ as soon as $\|\boldsymbol{\omega}\| \neq 0$. $\$\!\!\!\$^{\boldsymbol{\omega}}$ has only singularities of the form:

$$\omega_1 + \cdots \omega_i = 0 = \omega_{i+1} + \cdots + \omega_r$$

whereas $\$^{\omega}$ may have singularities of the form:

$$\omega_i + \omega_{i+1} + \cdots + \omega_j = 0 \quad (i, j \in \{1, \ldots, r\})$$

$\$\!\!\!\$^{\boldsymbol{\omega}}$ has *rational* coefficients, whereas $\$^{\boldsymbol{\omega}}$ has only *integral* coefficients, and those tend to be much larger (see Tables at the end of §11). Above all, $\$\!\!\!\$^{\bullet}$ is an incomparably more complex object than $\$^{\bullet}$, as borne out by the respective definitions of these two moulds; their modes of calculation; and the shape of their ∇_{ω_0}-derivatives. That impression will further deepen in the next section, when studying certain useful *generating functions* (known as *amplifications* and *coamplifications*) attached to $\$^{\bullet}$ and $\$\!\!\!\$^{\bullet}$.

5. Amplification of the moulds $\$^{\bullet}$ and $\$\!\!\!\$^{\bullet}$.

Moulds and their amplification.

When investigating the convergence/divergence properties of mould-comould expansions $\sum M^{\boldsymbol{\omega}} \mathbb{B}_{\boldsymbol{\omega}}$, one is often led to regroup all terms that correspond to sequences $\boldsymbol{\omega}'$ obtained from *one* given sequence $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ interspersed with any number of copies of a given element ω_0 which is usually 0, and in our case will always be 0. The natural way to study such regroupings is to introduce generating functions:

$$\sum M^{0^{(n_0)}, \omega_1, 0^{(n_1)}, \omega_2, 0^{(n_2)}, \ldots, \omega_r, 0^{(n_r)}} b_0^{n_0} b_1^{n_1} \cdots b_r^{n_r} \tag{5.1}$$

with $b_0, b_1, \ldots, b_r$ denoting independent complex variables, and with the symbols $0^{(n_i)}$ standing for sequences $(0, \ldots, 0)$ of n_i consecutive zeros. Now, if the mould $M^\bullet$ happens to be *alternal* (or again if it is *symmetral* but with $M^0 = 0$), the alternality (resp. symmetrality) relation (2.2), when applied to the pair $\boldsymbol{\omega}^1$, $\boldsymbol{\omega}^2$ with:

$$\boldsymbol{\omega}^1 = (0) \; ; \quad \boldsymbol{\omega}^2 = (0^{(n_0)}, \omega_1, 0^{(n_1)}, \omega_2, 0^{(n_2)}, \dots, \omega_r, 0^{(n_r)}) \tag{5.2}$$

yields rightaway:

$$\sum_{0 \le i \le r} (1+n_i) M^{0^{(n_0)}, \omega_1, 0^{(n_1)}, \dots, \omega_i, 0^{(1+n_i)}, \omega_{i+1}, \dots, \omega_r, 0^{(n_r)}} \equiv 0. \tag{5.3}$$

As a consequence, the generating function (5.1) is seen to depend only on the *differences*:

$$a_1 = b_1 - b_0, \; a_2 = b_2 - b_1, \dots, a_r = b_r - b_{r-1}. \tag{5.4}$$

This motivates the introduction of sequences $\boldsymbol{\varpi}$ of the form:

$$\boldsymbol{\varpi} = (\varpi_1, \dots, \varpi_r) = \begin{pmatrix} \omega_1, \dots, \omega_r \\ a_1, \dots, a_r \end{pmatrix} \tag{5.5}$$

and of an $\boldsymbol{\varpi}$-indexed mould:

$$\begin{aligned} M_{\text{amp}}^{\varpi_1, \dots, \varpi_r} &\overset{\text{def}}{=} \sum_{n_i \ge 0} M^{\omega_1, 0^{(n_1)}, \dots, \omega_r, 0^{(n_r)}} (a_1)^{n_1} \cdot \\ &\qquad\qquad \cdot (a_1 + a_2)^{n_2} \cdots (a_1 + \cdots + a_r)^{n_r} \\ &\overset{\text{def}}{=} \sum_{n_i \ge 0} M^{0^{(n_1)}, \omega_1, \dots, 0^{(n_r)}, \omega_r} (-1)^{n_1 + \cdots + n_r} (a_1 + \cdots + a_r)^{n_1} \cdot \\ &\qquad\qquad \cdot (a_2 + \cdots + a_r)^{n_2} \cdots (a_r)^{n_r}. \end{aligned} \tag{5.6}$$

The mould $M^\bullet_{\text{amp}}$ thus defined is known as the *amplification* of $M^\bullet$. It is automatically alternal if $M^\bullet$ is alternal (resp. symmetral if $M^\bullet$ is symmetral *and* $M^0 = 0$).

As it happens, most natural moulds $M^\bullet$ possess *convergent* amplifications $M^\bullet_{\text{amp}}$. More precisely, for a *fixed* sequence $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ and a *variable* sequence $\mathbf{a} = (a_1, \dots, a_r)$, the generating function $M^{\boldsymbol{\varpi}}_{\text{amp}}$ does not only converge for small values of $\mathbf{a}$ but, *as a rule*, the corresponding analytic germ can also be continued *endlessly* (*i.e.* along almost any broken line drawn in $\mathbb{C}^r$ and originating from 0), and thus gives rise to an analytic function of $\mathbf{a} = (a_i)$, uniform or multiform, but defined everywhere on $\mathbb{C}^r$, except on a singular set of complex dimension $< r$.

If we now revert to the topic of mould-comould contractions $\sum M^{\boldsymbol{\omega}} \mathbb{B}_{\boldsymbol{\omega}}$, the fact that the terms $\mathbb{B}_{\boldsymbol{\omega}}$ (being usually concatenations of $r(\boldsymbol{\omega})$ derivations) tend to grow (in norm) roughly like $r(\boldsymbol{\omega})!$, means that the sum $\sum M^{\boldsymbol{\omega}} \mathbb{B}_{\boldsymbol{\omega}}$ usually diverges (in norm). Nevertheless, most of the time, *one* Borel transform $z \to \zeta$ (relative to a suitable variable z) cancels off the noisome factorial $r(\boldsymbol{\omega})!$, and the *endless continuability* of $M^{\boldsymbol{\varpi}}_{\text{amp}}$

translates into the *resurgence* of $\sum M^{\boldsymbol{\omega}}\mathbb{B}_{\boldsymbol{\omega}}$ relative to the z variable. We shall soon enough (see §8 and §9) come across striking examples of this very general phenomenon, but first we must investigate the *amplifications* of our key moulds $\$^\bullet$ and $\$\!\!\$^\bullet$.

Amplification of the $\$^\bullet$ mould.

In (2.14) and (3.8) we introduced *derivations* ∇ and ∇_{ω_0} operating on $\boldsymbol{\omega}$-indexed moulds. We obtain analogous derivations ∇ and ∇_{ϖ_0} operating on $\boldsymbol{\varpi}$-indexed moulds, by replacing the sum $\|\boldsymbol{\omega}\|$ in (2.14) by $\|\boldsymbol{\varpi}\| = \|\boldsymbol{\omega}\| + \|\mathbf{a}\|$, and each term $\omega_i M^{\cdots,\omega_i,\cdots}$ on the right-hand side of (3.8) by the term $(\omega_i + a_i)M^{\cdots,\varpi_i,\cdots}$.

PROPOSITION 5.1 (Rationality of $\$^\bullet_{\text{amp}}$). — *The amplification* $\$^\bullet_{\text{amp}}$ *can be calculated inductively by means of the relations:*

$$\nabla_{\varpi_0}\$^\bullet_{\text{amp}} = I^\bullet_{\varpi_0} \times \$^\bullet_{\text{amp}} - \$^\bullet_{\text{amp}} \times I^\bullet_{\varpi_0} \quad (\forall \varpi_0) \tag{5.7}$$

$$\nabla\$^\bullet_{\text{amp}} = I^\bullet \times \$^\bullet_{\text{amp}} - \$^\bullet_{\text{amp}} \times I^\bullet \tag{5.8}$$

which have the same outward form as the induction (3.16), (3.16*) *for* $\$^\bullet$, *but with an induction-starting condition:*

$$\$^{\varpi_1}_{\text{amp}} \equiv a_1(\omega_1 + a_1)^{-1} \quad \left(\forall \varpi_1;\ \varpi_1 = \begin{pmatrix}\omega_1\\ a_1\end{pmatrix}\right) \tag{5.9}$$

which is valid for both $\omega_1 \neq 0$ *and* $\omega_1 = 0$, *unlike the corresponding condition for* $\$^\bullet$, *for which we had a dichotomy:*

$$\$^{\omega_1} \equiv 0 \textit{ if } \omega_1 \neq 0\ ;\quad \$^{\omega_1} = 1 \textit{ if } \omega_1 = 0. \tag{5.9bis}$$

As a consequence, each $\$^{\boldsymbol{\varpi}}_{\text{amp}}$ *is a rational function of the variables* ω_i *and* a_i, *with singular loci of the form:*

$$(\omega_i + \omega_{i+1} + \cdots + \omega_j) + (a_i + a_{i+1} + \cdots + a_j) = 0 \quad (1 \leq i \leq j \leq r) \tag{5.10}$$

but the analytical expression of $\$^{\boldsymbol{\varpi}}_{\text{amp}}$, *unlike that of* $\$^{\boldsymbol{\omega}}$, *doesn't depend on the actual degeneracy pattern of* $\boldsymbol{\omega}$. *It is given explicitely by:*

$$\$^\bullet_{\text{amp}} = I^\bullet + (\mathcal{S}^\bullet_{\text{amp}}) \times (\nabla^* \mathcal{S}^\bullet_{\text{amp}}) \tag{5.11}$$

with:

$$\mathcal{S}^{\varpi_1,\ldots,\varpi_r}_{\text{amp}} \overset{\text{def}}{=} \mathcal{S}^{\omega_1+a_1,\ldots,\omega_r+a_r} \qquad \textit{(see (2.20))} \tag{5.11*}$$

$$S^{\varpi_1,\ldots,\varpi_r}_{\text{amp}} \overset{\text{def}}{=} S^{\omega_1+a_1,\ldots,\omega_r+a_r} \qquad \textit{(see (2.19))} \tag{5.11**}$$

$$\nabla^* S^{\varpi_1,\ldots,\varpi_r}_{\text{amp}} \overset{\text{def}}{=} (\omega_1 + \cdots + \omega_r) S^{\varpi_1,\ldots,\varpi_r}_{\text{amp}}. \tag{5.11***}$$

Thus, putting $\eta_i \overset{\text{def}}{=} \omega_i + a_i$; $\eta_{ij} \overset{\text{def}}{=} \omega_i + \omega_j + a_i + a_j$, etc., we get:

$$\begin{aligned} \$^{\varpi_1,\varpi_2}_{\text{amp}} &\equiv +\omega_1(\eta_1\eta_{12})^{-1} - \omega_2(\eta_2\eta_{12})^{-1} \\ \$^{\varpi_1,\varpi_2,\varpi_3}_{\text{amp}} &\equiv -\omega_1(\eta_1\eta_{12}\eta_{123})^{-1} + \omega_2(\eta_2\eta_{12}\eta_{123})^{-1} \\ &\quad + \omega_2(\eta_2\eta_{23}\eta_{123})^{-1} - \omega_3(\eta_3\eta_{23}\eta_{123})^{-1} \end{aligned}$$

etc.

But due to cancellations within (5.11), for any fixed sequence $\boldsymbol{\omega}$, no matter how degenerate, $\$^{\boldsymbol{\varpi}}_{\text{amp}}$ is always a regular function of the sequence $\mathbf{a}$ at the origin $\mathbf{a} = 0$ of $\mathbb{C}^r$, whereas the factors $S^{\boldsymbol{\varpi}}_{\text{amp}}$ and $S^{\boldsymbol{\varpi}}_{\text{amp}}$ on their own, may clearly possess poles at $\mathbf{a} = 0$. In other words, for any fixed $\boldsymbol{\omega}$, $\$^{\boldsymbol{\varpi}}_{\text{amp}}$ has only singular loci of the form (5.10) *and* with $\omega_i + \cdots \omega_j \neq 0$.

Proof of Proposition 5.1. — The induction (3.16*) for $\$^{\bullet}$ yields:

$$\omega_i \, \$^{\omega_1,0^{(n_1)},\ldots,\omega_r,0^{(n_r)}} = -\$^{\ldots,0^{(n_{i-1}-1)},\omega_i,\ldots} + \$^{\ldots,\omega_i,0^{(n_i-1)},\ldots} \tag{5.12}$$

if $1 \leq n_{i-1}$, $1 \leq n_i$ and $0 < i < r$. These relations (along with their analogues in the fringe cases when $i = 1$ or r or when some of the components n_i vanish) translate precisely into the rules (5.7) for the ∇_{ω_0}-derivatives of $\$^{\bullet}_{\text{amp}}$. Adding these identities for $\varpi_0 = \varpi_1, \ldots, \varpi_0 = \varpi_r$, we find the rule (5.8) for the ∇-derivatives. Written out in full, the latter reads:

$$(\|\boldsymbol{\omega}\| + \|\mathbf{a}\|) \, \$^{\varpi_1,\ldots,\varpi_r}_{\text{amp}} = \$^{\varpi_2,\ldots,\varpi_r}_{\text{amp}} - \$^{\varpi_1,\ldots,\varpi_{r-1}}_{\text{amp}} \tag{5.13}$$

and since we may always divide by the function $\|\boldsymbol{\omega}\| + \|\mathbf{a}\|$, even when $\|\boldsymbol{\omega}\| = 0$, (5.13) is an effective recursion for calculating $\$^{\bullet}_{\text{amp}}$, and leads rightaway to (5.11).

Moreover, for any fixed $\boldsymbol{\omega}$, whatever its degeneracy pattern, we have:

$$\lim_{a \to 0} \$^{\boldsymbol{\varpi}}_{\text{amp}} = \$^{\boldsymbol{\omega}} \tag{5.14}$$

and this is even the simplest *direct* means for calculating a *given* $\$^{\boldsymbol{\omega}}$.

Amplification of the $\$^{\bullet}$ mould.

PROPOSITION 5.2 (Endless analyticity of $\$^{\boldsymbol{\varpi}}_{\text{amp}}$). — *For any fixed sequence* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$, *the amplification* $\$^{\boldsymbol{\varpi}}_{\text{amp}}$ *is an analytic function of* $\mathbf{a} = (a_1, \ldots, a_r)$, *defined almost everywhere on* $\mathbb{C}^r$, *and with ramifications if* $r(\boldsymbol{\varpi}) \geq 3$.

If $\text{van}(\boldsymbol{\omega}) = 0$, *i.e.* if $\|\boldsymbol{\omega}\| \neq 0$ (see after (2.18)) there is of course nothing to prove, since in that case both $\$^{\boldsymbol{\omega}}$ and its amplification $\$^{\boldsymbol{\varpi}}_{\text{amp}}$ are

$\equiv 0$. To establish Proposition 5.2 in the non-trivial case (when $\mathrm{van}(\boldsymbol{\omega}) \geq 1$), we need to know the *power series expansions* of $\$^{\varpi}_{\mathrm{amp}}$ at $\mathbf{a} = 0$, as well as *integral representations* valid in the large. We shall first calculate the power series expansions in the case when $\mathrm{van}(\boldsymbol{\omega}) = 1$, *i.e.* when $\|\boldsymbol{\omega}\| = 0$ but all partial sums $\check{\omega}_i$ and $\hat{\omega}_i$ (other than $\check{\omega}_r$ and $\hat{\omega}_1$, which by definition coincide with $\|\boldsymbol{\omega}\|$) are $\neq 0$.

PROPOSITION 5.3 (Power series expansions of $\$^{\varpi}_{\mathrm{amp}}$). — *If* $\mathrm{van}(\boldsymbol{\omega}) = 1$, *the power series expansion of* $\$^{\varpi}_{\mathrm{amp}}$ *(as a function of* $\mathbf{a}$*) admits no compact expression in terms of the original variables* $a_1, a_2, \ldots, a_r$ *but it can be easily calculated from either of the following expansions:*

$$(5.15)\quad \$^{\varpi}_{\mathrm{amp}} = \|\mathbf{a}\|^{-(r-1)} \sum_{1\leq i\leq r-1} \sum_{\left\{\begin{smallmatrix} 0\leq p_i \\ 1\leq n_i \end{smallmatrix}\right.} (x_1)^{n_1-1}(x_2)^{n_2-1}\cdots \\ \cdots(x_{r-1})^{n_{r-1}-1}(X_i)^{p_i}P_{i,1}P_{i,2}\cdots P_{i,r}$$

$$(5.16)\quad \$^{\varpi}_{\mathrm{amp}} = (-1)^{r-1}\|\mathbf{a}\|^{-(r-1)} \sum_{2\leq i\leq r} \sum_{\left\{\begin{smallmatrix} 0\leq p_i \\ 1\leq n_i \end{smallmatrix}\right.} (Y_i)^{p_i}(y_2)^{n_2-1}\cdots \\ \cdots(y_3)^{n_3-1}\cdots(y_r)^{n_r-1}Q_{i,1}Q_{i,2}\cdots Q_{i,r}$$

which involve the (non-independent) variables:

$$\begin{array}{llll}
x_i & \overset{\mathrm{def}}{=} \check{a}_i\|\mathbf{a}\|^{-1} & = (a_1+\cdots+a_i)(a_1+\cdots+a_r)^{-1} & (1\leq i\leq r-1) \\
X_i & \overset{\mathrm{def}}{=} \exp(\|\mathbf{a}\|/\check{\omega}_i) & = \exp((a_1+\cdots+a_r)(\omega_1+\cdots+\omega_i)^{-1}) & (1\leq i\leq r-1) \\
y_i & \overset{\mathrm{def}}{=} \hat{a}_i\|\mathbf{a}\|^{-1} & = (a_i+\cdots+a_r)(a_1+\cdots+a_r)^{-1} & (2\leq i\leq r) \\
Y_i & \overset{\mathrm{def}}{=} \exp(\|\mathbf{a}\|/\hat{\omega}_i) & = \exp((a_1+\cdots+a_r)(\omega_i+\cdots+\omega_r)^{-1}) & (2\leq i\leq r)
\end{array}$$

and the following coefficients:

$$(5.17)\quad P_{i,j} = \frac{\Gamma(1+\check{n}_{j-1}-p_i(\check{\omega}_j/\check{\omega}_i))}{\Gamma(1+\check{n}_{j-1}-p_i(\check{\omega}_{j-1}/\check{\omega}_i))} \qquad \left\{\begin{array}{l} 1\leq i\leq r-1 \\ 1\leq j\leq r \\ i\neq j \end{array}\right.$$

$$(5.17^*)\quad P_{i,i} = \frac{(-1)^{p_i-\check{n}_{i-1}}}{p_i\Gamma(1+\check{n}_{i-1}-p_i(\check{\omega}_{i-1}/\check{\omega}_i))\Gamma(p_i-\check{n}_{i-1})} \qquad (1\leq i\leq r-1)$$

$$(5.18)\quad Q_{i,j} = \frac{\Gamma(1+\hat{n}_{j+1}-p_i(\hat{\omega}_j/\hat{\omega}_i))}{\Gamma(1+\hat{n}_{j+1}-p_i(\hat{\omega}_{j+1}/\hat{\omega}_i))} \qquad \left\{\begin{array}{l} 2\leq i\leq r \\ 1\leq j\leq r \\ i\neq j \end{array}\right.$$

$$(5.18^*)\quad Q_{i,i} = \frac{(-1)^{\hat{n}_{i+1}-p_i}}{p_i\Gamma(1+\hat{n}_{i+1}-p_i(\hat{\omega}_{j+1}/\hat{\omega}_i))\Gamma(p_i-\hat{n}_{i+1})} \qquad (2\leq i\leq r)$$

with the usual notations:

$$(5.19)\qquad \check{\omega}_i = \omega_1+\cdots+\omega_i\ ;\quad \check{n}_i = n_1+\cdots+n_i$$

$$(5.20)\qquad \hat{\omega}_i = \omega_i+\cdots+\omega_r\ ;\quad \hat{n}_i = n_i+\cdots+n_r$$

supplemented by the natural convention:

$$\check{\omega}_0 = 0 \ ; \ \ \check{n}_0 = 0 \ ; \ \ \hat{\omega}_{r+1} = 0 \ ; \ \ \hat{n}_{r+1} = 0. \tag{5.21}$$

Remark 1. — There exist similar expansions for the case $\mathrm{van}(\boldsymbol{\omega}) \geq 2$, but they involve a larger number of "exponential" variables, namely:

$$X_{i,j} \overset{\text{def}}{=} \exp(\check{a}_i/\check{\omega}_j) \quad \text{with } i \in \check{I} \text{ but } j \notin \check{I} \tag{5.22}$$

$$Y_{i,j} \overset{\text{def}}{=} \exp(\hat{a}_i/\hat{\omega}_j) \quad \text{with } i \in \hat{I} \text{ but } j \notin \hat{I} \tag{5.23}$$

where $\check{I}$ (resp. $\hat{I}$) denotes the set of all indices i such that $\check{\omega}_i = 0$ (resp. $\hat{\omega}_j = 0$).

Remark 2. — Instead of extending the sums (5.15) and (5.16) to all $p_i \geq 0$, we may restrict them to the intervals:

$$1 + \check{n}_{i-1} \leq p_i \leq \check{n}_i \ (\text{if } 1 < i) \text{ and } 0 \leq p_1 \leq \check{n}_1 \ (\text{if } i = 1) \tag{5.24}$$

$$1 + \hat{n}_{i+1} \leq p_i \leq \hat{n}_i \ (\text{if } i < r) \text{ and } 0 \leq p_r \leq \hat{n}_r \ (\text{if } i = r) \tag{5.25}$$

because, for other values of p_i, the coefficients $P_{i,i}P_{i,i+1}$ and $Q_{i,i-1}Q_{i,i}$ vanish. But beware that, in spite of the convention (5.21), we *must* include in (5.15) (resp. (5.16)) the term corresponding to $p_1 = 0$ (resp. $p_r = 0$).

