ANALYTIC POTENTIAL THEORY OVER THE $p$-ADICS

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In these notes we develop the potential theory of the $p$-adic analogue of the symmetric stable distributions. We do this purely analytically and in an explicit manner. In §1 we recall Weil's formulation [11] of (the local part of) Tate's thesis [3] in terms of $\alpha$-homogeneous distributions, and relate these in §2 to probability. We prove that the function $|x|^\alpha$ is negative definite over $\mathbb{Q}_p$ for $\alpha \in (0, \infty)$, generating a semi-group of probability measures $\mu_t^\alpha$, explicitly given by

$$\mu_t^\alpha(dx) = \sum_{n \geq 1} \frac{(-t)^n}{n!} \frac{\zeta_p(1+n\alpha)}{\zeta_p(-n\alpha)} |x|^{-(1+n\alpha)} \, dx$$

(this formula being an analogue of a formula of Feller [4] for the $\alpha$-symmetric stable distribution over the reals $\mathbb{R}$). For $\alpha \in (0,1)$, $\mu_t^\alpha$ is transient, and its potential is the Riesz potential

$$k_t^\alpha(x) = \int_0^\infty \mu_t^\alpha(x) \, dt = \frac{\zeta_p(1-\alpha)}{\zeta_p(\alpha)} |x|^{\alpha-1}.$$ 

When we approach the boundary $\alpha \to \infty$ (in analogy with the real case $\alpha \to 2$) we obtain the "normal law", a very simple process which degenerates and "lives" on $\mathbb{Q}_p/\mathbb{Z}_p$. In §3 we recall the analytic properties of the Riesz potentials [10], their distributional meromorphic continuation, and the Riesz Reproduction formula. In §4 we begin to develop the potential theory of $\mathbb{R}^\alpha_p$, $\alpha \in (0,1)$. In view of §2 we can identify our potentials with probabilistic potentials and deduce the various potential theoretic principles in one blow [1], [2], [7], [9]; we prefer, nevertheless, to develop these principles purely analytically, and to grasp things along the way in a most explicit manner. We note, for example, that the Harnack inequalities

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become in our $p$-adic situation trivial equalities. We prove the principles of Descent, Dichotomy, Maximum, Regularization, and Uniqueness. We follow mostly the analogous real case as developed in [8], the $p$-adic setting offering many simplifications. In §5 we consider the finite energy measures leading to the proof that they form a complete positive cone in the Hilbert space of finite energy distributions. In §6 we prove the existence and various characterizations of the equilibrium measure, and show that the $\alpha$-capacity satisfies the usual properties and, moreover, is given explicitly via the $\alpha$-diameter:

$$\text{cap}_\alpha(K) = \frac{\zeta_p(\alpha)}{\zeta_p(1-\alpha)} \lim_{N \to \infty} \left[ \min_{x_1, \ldots, x_N \in K} \frac{1}{n(n-1)} \sum_{i<j} |x_i - x_j|^{\alpha-1} \right]^{-1}. $$

In §7, we approach Balayage and the Green measure using the Keldish transform [8]; i.e. instead of using Cartan’s method of projections in the Hilbert space of finite energy distributions (which equally works well), we use the more geometrical situation of the analysis of the $\text{PGL}_2(\mathbb{Q}_p)$-action on $\mathbb{P}(\mathbb{Q}_p)$ and on equilibrium measures. We calculate explicitly the Green measures of balls and their complements. Throughout §4 to §7 there are various strengthenings of the uniqueness principle which demonstrate the increasing grasp we have over our potentials. In §8, we develop the concepts of $\alpha$-(super)-harmonic functions. Explicitly, a function $f : \mathbb{Q}_p \to \mathbb{C}$ is $\alpha$-harmonic at $x \in \mathbb{Q}_p$ if for all $N$ sufficiently large:

$$f(x) = \frac{1}{\zeta_p(\alpha)} \int_{|y| \geq 1} f(x + p^N y) \frac{d^*y}{|y|^\alpha}. $$

We give an explicit solution to Dirichlet problems, prove the Riesz representation theorem, and prove our last principles for potentials: Domination and Harmonic minorant, concluding with a convexity property of the $\alpha$-capacity. We note that with slight modifications one can carry over the whole discussion to the case of an arbitrary finite dimensional vector space over an arbitrary non-archimedean local field.