Remark 3. — Applying the convention (5.21) to (5.17) and (5.18), we find for $P_{i,1}$ and $Q_{i,r}$ the simplified expression:

$$P_{i,1} = \Gamma(1 - p_i(\check{\omega}_1/\check{\omega}_i)) \ ; \ \ Q_{i,r} = \Gamma(1 - p_i(\hat{\omega}_r/\hat{\omega}_i)). \tag{5.26}$$

On the other hand, the values for $P_{i,i}$ and $Q_{i,i}$ as given in (5.17*), (5.18*) depart from the rule (5.17), (5.18) but remain close to it. Indeed, if we take $\check{\omega}_i/\check{\omega}_i$ and $\hat{\omega}_i/\hat{\omega}_i$ equal to $1 + \varepsilon$ instead of 1, and let ε go to 0, we find:

$$(P_{i,i} \text{ as given by } (5.17^*)) = \lim_{\varepsilon \to 0}(\varepsilon P_{i,i} \text{ as given by } (5.17)) \tag{5.27}$$

$$(Q_{i,i} \text{ as given by } (5.18^*)) = \lim_{\varepsilon \to 0}(\varepsilon Q_{i,i} \text{ as given by } (5.18)). \tag{5.28}$$

Remark 4. — As soon as we translate the expansions (5.15) or (5.16) into power series of the original variables $a_1, \ldots, a_r$, the negative powers $\|a\|^{-r}$ disappear, and so do the apparent poles of the form:

$$((m_j/\check{\omega}_j) - (m_i/\check{\omega}_i))^{-1} \text{ or } ((m_j/\hat{\omega}_j) - (m_i/\hat{\omega}_i))^{-1} \tag{5.29}$$

which are contributed by the gamma functions sitting in the numerators of the coefficients $P_{i,j}$ or $Q_{i,j}$. What we are left with is an entire power series in the variables a_i and $(\breve{\omega}_i)^{-1}$ (resp. a_i and $(\hat{\omega}_i)^{-1}$), with the obvious homogeneousness:

$$\$_{\text{amp}}^{\varpi_1^*,\ldots,\varpi_r^*} = t^{-(r-1)}\, \$_{\text{amp}}^{\varpi_1,\ldots,\varpi_r} \text{ if } \omega_i^* = t\omega_i,\ a_i^* = ta_i. \tag{5.30}$$

Thus, if we take $r = 3$ and calculate the first terms in (5.16), we find:

$$\$_{\text{amp}}^{\varpi_1,\varpi_2,\varpi_3} = \sum_{0\leq m_2} \sum_{0\leq m_3} (\hat{a}_1)^{-2-m_2-m_3}(\hat{a}_2)^{m_2}(\hat{a}_3)^{m_3} A_{m_2,m_3} \tag{5.31}$$

with $\omega_1 + \omega_2 + \omega_3 = 0$; $\omega_i \neq 0$; $\hat{a}_1 = a_1 + a_2 + a_3$, $\hat{a}_2 = a_2 + a_3$, $\hat{a}_3 = a_3$, and:

$$A_{0,0} = (\hat{\omega}_2 - 2\hat{\omega}_3)^{-1}[-\hat{\omega}_2 e^{2\hat{a}_1/\hat{\omega}_2} + 2\hat{\omega}_3 e^{\hat{a}_1/\hat{\omega}_3} + \hat{\omega}_2 - 2\hat{\omega}_3]$$

$$A_{1,0} = \begin{cases} (\hat{\omega}_2 - 2\hat{\omega}_3)^{-1}(\hat{\omega}_2 - 3\hat{\omega}_3)^{-1}[+2\hat{\omega}_2(\hat{\omega}_2 - 2\hat{\omega}_3)e^{3\hat{a}_1/\hat{\omega}_2} \ldots \\ -3\hat{\omega}_2(\hat{\omega}_2 - 3\hat{\omega}_3)e^{2\hat{a}_1/\hat{\omega}_2} - 6(\hat{\omega}_3)^2 e^{\hat{a}_1/\hat{\omega}_3} + (\hat{\omega}_2 - 2\hat{\omega}_3)(\hat{\omega}_2 - 3\hat{\omega}_3)] \end{cases}$$

$$A_{0,1} = \begin{cases} (2\hat{\omega}_2 - 3\hat{\omega}_3)^{-1}(\hat{\omega}_2 - 3\hat{\omega}_3)^{-1}[-2(\hat{\omega}_2)^2 e^{3\hat{a}_1/\hat{\omega}_2} \ldots \\ -3\hat{\omega}_3(\hat{\omega}_2 - 3\hat{\omega}_3)e^{2\hat{a}_1/\hat{\omega}_3} + 6\hat{\omega}_3(2\hat{\omega}_2 - 3\hat{\omega}_3)e^{\hat{a}_1/\hat{\omega}_3} \ldots \\ +(2\hat{\omega}_2 - 3\hat{\omega}_3)(\hat{\omega}_2 - 3\hat{\omega}_3)] \end{cases}$$

etc. But after expanding the exponentials and doing away with illusory poles, we find the following expressions, whose *alternality* is easy to check:

$$\$_{\text{amp}}^{\varpi_1,\varpi_2,\varpi_3} = \sum_{0\leq i} \sum_{0\leq j} (\hat{\omega}_2)^{-1-i}(\hat{\omega}_3)^{-1-j} B_{i,j} \tag{5.32}$$

with:

$$\begin{aligned} B_{0,0} &= 1 \\ B_{1,0} &= (1/3)(2a_1 - a_2 - a_3) \\ B_{0,1} &= (1/3)(a_1 + a_2 - 2a_3) \\ B_{2,0} &= (1/12)(4a_1^2 + a_2^2 + a_3^2 - 7a_1a_2 - 7a_1a_3 + 2a_2a_3) \\ B_{1,1} &= (1/12)(2a_1^2 - a_2^2 + 2a_3^2 + a_1a_2 - 8a_1a_3 + a_2a_3) \\ B_{0,2} &= (1/12)(a_1^2 + a_2^2 + 4a_3^2 + 2a_1a_2 - 7a_1a_3 - 7a_2a_3) \end{aligned}$$

etc.

Remark 5. — Only for $r \leq 2$ does the amplification $\$_{\text{amp}}^{\varpi}$ assume the form of a simple function. For $r = 1$, it is utterly trivial, since $\$_{\text{amp}}^{\varpi_1} \equiv 1$ (resp. $\equiv 0$) if $\omega_1 = 0$ (resp. $\neq 0$). For $r = 2$, the expansions (5.15) and (5.16) lead respectively to the following, clearly equivalent expressions:

$$\begin{aligned} \$_{\text{amp}}^{\varpi_1,\varpi_2} &= (1 - e^{a_{12}/\omega_1})(a_1 e^{a_{12}/\omega_1} + a_2)^{-1} \\ &= (e^{a_{12}/\omega_2} - 1)(a_1 + a_2 e^{a_{12}/\omega_2})^{-1} \end{aligned} \tag{5.33}$$

with $a_{12} = a_1 + a_2$, $\omega_1 \neq 0$, $\omega_2 \neq 0$ but $\omega_1 + \omega_2 = 0$.

Remark 6. — For $r \geq 3$, the amplification $\$^{\boldsymbol{\varpi}}_{\text{amp}}$ is of a far more complex nature, with features reminiscent not only of the hypergeometric functions (as obvious from the shape of coefficients P_{ij} and Q_{ij}) but also of the hyperlogarithms. This latter, more recondite kinship shows in the following fact. If we regard the variables x_i and X_i (resp. y_i and Y_i) as being independent, and denote by $\varphi_i(X_i)$ (resp. $\psi_i(Y_i)$) the power series inside the sum (5.15) (resp. (5.16)) that involves the variable X_i (resp. Y_i) as well as the corresponding function, the *alternality relations* for the mould $\$^{\bullet}_{\text{amp}}$ translate into *functional equations* of "logarithmic type", which relate $\varphi_i(X'X'')$ to the various $\varphi_j(X')$ and $\varphi_k(X'')$; or $\psi_i(Y'Y'')$ to the $\psi_j(Y')$ and $\psi_k(Y'')$; or again $\varphi_i(X)$ to the $\psi_j(X^{-1})$.

Short proof of Proposition 5.2 and 5.3.

Due to alternality we have:

$$\$^{\varpi_1,\varpi_2,\ldots,\varpi_r}_{\text{amp}} = (-1)^{r-1}\,\$^{\varpi_r,\ldots,\varpi_2,\varpi_1}_{\text{amp}} \tag{5.34}$$

so that (5.15) is clearly equivalent to (5.16). We shall establish the latter formula. To that end, it is convenient to start from this definition of the amplification:

$$\$^{\boldsymbol{\varpi}}_{\text{amp}} = \sum_{n} \$^{\boldsymbol{\omega}^{\mathbf{n}}}(-\hat{a}_1)^{n_1-1}(-\hat{a}_2)^{n_2-1}\cdots(-\hat{a}_r)^{n_r-1} \tag{5.35}$$

with $\boldsymbol{\varpi} = (\varpi_1, \ldots, \varpi_r)$, $\mathbf{n} = (n_1, \ldots, n_r)$; $n_i \geq 1$; and:

$$\boldsymbol{\omega}^{\mathbf{n}} \overset{\text{def}}{=} (0^{(n_1-1)}, \omega_1, 0^{(n_2-1)}, \omega_2, \ldots, 0^{(n_r-1)}, \omega_r). \tag{5.36}$$

Then we calculate $\$^{\boldsymbol{\omega}^{\mathbf{n}}}$ by the standard rule:

$$\$^{\boldsymbol{\omega}^{\mathbf{n}}} = \mathrm{Ral}^{\boldsymbol{\omega}^{\mathbf{n}}} \cdot T^{\boldsymbol{\omega}^{\mathbf{n}}}_{\text{rest}} \quad \text{(see (4.33))}. \tag{5.37}$$

To obtain the "restriction" $T^{\boldsymbol{\omega}^{\mathbf{n}}}_{\text{rest}}$, we must replace each zero in the sequence $\boldsymbol{\omega}^{\mathbf{n}}$ by an auxiliary variable η_i. In other words, we must write $\boldsymbol{\omega}^{\mathbf{n}}$ in the form:

$$\boldsymbol{\omega}^{\mathbf{n}} = (\eta_1, \eta_2, \ldots, \eta_{\check{n}_1}, \ldots, \eta_{\check{n}_2}, \ldots, \eta_{\check{n}_r}) \tag{5.38}$$

with $\check{n}_i = n_1 + \cdots + n_i$ and $\eta_j = 0$ except if $j = \check{n}_i$, in which case $\eta_j = \omega_i$. Applying (4.8) we get the factorization:

$$T^{\boldsymbol{\omega}^{\mathbf{n}}}_{\text{rest}} = T_2 T_3 \cdots T_r \tag{5.39}$$

with $T_i = \prod_{1+\check{n}_{i-1} \leq j \leq \check{n}_i} (\hat{\eta}_j)^{-1}$ and $\hat{\eta}_j = \eta_j + \eta_{j+1} + \cdots + \eta_{\check{n}_r}$. As for the differential operator $\mathrm{Ral}^{\boldsymbol{\omega}^{\mathbf{n}}}$, the rule (4.30) shows it to be of the form:

$$\mathrm{Ral}^{\boldsymbol{\omega}^{\mathbf{n}}} = \mathrm{ral}^{w_1, w_2, \ldots, w_{n_1}} \tag{5.40}$$

with $w_i = \binom{u_i}{v_i}$ and:

(5.41) $u_i{=}\partial_{\eta_i}$ if $1 \leq i \leq n_1$ and $u_{n_1} = \partial_{\eta_{n_1}}{+}\partial_{\eta_{1+n_1}}{+}\cdots\partial_{\eta_{n_1+n_2+\cdots n_r}}$

(5.42) $v_i{=}1$ if $1 \leq i \leq n_1$ and $v_{n_1} = 1{+}n_2{+}n_3{+}\cdots n_r = 1{+}\hat{n}_2$.

However, there being no factor T_1 in (5.39) nor, by the same token, any variable η_i of index $i < n_1$, each one of the operators u_i (for $1 \leq i < n_1$) annihilates $T^{\boldsymbol{\omega}^{\mathbf{n}}}_{\text{rest}}$, leaving only u_{n_1} to act non trivially. So in $\mathrm{ral}^{w_1,\ldots,w_{n_1}}$ we may ignore all terms but u_{n_1}. But from the induction (4.13) we easily infer that:

$$(5.43)\quad \mathrm{ral}^{w_1,w_2,\ldots,w_{n_1}} = \frac{(u_{n_1})^{n_1-1}}{(n_1{-}1)!}\;\frac{1}{(v_2{+}v_3{+}\cdots v_{n_1})(v_3{+}\cdots v_{n_1})\cdots(v_{n_1})}$$

modulo the terms $u_1, u_2, \ldots, u_{n_1-1}$. In view of (5.42) this reads:

$$(5.44)\qquad \mathrm{ral}^{w_1,\ldots,w_{n_1}} = \frac{\hat{n}_2!}{(\hat{n}_1-1)!}\frac{(u_{n_1})^{n_1-1}}{(n_1-1)!}\ (\text{modulo } u_1, u_2, \ldots, u_{n_1-1}).$$

Using (5.44) and applying $\mathrm{Rad}^{\boldsymbol{\omega}^{\mathbf{n}}}$ to the various factors T_i of $T^{\boldsymbol{\omega}^{\mathbf{n}}}_{\text{rest}}$, we find:

$$(5.45)\quad \$^{\boldsymbol{\omega}^{\mathbf{n}}} = \frac{(-1)^{n_1-1}}{(\hat{n}_1-1)!}(\hat{n}_2!) \sum_{\substack{\left\{\begin{smallmatrix} s_i \geq 0 \\ s_2+\cdots s_r = n_1-1\end{smallmatrix}\right. \\ s_2+\cdots s_r = n_1-1}} \prod_{(2\leq i\leq r)} \left\{\frac{(u_{n_1})^{s_i}}{(s_i!)}\cdot T_i\right\}.$$

After differentiating *and* annihilating (in *this* order) all auxiliary variables η_j of index j *not* of the form $j = \check{n}_i$, (5.45) becomes:

$$(5.46)\quad \$^{\boldsymbol{\omega}^{\mathbf{n}}} = \frac{(-1)^{n_1-1}}{(\hat{n}_1-1)!}(\hat{n}_2!) \sum_{\left\{\begin{smallmatrix} s_i \geq 0 \\ s_2+\cdots s_r = n_1-1\end{smallmatrix}\right.} \theta^{s_i}_{1+\hat{n}_{i+1},\hat{n}_i}\cdot(\hat{\omega}_i)^{-(s_i+n_i)}$$

with integers $\theta^{\bullet}_{\bullet\bullet}$ defined by

$$(5.47)\qquad \theta^{\sigma}_{\tau_1,\tau_2} \overset{\text{def}}{=} \sum(\tau_1)^{\sigma_0}(1+\tau_1)^{\sigma_1}(2+\tau_1)^{\sigma_2}(3+\tau_1)^{\sigma_3}\cdots(\tau_2)^{\sigma_{\tau_2-\tau_1}}$$

for $0 \leq \sigma$, $0 \leq \tau_1 \leq \tau_2$, and a sum extending to all integers $\sigma_i \geq 0$ such that $\sigma_0+\sigma_1+\cdots\sigma_{\tau_2-\tau_1} = \sigma$. However, we easily find (5.47) to be equivalent to:

$$(5.48)\qquad \theta^{\sigma}_{\tau_1,\tau_2} = \sum_{\tau_1\leq\tau\leq\tau_2} \frac{\tau^{\sigma+\tau_2-\tau_1}}{(\tau-\tau_1)!(\tau_2-\tau)!}(-1)^{\tau_2-\tau}.$$

Plugging (5.48) into (5.46) and (5.46) into (5.35), we may now proceed to sum inside (5.35), *first* over all values of $s_2, s_3, \ldots, s_r$ whose sum is $n_1 - 1$, by using the identity:

$$(5.49)\qquad \sum_{\left\{\begin{smallmatrix} s_i \geq 0 \\ s_2+s_3+\cdots s_r = n_1-1\end{smallmatrix}\right.} \alpha_2^{s_2}\alpha_3^{s_3}\cdots\alpha_r^{s_r} = \sum_{2\leq i\leq r} \alpha_i^{r-2+n_1} \prod_{\left\{\begin{smallmatrix} 2\leq j\leq r \\ j\neq i\end{smallmatrix}\right.} (\alpha_i-\alpha_j)^{-1}$$

and *then* over all multiintegers $\mathbf{n} = (n_2, n_3, \ldots, n_r)$ of components $n_i \geq 1$; with functions $R_*^{(\mathbf{n})}$ and $R_i^{(\mathbf{n})}$ of the form:

$$R_*^{(\mathbf{n})} = \|\mathbf{a}\|^{-\hat{n}_2}(\hat{n}_2!) \prod_{2 \leq j \leq r} (\hat{a}_j/\hat{\omega}_j)^{n_j-1} \tag{5.51}$$

$$R_i^{(\mathbf{n})} = e^{\|\mathbf{a}\|/\hat{\omega}_i} \cdot (\hat{\omega}_i)^{2-r+\hat{n}_2} \prod_{\left\{\substack{2 \leq j \leq r \\ j \neq i}\right.} ((1/\hat{\omega}_i) - (1/\hat{\omega}_j))^{-1} \tag{5.52}$$

and lastly with operators $K_j^{(\mathbf{n})}$ of the form:

$$K_j^{(\mathbf{n})} = \sum_{1+\hat{n}_{j+1} \leq m \leq \hat{n}_j} \frac{(-1)^{m-1-\hat{n}_{j+1}}}{(m-1-\hat{n}_{j+1})!(m-\hat{n}_j)!} \operatorname{dil}_j^{(m)} \tag{5.53}$$

where $\operatorname{dil}_j^{(m)}$ denotes a dilatation operator acting on the sole variable $\hat{\omega}_j$:

$$\operatorname{dil}_j^{(m)} \varphi(\hat{\omega}_1, \ldots, \hat{\omega}_j, \ldots, \hat{\omega}_r) \stackrel{\text{def}}{=} \varphi(\hat{\omega}_1, \ldots, \hat{\omega}_j/m, \ldots, \hat{\omega}_r). \tag{5.54}$$

We further transform the sum (5.50) by applying, to each given summand $R_i^{(\mathbf{n})}$, all operators $K_j^{(\mathbf{n})}$ of index $j \neq i$, and by using the easily proven identity:

$$\left\{\begin{aligned} & K_j^{(\mathbf{n})} \cdot \{(\hat{\omega}_j)^{1-n_j}((1/\hat{\omega}_i) - (1/\hat{\omega}_j))^{-1}\} \\ & \quad \equiv -(\hat{\omega}_i\hat{\omega}_j)^{1-n_j} \prod_{(1+\hat{n}_{i+1} \leq m \leq \hat{n}_i)} ((1/\hat{\omega}_i) - (m/\hat{\omega}_j))^{-1} \\ & \quad \equiv -(\hat{\omega}_j)(\hat{\omega}_i)^{1-n_j}\Gamma(1 + \hat{n}_{j+1} - (\hat{\omega}_j/\hat{\omega}_j))/\Gamma(1 + \hat{n}_j - (\hat{\omega}_j/\hat{\omega}_i)). \end{aligned}\right. \tag{5.55}$$

Then, as a last step, we apply the one still unused operator, namely $K_i^{(\mathbf{n})}$. Eventually, after switching to the variables y_i and Y_i of Proposition 5.3, we find that in (5.50) the coefficient in front of:

$$(-1)^{r-1}\|a\|^{-(r-1)}(Y_i)^{p_i}(y_2)^{n_2-1}(y_3)^{n_3-1}\cdots(y_r)^{n_r-1}$$

is none other than:

$$\frac{(-1)^{\hat{n}_{i+1}-p_i}}{p_i\Gamma(1+\hat{n}_{i+1}-p_i)\Gamma(p_i-\hat{n}_i)} \prod_{\left\{\substack{2 \leq j \leq r \\ j \neq i}\right.} \frac{\Gamma(1+\hat{n}_{j+1}-p_i(\hat{\omega}_j/\hat{\omega}_i))}{\Gamma(1+\hat{n}_j-p_i(\hat{\omega}_j/\hat{\omega}_i))}$$

which tallies exactly with the coefficient $Q_{i1}Q_{i2}\cdots Q_{ir}$ of (5.16).

This completes the proof of Proposition 5.3.

Now, for any fixed sequence $\boldsymbol{\omega}$, the expansion (5.15) is clearly a power series with positive (but finite) radius of convergence in the variables x_j and X_i, and so too in the variables a_j. Its sum is therefore an analytic germ at the origin 0 of $\mathbb{C}^r$ and, in order to prove Proposition 5.2, we have to show that this germ admits an endless analytic continuation, with a singular set of zero measure.

To do this, it is more convenient to reason on auxiliary power series Φ of the form:

(5.56)
$$\Phi\begin{pmatrix}\alpha_1,\dots,\alpha_{r-1},\alpha_r\\ x_1,\dots,x_{r-1},x_*\end{pmatrix} \overset{\text{def}}{=} \sum_{\begin{cases}1\le n_i\\ 0\le p\end{cases}} x_1^{n_1-1}\cdots x_{r-1}^{n_{r-1}-1}x_*^p R_1\cdots R_{r-1}R_*$$

with coefficients:

$$(5.57)\quad R_j \overset{\text{def}}{=} \frac{\Gamma(1+\check{n}_j+p\check{\alpha}_{j+1})}{\Gamma(1+\check{n}_j+p\check{\alpha}_j)\Gamma(1+p\alpha_{j+1})},\quad (j=1,2,\dots,r-1)$$

$$(5.58)\quad R_* \overset{\text{def}}{=} \frac{\Gamma(1+p\alpha_1)\Gamma(1+p\alpha_2)\cdots\Gamma(1+p\alpha_r)}{\Gamma(r+p\alpha_1+p\alpha_2+\cdots+p\alpha_r)}.$$

Here $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_r)$ is *any* sequence of complex numbers *independent over* $\mathbb{Z}$(as usual, $\check{\alpha}_j=\alpha_1+\cdots+\alpha_j$) and $x_1,\dots,x_{r-1},x_*$ are regarded as variables. The last one is denoted by x_* rather than x_r, because it is not at all on a par with the rest, as we shall see in a moment.

But right now, to motivate the introduction of Φ, let us observe that if we put:

$$(5.59)\qquad x_*=X_i\ ;\quad \alpha_j\equiv\varepsilon-p\omega_j/\check{\omega}_i\quad (j=1,2,\dots,r)$$

and let ε go to 0, then due to (5.27):

$$(5.60)\qquad \varepsilon\Gamma(r)\Phi\longrightarrow \$^{\boldsymbol{\varpi}}_{\text{amp},i}\quad (\text{as } \varepsilon\to 0)$$

where $\$^{\boldsymbol{\varpi}}_{\text{amp},i}$ denotes of course *the* inside series in (5.15) that involves the variable X_i.

So we may proceed with Φ and assume for a start that $\mathrm{Re}(\alpha_j)>0$ for each j. Under that assumption, and by classical gamma function theory, we find for the coefficients R_j and R_* the convergent integral representations:

$$(5.61)\qquad R_j=\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}(u_j)^{-1-\check{n}_j-p\check{\alpha}_j}(1-u_j)^{-1-p\alpha_{j+1}}du_j$$

$$(5.62)\qquad R_*=\int_0^1 w_1^{p\alpha_1}w_2^{p\alpha_2}\cdots w_r^{p\alpha_r}dw_1dw_2\cdots dw_{r-1}$$

with $w_1+w_2+\cdots+w_r\equiv 1$.

If we plug these integral representations into (5.56); then sum over $n_1,n_2,\dots,n_{r-1},p$; and then change from u_j to v_j with:

$$(5.63)\qquad v_j\overset{\text{def}}{=}(1-u_{j-1})u_ju_{j+1}\cdots u_{r-1}\quad (\text{if } 2\le j\le r-1)$$

$$(5.63^*)\qquad v_1\overset{\text{def}}{=}u_1u_2\cdots u_{r-1}\ ;\quad v_r\overset{\text{def}}{=}1-u_{r-1}$$

we find for Φ the following integral representation:

$$\Phi = \int H dv_1 \cdots dv_{r-1} dw_1 \cdots dw_{r-1} \tag{5.64}$$

with a simple integrand $H \equiv H_1 \cdots H_{r-1} H_*$:

$$H_j \equiv (2\pi i)^{-1} v_j^{-1} (v_1 + v_2 + \cdots + v_j - x_j)^{-1} \tag{5.65}$$

$$H_* \equiv (1 - x_* (w_1/v_1)^{\alpha_1} \cdots (w_r/v_r)^{\alpha_r})^{-1} \tag{5.66}$$

and with variables v_j, w_j bound by:

$$v_1 + \cdots + v_r \equiv 1 \ ; \quad w_1 + \cdots + w_r \equiv 1. \tag{5.67}$$

The w_j range over the same *finite* multipath of integration as in (5.62), while the v_j range over an *infinite* multipath deducible from that of the u_j under (5.63), (5.63*).

For x_j and x_* small enough, the integrand H doesn't vanish: the series (5.56) and the integral (5.64) converge to one and the same germ Φ, which clearly has the property of endless analytic continuation, with a singular set of zero measure.

It is but an easy step to see that this property survives even without the assumption $\mathrm{Re}(\alpha_j) > 0$, and that the limit (5.60) also leaves it in force. However, if we object to taking limits, we can also find for each single $\$^{\boldsymbol{\varpi}}_{\mathrm{amp},i}$ a direct integral representation akin to (5.64), though slightly less tidy, and conclude in this way. The advantage of proceeding as we did lies not only in the greater simplicity of Φ, but also in the fact that, via the specialization (5.59) and the limiting process (5.60), Φ disposes *at once* of *all* $\$^{\boldsymbol{\varpi}}_{\mathrm{amp},i}$, for all i. But whatever the means chosen, the argument establishes Proposition 5.2.

Coamplification of moulds.

Let us revert to the case of a general alternal mould $M^\bullet$ with its amplification $M^\bullet_{\mathrm{amp}}$ and the corresponding power series:

$$M_{\mathrm{amp}}^{\varpi_1,\ldots,\varpi_r} = \sum_{n_i \geq 0} M_{n_1,\ldots,n_r}^{\omega_1,\ldots,\omega_r} a_1^{n_1} \cdots a_r^{n_r} \quad (\text{with } \varpi_i = \tbinom{\omega_i}{a_i} \in \mathbb{C}^2). \tag{5.68}$$

For reasons that shall become apparent in §8 and §9, it is often useful to attach to $M^\bullet$ yet another alternal mould, the *coamplification* $M^\bullet_{\mathrm{coamp}}(z)$, which is indexed by sequences $\boldsymbol{\eta}$ similar in form to $\boldsymbol{\varpi}$:

$$\boldsymbol{\eta} = (\eta_1, \ldots, \eta_r) \ \text{ with } \ \eta_i = \begin{pmatrix} \omega_i \\ \sigma_i \end{pmatrix} \ ; \ \omega_i \in \mathbb{C}, \sigma_i \in \mathbb{C}. \tag{5.69}$$

but which takes its values in the algebra of formal power series of z^{-1} (z being regarded as *large*):

$$(5.70) \qquad M_{\text{coamp}}^{\eta_1,\ldots,\eta_r}(z) \in z^{-(\sigma_1+\cdots\sigma_r)}\mathbb{C}[\![z^{-1}]\!] \text{ if } \eta_i = \binom{\omega_i}{\sigma_i}.$$

The coamplification admits of a concise definition:

$$(5.71) \qquad M_{\text{coamp}}^{\eta_1,\ldots,\eta_r}(z) \overset{\text{def}}{=} \{M_{\text{amp}}^{\varpi_1,\ldots,\varpi_r} z_1^{-\sigma_1}\cdots z_r^{-\sigma_r}\}_{z_i\equiv z}$$

(with $\eta_i = \binom{\omega_i}{\sigma_i}$, $\varpi_i = \binom{\omega_i}{\partial_i}$).

In plain words: we turn M_{amp}^{ϖ} into an *operator* by replacing each variable a_i in ϖ_i by the derivation $\partial_i \equiv \partial_{z_i}$; then we let this operator M_{amp}^{ϖ} act on the monomial $z_1^{-\sigma_1}\cdots z_r^{-\sigma_r}$; and lastly we replace each z_i by z.

With the help of the expansion (5.68) we get the more explicit formula (where $\partial \equiv \partial_z$):

$$\begin{aligned}(5.72)\ M_{\text{coamp}}^{\eta_1,\ldots,\eta_r}(z) &= \sum_{n_i\geq 0} M_{n_1,\ldots,n_r}^{\omega_1,\ldots,\omega_r}(\partial^{n_1}z^{-\sigma_1})\cdots(\partial^{n_r}z^{-\sigma_r})\\ &= \sum_{n_i\geq 0} M_{n_1,\ldots,n_r}^{\omega_1,\ldots,\omega_r}(-1)^{n_1+\cdots+n_r}z^{-(\sigma_1+\cdots\sigma_r)-(n_1+\cdots n_r)}\\ &\qquad\qquad \prod_{1\leq i\leq r}\frac{\Gamma(\sigma_i+n_i)}{\Gamma(\sigma_i)}.\end{aligned}$$

Unless the power series (5.68) has infinite multiradius of convergence, (5.72) usually diverges as a series of z^{-1}. But if we subject it to the Borel transform:

$$(5.73) \qquad z^{-\sigma} \longmapsto \zeta^{\sigma-1}/\Gamma(\sigma)\ ;\quad M_{\text{coamp}}^{\eta_1,\ldots,\eta_r}(z) \longmapsto \widehat{M}_{\text{coamp}}^{\eta_1,\ldots,\eta_r}(\zeta)$$

$$(5.73^*) \qquad z \text{ large}\ ;\quad \zeta \text{ small},$$

and remember that Borel turns the derivation $\partial = \partial_z$ into multiplication by $(-\zeta)$, we find for ζ close to 0:

$$(5.74) \qquad \widehat{M}_{\text{coamp}}^{\eta_1,\ldots,\eta_r}(\zeta) = \int_0^\zeta M_{\text{amp}}^{\varpi_1,\ldots,\varpi_r}\cdot\frac{\zeta_1^{\sigma_1-1}}{\Gamma(\sigma_1)}\cdots\frac{\zeta_r^{\sigma_r-1}}{\Gamma(\sigma_r)}d\zeta_1\cdots d\zeta_{r-1}$$

with $\eta_i = \binom{\omega_i}{\sigma_i}$ as usual but with $\varpi_i = \binom{\omega_i}{-\zeta_i}$, and with integration along the complex multipath symbolized by:

$$(5.75) \qquad \{0<\zeta_i<\zeta\ (\forall i)\ ;\quad \zeta_1+\zeta_2+\cdots+\zeta_r\equiv\zeta\}$$

so that (5.74), despite the missing $d\zeta_r$, is perfectly symmetrical in $\zeta_1,\zeta_2,\ldots,\zeta_r$. Since $M_{\text{amp}}^{\bullet}$, as we saw in the case $M^\bullet = \$^\bullet$ or $\$\!\!\!\!\|^\bullet$, often tends to be more easily expressible in terms of the variables $\check{\zeta}_i$ or $\hat{\zeta}_i$, we may also replace $(d\zeta_1 \cdots d\zeta_{r-1})$ by $(d\check{\zeta}_1 \cdots d\check{\zeta}_{r-1})$ or $(d\hat{\zeta}_2 \cdots d\hat{\zeta}_r)$ and integrate along the multipaths:

$$(5.75^*) \quad \{0 < \check{\zeta}_1 < \check{\zeta}_2 < \cdots < \check{\zeta}_{r-1} < \zeta\}(\text{with } \check{\zeta}_i \overset{\text{def}}{=} \zeta_1 + \cdots + \zeta_i)$$

$$(5.75^{**}) \quad \{0 < \hat{\zeta}_r < \hat{\zeta}_{r-1} < \cdots < \hat{\zeta}_2 < \zeta\}(\text{with } \hat{\zeta}_i = \zeta_i + \cdots + \zeta_r).$$

But whatever the mode of integration, it is plain that if we have *endless analyticity* (in the sense of Proposition 5.2) of $M^{\varpi_1,\ldots,\varpi_r}_{\text{amp}}$ as a function of its several variables ζ_i (recall that here $\varpi_i = \left(\begin{smallmatrix} \omega_i \\ -\zeta_i \end{smallmatrix}\right)$), we automatically have *endless analyticity* of $\widehat{M}^{\eta_1,\ldots,\eta_r}_{\text{coamp}}(\zeta)$ as a function of its one variable ζ; and therefore *resurgence* of $M^{\eta_1,\ldots,\eta_r}_{\text{coamp}}(z)$ as a *formal power* series of z^{-1}.

We should note that definition (5.74) works well for complex numbers σ_i such that $\text{Re}(\sigma_i) > 0$. But even for general complex numbers σ_i, the *minors* $\widehat{M}^{\bullet}_{\text{coamp}}(\zeta)$ may still be defined unambiguously *via* the *majors* $\check{M}^{\bullet}_{\text{coamp}}(\zeta)$. For the notions of *minor* and *major* of a *resurgent function*, we refer to [E5] or [E7] or [E10](*)

In §8 and §9, we shall turn to good profit the resurgence properties of the coamplifications $\$^{\bullet}_{\text{coamp}}$ and $\not{\$}^{\bullet}_{\text{coamp}}$.

6. The alternel moulds $\$^{\bullet}$ and $\not{\$}^{\bullet}$ in the context of symmetrel compensation.

At the beginning of §2, we recalled the twin notions of *symmetrel/alternel* moulds, which are akin to *symmetral/alternal* moulds, but intervene in different contexts: the latter (mainly) in the study of vector fields, the former (mainly) in that of diffeomorphisms.

In the present instance, and parallel to the symmetral/alternal moulds:

$$(6.1) \qquad S^{\bullet}, \not{S}^{\bullet}; S^{\bullet}_{\text{ext}}, \not{S}^{\bullet}_{\text{ext}}, S^{\bullet}_{\text{co}}(t), \not{S}^{\bullet}_{\text{co}}(t); S^{\bullet}_{\text{aco}}(t), \not{S}^{\bullet}_{\text{aco}}(t), \quad (\text{symmetral})$$

$$(6.1^*) \qquad T^{\bullet}, \$^{\bullet}, \not{\$}^{\bullet} \quad (\text{alternal})$$

(*) *Majors* and *minors* are signalled respectively by $\vee$ and $\wedge$. Needless to say, this has nothing to do with the use of these symboles in the notations $\check{\omega}_i$ and $\hat{\omega}_i$ for the partial sums, *forward* and *backward*, of a sequence $\boldsymbol{\omega}$.

we require a series of 8 symmetrel and 3 alternel moulds, which we denote by the same symbols, but with cursive letters:

$$\mathcal{S}^\bullet, \not{\mathcal{S}}^\bullet; \mathcal{S}^\bullet_{\text{ext}}, \not{\mathcal{S}}^\bullet_{\text{ext}}, \mathcal{S}^\bullet_{\text{co}}(t), \not{\mathcal{S}}^\bullet_{\text{co}}(t); \mathcal{S}^\bullet_{\text{aco}}(t), \not{\mathcal{S}}^\bullet_{\text{aco}}(t) \quad \text{(symmetrel)} \tag{6.2}$$

$$\mathcal{T}^\bullet, \$^\bullet, \not{\$}^\bullet \quad \text{(alternel).} \tag{6.2*}$$

We define them in much the same way as their models, but with automorphisms $\exp(\nabla)$ and $\exp(\nabla - t\partial_t)$ in place of the derivations ∇ and $\nabla - t\partial_t$. More precisely, we begin with the counterparts of $S^\bullet$, $\not{S}^\bullet$, $T^\bullet$, which are defined (almost everywhere) by the formulae:

$$\begin{aligned}
&(6.3) && \mathcal{S}^{\boldsymbol{\omega}} &&= e^{-\|\boldsymbol{\omega}\|}(e^{-\check{\omega}_1}-1)^{-1}\cdots(e^{-\check{\omega}_r}-1)^{-1} && \text{with } \check{\omega}_i=\omega_1+\cdots+\omega_i\\
&(6.4) && \not{\mathcal{S}}^{\boldsymbol{\omega}} &&= (e^{\hat{\omega}_1}-1)^{-1}\cdots(e^{\hat{\omega}_r}-1)^{-1} && \text{with } \hat{\omega}_i=\omega_i+\cdots+\omega_r\\
&(6.5) && \mathcal{T}^{\boldsymbol{\omega}} &&= 0 && \text{if } \|\boldsymbol{\omega}\| \neq 0\\
&(6.5^*) && \mathcal{T}^{\boldsymbol{\omega}} &&= (e^{\hat{\omega}_2}-1)^{-1}(e^{\hat{\omega}_3}-1)^{-1}\cdots(e^{\hat{\omega}_r}-1)^{-1} && \text{if } \|\boldsymbol{\omega}\| = 0.
\end{aligned}$$

The analogues of (2.23), (2.24) read:

$$e^\nabla \cdot \mathcal{S}^\bullet = \mathcal{S}^\bullet \times (1^\bullet + I^\bullet)^{-1} \tag{6.6}$$

$$e^\nabla \cdot \not{\mathcal{S}}^\bullet = (1^\bullet + I^\bullet) \times \not{\mathcal{S}}^\bullet . \tag{6.7}$$

We then introduce an auxiliary variable $t \in \mathbb{C}_\bullet$ ($\mathbb{C}_\bullet$ denotes the Riemann surface of $\log t$) and construct the *symmetrel compensators*:

$$\mathcal{S}^\bullet_{\text{co}}(t) \overset{\text{def}}{=} (t^\nabla \not{\mathcal{S}}^\bullet) \times (\mathcal{S}^\bullet) \tag{6.8}$$

$$\not{\mathcal{S}}^\bullet_{\text{co}}(t) \overset{\text{def}}{=} (\not{\mathcal{S}}^\bullet) \times (t^\nabla \mathcal{S}^\bullet) \tag{6.9}$$

which, unlike $\mathcal{S}^\bullet$ and $\not{\mathcal{S}}^\bullet$, are defined for all sequences $\boldsymbol{\omega}$. In the case of *degenerate sequences* $\boldsymbol{\omega}$ (see (2.17), (2.18)), we denote by $\mathcal{S}^{\boldsymbol{\omega}}_{\text{aco}}(t)$ and $\not{\mathcal{S}}^{\boldsymbol{\omega}}_{\text{aco}}(t)$ the logarithm-free parts of $\mathcal{S}^{\boldsymbol{\omega}}_{\text{co}}(t)$ and $\not{\mathcal{S}}^{\boldsymbol{\omega}}_{\text{co}}(t)$. This leads smoothly to the *lateral* and *central* decompositions of symmetrel compensators, which faithfully mirror the symmetral models on which they are patterned. Thus:

$$\begin{aligned}
(6.10) \quad \mathcal{S}^\bullet_{\text{co}}(t) &= \mathcal{S}^\bullet_{\text{aco}}(t) \times \exp((\log t)\,\$^\bullet) && \text{(right-lateral)}\\
&= \exp((\log t)(t^\nabla\,\$^\bullet)) \times \mathcal{S}^\bullet_{\text{aco}}(t) && \text{(left-lateral)}\\
&= (t^\nabla\,\not{\mathcal{S}}^\bullet_{\text{ext}}) \times \exp((\log t)\,\not{\$}^\bullet) \times \mathcal{S}^\bullet_{\text{ext}} && \text{(central).}
\end{aligned}$$

The proofs also mimick the earlier arguments (see at the end of §2 and §3) and rely on mould-comould contractions $\sum M^\bullet \mathbb{B}^{\cdot}_\bullet$, but since $M^\bullet$ is now either symmetrel or alternel, it should (always) be contracted with a cosymmetrel comould $\mathbb{B}_\bullet$, *i.e.* one that obeys a cosymmetrel coproduct:

$$\text{cop}(\mathbb{B}_{\boldsymbol{\omega}}) = \sum_{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2} \mathbb{B}_{\boldsymbol{\omega}^1} \otimes \mathbb{B}_{\boldsymbol{\omega}^2} \tag{6.11}$$

with $\boldsymbol{\omega}$ obtainable by *contracting shuffling* (see (2.4*), (2.5*)) from $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$.

The only point in need of elaboration is the construction of the canonical "tough" moulds $\mathcal{S}^\bullet_{\text{ext}}, \mathscr{S}^\bullet_{\text{ext}}, \$^\bullet$ that appear in the *central decomposition.* As in §4, we follow a two-stepped procedure:

$$(\mathcal{S}^\bullet, \mathscr{S}^\bullet, \mathcal{T}^\bullet) \xrightarrow{\text{rest}} (\mathcal{S}^\bullet_{\text{rest}}, \mathscr{S}^\bullet_{\text{rest}}, \mathcal{T}^\bullet_{\text{rest}}) \xrightarrow{\text{diff}} (\mathcal{S}^\bullet_{\text{ext}}, \mathscr{S}^\bullet_{\text{ext}}, \$^\bullet).$$

We define, predictably enough, the "restrictions" $\mathcal{S}^\bullet_{\text{rest}}, \mathscr{S}^\bullet_{\text{rest}}, \mathcal{T}^\bullet_{\text{rest}}$ by the earlier formulae (6.3), (6.4), (6.5–5*), but under omission of the factors for which $\breve{\omega}_i = 0$ (resp. $\hat{\omega}_i = 0$). Then we subject the "restrictions" to suitable differential operators $\text{Red}^{\boldsymbol{\omega}}$, $\text{Ređ}^{\boldsymbol{\omega}}$, $\text{Rel}^{\boldsymbol{\omega}}$ that are defined as in (4.28), (4.29), (4.30), but relative to new auxiliary moulds $\text{red}^{\mathbf{w}}$, $\text{ređ}^{\mathbf{w}}$, $\text{rel}^{\mathbf{w}}$, which instead of verifying (4.16), (4.17), interrelate as follows:

(6.12) $$1^\bullet = \text{red}^\bullet \times \text{ređ}^\bullet$$

(6.13) $$\text{rel}^\bullet = \text{red}^\bullet \times J^\bullet \times \text{ređ}^\bullet \quad (J^\bullet \text{ as in } (2.10))$$

and are defined by an induction markedly different from (4.11), (4.12), (4.13). That new induction reads for $1 \le r$ (resp. $2 \le i \le r$):

$$\begin{aligned}
\left(e^{v_1\partial_{u_1}+v_2\partial_{u_2}+\cdots v_r\partial_{u_r}} - 1\right)\text{ređ}^{w_1,\dots,w_r} &= \text{ređ}^{w_2,w_3,\dots,w_r}\\
\left(e^{v_i\partial_{u_i}+v_{i+1}\partial_{u_{i+1}}+\cdots v_r\partial_{u_r}} - 1\right)\text{ređ}^{w_1,\dots,w_r} &= \text{ređ}^{w_1,\dots,w_{i-1}+w_i,\dots,w_r}\\
\left(e^{v_1\partial_{u_1}+v_2\partial_{u_2}+\cdots v_r\partial_{u_r}} - 1\right)\text{rel}^{w_1,\dots,w_r} &= 0\\
\left(e^{v_i\partial_{u_i}+v_{i+1}\partial_{u_{i+1}}+\cdots v_r\partial_{u_r}} - 1\right)\text{rel}^{w_1,\dots,w_r} &= \text{rel}^{w_1,\dots,w_{i-1}+w_i,\dots,w_r}
\end{aligned}$$

(there are similar formulae for $\text{red}^\bullet$). Thus, whereas the old induction (4.11), (4.12), (4.13) involved *derivations* $v_i\partial_{u_i}$, the new induction involves *automorphisms* $\exp(v_i\partial_{u_i})$, *i.e. shifts* $u_i \mapsto u_i + v_i$ on the u_i variables. As a result, $\text{red}^{\mathbf{w}}$, $\text{ređ}^{\mathbf{w}}$, $\text{rel}^{\mathbf{w}}$ (unlike $\text{rad}^{\mathbf{w}}$, $\text{rađ}^{\mathbf{w}}$, $\text{ral}^{\mathbf{w}}$) are *non-homogeneous polynomials* of $u_1, u_2, \dots, u_r$, although their constant terms vanish, and their highest order parts coincide with those of the symmetral/alternal case. Indeed, if we mark with a lower index s the homogeneous part of degree s, we find:

(6.14) $$\text{red}^{\emptyset} = \text{ređ}^{\emptyset} = 1\ ;\ \ \text{rel}^{\emptyset} = 0$$

and for sequences $\mathbf{w} = (w_1, \dots, w_r)$ of any length $r \ge 1$:

(6.15) $$\text{red}^{\mathbf{w}} = \sum_{1\le s\le r} \text{red}^{\mathbf{w}}_s \quad \text{with } \text{red}^{\mathbf{w}}_r \equiv \text{rad}^{\mathbf{w}}$$

(6.16) $$\text{ređ}^{\mathbf{w}} = \sum_{1\le s\le r} \text{ređ}^{\mathbf{w}}_s \quad \text{with } \text{ređ}^{\mathbf{w}}_r \equiv \text{rađ}^{\mathbf{w}}$$

(6.17) $$\text{rel}^{\mathbf{w}} = \sum_{0\le s\le r-1} \text{rel}^{\mathbf{w}}_s \quad \text{with } \text{rel}^{\mathbf{w}}_0 \equiv J^{\mathbf{w}} \text{ and } \text{rel}^{\mathbf{w}}_{r-1} \equiv \text{ral}^{\mathbf{w}}\,.$$

For $1 \le r \le 3$ and with the usual short-hand $u_{ij} = u_i + u_j$, etc., the lower-order terms read:

$$
\begin{aligned}
\mathrm{red}_1^{w_1,w_2} &= -\,\text{ređ}_1^{w_1,w_2} = +\frac{1}{2}\frac{u_{12}}{v_{12}}\\
\mathrm{rel}_1^{w_1,w_2} &= -\frac{u_1}{v_1}+\frac{u_2}{v_2}\\
\mathrm{red}_1^{w_1,w_2,w_3} &= -\,\mathrm{red}_1^{w_1,w_2,w_3} = -\frac{1}{3}\frac{u_{123}}{v_{123}}\\
\mathrm{rel}_1^{w_1,w_2,w_3} &= +\frac{1}{2}\frac{u_1}{v_1}-\frac{1}{2}\frac{u_3}{v_3}+\frac{1}{2}\frac{u_{12}}{v_{12}}-\frac{1}{2}\frac{u_{23}}{v_{23}}\\
\mathrm{red}_2^{w_1,w_2,w_3} &= -\frac{1}{4}\frac{u_{123}^2}{v_{123}}\left\{\frac{1}{v_1}+\frac{1}{v_{12}}\right\}+\frac{1}{4}\frac{u_3^2}{v_3v_{12}}+\frac{1}{4}\frac{u_{23}^2}{v_1v_{23}}\\
\text{ređ}_2^{w_1,w_2,w_3} &= -\frac{1}{4}\frac{u_{123}^2}{v_{123}}\left\{\frac{1}{v_3}+\frac{1}{v_{23}}\right\}+\frac{1}{4}\frac{u_1^2}{v_1v_{23}}+\frac{1}{4}\frac{u_{12}^2}{v_3v_{12}}\\
\mathrm{rel}_2^{w_1,w_2,w_3} &= +\frac{1}{2}\frac{u_{23}^2}{v_3v_{12}}-\frac{1}{2}\frac{u_3^2}{v_2v_3}+\frac{1}{2}\frac{u_{12}^2}{v_1v_{12}}-\frac{1}{2}\frac{u_1^2}{v_1v_2}.
\end{aligned}
$$

7. The nilpotent part and distinguished form of a resonant vector field or diffeomorphism.

From now on, we are going to apply the mould apparatus of the previous sections to the study of the so-called *analytical local objects*. More precisely, we shall be dealing with *local analytical vector fields* (or *fields* for short) on $\mathbb{C}^\nu$ at 0:

$$X=\sum_{1\le i\le\nu} X_i(x)\partial_{x_i}\quad (X_i(0)\in 0;\ X_i(x)\in\mathbb{C}\{x\}) \tag{7.1}$$

and with *local analytic self-mappings* (or *diffeos*, short for diffeomorphisms) of $\mathbb{C}^\nu$ with 0 as fixed point:

$$f: x_i\longmapsto f_i(x)\quad (i=1,2,\dots,\nu;\ f_i(0)=0;\ f_i(x)\in\mathbb{C}\{x\}) \tag{7.2}$$

or again, equivalently, with the related *substitution operators* (capital-lettered):

$$F:\varphi\longmapsto F\varphi\overset{\text{def}}{=}\varphi\circ f\quad (\varphi(x)\text{ and }\varphi\circ f(x)\in\mathbb{C}\{x\}). \tag{7.3}$$

Throughout, we will assume the linear part to be *diagonalizable*, and work with "prepared forms", *i.e.* consider analytic charts where the linear part assumes diagonal shape. Thus, we will consider fields of the form:

(7.4) $X = X^{\mathrm{lin}} + \sum \mathbb{B}_n$

(7.4*) $X^{\mathrm{lin}} = \sum \lambda_i x_i \partial_{x_i} \ (1 \leq i \leq \nu,\ \lambda_i \in \mathbb{C})$

(7.4**) $\mathbb{B}_n = \mathbb{B}_{n_1,\ldots,n_\nu} =$ homogeneous part of degree n $(n_i \geq -1)$

and diffeos of the form:

(7.5) $F = \{1 + \sum \mathbb{B}_n\} F^{\mathrm{lin}}$

(7.5*) $F^{\mathrm{lin}} \varphi(x_1, \ldots, x_\nu) \overset{\mathrm{def}}{=} \varphi(\ell_1 x_1, \ldots, \ell_\nu x_\nu)\ (\forall \varphi; \ell_i \in \mathbb{C}^*)$

(7.5**) $\mathbb{B}_n = \mathbb{B}_{n_1,\ldots,n_\nu} =$ homogeneous part of degree n $(n_i \geq -1)$.