1. The local zeta function and homogeneous distributions.

We let $\psi$ denote the “canonical” character of $\mathbb{Q}_p$, given by

$$\psi(x) = e^{-2\pi i(x)} : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z} \overset{e^{-2\pi i x}}{\hookleftarrow} \mathbb{C}^*. $$

The self-dual Haar measure with respect to $\psi$ is given by $dx(\mathbb{Z}_p) = 1$. We let $S = \{ \varphi : \mathbb{Q}_p \to \mathbb{C}, \varphi$ locally constant and compactly supported$\}$, and its dual $S^* = \text{the space of temperate distributions}$.
$\mathbb{Q}_p^*$ acts on $\mathbb{Q}_p$, hence on $S$ via $\pi_a \varphi(x) = \varphi(ax)$ and hence on $S^*$ via $\pi_a^* \mu(a) = \mu(\pi_a \varphi)$, e.g. $\pi_a^* \delta = \delta$, $\pi_a^* \, dx = |a|^{-1} \, dx$.

We let $S^*[\alpha] = \{ \mu \in S^* | \pi_a^* \mu = |a|^{-\alpha} \mu, \, \forall a \in \mathbb{Q}_p^* \}$, the homogeneous distributions of order $\alpha \in \mathbb{C}/\mathbb{Z}$, e.g. $\delta \in S^*[0]$, $dx \in S^*_p[1]$.

We normalize the Haar measure $d^*x$ on $\mathbb{Q}_p^*$ by $d^*x(\mathbb{Z}_p^*) = 1$, so that $d^*x = (1-p^{-1})^{-1} \frac{dx}{|x|}$. By uniqueness of Haar measure, if $\mu \in S^*[\alpha]$ then $\mu|_{\mathbb{Q}_p^*} = \text{cons} \cdot |x|^\alpha \, d^*x$. Thus, if $\mu_1, \mu_2 \in S^*[\alpha]$, there exist constants $c_1, c_2 \in \mathbb{C}$ such that $c_1 \mu_1 + c_2 \mu_2$ has support $\{0\}$. The only distribution on $\mathbb{Q}_p$ with support $\{0\}$ is $\delta$. For $\Re \alpha > 0$ we have $|x|^\alpha-1 \in L^1_{\text{loc}}$ hence $M^\alpha \defeq |x|^\alpha \, d^*x \in S^*[\alpha]$, and by the above:

$$S^*[\alpha] = C \cdot M^\alpha, \quad \Re \alpha > 0.$$ 

The “additive vacuum” is $\phi = \text{characteristic function of } \mathbb{Z}_p$; $\phi \in S$, and we have : $M^\alpha(\phi) = (1-p^{-\alpha})^{-1} \defeq \zeta(\alpha)$, $\Re \alpha > 0$.

We denote the Fourier transform by $\mathcal{F} \varphi(y) = \int \varphi(x) \tilde{\psi}(xy) \, dx$ so that $\mathcal{F} \mathcal{F} = \pi_{-1}$, $\mathcal{F} \pi_a = |a|^{-1} \pi_{a-1} \mathcal{F}$, $\mathcal{F} \phi = \phi$. On distributions we have $\mathcal{F}^* \mu(\varphi) = \mu(\mathcal{F} \varphi)$, $\mathcal{F}^* \mathcal{F}^* = \pi_{-1}$, $\pi_a \mathcal{F}^* = |a|^{-1} \mathcal{F}^* \pi_{a-1}$; in particular, $\mathcal{F}^* : S^*[\alpha] \sim S^*[1-\alpha]$, and hence

$$S^*[\alpha] = C \cdot \mathcal{F}^* M^{1-\alpha}, \quad \Re \alpha < 1.$$ 

We let $\tilde{M}^\alpha = (1-p^{-1})M^\alpha = (1-p^{-\alpha})M^\alpha = \frac{1}{\zeta(\alpha)} \cdot M^\alpha$. For $\varphi \in S$, $\tilde{M}^\alpha(\varphi) = \int (\varphi(x)-\varphi(p^{-1}x))|x|^\alpha \, d^*x$, where the integral converges for all $\alpha$, since $\varphi(x)-\varphi(p^{-1}x) = 0$ for $x$ near 0 ($\varphi$ locally constant!). Hence, $M^\alpha = \zeta(\alpha) \cdot \tilde{M}^\alpha$ has meromorphic continuation to all $\alpha \in \mathbb{C}/2\pi i \mathbb{Z}$, with a unique simple pole at $\alpha = 0$. On the other hand, $\tilde{M}^\alpha$ is entire, and since $\mathcal{F}^* \mathcal{M}^\alpha = \text{cons} \cdot \tilde{M}^{1-\alpha}$, $\tilde{M}^\alpha(\phi) = 1$, $\mathcal{F} \phi = \phi$, we get $\mathcal{F}^* \tilde{M}^\alpha = \mathcal{M}^{1-\alpha}$ and hence $\mathcal{F}^* M^\alpha = \frac{\zeta(\alpha)}{\zeta(1-\alpha)} \mathcal{M}^{1-\alpha}$. An easy check gives $\tilde{M}^0 = \delta$, and hence

$$\text{Res}_{\alpha=0} M^\alpha = \frac{1}{\log p} \cdot \delta.$$

2. Probabilistic interpretation.

Consider the “Riesz kernel” \( k^\alpha(x) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} |x|^{\alpha-1} \). For \( \alpha \in (0, \infty) \), \( \mathcal{L}^\alpha(x) = -k^{-\alpha}(x)dx = \frac{p^{\alpha-1}}{1-p^{-(1+\alpha)}} \frac{dx}{|x|^{1+\alpha}} \) is a positive measure on \( \mathbb{Q}_p \setminus \{0\} \).

2.1. LEMMA.

(i) \[ \int_{|x| \leq p^k} \psi_p(x) \, dx = \begin{cases} p^k, & k \leq 0 \\ 0, & k > 0 \end{cases} \]

(ii) \[ \int_{|x| = p^k} \psi_p(x) \, dx = \begin{cases} p^k(1-p^{-1}), & k \leq 0 \\ -1, & k = 1 \\ 0, & k > 1 \end{cases} \]

(iii) \[ \int_{|x| \leq p^k} (1-\psi_p(xy)) \frac{dx}{|x|^{1+\alpha}} = \begin{cases} 0, & |y| \leq p^{-k} \\ p^{-\alpha}|y|^\alpha = p^{-\alpha}, & |y| = p^{1-k} \\ p^{-k\alpha}(1-p^{-1}), & |y| > p^{1-k}. \end{cases} \]

Proof. — Straightforward using (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii).

Using this lemma we get: \( |y|^\alpha = \int_{\mathbb{Q}_p \setminus \{0\}} (1 - \text{Re} \psi(xy)) d\mathcal{L}^\alpha(x) \), the “Levy representation” (cf. [1], Cor. 18.20, p. 184).

2.2. COROLLARY. — For \( \alpha \in (0, \infty) \), \( |y|^\alpha \) is a negative definite function with associated Levy measure \( \mathcal{L}^\alpha(x) \).

Hence we get a convolution semi-group of probability measures \( \mu_t^\alpha \), (the \( p \)-adic analogue of) the “symmetric stable semi-group of order \( \alpha \)”, given by:

\[ \mu_t^\alpha = \mathcal{F} e^{-t|y|^\alpha}, \; t \in [0, \infty), \; \alpha \in (0, \infty), \]

i.e. \( \mu_t^\alpha \) is a probability measure on \( \mathbb{Q}_p \), \( \mu_t^\alpha \ast \mu_t^\beta = \mu_{t_1+t_2}^\alpha, \; \mu_t^\alpha \underset{t \to 0}{\sim} \delta \) (where “\( \sim \)” denotes vague convergence : \( \mu_t \sim \mu \) iff \( \mu_t(\varphi) \to \mu(\varphi) \) for all \( \varphi \in C_c(\mathbb{Q}_p) \)).
Explicitly, we have
\[
\mu_t^\alpha = \int e^{-t|x|^\alpha} \psi(xy) \, dy
\]
\[
= \sum_{m \in \mathbb{Z}} e^{-tp^{-m\alpha}} \int_{|y|=p^{-m}} \psi(xy) \, dy \quad \text{(and using 2.1)}
\]
\[
= (1-p^{-1}) \sum_{m \geq 0} e^{-t|p^{-m}x|^{-\alpha} |p^{-m}x|^{-1}} e^{-t|px|^{-\alpha} |x|^{-1}}
\]
\[
= \sum_{k \geq 0} \frac{(-t)^k}{k!} |x|^{-k\alpha - 1} \left[ (1-p^{-1}) \sum_{m \geq 0} p^{-(k\alpha+1)m} - p^{k\alpha} \right]
\]
\[
= \sum_{n \geq 1} \frac{(-1)^n}{n!} k^{-n\alpha}(x),
\]
i.e. we have the following “Feller’s formula” (cf. [4] XVII, 6, Lemma 1 (6.8) p. 549):
\[
\mu_t^\alpha = \sum_{n \geq 1} \frac{(-t)^n}{n!} k^{-n\alpha}(x).
\]

For \( \alpha \in (0,1) \), \(|x|^{-\alpha} \in L_{\text{loc}}^1 \), hence \( \mu_t^\alpha \) is integrable, i.e. \( \int_0^\infty \mu_t \, dt \)
defines a potential kernel which is nothing but \( k^\alpha(x) \). Thus,
\[
k^\alpha(x) = \mathcal{F}(|x|^{-\alpha} \, dx) = \int_0^\infty \mu_t^\alpha(x) \, dt
\]