Of course, n-homogeneousness means that for each monomial x^n we have:

(7.6) $\mathbb{B}_n \cdot x^m = \beta_{n,m} x^{n+m}$ with $\beta_{n,m} \in \mathbb{C}$; $x^m = \prod x_i^{m_i}$; $x^n = \prod x_i^{n_i}$.

Note that, for any given $\mathbb{B}_n$, at most one component n_i may assume the value -1.

The eigenvalues λ_i or ℓ_i will be referred to as *multipliers*. We say that the local object (field or diffeo) is *resonant*, if there exist non-trivial relations of the form:

(7.7) $$\sum_{1 \leq i \leq \nu} m_i \lambda_i = 0 \ (\text{or } \lambda_j) \quad (m_i \in \mathbb{N})$$

(7.8) $$\prod_{1 \leq i \leq \nu} (\ell_i)^{m_i} = 1 \ (\text{or } \ell_j) \quad (m_i \in \mathbb{N}).$$

If (7.7) or (7.8) are "very nearly" fullfilled for an *infinity* of multiintegers m, that is to say, more precisely, if the multipliers do not meet A.D. Bryuno's *diophantine condition* (see [B], [M] or [E7], p. 78), we speak of *quasiresonance*.

Lastly, *nihilence* (which presupposes resonance) amounts to the existence of a "first integral", in the form of a (formal) power series $H(x) \in \mathbb{C}[[x]]$ with the invariance property:

(7.9) $X \cdot H(x) \equiv 0$ (for a field)

(7.10) $H \circ f(x) \overset{\mathrm{def}}{=} F \cdot H(x) \equiv H(x)$ (for a diffeo).

If the Taylor expansion of the object under consideration involves only *resonant* monomials or, what amounts to the same, if each homogeneous part $\mathbb{B}_n$ in (7.4) or (7.5) *commutes* with the linear part X^{lin} or F^{lin}, we say that the object is given in a *prenormal form* (or chart). If the number of these resonant monomials is *minimal* (with *formal invariants* as coefficients), we speak of a *normal form*.

For a *resonant vector field* X, there is a classical decomposition (see [B]):

$$X = X^{\text{dia}} + X^{\text{nil}} \quad \text{with } [X^{\text{dia}}, X^{\text{nil}}] = 0 \tag{7.11}$$

into a *diagonalizable part* X^{dia} and *nilpotent part* X^{nil}. The decomposition is fully characterized by *chart invariance* meaning that for any substitution operator Θ expressive of a change of variables we have:

$$(\Theta X \Theta^{-1})^{\text{dia}} = \Theta X^{\text{dia}} \Theta^{-1} \tag{7.12}$$

$$(\Theta X \Theta^{-1})^{\text{nil}} = \Theta X^{\text{nil}} \Theta^{-1} \tag{7.13}$$

and by the condition that in *one*, and therefore *every, prenormal chart*, X^{dia} should reduce to the linear diagonal part X^{lin} (and X^{nil} should contain only higher-order resonant monomials).

We have a similar decomposition for all resonant diffeos, but for simplicity we restrict ourselves to *torsion-free diffeos*, *i.e.* to diffeos whose eigenvalues ℓ_i admit a system of logarithms $\lambda_i = \log \ell_i \in \mathbb{C}$, $(i = 1, 2, \dots, \nu)$ such that any *multiplicative* resonance relation (7.8) translates into a corresponding additive resonance relation (7.7). (Even if F is not torsion-free, suitable iterates F^p are.) For any torsion-free diffeo F, we have the decomposition (in operatorial notation):

$$F = F^{\text{dia}} F^{\text{nil}} = F^{\text{nil}} F^{\text{dia}} \tag{7.14}$$

characterized by chart-invariance:

$$(\Theta F \Theta^{-1})^{\text{dia}} = \Theta F^{\text{dia}} \Theta^{-1} \tag{7.15}$$

$$(\Theta F \Theta^{-1})^{\text{nil}} = \Theta F^{\text{nil}} \Theta^{-1} \tag{7.16}$$

and by the condition that in *one*, and therefore *any* prenormal chart, F^{dia} should reduce to F^{lin}.

The existence of *prenormal charts* is immediate to establish (by inductive coefficient identification) and the *consistency* of the above definition (for the diagonalizable and nilpotent part) follows from the fact that any substitution operator Θ that takes us from one prenormal chart to another, automatically commutes with the object's linear part X^{lin} or F^{lin}.

PROPOSITION 7.1 (Analytical expression of the nilpotent part and distinguished form of a vector field). — *Any resonant vector field* $X = X^{\text{lin}} + \sum \mathbb{B}_n$ *decomposes intrinsically into* $X^{\text{dia}} + X^{\text{nil}}$ *with:*

$$X^{\text{nil}} = \sum \$^{\bullet}\, \mathbb{B}_{\bullet} \quad (= \textit{nilpotent part}) \tag{7.17}$$

and it admits a canonical (though non-intrinsic) prenormal form:

$$X^{\mathrm{dist}} = X^{\mathrm{lin}} + \sum \$^{\bullet}\,\mathbb{B}_{\bullet} \quad \text{(=\textit{distinguished form})} \tag{7.18}$$

to which it is conjugate:

$$X = \Theta_{\mathrm{ext}} X^{\mathrm{dist}} \Theta_{\mathrm{ext}}^{-1} \tag{7.19}$$

under the reciprocal changes of variables:

$$\Theta_{\mathrm{ext}} = \sum S^{\bullet}_{\mathrm{ext}} \mathbb{B}_{\bullet} \tag{7.20}$$

$$\Theta_{\mathrm{ext}}^{-1} = \sum \not{S}^{\bullet}_{\mathrm{ext}}\, \mathbb{B}_{\bullet}. \tag{7.20*}$$

PROPOSITION 7.2 (Analytical expression of the nilpotent part and distinguished form of a diffeo). — *Any resonant, torsion-free diffeo $F = \{1 + \sum \mathbb{B}_n\} F^{\mathrm{lin}}$ decomposes intrinsically into $F^{\mathrm{nil}} F^{\mathrm{dia}} = F^{\mathrm{dia}} F^{\mathrm{nil}}$ with:*

$$F^{\mathrm{nil}} = \exp(X^{\mathrm{nil}}) = \exp(\textstyle\sum \$^{\bullet}\,\mathbb{B}_{\bullet}) \quad \text{(=\textit{nilpotent part})} \tag{7.21}$$

and it admits a canonical (though non-intrinsic) prenormal from:

$$\begin{aligned} F^{\mathrm{dist}} &= F^{\mathrm{lin}} \cdot \exp(X^{\mathrm{dist}}) \\ &= \exp(X^{\mathrm{dist}}) \cdot F^{\mathrm{lin}} \quad \text{(= \textit{distinguished form})} \end{aligned} \tag{7.22}$$

$$X^{\mathrm{dist}} = \sum \$^{\bullet}\,\mathbb{B}_{\bullet} \tag{7.22*}$$

to which it is conjugate:

$$F = \Theta_{\mathrm{ext}} F^{\mathrm{dist}} \Theta_{\mathrm{ext}}^{-1} \tag{7.23}$$

under the reciprocal changes of variables:

$$\Theta_{\mathrm{ext}} = \sum \mathcal{S}^{\bullet}_{\mathrm{ext}} \mathbb{B}_{\bullet} \tag{7.24}$$

$$\Theta_{\mathrm{ext}}^{-1} = \sum \not{\mathcal{S}}^{\bullet}_{\mathrm{ext}}\, \mathbb{B}_{\bullet}. \tag{7.24*}$$

Remark 1. — All the above formulae involve mould-comould contractions of type:

$$\sum M^{\bullet} \mathbb{B}_{\bullet} = \sum_{r \geq 0} \sum_{n_i} M^{\omega_1,\ldots,\omega_r} \mathbb{B}_{n_1,\ldots,n_r} \tag{7.25}$$

with indices:

$$n_i = (n_{i1}, \ldots, n_{i\nu});\ \omega_i = \langle n_i, \lambda \rangle = n_{i1}\lambda_1 + \cdots n_{i\nu}\lambda_\nu \tag{7.26}$$

relative to the spectrum λ_i of the field (resp. $\lambda_i = \log \ell_i$ for a diffeo) and to the cosymmetral (resp. cosymmetrel) comould:

$$\mathbb{B}_{n_1,\ldots,n_r} \overset{\mathrm{def}}{=} \mathbb{B}_{n_r} \cdots \mathbb{B}_{n_2} \mathbb{B}_{n_1} \quad (n_i \in \mathbb{N}^{\nu}_{*}) \tag{7.27}$$

constructed from the homogeneous parts $\mathbb{B}_n$ of the vector field X (resp. diffeo F). In the case of a diffeo, we may note that the various moulds $M^{\omega_1,\dots,\omega_r}$ being used are *rational functions* of $e^{\omega_1},\dots,e^{\omega_r}$ and therefore independent of the determination $\lambda_i = \log \ell_i$, provided this determination is *coherent* (*i.e.* respectful of all resonance relations; see above (7.14)), so that the degeneracy type or vanishing pattern (see (2.17) and below) of a sequence $\boldsymbol{\omega} = (\omega_1,\dots,\omega_r)$ associated to $\mathbf{n} = (n_1,\dots,n_r)$ depends on $\mathbf{n}$ alone (not on the determination).

Remark 2. — *Like* the homogeneous parts $\mathbb{B}_n$, but *unlike* the nilpotent part X^{nil} or F^{nil}, the distinguished form X^{dist} or F^{dist} and the corresponding changes of variables Θ_{ext} and $\Theta_{\mathrm{ext}}^{-1}$ are *not intrinsic*, *i.e.* not chart-independent, because the moulds $\$\!\!\!\|^{\,\bullet}$, $S^\bullet_{\mathrm{ext}}$, $S\!\!\!/^\bullet_{\mathrm{ext}}$ don't behave like the moulds $\$^\bullet$, $S^\bullet$, $S\!\!\!/^\bullet$ under ∇_{ω_0}-derivation: compare (4.57), (4.58), (4.59) with (3.10*), (3.11*), (3.16*). Nonetheless, for a given chart, the distinguished form is well-defined, with a transparent analytical expression (7.18) or (7.21), and there is no denying that it is "canonical": it is just as canonical among the various prenormal forms, as the mould $\$\!\!\!\|^{\,\bullet}$ satisfying (2.42) is among the various solutions of (2.35). The distinguished form is especially valuable in two cases:

(i) For local objects with *multiple resonance*, because such objects tend to possess several (finitely many) *normal forms*, each of them marred by a degree of arbitrariness, and riddled with an infinite number of coefficients (since multiple resonance induces an infinite number of *formal invariants*).

(ii) For objects endowed with an *additional structure*, e.g. symplectic or volume-preserving, especially with *extrinsic resonance* (*i.e.* with more degrees of resonance than those induced by symplecticity or volume-preservation) because in that case the conjugating change of variables Θ_{ext} that goes together with the distinguished form, is itself symplectic or volume-preserving, as apparent from its expansion (7.20) or (7.24).

In view of the importance of the *distinguished form*, the lack of a simple characterization for it is rather frustrating. Mere rationality conditions like (2.42) would not do. The closest one might come to such a characterization would be by investigating the effect of an infinitesimal change of chart:

$$X \longmapsto X + \varepsilon[Y, X] + o(\varepsilon) \quad (Y \text{ fixed}) \tag{7.28}$$

$$X^{\mathrm{dist}} \longmapsto X^{\mathrm{dist}} + \varepsilon[Y^*, X^{\mathrm{dist}}] + o(\varepsilon) \tag{7.29}$$

because, due to equation (4.59), Y^* has a simple expression in terms of X, Y *and* the moulds $\$^{\bullet}_{\omega_0}$. But since successive derivations ∇_{ω_0}, $\nabla_{\omega_0'}$, $\nabla_{\omega_0''}$, etc., when applied to $\$^{\bullet}$, seem to generate ever new moulds, the prospects for a useful characterization (such as a simple link between X, Y, X^{dist}, Y^*, *without the involvement of any mould*) appear to be very remote.

The truth of the matter seems to be that the *distinguished form* belongs to those notions that admit of no other workable definition than *analytical ones*, and this peculiarity will find its reflection in the very distinctive type of *divergence* and *resurgence* that *distinguished forms* exhibit (see §9).

Proof of Proposition 7.1 and 7.2. — Let us deal with vector fields first. We closely follow the proofs of Proposition 2.2 and 2.3, at the end of §2, except for two things. *First*, we can, right at the outset, make use of the mould factorizations (2.33)–(2.36), whereas the whole point of the earlier proof was to establish those factorizations. *Second*, the mould-comould contractions, instead of involving the comould (2.53) made up from elements of the free Lie algebra $\mathcal{L}$, now involve the comould (7.27) built from the homogeneous parts $\mathbb{B}_n$ of the vector field X. But since we may now take the *lateral and central factorizations* (2.33)–(2.36) for granted, the freedom of $\mathcal{L}$ matters no longer, and the only material points are the *cosymmetralness* of $\mathbb{B}_{\bullet}$, along with the *gradedness* property (which is now relative to the scalar product $\omega = \langle \lambda, n \rangle$ with $\lambda \in \mathbb{C}^{\nu}$ and $n \in \mathbb{N}^{\nu}_{*}$) so that we can duplicate all the steps of the earlier proof.

More precisely, for any *non-resonant* vector field X, equation (2.61) provides an explicit linearization of X, with local coordinate changes (2.58), (2.59) that are not merely formal, but also convergent (*i.e.* analytic) if X is non-quasiresonant as well as being non-resonant. If, however, X is quasiresonant, the only way to restore convergence is by means of the *compensation* technique, *i.e.* by introducing one or several variables t and allowing *non-entire powers* of those variables. Now, when one studies the quasiresonant case for its own sake, as in [E8], it is advisable to work exclusively with *real positive* powers of t (so as to handle only infinitesimal quantities) and this may call for the introduction of several (upto three) new "ramified" variables. Here, however, we are interested in quasiresonance merely as a stepping-stone to resonance, and for our purpose one additional variable t is enough, even if that may entail working with negative or non-real powers $t^{\|\boldsymbol{\omega}\|}$ of t. The "compensated" linearization equation is none other than (2.64), and the corresponding coordinate changes are given by (2.62), (2.63). "Compensated" linearization, however, unlike plain linearization, survives even in the limit-case of *resonance*, and there the careful separation of the logarithmic and logarithm-free parts in (2.62), (2.63) leads successively to the conjugacy relations (2.66), (2.71), (2.72), (2.78), (2.79), which establish the analytical expression (7.17) for X^{nil}.

There is also a more direct, if less natural, way of establishing the formal expansion (7.17) of X^{nil}. Using the fact that $\$^{\boldsymbol{\omega}} = 1$ (resp. 0) if $\boldsymbol{\omega}$ is a sequence consisting of *one* (resp. *several*) *zeros* (see (3.17)), we see at once that, *in any prenormal chart*:

$$X^{\text{dia}} = X^{\text{lin}};\ X^{\text{nil}} = X - X^{\text{lin}}. \tag{7.30}$$

Then, using the formula (3.16*) for the ∇_{ω_0}-derivatives of $\$^{\bullet}$, and reasoning as in §3 (see towards the end, after (3.36)), we observe that, under any change of coordinates Θ, the *formal vector field* X^{nil} *as defined by* (7.17) *transforms precisely as in* (7.13). Both properties, taken together, show that the sum (7.17), calculated in any chart, *is* indeed the nilpotent part of X.

Paradoxically, the results pertaining to X^{dist} are quicker to prove than those pertaining to X^{nil}. Indeed, the conjugacy equation (2.39) between the moulds $\$^{\bullet}$ and $\$\!\!|^{\bullet}$ immediately translates, due to the inversion (2.65), into the conjugacy equation:

$$\left(\sum \$^{\bullet}\mathbb{B}_{\bullet}\right)\Theta_{\text{ext}} = \Theta_{\text{ext}}\left(\sum \$\!\!|^{\bullet}\mathbb{B}_{\bullet}\right). \tag{7.31}$$

8. Divergence and resurgence of the nilpotent part.

It has been known for a long time (see [B]) that the *nilpotent part* of a resonant vector field (and *a fortiori* of a diffeo) is generically divergent. For an exhaustive description of that divergence, we require the notion of *resurgent function* and *alien derivation* (see for ex. [E1], [E5], [E7], [E10]) and the *Bridge Equation* (see [E3], [E6], [E7]) which in its usual form reads:

$$\dot{\Delta}_{\omega}= \tilde{x}(z,u) = \mathbb{A}_{\omega}\tilde{x}(z,u) \quad (\forall\, \dot{\omega}\in \Omega) \tag{8.1}$$

and involves the following three ingredients.

First, we have a so-called *formal integral*:

$$\tilde{x}(z,u) = \{\tilde{x}_1(z,u_1,\dots,u_{\nu-1}),\dots,\tilde{x}_{\nu}(z,u_1,\dots,u_{\nu-1})\} \tag{8.2}$$

which is a *general* (*i.e.* parameter-saturated) *formal* solution of the differential system associated with the field $X = \sum X_i \partial_{x_i}$:

$$\partial_z \tilde{x}_i(z,u) = X_i(\tilde{x}(z,u)) \quad (i = 1, \dots, \nu) \tag{8.3}$$

or of the system of difference equations associated with a diffeo $f : x_i \mapsto f_i(x)$:

$$\tilde{x}_i(z+1,u) = f_i(\tilde{x}(z,u)) \quad (i = 1, \dots, \nu). \tag{8.4}$$

It thus provides a *formal* (hence the *twiddles*, which from now on will signal formalness) *non-entire chart* $(z, u_1, \dots, u_{\nu-1})$ in which the object assumes the simplest conceivable form, namely:

$$X = \frac{\partial}{\partial z} \quad \text{or} \quad f : z \longmapsto z+1. \tag{8.5}$$

Second, we have the symbols $\overset{\bullet}{\Delta}_\omega$ on the left-hand side of (8.1), which denote (pointed) *alien derivations* of index $\omega \in \mathbb{C}_\bullet$ (with projection $\overset{\bullet}{\omega}$ on $\mathbb{C}$). For a straightforward definition , see [E1] or [E7] or [E10]. The *raison d'être* of alien derivations is to analyse *divergence* and measure *singularities*. Indeed, *divergent-but-resurgent* power series $\widetilde{\varphi}(z) = \sum a_n z^{-n}$ have *endlessly continuable* Borel transforms $\widehat{\varphi}(\zeta) = \sum a_n \zeta^{n-1}/\Gamma(n)$, and the *singularities* of $\widehat{\varphi}(\zeta)$, which are responsible for the *divergence* of $\widetilde{\varphi}(z)$, are described with complete accuracy by the successive alien derivatives $\overset{\bullet}{\Delta}_{\omega_1} \widetilde{\varphi}(z)$, $\overset{\bullet}{\Delta}_{\omega_2} \overset{\bullet}{\Delta}_{\omega_1} \widetilde{\varphi}(z)$, etc.

Third, we have the symbols $\mathbb{A}_\omega$ on the right-hand side of (8.1), which denote *ordinary differential operators* in z and u. These are (completely and constructively) determined by the Bridge Equation, but are subject to no other *a priori* constraints than:

(i) preserving the general form of $\tilde{x}(z,u)$

(ii) satisfying the commutativity relations:

$$[\mathbb{A}_\omega, \partial] = 0 \quad \text{for a field} \quad (\partial = \partial_z) \tag{8.6}$$

$$[\mathbb{A}, \exp \partial] = 0 \quad \text{for a diffeo} \quad (\exp \partial = \text{ unit shift } z \mapsto z+1). \tag{8.7}$$

For a *vector field*, the operators $\mathbb{A}_\omega$ always assume the form:

$$\mathbb{A}_\omega = u^n \{ A^0_\omega \partial_z + \sum A^i_\omega u_i \partial_{u_i} \} \tag{8.8}$$

with indices ω ranging over an enumerable set Ω generated by the multipliers λ_i:

$$\overset{\bullet}{\omega} = \sum n_i \lambda_i; \; u^n = \prod u_i^{n_i} \quad (\omega \in \mathbb{C}_\bullet; \; \overset{\bullet}{\omega} \in \mathbb{C}; \; n_i \geq -1) \tag{8.9}$$

and for a diffeo it assumes the form:

$$\mathbb{A}_\omega = u^n e^{-n_0\lambda_0 z}\{A_\omega^0 \partial_z + \sum A_\omega^i u_i \partial_{u_i}\} \tag{8.10}$$

with indices ω ranging over a set Ω generated by the multipliers $\lambda_i = \log \ell_i$:

$$\dot{\omega} = n_0\lambda_0 + \sum n_i\lambda_i;\ \lambda_0 \overset{\text{def}}{=} 2\pi i;\ u^n = \prod u_i^{n_i}\ (i \neq 0)\ (n_i \geq -1) \tag{8.11}$$

relative to a *coherent determination* (see after (7.13)) of $\log \ell_i$. Note that in (8.9) and (8.11) *at most one* component n_i may be $= -1$, all others being ≥ 0.

Moreover, the operators $\mathbb{A}_\omega$ are *analytic invariants* of the object (diffeo or field) under investigation. In the case of one (resp. several) degrees of resonance, the formal integral $\tilde{x}(z,u)$ is essentially unique (resp. there exist essentially a finite number of them, each with its own invariants $\mathbb{A}_\omega$) and the coefficients A_ω^i of the operators $\mathbb{A}_\omega$ are scalar-valued (resp. dependent on some of the parameters u_i).

If we now resort to the formal change of variables $x_i = \tilde{x}_i(z,u)$ and denote by:

$$\mathcal{A}_\omega = \sum \mathcal{A}_\omega^i(x)\partial_{x_i} \qquad (i = 1, \dots, \nu) \tag{8.12}$$

the operators $\mathbb{A}_\omega$ expressed in the original, analytic chart $x = (x_i)$, we are in a position to analyse the divergence of the *diagonalizable* and *nilpotent* parts of local objects with the help of *resurgence equations*. We use the same notations as in Proposition 7.1 and 7.2.

PROPOSITION 8.1 (The Bridge Equation for the diagonalizable and nilpotent part). — *For a resonant vector field X, we have two systems of resurgence equations:*

$$[\dot{\Delta}_\omega, X^{\text{dia}}] = -[\mathcal{A}_\omega, X^{\text{dia}}] = +\,\dot{\omega}\,\mathcal{A}_\omega \tag{8.13}$$

$$[\dot{\Delta}_\omega, X^{\text{nil}}] = -[\mathcal{A}_\omega, X^{\text{nil}}] = -\,\dot{\omega}\,\mathcal{A}_\omega \tag{8.13*}$$

with ω of projection $\dot{\omega}$ as in (8.9).

For a resonant diffeo F, we have four systems:

$$[\dot{\Delta}_\omega, F^{\text{dia}}] = -[\mathcal{A}_\omega, F^{\text{dia}}] = (e^{\dot{\omega}} - 1)\mathcal{A}_\omega F^{\text{dia}} \tag{8.14}$$

$$[\dot{\Delta}_\omega, F^{\text{nil}}] = -[\mathcal{A}_\omega, F^{\text{nil}}] = (e^{\dot{\omega}} - 1)\mathcal{A}_\omega F^{\text{nil}} \tag{8.14*}$$

$$[\dot{\Delta}_\omega, X^{\text{dia}}] = -[\mathcal{A}_\omega, X^{\text{dia}}] = +\,\dot{\omega}\,\mathcal{A}_\omega \tag{8.15}$$

$$[\dot{\Delta}_\omega, X^{\text{nil}}] = -[\mathcal{A}_\omega, X^{\text{nil}}] = (n_0\lambda_0 - \dot{\omega})\mathcal{A}_\omega \tag{8.15*}$$

with ω, $\overset{\bullet}{\omega}$ *and* $n_0\lambda_0$ *as in (8.11).*

Before proceeding with the proof, a few words of elucidation are in order.

Remark 1 ((Interpretation of the Bridge Equation). — In the above equations, the alien derivations $\overset{\bullet}{\Delta}_\omega$ are of course relative, *not* to the variable z of the normalizing (z,u) chart (see (8.3), (8.4)), but to a variable z_* which, in the case of *one single degree of resonance* $\sum \lambda_i m_i = 0$ (with mutually prime integers m_i) always assumes the form:

$$z_* = x^{-pm} = x_1^{-pm_1} \cdots x_\nu^{-pm_\nu} \tag{8.16}$$

for some well-defined integer $p \geq 1$ (generically, $p = 1$).

In the case of multiple resonance, there are several such z_* (as many as there are formal integrals $\tilde{x}(z,u)$). However, the variables z and z_*, though distinct, are *formally equivalent* (in the sense that $z \sim z_*$ formally when z goes to infinity) when we relate them under the formal change of coordinates:

$$x_i \equiv \tilde{x}_i(z,u); \quad z_* \equiv z_*(\tilde{x}_i(z,u)). \tag{8.17}$$

Remark 2 (Consistency of the Bridge Equation). — Adding (8.13) and (8.13*) we find for a field:

$$[\overset{\bullet}{\Delta}_\omega, X] \equiv 0. \tag{8.18}$$

Similarly, applying $\overset{\bullet}{\Delta}_\omega$ to $F \equiv F^{\mathrm{dia}}F^{\mathrm{nil}}$ we find for a diffeo:

$$\begin{aligned}[\overset{\bullet}{\Delta}_\omega, F] &= [\overset{\bullet}{\Delta}_\omega, F^{\mathrm{dia}}]F^{\mathrm{nil}} + F^{\mathrm{dia}}[\overset{\bullet}{\Delta}_\omega, F^{\mathrm{nil}}]\\ &= (e^{\overset{\bullet}{\omega}} - 1)\mathcal{A}_\omega F + F^{\mathrm{dia}}(e^{-\overset{\bullet}{\omega}} - 1)\mathcal{A}_\omega F^{\mathrm{nil}}\\ &= (e^{\overset{\bullet}{\omega}} - 1)\mathcal{A}_\omega F + (e^{-\overset{\bullet}{\omega}} - 1)e^{\overset{\bullet}{\omega}}\mathcal{A}_\omega F\end{aligned}$$

so that here also we have:

$$[\overset{\bullet}{\Delta}_\omega, F] \equiv 0. \tag{8.19}$$

This is no surprise: (8.18) and (8.19) merely reflect the *analyticity* of the vector field X or diffeo F in the original (x_i)-chart. On the contrary, for a diffeo F of infinitesimal generator X (relative to some *coherent determination* of the various $\log \ell_i$; see above after (7.13)) we find after adding (8.15) and (8.15*):

$$[\overset{\bullet}{\Delta}_\omega, X] = -[\mathcal{A}_\omega, X] = n_0\lambda_0\mathcal{A}_\omega \tag{8.20}$$

with a third term that vanishes if $\overset{\bullet}{\omega}$ has no component $n_0\lambda_0$ (see (8.11)) but otherwise is generically $\neq 0$. This non-vanishing of $[\overset{\bullet}{\Delta}_\omega, X]$, in turn, simply reflects the generic divergence of the *infinitesimal generators* X of resonant diffeos F.

Remark 3 (Double completeness of the Bridge Equation). — Apart from being *consistent* (we couldn't expect less!), the Bridge Equation is also *complete*, and that too in a double sense. *First*, its form is such that it can be indefinitely iterated. In other words, we are in one of those cases when it is enough to know the *first order* alien derivatives to be capable of *recovering all* alien derivatives, of *all* orders. Thus, for a field X with its invariants $\mathcal{A}_\omega$ expressed in the (x_i)-chart, we find:

$$[\overset{\bullet}{\Delta}_{\omega_1}, \mathcal{A}_{\omega_0}] = -[\mathcal{A}_{\omega_1}, \mathcal{A}_{\omega_0}] \qquad (\forall \omega_0, \omega_1) \tag{8.21}$$

$$[[\overset{\bullet}{\Delta}_{\omega_2}, \overset{\bullet}{\Delta}_{\omega_1}], \mathcal{A}_{\omega_0}] \equiv +[[\mathcal{A}_{\omega_2}, \mathcal{A}_{\omega_1}], \mathcal{A}_{\omega_0}] \quad (\forall \omega_0, \omega_1, \omega_2) \tag{8.22}$$

etc., so that:

$$[[\overset{\bullet}{\Delta}_{\omega_2}, \overset{\bullet}{\Delta}_{\omega_1}]X^{\mathrm{dia}}] = -(\overset{\bullet}{\omega}_1 + \overset{\bullet}{\omega}_2)[\mathcal{A}_{\omega_2}, \mathcal{A}_{\omega_1}] \tag{8.23}$$

$$[[\overset{\bullet}{\Delta}_{\omega_2}, \overset{\bullet}{\Delta}_{\omega_1}]X^{\mathrm{nil}}] = +(\overset{\bullet}{\omega}_1 + \overset{\bullet}{\omega}_2)[\mathcal{A}_{\omega_2}, \mathcal{A}_{\omega_1}] \tag{8.23*}$$

etc. (compare with (8.13), (8.13*) and note the *reversal* of signs).

But more than that: we can also express all successive alien derivatives, *without brackets*:

$$\overset{\bullet}{\Delta}_{\omega_r} \cdots \overset{\bullet}{\Delta}_{\omega_2} \overset{\bullet}{\Delta}_{\omega_1} \; X^{\mathrm{dia}} \quad (\text{or } X^{\mathrm{nil}}) \tag{8.24}$$

in terms of the sole invariants $\mathcal{A}_{\omega_1}, \mathcal{A}_{\omega_2}, \ldots, \mathcal{A}_{\omega_r}$ and X^{dia} (or X^{nil}), which means that the Bridge Equation formalism encapsulates all the information needed to understand the divergence-cum-resurgence of the diagonalizable and nilpotent parts (at least in the absence of quasiresonance or nihilence) and, in particular, to describe the highly intricate behaviour of *their Borel transforms* ($z_* \to \zeta_*$) on all the leaves of their severely ramified Riemann surfaces.

The "second completeness", which of course is intimately connected with the first, has to do with the collection $\{\mathcal{A}_\omega;\ \overset{\bullet}{\omega} \in \Omega\}$ of *invariants* produced by the Bridge Equation: they happen to constitute a *complete* system of *holomorphic invariants*, and also (barring quasiresonance or nihilence) a complete system of *analytic invariants*. See for instance [E3] or [E7]. We recall that *analytic invariants* are invariants relative to *analytic* changes of coordinates; while the nearly homonymous *holomorphic invariants* are a special subclass of analytic invariants — those namely which depend *holomorphically* on the object X or F, *i.e.* on its Taylor coefficients (except for the first few coefficients that determine the resonance pattern, the level p, etc.).

Remark 4 (Differences between fields and diffeos). — For a field X, we have the two systems of resurgence equations (8.13) and (8.13*), each yielding *all* the invariants $\mathcal{A}_\omega$. But for a diffeo F, we have four systems: (8.15), (8.15*), which again yield *all* the invariants; and (8.14), (8.14*), which yield only the invariants of index $\overset{\bullet}{\omega} \neq 0 \pmod{2\pi i}$.

Remark 5 (Comparison with the classical Bridge Equation). — Although, from the point of view of analysis, the classical form (8.1) of the Bridge Equation, which involves the formal integral $\tilde{x}(z,u)$, and the other classical form:

$$[\overset{\bullet}{\Delta}_\omega, \Theta_{\text{nor}}] = -\mathcal{A}_\omega \Theta_{\text{nor}} = -\Theta_{\text{nor}} \mathbb{A}_\omega \tag{8.25}$$

(see [E7]), which involves the normalizing change of coordinates Θ_{nor}:

$$X = \Theta_{\text{nor}} X^{\text{nor}} \Theta_{\text{nor}}^{-1} (\text{resp. } F = \Theta_{\text{nor}} F^{\text{nor}} \Theta_{\text{nor}}^{-1}) \tag{8.26}$$

are both equivalent to the Bridge Equation of Proposition 8.1, which involves the diagonalizable or nilpotent part, the new variant has its special merits, because its ingredients X^{dia}, X^{nil}, F^{dia}, F^{nil} are expressible in terms of formal but *entire* power series (unlike the non-entire formal series inside $\tilde{x}(z,u)$) and are also *intrinsic* (unlike Θ_{nor}, which depends on the choice of the normal form X^{nor} or F^{nor}, to which there attaches a degree of arbitrariness, especially in the case of multiple resonance).

Admittedly, for the actual calculation of the alien derivatives, we must, here also, introduce some variable z_* like (8.16), which has the effect of ultimately destroying the "entireness" of our objects, but this *doesn't show in the Bridge Equation itself*.

In any case, the end result remains unaffected, and this brings home, once again, the flexibility of *alien calculus, which enables us to "read" all the invariants* $\mathbb{A}_\omega$*, in a simple, constructive manner, on practically any divergent object deduced in a natural way from* X *or* F.

Things change, however, when the object in question is defined by *analytical* rather than *geometric* means, as in the case of the *distinguished forms* X^{dist} and F^{dist}: we shall see in §9 that we still have resurgence, but of a different nature altogether.

First proof of Proposition 3.1. — Let us begin with vector fields. In any of the *normal charts* referred to in (8.5) and Remark 5 (above), the fields X, X^{dia}, X^{nil} reduce respectively to ∂_z, X^{lin}, $\partial_z - X^{\text{lin}}$, so that, in addition to (8.26), we have the conjugacies

$$X^{\text{dia}} = \Theta_{\text{nor}} X^{\text{lin}} \Theta_{\text{nor}}^{-1} \tag{8.27}$$

$$X^{\text{nil}} = \Theta_{\text{nor}} (\partial_z - X^{\text{lin}}) \Theta_{\text{nor}}^{-1}. \tag{8.28}$$

But the normalizing transformation Θ_{nor} verifies the resurgence equations (8.25). So its inverse Θ_{nor}^{-1} verifies:

$$[\overset{\bullet}{\Delta}_\omega, \Theta_{\text{nor}}^{-1}] = \mathbb{A}_\omega \Theta_{\text{nor}}^{-1} = \Theta_{\text{nor}}^{-1} \mathcal{A}_\omega. \tag{8.29}$$

Applying the alien derivation $\overset{\bullet}{\Delta}_\omega$ to (8.27), we find:

$$\begin{aligned} [\overset{\bullet}{\Delta}_\omega, X^{\text{dia}}] = &+ [\overset{\bullet}{\Delta}_\omega, \Theta_{\text{nor}}] X^{\text{lin}} \Theta_{\text{nor}}^{-1} \\ &+ \Theta_{\text{nor}} [\overset{\bullet}{\Delta}_\omega, X^{\text{lin}}] \Theta_{\text{nor}}^{-1} \\ &+ \Theta_{\text{nor}} X^{\text{lin}} [\overset{\bullet}{\Delta}_\omega, \Theta_{\text{nor}}^{-1}]. \end{aligned} \tag{8.31}$$

In view of (8.26) and (8.29), and since $\overset{\bullet}{\Delta}_\omega$ commutes with X^{lin}, this becomes:

$$\begin{aligned} [\overset{\bullet}{\Delta}_\omega, X^{\text{dia}}] &= -\mathcal{A}_\omega \Theta_{\text{nor}} X^{\text{lin}} \Theta_{\text{nor}}^{-1} + \Theta_{\text{nor}} X^{\text{lin}} \Theta_{\text{nor}}^{-1} \mathcal{A}_\omega \\ &= -[\mathcal{A}_\omega, X^{\text{dia}}] \end{aligned} \tag{8.31}$$

which establishes (8.13). Equation (8.13*) follows in the same way from alien-differentiating (8.28) and using the commutation of $\overset{\bullet}{\Delta}_\omega$ with ∂_z. The argument is exactly the same for diffeos.

Sketch of a second proof. — The shortness of the first proof is slightly deceptive, because it presupposes equation (8.26) and all the work that goes into establishing it (see [E3] or [E7]). But Proposition 8.1 may also be proven *from scratch*, directly from the mould expansion (7.17) for X^{nil}. We shall explain the method in some detail in §9, to study the resurgence of X^{dist} from its own mould expansion, because for X^{dist} there seems to be no other approach. For X^{nil}, however, this "direct" method is just one among others, and so we shall be very brief.

Using the same notations as in §9, but with $\$^{\bullet}$ instead of $\$\!\!\!\$^{\bullet}$, we find for X^{nil} the formal expansion:

$$X^{\text{nil}} = \partial_z + \sum_{r \geq 1} \sum_{\eta_i} \frac{1}{r} \$_{\text{coamp}}^{\eta_1, \ldots, \eta_r}(z) e^{(\omega_1 + \cdots \omega_r) z} [\mathbb{D}_{\eta_r} \cdots [\mathbb{D}_{\eta_2}, \mathbb{D}_{\eta_1}]] \tag{8.32}$$

deduced from (7.17) essentially by regrouping all terms that differ only by the number of occurrences of the *lowest homogeneous component* $\mathbb{B}_m$ of X (see (9.5), (9.16)). Here, $\eta_i = \binom{\omega_i}{\eta_i}$ as in (5.70) and (9.17*) but $\omega_1 + \cdots + \omega_r$ may be $\neq 0$, which explains why (8.32), unlike its counterpart (9.20), has an exponential term. As for the operators $\mathbb{D}_{\eta_i}$, they are deduced in a simple way from the homogeneous components $\mathbb{B}_n$ of X (see (9.18)). But the crucial ingredient of (8.32) is of course the *coamplification* $\$^\bullet_{\text{coamp}}(z)$ of the mould $\$^\bullet$, which is defined as in (5.71), (5.72) and contains *in nuce* all the resurgence properties of X^{nil}. Indeed, the mould equation (5.11) valid for the *amplification* $\$^\bullet_{\text{amp}}$ translates into an entirely similar mould equation for the *coamplification* $\$^\bullet_{\text{coamp}}$:

$$(8.33) \qquad \$^\bullet_{\text{coamp}}(z) = I^\bullet + (S^\bullet_{\text{coamp}}(z)) \times (\nabla^* S^\bullet_{\text{coamp}}(z)).$$

The factors $S^\bullet_{\text{coamp}}(z)$ and $S^\bullet_{\text{coamp}}(z)$ are *symmetral* moulds, and *resurgent* in z. In fact, they coincide with the classical *resurgence monomials* $\mathcal{V}^\bullet(z)$ and $\mathcal{V}\!\!\!{-}^\bullet(z)$, whose *alien derivatives* involve the scalar-valued moulds $V^\bullet_{\omega_0}$ of "hyperlogarithmic" type:

$$(8.34) \qquad \Delta_{\omega_0}\mathcal{V}^\bullet(z) = +V^\bullet_{\omega_0} \times \mathcal{V}^\bullet(z)$$

$$(8.35) \qquad \Delta_{\omega_0}\mathcal{V}\!\!\!{-}^\bullet(z) = -\mathcal{V}\!\!\!{-}^\bullet(z) \times V^\bullet_{\omega_0}.$$

For details, see for instance [E7], §6, p. 104–108.

In this way, the Bridge Equation for X^{nil} (and so too for X^{dia}) can be recovered directly from (8.32) with the help of (8.33), (8.34), (8.35) and some analysis. We even get as a premium a nice expression of $\mathcal{A}_\omega$ in terms of $V^\bullet_\omega$ and $\$^\bullet$, which mirrors the classical expression (see [E7], p. 116, (7.62)) of $\mathbb{A}_\omega$ in terms of $V^\bullet_\omega$ alone.

9. Divergence and resurgence of the distinguished form.

The present section aims at suggestiveness rather than completeness. It is meant as an appetizer — a means of whetting the reader's curiosity for the stunning breadth and variety of resurgence phenomena that anyone trafficking in divergent series is bound to encounter at every step.

We restrict ourselves to *vector fields* (although the picture isn't much different for *diffeos*), and, to simplify still further, we assume that there is only one degree of resonance:

$$(9.1) \quad \sum m_i\lambda_i = 0 \quad (m_i \geq 0 \text{ and the non-zero } m_i \text{ are mutually prime}).$$

As usual (see §8 or [E7]) we denote by Ω the set of all complex numbers ω of the form (9.2) or (9.2*):

$$\omega = \sum n_i\lambda_i \qquad (n_i \in \mathbb{N}) \tag{9.2}$$

$$\omega = -\lambda_j + \sum n_i\lambda_i \qquad (i \neq j,\ n_i \in \mathbb{N}) \tag{9.2*}$$

but (in order to avoid the complications that come from everywhere-dense singularities in the Borel plane) we assume Ω to be discrete. For definiteness, we may think of the case (9.3) or (9.4):

$$m_1\lambda_1 + m_2\lambda_2 = 0, \text{ with } m_i \geq 1 \text{ and } \operatorname{Re}(\lambda_j) > 0 \text{ for } j = 3,4,\ldots,\nu \tag{9.3}$$

(9.4)

$$m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 = 0, \text{ with } m_i \geq 1 \text{ and } \operatorname{Re}(\lambda_j) > 0 \text{ for } j = 4,5,\ldots,\nu.$$

In (9.4) we assume $\lambda_1, \lambda_2, \lambda_3$ to be *non-aligned.*

Lastly, we assume that X admits the simplest possible (non linear) *normal form* compatible with its resonance type, namely:

$$X^{\text{nor}} = X^{\text{lin}} + \mathbb{B}_m;\ X^{\text{lin}} = \sum \lambda_i x_i \partial_{x_i};\ \mathbb{B}_m = x^m \sum \tau_i x_i \partial_{x_i} \tag{9.5}$$

with:

$$x^m = x_1^{m_1} \cdots x_\nu^{m_\nu};\ 0 = \langle m, \lambda \rangle = \sum m_i\lambda_i;\ -1 = \langle m, \tau \rangle = \sum m_i\tau_i \tag{9.5*}$$

and we agree to express X in a *prepared chart*; *i.e.* an analytic chart that "isolates" the normal form:

$$X = X^{\text{lin}} + \mathbb{B}_m + \sum_n \mathbb{B}_n \text{ with } n_1 + \cdots + n_\nu > 2(m_1 + \cdots + m_\nu). \tag{9.5**}$$

These assumptions aren't essential, but they will make life easier.

PROPOSITION 9.1 (Divergence of the distinguished form). — *The distinguished form X^{dist} of the resonant vector field X is resurgent with respect to the same variable $z_* \equiv x^{-m}$ as the nilpotent part X^{nil}, but with a resurgence "lattice" Ω^{dist} much larger than Ω:*

$$\Omega^{\text{dist}} = 2\pi i\mathbb{Z}^* \cdot \Omega \quad (\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}) \tag{9.6}$$

and with alien derivatives $[\overset{\bullet}{\Delta}_\omega, X^{\text{dist}}]$ which are strikingly different from the earlier derivatives $[\overset{\bullet}{\Delta}_\omega, X^{\text{nil}}]$ in so far as:

(i) *they do not involve the holomorphic invariants of X, whether in the form $\mathbb{A}_\omega$ or $\mathcal{A}_\omega$;*

(ii) *they are expressible as "bilateral" power series of the form $\sum a_n(z_*)^{-n-\tau(\omega)}$, with n running through the whole of $\mathbb{Z}$, not just $\mathbb{N}$;*

(iii) *they make it possible (save in one trivial case, mentioned below) to reconstruct the actual field X, in its original analytic chart, from the sole knowledge of X^{dist} and its alien derivatives.*

Main steps of the proof. — We begin with three easy lemmas, which involve the following ingredients:

(i) an alternal mould $M^\bullet$ with its amplification $M^\bullet_{\text{amp}}$:

$$(9.7)\quad M^\bullet = \{M^{\omega_1,\dots,\omega_r};\omega_i \in \mathbb{C}\};\ M^\bullet_{\text{amp}} = \{M_{\text{amp}}^{\varpi_1,\dots,\varpi_r};\varpi_i = \begin{pmatrix}\omega_i\\ a_i\end{pmatrix} \in \mathbb{C}^2\};$$

(ii) a free associative algebra $\mathcal{A}$ (resp. a free Lie algebra $\mathcal{L}$) generated by elements $\beta_0,\beta_1,\beta_2,\dots$, with the following notations:

$$(9.8)\qquad \bar\beta_0\beta \overset{\text{def}}{=} [\beta_0,\beta] \qquad\qquad (\forall\beta \text{ in } \mathcal{L} \text{ or } \mathcal{B})$$

$$(9.9)\qquad \beta_i^{[n]} \overset{\text{def}}{=} (\bar\beta_0)^n\cdot\beta_i = [\beta_0\cdots[\beta_0[\beta_0,\beta_i]]] \quad (\beta_0 \text{ repeated } n \text{ times}).$$

LEMMA 9.1. — *For any two sequences $\boldsymbol{\omega}^1,\boldsymbol{\omega}^2$ of length r_1,r_2 and any $\omega_0 \in \mathbb{C}$, we have:*

(9.10)
$$M^{\boldsymbol{\omega}^1\omega_0\boldsymbol{\omega}^2} = (-1)^{r_1}\sum_{\boldsymbol{\omega}\in\text{sh}(\widetilde{\boldsymbol{\omega}}^1,\boldsymbol{\omega}^2)} M^{\omega_0\boldsymbol{\omega}} = (-1)^{r_2}\sum_{\boldsymbol{\omega}\in\text{sh}(\boldsymbol{\omega}^1,\widetilde{\boldsymbol{\omega}}^2)} M^{\boldsymbol{\omega}\omega_0}.$$

Here the notations $\boldsymbol{\omega}^1\omega_0\boldsymbol{\omega}^2$, or $\boldsymbol{\omega}\omega_0$, or $\omega_0\boldsymbol{\omega}$, stand for the usual juxtaposition of sequences, and the tilda denotes order reversal:

$$(9.11)\qquad (\omega_1,\omega_2,\dots,\omega_r)^\sim \overset{\text{def}}{=} (\omega_r,\dots,\omega_2,\omega_1).$$

The first (resp. second) sum in (9.10) extends to all sequences $\boldsymbol{\omega}$ obtained by *shuffling* $\widetilde{\boldsymbol{\omega}}^1$ with $\boldsymbol{\omega}^2$ (resp. $\boldsymbol{\omega}^1$ with $\widetilde{\boldsymbol{\omega}}^2$). (See (2.2)). These identities can be checked inductively on r_1 (or r_2) under repeated use of the identity (2.2) for alternal moulds (with 0 on the right-hand side).

LEMMA 9.2. — *For any given sequence $\boldsymbol{\omega} = (\omega_1,\dots,\omega_r)$, each of these three finite sums defines the same element of $\mathcal{L}$:*

$$(9.12^*)\qquad \sum_\sigma M^{\omega_{\sigma(1)},\dots,\omega_{\sigma(r)}}\beta_{\sigma(r)}\cdots\beta_{\sigma(2)}\beta_{\sigma(1)}$$

$$(9.12^{**})\qquad r^{-1}\sum_\sigma M^{\omega_{\sigma(1)},\dots,\omega_{\sigma(r)}}[\beta_{\sigma(r)}\cdots[\beta_{\sigma(2)}\beta_{\sigma(1)}]]$$

$$(9.12^{***})\qquad \sum_{\sigma(1)=1} M^{\omega_1,\omega_{\sigma(2)},\dots,\omega_{\sigma(r)}}[\beta_{\sigma(r)}\cdots[\beta_{\sigma(2)}\beta_1]].$$

All three sums are over all permutations σ of the set $\{1, 2, \ldots, r\}$, except that in (9.12***) we allow only permutations that leave 1 fixed. Though each summand in (9.13*) belongs to $\mathcal{B}$ rather than $\mathcal{L}$, due to the alternality of $M^{\bullet}$, the sum belongs to $\mathcal{L}$, and so it admits the classical projection (9.12**) in terms of Lie brackets, with a factor r^{-1} reflecting the homogeneous degree. Then we resort to identity (9.10) with $\omega_0 = \omega_1$ to rewrite each summand of (9.12**) in terms of sequences beginning with ω_1, and we check that (9.12**) transforms into (9.12***), *without* the factor r^{-1}.

LEMMA 9.3. — *For any given sequence $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$, each of these four infinite sums defines the same element of $\overline{\mathcal{L}}$ (the natural closure of $\mathcal{L}$):*

(9.13)
$$\sum_{\sigma} \sum_{n_i \geq 0} M^{0^{(n_0)}, \omega_{\sigma(1)}, 0^{(n_1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(r)}, 0^{(n_r)}} (\beta_0)^{n_r} \beta_r \cdots \beta_2 (\beta_0)^{n_1} \beta_1 (\beta_0)^{n_0}$$

$$(9.13^*) \qquad \sum_{\sigma} \sum_{n_i \geq 0} M_{n_1, n_2, \ldots, n_r}^{\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(r)}} \beta_{\sigma(r)}^{[n_r]} \cdots \beta_{\sigma(2)}^{[n_2]} \beta_{\sigma(1)}^{[n_1]}$$

$$(9.13^{**}) \qquad r^{-1} \sum_{\sigma} \sum_{n_i \geq 0} M_{n_1, n_2, \ldots, n_r}^{\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(r)}} [\beta_{\sigma(r)}^{[n_r]} \cdots [\beta_{\sigma(2)}^{[n_2]}, \beta_{\sigma(1)}^{[n_1]}]]$$

$$(9.13^{***}) \qquad \sum_{\sigma(1)=1} \sum_{n_i \geq 0} M_{n_1, n_2, \ldots, n_r}^{\omega_1, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(r)}} [\beta_{\sigma(r)}^{[n_r]} \cdots [\beta_{\sigma(2)}^{[n_2]}, \beta_1^{[n_1]}]].$$

The sums are over all integers $n_i \geq 0$ and all permutations σ of $\{1, 2, \ldots, r\}$ or, in the last instance, of $\{2, \ldots, r\}$. The $\beta_i^{[n]}$ are defined as in (9.9) and the coefficients $M_{n_1, \ldots, n_r}^{\omega_1, \ldots, \omega_r}$ as in (5.68). It is plain that (9.13*), (9.13**), (9.13***) relate to each other exactly as (9.14*), (9.14**), (9.14***) do. So the only thing left to prove is the equivalence of (9.13) and (9.13*). This is done by rewriting (9.13) in terms of $(\bar{\beta}_0)^{n_i}$ rather than $(\beta_0)^{n_i}$, and by using the Leibniz identity:

$$(9.14) \qquad (\beta_i \cdots \beta_2 \beta_1)^{[n]} = \sum_{n_1 + n_2 + \cdots + n_i = n} \frac{n!}{n_1! n_2! \cdots n_i!} \beta_i^{[n_i]} \cdots \beta_2^{[n_2]} \beta_1^{[n_1]}$$

successively for $i = 1, 2, \ldots, r$.

We now revert to our vector field X and its decomposition (9.5**) into homogeneous components. If we introduce the non-entire chart (z, u) defined by (*):

$$(9.15) \qquad x_i = u_i z^{\tau_i} \exp(\lambda_i z) \qquad (i = 1, 2, \ldots, \nu;\ u^m = 1;\ x^m = z^{-1})$$

(*) Denoting the new chart (z_*, u_*) rather than (z, u) would be more consistent with the notations of Proposition 9.1, but all too cumbersome.

(with z large and u suitably small), and express the fields X^{lin}, X^{nor}, $\mathbb{B}_m$ and X in the new chart, we find:

$$(9.16)\quad X^{\text{lin}} = \sum \lambda_i u_i \partial_{u_i};\quad X^{\text{nor}} = \partial_z\ ;\quad \mathbb{B}_m = \partial_z - \sum_i \lambda_i u_i \partial_{u_i}$$

$$(9.17)\quad X = \partial_z + \mathbb{B}$$

$$(9.17^*)\quad \mathbb{B} = \sum_n \mathbb{B}_n = \sum_\eta e^{\omega z} z^{-\sigma} \mathbb{B}_\eta \quad \left(n \in \mathbb{N}^\nu_*;\ \eta = \binom{\omega}{\sigma} \in \mathbb{C}^2\right)$$

$$(9.17^{**})\quad \mathbb{B}_\eta = B^0_\eta \partial_z + \sum_i B^i_\eta u_i \partial_{u_i} \quad (B^i_\eta \in \mathbb{C}).$$

Each operator $\mathbb{B}_\eta$ is elementarily calculable from (at most) *two* homogeneous parts $\mathbb{B}_n$, and obviously commutes with ∂_z. But we require yet another set of operators $\mathbb{D}_\eta$, which have the same outward form (9.17***) as the $\mathbb{B}_\eta$, and derive from them according to the simple rule:

$$(9.18)\qquad \mathbb{D} = (1 + \mathbb{B} \cdot z)^{-1} \mathbb{B}$$

which relates the generating function $\mathbb{B}$ of the $\mathbb{B}_\eta$ to the analogous generating function for the $\mathbb{D}_\eta$:

$$(9.19)\qquad \mathbb{D} = \sum_\eta e^{\omega z} z^{-\sigma} \mathbb{D}_\eta \qquad \left(\eta = \binom{\omega}{\sigma} \in \mathbb{C}^2\right).$$

LEMMA 9.4. — *With the above notations, the distinguished form of X admits in the (z, u) chart the formal expansion:*

$$(9.20)\qquad X^{\text{dist}} = \partial_z + \sum_{r \ge 1} \sum_{\eta_i} \frac{1}{r}\, \$^{\eta_1,\dots,\eta_r}_{\text{coamp}}(z) [\mathbb{D}_{\eta_r} \cdots [\mathbb{D}_{\eta_2}, \mathbb{D}_{\eta_1}]]$$

where $\$^{\bullet}_{\text{coamp}}(z)$ denotes the coamplification of the mould $\$^{\bullet}$, defined as in (5.71), (5.72).

To establish (9.20), we isolate in (9.5**) the linear part X^{lin} and the *lowest homogeneous component* $\mathbb{B}_m$, which in the (z, u) chart assume the form (9.16). Then we fall back on the expansion (7.18) *that actually defines* X^{dist}, and regroup therein *all terms* that *differ only by the number of occurences of* $\mathbb{B}_m$. Then we fix a sequence $(\eta_1, \dots, \eta_r)$ and apply Lemma 9.3 with:

$$(9.21)\qquad \beta_0 = \mathbb{B}_m;\ \ \beta_1 = \mathbb{B}_{\eta_1};\ \ \beta_2 = \mathbb{B}_{\eta_2}; \cdots;\ \ \beta_r = \mathbb{B}_{\eta_r}.$$

More precisely, we use the identity of (9.13) and (9.13**), together with the remark that:

$$(9.22)\qquad (\bar{\beta}_0)^n (u^{n(\omega)} e^{\omega z} \cdot z^{-\sigma}) = u^{n(\omega)} e^{\omega z} (\partial^n z^{-\sigma}).$$

In the end, everything turns out to be expressible in terms of the *coamplification* of $\$^{\bullet}$ and of the operators $\mathbb{D}_{\eta}$. A little effort is required to justify the change from $\mathbb{B}_{\eta}$ to $\mathbb{D}_{\eta}$, by means of suitable rearrangements. We may proceed as in [E7], p. 114. But this point is inessential, and if we want to concentrate purely on the *analysis difficulties*, we can think of the situation when, except for the lowest homogeneous component $\mathbb{B}_{m}$, all the other $\mathbb{B}_{n}$ annihilate the resonant monomial $x^{m} \equiv z^{-1}$, in which case $\mathbb{B}_{\eta} \equiv \mathbb{D}_{\eta}$ for all η.

At this stage, two more things are required to complete the proof of Proposition 9.1:

(i) We must show that each single term $\$^{\eta_1,\dots,\eta_r}_{\text{coamp}}(z)$, as a formal power series in $z^{-(\sigma_1+\dots+\sigma_r)}\mathbb{C}[[z^{-1}]]$, is *resurgent* in z, with a Borel transform $\widehat{\$}^{\bullet}_{\text{coamp}}(\zeta)$ that has no singularities outside the set Ω^{dist} introduced in (9.6); and then calculate the alien derivatives $\Delta_{\omega}\,\$^{\bullet}_{\text{coamp}}(z)$ for $\overset{\bullet}{\omega}$ in Ω^{dist}.

(ii) We must check that the term-by-term Borel transform $z \to \zeta$ of expansion (9.20) converges uniformly on each compact set of the universal covering of $\mathbb{C} \overset{\bullet}{-} \Omega^{\text{dist}}$.

We shall leave the *second point* alone (it is routine work but stupefyingly boring) and shall settle only part of the *first point*. The resurgent quality of $\$^{\eta_1,\dots,\eta_r}_{\text{coamp}}(z)$ follows from the endless analyticity of $\widehat{\$}^{\eta_1,\dots,\eta_r}_{\text{coamp}}(\zeta)$ as a function of ζ, which itself follows (as observed after (5.74), (5.75)) from the endless analyticity of $\$^{\varpi_1,\dots,\varpi_r}_{\text{amp}}$ as a function of $a_1,\dots,a_r$, which in turn follows from the integral representations (5.64). The precise shape (9.6) of Ω^{dist} is also deducible therefrom. So the only point left to elucidate is the nature of the *resurgence equations* verified by the mould $\$^{\bullet}_{\text{coamp}}(z)$. We shall describe these only in the simplest non-trivial case. This rules out sequences $\boldsymbol{\eta} = (\eta_1)$ of length 1, since for such sequences:

$$(9.23)\quad \$^{\eta_1}_{\text{coamp}}(z) \equiv 1 \text{ (resp. 0) if } \eta_1 = \begin{pmatrix}\omega_1\\ \sigma_1\end{pmatrix} \text{ with } \omega_1 = 0 (\text{resp. } \omega_1 \neq 0)$$

so that the *linear part* in (9.20) is in fact trivial, and merely reintroduces the whole collection of *resonant terms* $\mathbb{B}_{n}$ (with $\langle n, \lambda\rangle = 0$) present in (7.4). Thus the simplest non-trivial terms in (9.20) are *bilinear*, and correspond to sequences $\boldsymbol{\eta} = (\eta_1, \eta_2)$ of length $r = 2$, with of course $\omega_1 + \omega_2 = 0$. To further simplify, let us assume that σ_1, σ_2 are both in $\mathbb{N}^*$. The resurgence properties of $\$^{\eta_1,\eta_2}_{\text{coamp}}(z)$ are completely described by the following lemma.

LEMMA 9.5. — *Let σ_1, σ_2 be integers ≥ 1 and let $\omega_1 + \omega_2 = 0$ but*

$\omega_i \neq 0$, so that $\$^{\omega_1,\omega_2} = -(\omega_1)^{-1} = (\omega_2)^{-1}$. Then:

$$(9.24)\quad \widehat{\$}^{\eta_1,\eta_2}_{\text{coamp}}(\zeta) = \zeta^{\sigma_1+\sigma_2-2}\frac{(1-e^{-\zeta/\omega_1})^{1-\sigma_1}}{\Gamma(\sigma_1)}\frac{(1-e^{-\zeta/\omega_2})^{1-\sigma_2}}{\Gamma(\sigma_2)} \{\zeta\,\$^{\omega_1,\omega_2} + P^{\eta_1,\eta_2}(\zeta)\}$$

where $P^{\eta_1,\eta_2}(\zeta)$ is a polynomial in the variables $\exp(-\zeta/\omega_1)$ *and* $\exp(-\zeta/\omega_2)$*, of degree* (σ_1-1) *and* (σ_2-1)*, and such that:*

$$(9.25)\qquad \zeta\,\$^{\omega_1,\omega_2} + P^{\eta_1,\eta_2}(\zeta) = \mathcal{O}(\zeta^{\sigma_1+\sigma_2-1}) \text{ as } \zeta \to 0.$$

Proof of Lemma 9.5. — By (5.74) we have:

$$(9.26)\qquad \widehat{\$}^{\eta_1,\eta_2}_{\text{coamp}}(\zeta) = \int_0^{\zeta} \$^{\varpi_1,\varpi_2}_{\text{amp}} \frac{(\zeta-\zeta_2)^{\sigma_1-1}}{\Gamma(\sigma_1)}\frac{(\zeta_2)^{\sigma_2-1}}{\Gamma(\sigma_2)}d\zeta_2$$

with $\varpi_i = \binom{\omega_i}{-\zeta_i}$ and for ζ close to 0. But due to (5.33):

$$(9.26^*)\qquad \$^{\varpi_1,\varpi_2}_{\text{amp}} = -(\zeta_2 - B)^{-1} \text{ with } B = \zeta(1-\exp(-\zeta/\omega_2))^{-1}.$$

So we find at first for $(\sigma_1,\sigma_2)=(1,1)$:

$$(9.27)\qquad \widehat{\$}^{\eta_1,\eta_2}_{\text{coamp}}(\zeta) = \zeta\,\$^{\omega_1,\omega_2} = \zeta/\omega_2 \qquad (\sigma_1=\sigma_2=1).$$

Next, by using the identity:

$$(9.28)\quad (\zeta_2^{\sigma_2-1} - B^{\sigma_2-1})(\zeta-B)^{-1} = \sum \zeta_2^p B^q \quad (\text{with } p+q=\sigma_2-2)$$

we can reduce the case $(1,\sigma_2)$ to the case $(1,1)$. Lastly, by expanding $(\zeta-\zeta_2)^{\sigma_1-1}$ inside (9.26) into a sum of powers of ζ_2, we reduce the general case (σ_1,σ_2) to a superposition of cases $(1,\sigma_2')$. As for the behaviour of (9.25) when $\zeta \to 0$, it follows from the fact that $\$^{\varpi_1,\varpi_2}_{\text{amp}} = \$^{\omega_1,\omega_2}$ for $\zeta_1=\zeta_2=0$, so that:

$$(9.29)\qquad \widehat{\$}^{\eta_1,\eta_2}_{\text{coamp}}(\zeta) = \$^{\omega_1,\omega_2}\frac{\zeta^{\sigma_1+\sigma_2-1}}{\Gamma(\sigma_1+\sigma_2)}\{1+o(\zeta)\} \text{ as } \zeta\to 0.$$

We may now use Lemma 9.5 to understand the resurgence properties. The only singularities of the function (9.24) correspond to points $\zeta=\omega^*$ of the form:

$$(9.30)\qquad \omega^* = 2\pi i k\omega_2 = -2\pi i k \omega_1 \quad (\text{with } k\in\mathbb{Z}^*).$$

Combining the *periodicity* of $P^{\eta_1,\eta_2}(\zeta)$ with the estimate (9.25), we see that for ζ^* small and $\zeta=\omega^*+\zeta^*$:

$$(9.31)\quad \widehat{\$}^{\eta_1,\eta_2}_{\text{coamp}}(\omega^*+\zeta^*) = (2\pi i k)(\omega^*+\zeta^*)^{\sigma_1+\sigma_2-2}\frac{(1-e^{-\zeta^*/\omega_1})^{1-\sigma_1}}{\Gamma(\sigma_1)} \frac{(1-e^{-\zeta^*/\omega_2})^{1-\sigma_2}}{\Gamma(\sigma_2)} \ (\text{mod. REG})$$

modulo the space REG $= \mathbb{C}\{\{\zeta^*\}\}$ *of all regular analytic germs of* ζ^*. Going back to the z variable, we find an alien derivative:

$$\Delta_{\omega^*}\, \$\!\!\!\|^{\eta_1,\eta_2}_{\text{coamp}}(z) = Q^{\eta_1,\eta_2}_{\omega^*}(z) \in \mathbb{C}[z] \quad (\text{not } z^{-1}!) \tag{9.32}$$

with a polynomial $Q(z)$ of degree exactly $\sigma_1 + \sigma_2 - 3$ (except when $\sigma_1 = \sigma_2 = 1$, in which case $Q \equiv 0$) and of simple coefficients (involving no transcendental constants).

Things would change slightly for non-integral values of σ_1, σ_2, but the leading term in Q would still be $z^{\sigma_1+\sigma_2-3}$ with $\sigma_1 + \sigma_2 \in \mathbb{N}$, because each sequence $\eta_1, \dots, \eta_r$ in (9.20) corresponds to a sequence $\omega_1, \dots, \omega_r$ such that $\sum \omega_i = 0$, and therefore to a sequence $\sigma_1, \dots, \sigma_r$ such that $\sum \sigma_i \in \mathbb{N}$, even though the individual σ_i may not be in $\mathbb{N}$ or even $\mathbb{Z}$.

Thus, even for a sequence length $r = 2$, we may glimpse at several striking differences between the resurgence properties of the mould $\$^{\bullet}_{\text{coamp}}(z)$ which is relevant for X^{nil}, and those of the mould $\$\!\!\!\|^{\bullet}_{\text{coamp}}(z)$, which is relevant for X^{dist}.

First, whereas the alien derivatives of $\$^{\bullet}_{\text{coamp}}(z)$ involved transcendental constants $V^{\bullet}_{\omega_0}$ (see (8.34), (8.35)), those of $\$\!\!\!\|^{\bullet}_{\text{coamp}}(z)$ do not.

Second, for a given resonant field X and any fixed ω_i in Ω, there is a *lower limit* $\tau(\omega_i)$, but in general no *upper limit* to the values which the corresponding σ_i may assume in the pairs $\eta_i = \binom{\omega_i}{\sigma_i}$ that index the expansions (9.17*) and (9.19): generically, these σ_i will run through an entire set of the form $\tau(\omega_i) + \mathbb{N}$. So, looking back at (9.32) and the degree of Q, we see that for a generic resonant vector field X, and any point ω^* in Ω^{dist}, the Borel transform $z \to \zeta$ of X^{dist} will possess an *essential singularity* at ω^*. In the z variable, this means that the alien derivatives of X^{dist} will involve truly *bilateral* power series of the form $\sum a_n z^{-n-\tau(\omega^*)}$, with n running through the whole of $\mathbb{Z}$, not just $\mathbb{N}$ as in the case of X^{nil}.

These peculiarities, and many more which we gloss over, seem to be a standing feature with divergent power series that are "*constructed*" (by analy*tical* means) rather than "*found*" (as formal solutions of analy*tic* equations, or as formal expansions into power series of small singular parameters). For another instance of "artificial resurgence", coming from a rather different context but displaying very similar features, see [E3], p. 537–550 (especially Prop. 11.3.7).

To conclude this section, let us emphasize that the interest of the *distinguished form* X^{dist} lies, not in its relative simplicity compared with other *prenormal forms* (*i.e.* forms having only *resonant terms* in them), but in its handy *analytical expansion* (7.18) and *very special resurgence properties* (see above). But the *distinguished form* X^{dist} doesn't claim to be, and is not, a particularly "simple" prenormal form — quite the opposite, in fact. Indeed, according to [E7] (see p. 152, after (11.30)), in order for a *resonant analytic* vector field X to admit an *analytic prenormal form* or, more accurately, to be conjugate to an *analytic* X^{prenor} under an *analytic* change of variables Θ:

$$X = \Theta X^{\text{prenor}} \Theta^{-1} \tag{9.33}$$

it is *necessary* (and, barring quasiresonance or nihilence, *sufficient*) that the ∂_z-component of all *holomorphic invariants should vanish*, that is to say:

$$\mathbb{A}_\omega \cdot z \equiv 0 \qquad (\forall\, \overset{\bullet}{\omega} \in \Omega) \tag{9.34}$$

but *even when that condition is fulfilled*, the distinguished form X^{dist} is generically non-analytic.

10. Explicit criteria for linearizability or nihilence. Remark on the size and splitting of Lie ideals.

Let the $\mathbb{B}_{n_0}$ be elements of a graded Lie algebra, with indices n_0 in some abstract set $\mathcal{N}$ and with a gradation $\text{grad}(\mathbb{B}_{n_0}) = \omega_0 = \omega(n_0) \in \mathbb{C}$.

For any (fully) ordered sequence $\mathbf{n} = (n_1, \ldots, n_r)$ with $n_i \in \mathcal{N}$ and its image $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ under $n_i \mapsto \omega_i = \omega(n_i)$, we denote by $\underline{\mathbf{n}}$ and $\underline{\boldsymbol{\omega}}$ the corresponding *unordered sequences*, and we put:

$$\mathbb{B}_{\mathbf{n}} = \mathbb{B}_{n_1,\ldots,n_r} \overset{\text{def}}{=} \mathbb{B}_{n_r} \cdots \mathbb{B}_{n_2} \mathbb{B}_{n_1} \tag{10.1}$$

$$\mathbb{B}^{\$}_{\underline{\mathbf{n}}} \overset{\text{def}}{=} \sum_{\underline{\mathbf{n}}^*=\underline{\mathbf{n}}}^{*} \$^{\boldsymbol{\omega}^*}\, \mathbb{B}_{\mathbf{n}^*} = \sum_{\sigma} \$^{\omega_{\sigma(1)},\ldots,\omega_{\sigma(r)}}\, \mathbb{B}_{n_{\sigma(1)},\ldots,n_{\sigma(r)}} \tag{10.2}$$

$$\mathbb{B}^{\$\!\!\!|}_{\underline{\mathbf{n}}} \overset{\text{def}}{=} \sum_{\underline{\mathbf{n}}^*=\underline{\mathbf{n}}}^{*} \$\!\!\!|^{\,\boldsymbol{\omega}^*}\, \mathbb{B}_{\mathbf{n}^*} = \sum_{\sigma} \$\!\!\!|^{\,\omega_{\sigma(1)},\ldots,\omega_{\sigma(r)}}\, \mathbb{B}_{n_{\sigma(1)},\ldots,n_{\sigma(r)}} \tag{10.3}$$

with sums $\sum^*$ over all sequences $\mathbf{n}^*$ equivalent to $\mathbf{n}$ *upto order*, or with sums $\sum$ over all permutations σ of the set $\{1, 2, \ldots, r\}$. Due to the alternality of $\$^\bullet$ and $\$\!\!\!|^{\,\bullet}$, both $\mathbb{B}^{\$}_{\underline{\mathbf{n}}}$ and $\mathbb{B}^{\$\!\!\!|}_{\underline{\mathbf{n}}}$ are (contrary to the individual summands $\mathbb{B}_{\mathbf{n}^*}$) *Lie elements*, with gradation $\|\underline{\boldsymbol{\omega}}\| = \omega_1 + \cdots + \omega_r$. Clearly, $\mathbb{B}^{\$\!\!\!|}_{\underline{\mathbf{n}}} = 0$ if $\|\underline{\boldsymbol{\omega}}\| \neq 0$.

We now fix a sequence $\lambda = (\lambda_1, \ldots, \lambda_\nu) \in \mathbb{C}^\nu$ and we *specialize* both $\mathcal{N}$ and $\omega(n_0)$ as follows:

$$\mathcal{N} \overset{\text{def}}{=} \mathbb{N}^\nu_*;\ \ \omega(n_0) \overset{\text{def}}{=} \langle n_0, \lambda \rangle = \sum n_{0,i}\lambda_i \qquad (\forall n_0 \in \mathcal{N}). \tag{10.4}$$

As in §§7,8,9, $\mathbb{N}^{\nu}_{*}$ denotes the set of all multiintegers $n = (n_1, \ldots, n_\nu)$ such that $n_1 + \cdots + n_\nu \geq 1$ and $n_i \geq 0$ (except for *at most* one component n_i, that may assume the value -1) and $\langle n, \lambda \rangle$ denotes the usual scalar product.

For any resonant vector field of spectrum λ:

$$X = X^{\text{lin}} + \sum \mathbb{B}_{n_0} \quad (n_0 \in \mathcal{N};\ X^{\text{lin}} = \sum \lambda_i x_i \partial_{x_i}) \tag{10.5}$$

the *nilpotent part* and *distinguished form* decompose into series of n_0-homogeneous components:

$$X^{\text{nil}} = \sum X^{\text{nil}}_{n_0} \qquad (n_0 \in \mathcal{N};\ \omega_0 = \langle n_0, \lambda \rangle \in \mathbb{C}) \tag{10.6}$$

$$X^{\text{dist}} = X^{\text{lin}} + \sum X^{\text{dist}}_{n_0} \qquad (n_0 \in \mathcal{N};\ \omega_0 = \langle n_0, \lambda \rangle = 0) \tag{10.7}$$

which in turn, due to Proposition 7.1, are expressible as *finite* sums of $\$$-contractions or $\$\!\!\!\$$-contractions:

$$X^{\text{nil}}_{n_0} = \sum_{\|\underline{\mathbf{n}}\| = n_0} \mathbb{B}^{\$}_{\underline{\mathbf{n}}} \quad (n_0 \in \mathcal{N}) \tag{10.8}$$

$$X^{\text{dist}}_{n_0} = \sum_{\|\underline{\mathbf{n}}\| = n_0} \mathbb{B}^{\$\!\!\!\$}_{\underline{\mathbf{n}}} \quad (n_0 \in \mathcal{N}) \tag{10.9}$$

with of course:

$$\underline{\mathbf{n}} = (n_1, \ldots, n_r) \in \underline{\mathcal{N}}^r = \mathcal{N}^r \text{ symmetrized} \tag{10.10}$$

$$\|\underline{\mathbf{n}}\| = n_1 + n_2 + \cdots + n_r \in \mathcal{N}. \tag{10.10*}$$

Let us further denote by $\mathcal{N}_0$ the set of all $n \in \mathcal{N}$ such that $\langle n, \lambda \rangle = 0$. Monomials x^n with $n \in \mathcal{N}_0$ will be referred to as *resonant monomials*.

The preceding notations enable us to write down explicit criteria for various properties, such as *linearizability*, or *nihilence*, or *total nihilence* (we say that a resonant vector field X is totally nihilent if the number of independent formal integrals $H(x) \in \mathbb{C}[[x]]$ is equal to the resonance degree of, what amounts to the same, if in *one*, and therefore *any*, prenormal chart, X *annihilates all resonant monomials*).

For any resonant vector field X, we have the obvious criteria:

$$\{X \text{ formally linearizable}\} \Longleftrightarrow \{X^{\text{nil}}_n \equiv 0, \forall n \in \mathcal{N}\} \tag{10.11}$$

$$\{X \text{ formally linearizable}\} \Longleftrightarrow \{X^{\text{dist}}_n \equiv 0, \forall n \in \mathcal{N}\} \tag{10.12}$$

$$\{X \text{ totally nihilent}\} \Longleftrightarrow \{X^{\text{dist}}_n \cdot x^m \equiv 0, \forall n \in \mathcal{N}_0, \forall m \in \mathcal{N}_0\}. \tag{10.13}$$

These criteria verge on the tautological, but in combination with the decompositions (10.8), (10.9) they open up interesting vistas. Indeed, if on top of resonance we now assume that our vector field is *polynomial of* degree d (*i.e.* with a finite number of homogeneous components $\mathbb{B}_n$, for $\|\mathbf{n}\| \leq d$) two important questions arise:

Q_1) What is the smallest number $\operatorname{lin}(d,\lambda)$ of relations $X_n^{\mathrm{nil}} = 0$ (or $X_n^{\mathrm{dist}} = 0$) that guarantee formal linearizability?

Q_2) What is the smallest number $\operatorname{nil}(d,\lambda)$ of relations $X_n^{\mathrm{dist}} \cdot x^m = 0$ that guarantee total nihilence?

Obviously, both numbers are *finite*, since the formal linearizability (resp. total nihilence) of a polynomial X is equivalent to the annihilation of a suitable ideal $\mathbb{I}$ in the algebra $\mathcal{A}$ generated by the (finitely many) Taylor coefficients of X, and since any such ideal is known to be *finitely generated*.

A special instance is the so-called *center problem*, namely the question of determining the smallest number of polynomial identities that guarantee the existence of a *center-focus* for a polynomial vector field on $\mathbb{R}^2$, of the form:

$$X = x_1\partial_{x_2} - x_2\partial_{x_1} + (\cdots) \qquad (\deg X = d). \tag{10.14}$$

After changing to "isotropic coordinates" we get:

$$X = iy_1\partial_{y_1} - iy_2\partial_{y_2} + (\cdots) \qquad (\lambda = (i,-i)) \tag{10.14*}$$

and the problem "reduces" to finding the number $\operatorname{nil}(d,\lambda)$ of conditions that ensure the nihilence (necessarily *total* in this case) of X. See on the subject [S1] and [S2].

However, *in the center problem as in the general case, the numbers* $\operatorname{lin}(d,\lambda)$ *and* $\operatorname{nil}(d,\lambda)$ *remain unknown* (except for the lowest values of d, such as $d = 2$ in the center problem) and seem to be very elusive, not least because the ideals $\mathbb{I}$ mentioned above are *unwieldy, unstructured*, and lacking in truly *canonical bases*.

But the considerations at the beginning of this section suggest another, possibly more promising approach, namely:

(i) to replace the commutative ideals $\mathbb{I}$ mentioned a moment ago, by the Lie ideals $\mathbb{J}$ generated by the homogeneous components $X_{n_0}^{\mathrm{nil}}$ or $X_{n_0}^{\mathrm{dist}}$ of (10.6) or (10.7);

(ii) to use the explicit decompositions (10.8), (10.9) of these homogeneous components into the elementary Lie elements $\mathbb{B}_{\underline{\mathbf{n}}}^{\$}$ and $\mathbb{B}_{\underline{\mathbf{n}}}^{\$\!\!\!\mid}$ directly constructed from the homogeneous components $\mathbb{B}_{n_0}$ of X (see (10.2), (10.3));

(iii) to investigate the *splitting properties* of the Lie elements $\mathbb{B}_{\underline{\mathbf{n}}}^{\$}$ and $\mathbb{B}_{\underline{\mathbf{n}}}^{\$\!\!\!\mid}$ (see below).

The reasons for reposing some hope in this approach are three:

First, the Lie ideal $\mathbb{J}$ is more tightly structured, and closer to the nature of X, than the commutative ideal $\mathbb{I}$.

Second, the components $X^{\mathrm{nil}}_{n_0}$ and $X^{\mathrm{dist}}_{n_0}$, while more intrinsic than anything in the commutative ideal $\mathbb{I}$, are still highly "composite". The truly simple objects to focus on are the Lie elements $\mathbb{B}^{\$}_{\underline{\mathbf{n}}}$ and $\mathbb{B}^{\not{\$}}_{\underline{\mathbf{n}}}$, and any regularity that the components $X^{\mathrm{nil}}_{n_0}$ or $X^{\mathrm{dist}}_{n_0}$ may possess, is probably *inherited from*, and more easily *detectable on*, the Lie elements $\mathbb{B}^{\$}_{\underline{\mathbf{n}}}$ and $\mathbb{B}^{\not{\$}}_{\underline{\mathbf{n}}}$.

Third, if we define the upper (resp. lower) degeneracy of an unordered sequence $\underline{\boldsymbol{\omega}} = (\omega_1, \ldots, \omega_r)$ as being equal to the highest (resp. lowest) degeneracy attained by the ordered sequences $\boldsymbol{\omega}$ corresponding to $\underline{\boldsymbol{\omega}}$, and if we fix the degree d and spectrum λ, we observe that the blocks $\mathbb{B}^{\$}_{\underline{\mathbf{n}}}$ and $\mathbb{B}^{\not{\$}}_{\underline{\mathbf{n}}}$ tend to *split* (*i.e.* to decompose into Lie brackets of simpler blocks) more and more as the upper *or* lower degeneracy of the sequence $\underline{\boldsymbol{\omega}}$ (associated with $\underline{\mathbf{n}}$) increases.

As a very elementary instance of this phenomenon, let us mention the following fact:

LEMMA 10.1. — *Any $\mathbb{B}^{\$}_{\underline{\mathbf{n}}}$, indexed by any sequence $\underline{\mathbf{n}} = (n_1, \ldots, n_r)$, belongs to the Lie ideal generated by the blocks $\mathbb{B}^{\$}_{\underline{\mathbf{m}}}$ indexed by sequences $\underline{\mathbf{m}} = (m_1, \ldots, m_s)$ such that:*

$$0 = \langle \|\mathbf{m}\|, \lambda \rangle \overset{\mathrm{def}}{=} \langle m_1, \lambda \rangle + \cdots + \langle m_s, \lambda \rangle. \tag{10.15}$$

Proof. — From the identity (3.16) we deduce:

$$\|\underline{\boldsymbol{\omega}}\| \mathbb{B}_{\underline{\mathbf{n}}} = \sum_{1 \le i \le r} [\mathbb{B}_{\underline{\mathbf{n}}^i}, \mathbb{B}_{n_i}] \tag{10.16}$$

with $\underline{\boldsymbol{\omega}}$ corresponding to $\underline{\mathbf{n}}$, and with $\underline{\mathbf{n}}^i$ denoting the sequence $\underline{\mathbf{n}} = \underline{(n_1, \ldots, n_r)}$ deprived of its component n_i. Then we use (10.16) repeatedly, so as to decompose the elements $\mathbb{B}_{\underline{\mathbf{n}}^i}$ into brackets of $\mathbb{B}_{\underline{\mathbf{n}}^{ij}}$, then the $\mathbb{B}_{\underline{\mathbf{n}}^{ij}}$ into smaller elements, etc., until we get rid of all sequences $\underline{\mathbf{n}}^{ij\cdots}$ such that $\|\underline{\boldsymbol{\omega}}^{ij\cdots}\| \neq 0$. □

The above lemma is only a pointer towards a very general *splitting tendency*, which inheres in the Lie elements $\mathbb{B}^{\$}_{\underline{\mathbf{n}}}$ and $\mathbb{B}^{\not{\$}}_{\underline{\mathbf{n}}}$, and which would seem to warrant a systematic investigation. We intend to pursue this question in a follow-up paper.

11. Synopsis. Main formulae. Tables.

The basic notions about *moulds* (multiplication, symmetral/alternal, symmetrel/alternel, etc.) are recalled at the beginning of §1, along with the definitions of the *trivial moulds* $1^\bullet$, $I^\bullet$, $J^\bullet$, $I^\bullet_{\mathrm{ex}}$, $J^\bullet_{\mathrm{ex}}$ (see (2.1)–(2.12)) and *mould derivations* ∇ and ∇_{ω_0} (see (2.14) and (3.8)). Most moulds here are indexed by sequences $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ written in bold-face, with components ω_i in ordinary print. The *degeneracy* $\mathrm{dgn}(\boldsymbol{\omega})$ and *vanishing order* $\mathrm{van}(\boldsymbol{\omega})$ of a sequence $\boldsymbol{\omega}$ are defined before (2.17) and after (2.18).

The main moulds used in this paper are:

$S^\bullet$ and $\not{S}^\bullet$ (resp. $T^\bullet$): symmetral and mutually inverse (resp. alternal).

$S^\bullet_{\mathrm{co}}(t)$ and $\not{S}^\bullet_{\mathrm{co}}(t)$: symmetral and mutually inverse.

$S^\bullet_{\mathrm{ext}}$ and $\not{S}^\bullet_{\mathrm{ext}}$ (resp. $T^\bullet$): symmetral and mutually inverse.

$\$^\bullet$ and $\not{\$}^\bullet$: alternal and mutually conjugate.

$1^\bullet = S^\bullet \times \not{S}^\bullet = S^\bullet_{\mathrm{co}}(t) \times \not{S}^\bullet_{\mathrm{co}}(t) = S^\bullet_{\mathrm{ext}} \times \not{S}^\bullet_{\mathrm{ext}}$

$S^{\boldsymbol{\omega}}$, $\not{S}^{\boldsymbol{\omega}}$, $T^{\boldsymbol{\omega}}$ are defined for *almost every* sequence $\boldsymbol{\omega}$.

All other moulds are defined for every sequence $\boldsymbol{\omega}$.

Direct definition of $S^\bullet$, $\not{S}^\bullet$, $T^\bullet$:

$$S^{\omega_1,\dots,\omega_r} \overset{\mathrm{def}}{=} (-1)^r (\check{\omega}_1 \cdots \check{\omega}_r)^{-1} \quad \text{with } \check{\omega}_i \overset{\mathrm{def}}{=} \omega_1 + \cdots + \omega_i$$

$$\not{S}^{\omega_1,\dots,\omega_r} \overset{\mathrm{def}}{=} (\hat{\omega}_1 \cdots \hat{\omega}_r)^{-1} \quad \text{with } \hat{\omega}_i \overset{\mathrm{def}}{=} \omega_i + \cdots + \omega_r$$

$$T^{\omega_1,\dots,\omega_r} \overset{\mathrm{def}}{=} 0 \quad \text{if } 0 \neq \|\boldsymbol{\omega}\| \overset{\mathrm{def}}{=} \omega_1 + \cdots + \omega_r$$

$$T^{\omega_1,\dots,\omega_r} \overset{\mathrm{def}}{=} (\hat{\omega}_2 \hat{\omega}_3 \cdots \hat{\omega}_r)^{-1} = (-1)^{r-1} (\check{\omega}_1 \check{\omega}_2 \cdots \check{\omega}_{r-1})^{-1} \ \text{ if } 0 = \|\boldsymbol{\omega}\|.$$

Direct definition of the compensators $S^\bullet_{\mathrm{co}}(t)$ *and* $\not{S}^\bullet_{\mathrm{co}}(t)$:

$$S^\bullet_{\mathrm{co}}(t) \overset{\mathrm{def}}{=} (t^\nabla \not{S}^\bullet) \times (S^\bullet)$$

$$\not{S}^\bullet_{\mathrm{co}}(t) \overset{\mathrm{def}}{=} (\not{S}^\bullet) \times (t^\nabla S^\bullet)$$

with t on $\mathbb{C}_\bullet$ (Riemann surface of $\log t$) and t^∇ as in (2.14*).

Link between *symmetral* and *symmetric* compensators: see (2.29)–(2.32).

Lateral decomposition of compensators (see Proposition 2.2):

$$\begin{aligned} S^{\bullet}_{\mathrm{co}}(t) &= \exp((\log t)t^{\nabla}\,\$^{\bullet}) \times S^{\bullet}_{\mathrm{aco}}(t) \\ &= S^{\bullet}_{\mathrm{aco}}(t) \times \exp((\log t)\,\$^{\bullet}) \\ \not{S}^{\bullet}_{\mathrm{co}}(t) &= \exp(-(\log t)\,\$^{\bullet}) \times \not{S}^{\bullet}_{\mathrm{aco}}(t) \\ &= \not{S}^{\bullet}_{\mathrm{aco}}(t) \times \exp(-(\log t)t^{\nabla}\,\$^{\bullet}) \end{aligned}$$

with $\exp(\cdots)$ denoting the *mould exponential* (see (2.13*)) and with $S^{\bullet}_{\mathrm{aco}}(t)$, $\not{S}^{\bullet}_{\mathrm{aco}}(t)$ denoting the logarithm-free parts of $S^{\bullet}_{\mathrm{co}}(t)$, $\not{S}^{\bullet}_{\mathrm{co}}(t)$. The above relations *characterize the alternal* mould $\$^{\bullet}$.

Central decomposition of compensators (see Proposition 2.3):

$$S^{\bullet}_{\mathrm{co}}(t) = (t^{\nabla}\,S^{\bullet}_{\mathrm{ext}}) \times \exp((\log t)\,\$\!\!|^{\bullet}) \times (\not{S}^{\bullet}_{\mathrm{ext}})$$

$$\not{S}^{\bullet}_{\mathrm{co}}(t) = (S^{\bullet}_{\mathrm{ext}}) \times \exp(-(\log t)\,\$\!\!|^{\bullet}) \times (t^{\nabla}\not{S}^{\bullet}_{\mathrm{ext}}).$$

The above relations, together with the rationality conditions (2.40), (2.41), (2.42), *characterize* simultaneously the *alternal* mould $\$\!\!|^{\bullet}$ and the *symmetral, mutually inverse* moulds $S^{\bullet}_{\mathrm{ext}}$, $\not{S}^{\bullet}_{\mathrm{ext}}$.

Conjugacy of $\$^{\bullet}$ and $\$\!\!|^{\bullet}$.

$$\$^{\bullet} = \not{S}^{\bullet}_{\mathrm{ext}} \times \$\!\!|^{\bullet} \times S^{\bullet}_{\mathrm{ext}}$$

$$\$\!\!|^{\bullet} = S^{\bullet}_{\mathrm{ext}} \times \$^{\bullet} \times \not{S}^{\bullet}_{\mathrm{ext}}$$

$\$^{\boldsymbol{\omega}} \equiv 0$ as soon as $\mathrm{dgn}(\boldsymbol{\omega}) = 0$

$\$\!\!|^{\boldsymbol{\omega}} \equiv 0$ as soon as $\mathrm{van}(\boldsymbol{\omega}) = 0$ (*i.e.* when $\|\boldsymbol{\omega}\| \neq 0$).

∇ and ∇_{ω_0} derivatives of $\$^{\bullet}$ and $\$\!\!|^{\bullet}$ (see §3 and §4):

$$\nabla\,\$^{\bullet} = I^{\bullet} \times \$^{\bullet} - \$^{\bullet} \times I^{\bullet}$$

$$\nabla_{\omega_0}\,\$^{\bullet} = I^{\bullet}_{\omega_0} \times \$^{\bullet} - \$^{\bullet} \times I^{\bullet}_{\omega_0} \qquad \text{(with } I^{\bullet}_{\omega_0} \text{ as in (3.7))}$$

$$\nabla\,\$\!\!|^{\bullet} = 0$$

$$\nabla_{\omega_0}\,\$\!\!|^{\bullet} = \$\!\!|^{\bullet}_{\omega_0} \times \$\!\!|^{\bullet} - \$\!\!|^{\bullet} \times \$\!\!|^{\bullet}_{\omega_0} \qquad \text{(with } \$\!\!|^{\bullet}_{\omega_0} \textit{ alternal}\text{).}$$

Construction of the "tough" moulds $S^{\bullet}_{\mathrm{ext}}$, $\not{S}^{\bullet}_{\mathrm{ext}}$, $\$\!\!|^{\bullet}$ (see §4):

$$S^{\boldsymbol{\omega}}_{\mathrm{ext}} \overset{\mathrm{def}}{=} \mathrm{Rad}^{\boldsymbol{\omega}}\,S^{\boldsymbol{\omega}}_{\mathrm{rest}}$$

$$\not{S}^{\boldsymbol{\omega}}_{\mathrm{ext}} \overset{\mathrm{def}}{=} \mathrm{Ra}\not{\mathrm{d}}^{\boldsymbol{\omega}}\,\not{S}^{\boldsymbol{\omega}}_{\mathrm{rest}}$$

$$\$\!\!|^{\boldsymbol{\omega}} \overset{\mathrm{def}}{=} \mathrm{Ral}^{\boldsymbol{\omega}}\,T^{\boldsymbol{\omega}}_{\mathrm{rest}}$$

with "*restrictions*" $S^{\boldsymbol{\omega}}_{\text{rest}}$, $\not{S}^{\boldsymbol{\omega}}_{\text{rest}}$, $T^{\boldsymbol{\omega}}_{\text{rest}}$ defined as in (4.5), (4.6), (4.7) or (4.8) and with *differential operators* $\text{Rad}^{\boldsymbol{\omega}}$, $\text{Rađ}^{\boldsymbol{\omega}}$, $\text{Ral}^{\boldsymbol{\omega}}$ (of degrees $\overleftarrow{\text{van}}\,(\boldsymbol{\omega})$, $\overrightarrow{\text{van}}\,(\boldsymbol{\omega})$, $\text{van}(\boldsymbol{\omega})$) defined with the help of the auxiliary moulds $\text{rad}^\bullet$, $\text{rađ}^\bullet$, $\text{ral}^\bullet$, which in turn are characterized by the systems (4.11), (4.12), (4.13) and verify:

$$\begin{aligned} &1^\bullet = \text{rad}^\bullet \times \text{rađ}^\bullet && (\text{rad}^\bullet, \text{rađ}^\bullet\ \text{ symmetral})\\ &\text{ral}^\bullet = \text{rad}^\bullet \times I^\bullet \times \text{rađ}^\bullet && (\text{ral}^\bullet\ \text{ alternal}). \end{aligned}$$

The "restrictions" $S^\bullet_{\text{rest}}$, $\not{S}^\bullet_{\text{rest}}$, $T^\bullet_{\text{rest}}$ being elementary, all the complexity of the "tough" moulds is concentrated in the auxiliary moulds $\text{rad}^\bullet$, $\text{rađ}^\bullet$, $\text{ral}^\bullet$.

Amplification $M^\bullet_{\text{amp}}$ *and coamplification* $M^\bullet_{\text{coamp}}(z)$ *of an alternal mould* $M^\bullet$ *(see §5)*:

The *amplification* is indexed by sequences $\varpi_1, \ldots, \varpi_r$ with $\varpi_i = \binom{\omega_i}{a_i} \in \mathbb{C}^2$, and is defined as follows:

$$\begin{aligned} M^{\varpi_1,\ldots,\varpi_r}_{\text{amp}} &\overset{\text{def}}{=} \sum_{n_i \geq 0} M^{\omega_1, 0^{(n_1)}, \ldots, \omega_r, 0^{(n_r)}} (a_1)^{n_1} (a_1 + a_2)^{n_2} \cdots \\ &\qquad\qquad\qquad \cdots (a_1 + \cdots + a_r)^{n_r} \\ &= \sum_{n_i \geq 0} M^{0^{(n_1)}, \omega_1, \ldots, 0^{(n_r)}, \omega_r} (-1)^{n_1 + \cdots + n_r} (a_1 + \cdots + a_r)^{n_1} \cdots \\ &\qquad\qquad\qquad \cdots (a_2 + \cdots + a_r)^{n_2} \cdots (a_r)^{n_r} \\ &= \sum_{n_i \geq 0} M^{\omega_1, \ldots, \omega_r}_{n_1, \ldots, n_r} (a_1)^{n_1} \cdots (a_r)^{n_r}. \end{aligned}$$

The *coamplification* is indexed by sequences $\eta_1, \ldots, \eta_r$ with $\eta_i = \binom{\omega_i}{\sigma_i} \in \mathbb{C}^2$, and is defined as follows:

$$M^{\eta_1,\ldots,\eta_r}_{\text{coamp}}(z) \overset{\text{def}}{=} \sum_{n_i \geq 0} M^{\omega_1,\ldots,\omega_r}_{n_1,\ldots,n_r} (\partial^{n_1} z^{-\sigma_1}) \cdots (\partial^{n_r} z^{-\sigma_r})$$

with $\partial = \partial_z$. It assumes values in the space $\bigcup_\sigma z^{-\sigma}\mathbb{C}[\![z^{-1}]\!]$ of formal power series of z^{-1}.

For natural moulds $M^\bullet$, the amplification $M^\bullet_{\text{amp}}$ tends to be "endlessly analytic" in $\mathbf{a} = (a_1, \ldots, a_r)$ and the coamplification tends to be divergent *and* resurgent in z. Such is the case in particular when we take as $M^\bullet$ the mould $\$^\bullet$ or $\rlap{\$}\,\$^\bullet$: the respective *coamplifications* are resurgent, but of very different type (much less elementary for $\rlap{\$}\,\$^\bullet$ than for $\$^\bullet$).

The *symmetral/alternal* moulds:

$$S^\bullet,\ \not{S}^\bullet,\ S^\bullet_{\text{co}}(t),\ \not{S}^\bullet_{\text{co}}(t),\ S^\bullet_{\text{ext}}, \not{S}^\bullet_{\text{ext}},\ \$^\bullet,\ \rlap{\$}\,\$^\bullet$$

are particularly helpful in the study of *local vector fields*.

They possess *symmetrel/alternel* analogues:

$$\mathcal{S}^{\bullet},\ \mathcal{S}\!\!\!/^{\,\bullet},\ \mathcal{S}^{\bullet}_{\mathrm{co}}(t),\ \mathcal{S}\!\!\!/^{\,\bullet}_{\mathrm{co}}(t),\ \mathcal{S}^{\bullet}_{\mathrm{ext}}, \mathcal{S}\!\!\!/^{\,\bullet}_{\mathrm{ext}},\ \$^{\bullet},\ \$\!\!\!|^{\,\bullet}$$

which are particularly helpful in the study of *local diffeomorphisms*. These latter moulds are defined and investigated in §6.

Local, analytic, resonant vector fields X *on* $\mathbb{C}^{\nu}$:

$$\begin{cases} X = X^{\mathrm{lin}} + \sum\limits_{n} \mathbb{B}_n \\ X^{\mathrm{lin}} = \sum \lambda_i x_i \partial_{x_i} = \text{ resonant linear part} \\ \mathbb{B}_n = \text{ homogeneous component of degree } n = (n_1, \dots, n_{\nu}) \in \mathbb{N}^{\nu}_{*} \end{cases}$$

and *local, analytic, resonant* diffeomorphisms f of $\mathbb{C}^{\nu}$, viewed as substitution operators F:

$$\begin{cases} F = \Big\{1 + \sum\limits_{n} \mathbb{B}_n\Big\} F^{\mathrm{lin}} \\ F^{\mathrm{lin}}\varphi(x_1, \dots, x_{\nu}) \equiv \varphi(\ell_1 x_1, \dots, \ell_{\nu} x_{\nu}) \quad (\forall \varphi \in \mathbb{C}\{\{x\}\}) \\ F^{\mathrm{lin}} = \text{ resonant linear part} \\ \mathbb{B}_n = \text{ homogeneous component of degree } n = (n_1, \dots, n_{\nu}) \in \mathbb{N}^{\nu}_{*} \end{cases}$$

admit each an *intrinsic decomposition* into a *diagonalizable* and *nilpotent part*:

$$\begin{cases} X = X^{\mathrm{dia}} + X^{\mathrm{nil}} \quad (\text{with } [X^{\mathrm{dia}}, X^{\mathrm{nil}}] = 0) \\ F = F^{\mathrm{nil}} F^{\mathrm{dia}} = F^{\mathrm{dia}} F^{\mathrm{nil}} \end{cases}$$

as well as a non-intrinsic, but *canonical prenormal form* (made up solely of resonant monomials), known as the *distinguished form*:

$$\begin{cases} X = \Theta_{\mathrm{ext}} X^{\mathrm{dist}} \Theta^{-1}_{\mathrm{ext}} \quad (\text{fields}) \\ F = \Theta_{\mathrm{ext}} F^{\mathrm{dist}} \Theta^{-1}_{\mathrm{ext}} \quad (\text{diffeos}). \end{cases}$$

The *nilpotent part* has an explicit expansion:

$$\begin{cases} X^{\mathrm{nil}} = \sum \$^{\bullet}\, \mathbb{B}_{\bullet} & (\text{see §7}) \\ F^{\mathrm{nil}} = \exp(X^{\mathrm{nil}}) = \exp(\sum \$^{\bullet}\, \mathbb{B}_{\bullet}) & (\text{see §7}) \end{cases}$$

and so does the *distinguished form*:

$$\begin{cases} X^{\mathrm{dist}} = X^{\mathrm{lin}} + \sum \$\!\!\!|^{\,\bullet}\, \mathbb{B}_{\bullet} & (\text{see §7}) \\ F^{\mathrm{nil}} = F^{\mathrm{lin}} \exp(X^{\mathrm{nil}}) = F^{\mathrm{lin}} \exp(\sum \$\!\!\!|^{\,\bullet}\, \mathbb{B}_{\bullet}) & (\text{see §7}). \end{cases}$$

The *nilpotent parts* are generically *divergent*, but *resurgent*, and satisfy a variant of the *Bridge Equation*:

$$\begin{cases} [\overset{\bullet}{\Delta}_{\omega}, X^{\mathrm{nil}}] \equiv -\, \overset{\bullet}{\omega}\, \mathcal{A}_{\omega} & (\text{fields}) \\ [\overset{\bullet}{\Delta}_{\omega}, F^{\mathrm{nil}}] \equiv (e^{-\overset{\bullet}{\omega}} - 1) \mathcal{A}_{\omega} F^{\mathrm{nil}} & (\text{diffeos}) \\ [\overset{\bullet}{\Delta}_{\omega}, X^{\mathrm{nil}}] \equiv (n_0 \lambda_0 - \overset{\bullet}{\omega}) \mathcal{A}_{\omega} & (\text{diffeos again}) \end{cases} \quad .$$

with (pointed) *alien derivations* $\overset{\bullet}{\Delta}_\omega$ (*); with indexes ω of the form (8.9) for fields (resp (8.11) for diffeos); and with ordinary *differential operators* $\mathcal{A}_\omega$ which, when expressed in the *normal chart* (z, u) (where $X \equiv \partial_z$ and $F \equiv \exp(\partial_z)$) reduce to the *holomorphic invariants* $\mathbb{A}_\omega$ that appear in the classical *Bridge Equation*:

$$\overset{\bullet}{\Delta}_\omega \, \tilde{x}(z,u) = \mathbb{A}_\omega \tilde{x}(z,u) \qquad (\overset{\bullet}{\omega} \in \Omega)$$

along with the so-called *formal integral* $\tilde{x}(z,u)$ of X or F.

The *distinguished forms* X^{dist} and F^{dist} are resurgent, too, but with a richer "resurgence lattice" Ω^{dist}, and a markedly different *regimen of resurgence*: "rigid" and "universal" (see §9).

Lastly, for *non-ordered sequences* $\underline{\mathbf{n}} = (n_1, \ldots, n_r)$ with $n_i \in \mathbb{N}^\nu_*$, the Lie elements $\mathbb{B}^{\$}_{\underline{\mathbf{n}}}$ and $\mathbb{B}^{\$\!\!\$}_{\underline{\mathbf{n}}}$ constructed in (10.2) and (10.3) from the homogeneous components $\mathbb{B}_n$ of a resonant vector field X, provide *neat* and *explicit conditions* (necessary and sufficient) for the *linearizability* or *nihilence* of X. In the case of *polynomial* vector fields X, this may hopefully lead to *explicit bounds* for the codimensions of the corresponding "linearizability ideal" and "nihilence ideal".

We conclude this synopsis with tables listing the values of $\$^{\boldsymbol{\omega}}$, $\$\!\!\$^{\boldsymbol{\omega}}$, $S^{\boldsymbol{\omega}}_{\text{ext}}$, $\mathcal{S}^{\boldsymbol{\omega}}_{\text{ext}}$ for sequences $\boldsymbol{\omega}$ of length $r(\boldsymbol{\omega}) \leq 5$ and of various degeneracy types (we use the usual shorthand: ω_{ij} for $\omega_i + \omega_j$, etc.).

(*) not to be confused with the *mould derivations* ∇_{ω_0}.

$\boldsymbol{\omega}$	$\not S^{\boldsymbol{\omega}}$	$\not\not S^{\boldsymbol{\omega}}$	$S^{\boldsymbol{\omega}}_{\text{ext}}$	$\mathcal{S}^{\boldsymbol{\omega}}_{\text{ext}}$
(ω)	0	0	$-\omega^{-1}$	$+\omega^{-1}$
(0)	1	1	0	0
(ω_1,ω_2)	0	0	$+\omega_1^{-1}\omega_{12}^{-1}$	$+\omega_{12}^{-1}\omega_2^{-1}$
$(\omega,0)$	$+\omega^{-1}$	$+\omega^{-1}$	$+\omega^{-2}$	$-\omega^{-2}$
$(0,\omega)$	$-\omega^{-1}$	$-\omega^{-1}$	$-\omega^{-2}$	$+\omega^{-2}$
$(\omega,-\omega)$	$-\omega^{-1}$	$-\omega^{-1}$	$-\frac{1}{2}\omega^{-2}$	$-\frac{1}{2}\omega^{-2}$
$(0,0)$	0	0	0	0
$(\omega_1,\omega_2,\omega_3)$	0	0	$-\omega_1^{-1}\omega_{12}^{-1}\omega_{123}^{-1}$	$+\omega_{123}^{-1}\omega_{23}^{-1}\omega_3^{-1}$
$(\omega_1,\omega_2,\omega_3)_{\omega_{123}=0}$	$+\omega_{23}^{-1}\omega_3^{-1}$	$+\omega_{23}^{-1}\omega_3^{-1}$	$+\frac{2}{3}\omega_{12}^{-2}\omega_1^{-1}+\frac{1}{3}\omega_{12}^{-1}\omega_1^{-2}$	$-\frac{2}{3}\omega_{23}^{-2}\omega_3^{-1}-\frac{1}{3}\omega_{23}^{-1}\omega_3^{-2}$
$(\omega_1,\omega_2,\omega_3)_{\omega_{12}=0}$	$+\omega_1^{-1}\omega_3^{-1}$	0	$+\frac{1}{2}\omega_1^{-2}\omega_3^{-1}+\omega_1^{-1}\omega_3^{-2}$	$+\omega_{23}^{-1}\omega_3^{-2}$
$(\omega_1,\omega_2,\omega_3)_{\omega_{23}=0}$	$+\omega_1^{-1}\omega_3^{-1}$	0	$-\omega_1^{-2}\omega_{12}^{-1}$	$-\frac{1}{2}\omega_1^{-1}\omega_3^{-2}-\omega_1^{-2}\omega_3^{-1}$
$(\omega,-\omega,\omega)$	$+2\omega^{-2}$	0	$+\frac{3}{2}\omega^{-3}$	$-\frac{3}{2}\omega^{-3}$
$(\omega,0,0)$	$-\omega^{-2}$	0	$-\omega^{-3}$	$+\omega^{-3}$
$(0,\omega,0)$	$+2\omega^{-2}$	0	$+2\omega^{-3}$	$-2\omega^{-3}$
$(0,0,\omega)$	$-\omega^{-2}$	0	$-\omega^{-3}$	$+\omega^{-3}$
$(\omega,-\omega,0)$	$-\omega^{-2}$	$-\frac{1}{2}\omega^{-2}$	$-\frac{1}{6}\omega^{-3}$	$-\frac{5}{6}\omega^{-3}$
$(\omega,0,-\omega)$	$+2\omega^{-2}$	$+\omega^{-2}$	$+\omega^{-3}$	$+\omega^{-3}$
$(0,\omega,-\omega)$	$-\omega^{-2}$	$-\frac{1}{2}\omega^{-2}$	$-\frac{5}{6}\omega^{-3}$	$-\frac{1}{6}\omega^{-3}$
$(\omega,0,0,0)$	$+\omega^{-3}$	0	$+\omega^{-4}$	$-\omega^{-4}$
$(0,\omega,0,0)$	$-3\omega^{-3}$	0	$-3\omega^{-4}$	$+3\omega^{-4}$
$(0,0,\omega,0)$	$+3\omega^{-3}$	0	$+3\omega^{-4}$	$-3\omega^{-4}$
$(0,0,0,\omega)$	$-\omega^{-3}$	0	$-\omega^{-4}$	$+\omega^{-4}$
$(\omega,-\omega,0,0)$	$-\omega^{-3}$	$-\frac{1}{6}\omega^{-3}$	$-\frac{1}{24}\omega^{-4}$	$-\frac{23}{24}\omega^{-4}$
$(\omega,0,-\omega,0)$	$+3\omega^{-3}$	$+\omega^{-3}$	$+\frac{7}{12}\omega^{-4}$	$+\frac{29}{12}\omega^{-4}$
$(0,\omega,-\omega,0)$	$-\omega^{-3}$	$-\frac{2}{3}\omega^{-3}$	$-\frac{1}{2}\omega^{-4}$	$-\frac{1}{2}\omega^{-4}$
$(\omega,0,0,-\omega)$	$-3\omega^{-3}$	$-\omega^{-3}$	$-\frac{3}{2}\omega^{-4}$	$-\frac{3}{2}\omega^{-4}$
$(0,\omega,0,-\omega)$	$+3\omega^{-3}$	$+\omega^{-3}$	$+\frac{29}{12}\omega^{-4}$	$+\frac{7}{12}\omega^{-4}$
$(0,0,\omega,-\omega)$	$-\omega^{-3}$	$-\frac{1}{6}\omega^{-3}$	$-\frac{23}{24}\omega^{-4}$	$-\frac{1}{24}\omega^{-4}$

$\boldsymbol{\omega}$	$\$^{\boldsymbol{\omega}}$	$\|\!\$^{\boldsymbol{\omega}}$
$(\omega_1,\omega_2,\omega_3,\omega_4)\omega_{1234}=0$	$+\omega_{234}^{-1}\omega_{34}^{-1}\omega_4^{-1}$	$+\omega_{234}^{-1}\omega_{34}^{-1}\omega_4^{-1}$
$(\omega_1,\omega_2,\omega_3,\omega_4)\begin{cases}\omega_{23}=0\\\omega_{14}=0\end{cases}$	$+\omega_{34}^{-1}\omega_4^{-2}+\omega_3^{-1}\omega_4^{-2}$	$+\omega_4^{-2}\omega_{34}^{-1}$
$(\omega_1,\omega_2,\omega_3,\omega_4)\begin{cases}\omega_{12}=0\\\omega_{34}=0\end{cases}$	$-\omega_1^{-2}\omega_4^{-1}+\omega_1^{-1}\omega_4^{-1}$	$-\frac{1}{2}\omega_1^{-2}\omega_4^{-1}+\frac{1}{2}\omega_1^{-1}\omega_4^{-2}$
$(\omega_1,\omega_2,\omega_3,0)\omega_{123}=0$	$-\omega_{23}^{-2}\omega_3^{-1}-\omega_{23}^{-1}\omega_3^{-2}$	$-\frac{1}{3}\omega_{23}^{-2}\omega_3^{-1}-\frac{2}{3}\omega_{23}^{-1}\omega_3^{-2}$
$(\omega_1,\omega_2,0,\omega_4)\omega_{124}=0$	$+\omega_2^{-1}\omega_4^{-2}-\omega_1^{-1}\omega_4^{-2}$	$-\omega_1^{-1}\omega_4^{-2}$
$(\omega_1,0,\omega_3,\omega_4)\omega_{134}=0$	$+\omega_1^{-2}\omega_4^{-1}-\omega_1^{-2}\omega_3^{-1}$	$+\omega_1^{-2}\omega_4^{-1}$
$(0,\omega_2,\omega_3,\omega_4)\omega_{234}=0$	$-\omega_2^{-2}\omega_4^{-1}+\omega_2^{-1}\omega_4^{-2}$	$-\frac{2}{3}\omega_2^{-2}\omega_4^{-1}+\frac{1}{3}\omega_2^{-1}\omega_4^{-2}$
$(\omega,-\omega,0,0,0)$	$-\omega^{-4}$	$-\frac{1}{24}\omega^{-4}$
$(\omega,0,-\omega,0,0)$	$+4\omega^{-4}$	$+\frac{7}{12}\omega^{-4}$
$(0,\omega,-\omega,0,0)$	$-\omega^{-4}$	$-\frac{11}{24}\omega^{-4}$
$(\omega,0,0,-\omega,0)$	$-6\omega^{-4}$	$-\frac{3}{2}\omega^{-4}$
$(0,\omega,0,-\omega,0)$	$+4\omega^{-4}$	$+\frac{11}{6}\omega^{-4}$
$(0,0,\omega,-\omega,0)$	$-\omega^{-4}$	$-\frac{11}{24}\omega^{-4}$
$(\omega,0,0,0,-\omega)$	$+4\omega^{-4}$	$+\omega^{-4}$
$(0,\omega,0,0,-\omega)$	$-6\omega^{-4}$	$-\frac{3}{2}\omega^{-4}$
$(0,0,\omega,0,-\omega)$	$+4\omega^{-4}$	$+\frac{7}{12}\omega^{-4}$
$(0,0,0,\omega,-\omega)$	$-\omega^{-4}$	$-\frac{1}{24}\omega^{-4}$

BIBLIOGRAPHY

[B] A.D. BRYUNO (often transliterated as BRUNO or BRJUNO), Analytical forms of differential equations I, II, Trudy Mosk. Mat. Ob., 25 (1971), 119–262; 26 (1972), 199–239; English translation in Trans. Mosc. Math. Soc., 25 (1971), 26 (1972).

[E1] J. ECALLE, Les fonctions résurgentes, vol. 1, Algèbres de fonctions résurgentes, Publ. Math. Orsay, 1981.

[E2] J. ECALLE, Les fonctions résurgentes, vol. 2, Les fonctions résurgentes appliquées à l'itération, Publ. Math. Orsay, 1981.

[E3] J. ECALLE, Les fonctions résurgentes, vol. 3, L'équation du pont et la classification analytique des objets locaux, Publ. Math. Orsay, 1985.

[E4] J. ECALLE, Finitude des cycles-limites et accéléro-sommation de l'application de retour, in: Bifurcations of Planar Vector Fields (J.-P. Françoise, R. Roussarie, eds). Lecture Notes in Math. 1455, Springer-Verlag, Berlin-Heidelberg-New York (1990), 74–159.

[E5] J. ECALLE, Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac, Actualités Math. Hermann, Paris, 1992.

[E6] J. ECALLE, The bridge equation and its applications to local geometry, Proc. Internat. Conf. Dynam. Syst., Nagoya (Sept. 1990) (K. Shiraiwa, ed.), Adv. Ser. Dynam. Syst., 9.

[E7] J. Ecalle, Singularités non abordables par la géométrie, Ann. Inst. Fourier (Grenoble), 42 (1–2) (1992), 73–164.

[E8] J. Ecalle, Compensation of small denominators and ramified linearization of local objects, Prepubl. Math. Orsay (1992), n° 92–53; to appear in Astérisque, Proc. Conf. Dynam. Syst. (Rio de Janeiro, IMPA, Jan. 1992) (C. Camacho, ed.).

[E9] J. Ecalle, Alien calculus and its applications, three volumes in course of completion; scheduled to appear with Cambridge University Press.

[E10] J. Ecalle, Six lectures on transseries, analysable functions and the constructive proof of Dulac's conjecture, 75–184, in: Bifurcations and Periodic Orbits of Vector Fields (D. Schlomiuk, ed.), 1993, Kluwer Acad. Publ., NATO ASI Series, Series C, Vol. 408.

[M] J. Martinet, Normalisation des champs de vecteurs holomorphes, d'après A.D. Bryuno, Séminaire Bourbaki 1981–82, n° 582; Astérisque, 92–93 (1982), 59–73.

[S1] D. Schlomiuk, Algebraic particular integrals, integrability and the problem of the center, Trans. Amer. Math. Soc., 1993.

[S2] D. Schlomiuk, Algebraic and geometric aspects of the theory of polynomial vector fields, 429–467, in: Bifurcations and Periodic Orbits of Vector Fields (D. Schlomiuk, ed.), 1993, Kluwer Acad. Publ., NATO ASI Series, Series C, Vol. 408.

J. ÉCALLE,
Université de Paris-Sud
Mathématiques
Bâtiment 425
91405 Orsay (France)
&
D. SCHLOMIUK,
Université de Montréal
Département de Math. et Statistique
C.P. 6128
Montréal P.Q. (Canada).