Note that \( \mu_t^\alpha(x) \) is “symmetric”, i.e. it is invariant under \( \mathbb{Z}^*_p \)-action. The support of \( \mu_t^\alpha \) is all of \( \mathbb{Q}_p \), i.e. \( \mu_t^\alpha(x) > 0 \) for all \( x \in \mathbb{Q}_p \). We have
\[
\mu_t^\alpha(\phi) = (1-p^{-1}) \sum_{m \geq 0} p^{-m} e^{-tp^{-m\alpha}}.
\]

As \( \alpha \to \infty \), \( e^{-t|x|^\alpha} \to e^{-t\phi(x)} + (1-e^{-t})\phi\left(\frac{1}{p}x\right) \), pointwise, hence, upon taking Fourier transform, we get \( \mu_t^\alpha \sim \mu_t^\infty(x) = e^{-t\phi(x)} + (1-e^{-t})p^{-1}\phi(px) \), vaguely. \( \mu_t^\infty \) is a convolution semi-group of probability measures on \( \mathbb{Q}_p/\mathbb{Z}_p \), (the \( p \)-adic analogue of) “the normal distribution”, i.e. \( \mu_t^\infty \) is a probability measure on \( \mathbb{Q}_p/\mathbb{Z}_p \), \( \mu_t^\infty \ast \mu_t^\infty = \mu_{t+t}^\infty \), and \( \mu_t^\infty \sim \delta \) on \( \mathbb{Q}_p/\mathbb{Z}_p \).

Its infinitesimal generator, (the \( p \)-adic analogue of) “the Laplacian”, is
\[
\frac{\partial}{\partial t} \mu_t^\infty(x) = p^{-1} \phi(px) - \phi(x) = -\Delta(x).
\]
Letting $\phi^*(x)$ denote the characteristic function of $\mathbb{Z}_p^*$, we can rewrite the above in the following form, remembering that $\psi(x)$ induces duality between $\mathbb{Z}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p$; the Levy measure is $L^\infty = \frac{1}{p} \sum \delta_{a/p}$ on $\mathbb{Q}_p/\mathbb{Z}_p$; the associated negative definite function on $\mathbb{Z}_p$ is

$$
\phi^*(x) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} (1-\psi_p(xy))dL^\infty(y);
$$

and the associated "normal law" is given by $\mu_1^\infty = \mathcal{F}e^{-t\phi^*(x)}$, while its infinitesimal generator is $\Delta = \mathcal{F}\phi^*$.

### 3. The Riesz kernel.

We call the measure

$$
k^\alpha = \frac{\zeta(1-\alpha)}{\zeta(\alpha)}|x|^{\alpha-1}dx, \quad \alpha > 0, \quad \alpha \neq 1,
$$

the M. Riesz kernel. As a distribution, it has a meromorphic continuation to all $\alpha \neq 1 \pmod{2\pi i/\log p}$, given by

$$
k^\alpha(\varphi) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\varphi(0) + \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \left[ \int_{|x|>1} \frac{\varphi(x)dx}{|x|^{1-\alpha}} + \int_{|x|\leq 1} (\varphi(x)-\varphi(0)) \frac{dx}{|x|^{1-\alpha}} \right].
$$

In particular, for $\Re(\alpha) > 0$:

$$
k^\alpha(\varphi) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int \varphi(x)|x|^{\alpha-1}dx, \quad \alpha \neq 1 \pmod{2\pi i/\log p}.
$$

For $\alpha = 0$, $k^0(\varphi) = \varphi(0)$, i.e. $k^0 = \delta_0$.

For $\Re \alpha < 1$, $\mathcal{F}k^\alpha = |x|^{-\alpha} dx$; in particular, for $\alpha \in (0,1)$, both $k^\alpha$ and $\mathcal{F}k^\alpha$ are measures. Consider next $k^\alpha$ as an operator via convolution, namely for $\varphi$ locally constant and such that $\int_{|x|>1} |\varphi(x)| \frac{dx}{|x|^{1-\Re \alpha}} < \infty$,

we can form the convolution $k^\alpha \ast \varphi$:

$$
k^\alpha \ast \varphi(x_0) = \frac{\zeta(1-\alpha)}{\zeta(1)}\varphi(x_0) + \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \left[ \int_{|x|>1} \varphi(x_0 + x) \frac{dx}{|x|^{1-\alpha}} + \int_{|x|\leq 1} (\varphi(x_0 + x)-\varphi(x_0)) \frac{dx}{|x|^{1-\alpha}} \right].
$$
and again, for $\text{Re}(\alpha) > 0$,
\begin{equation}
k^{\alpha} * \varphi(x_0) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int \varphi(x_0 + x)|x|^\alpha \, dx,
\end{equation}
\begin{equation}
k^{-\alpha} * \varphi(x_0) = \frac{\zeta(1+\alpha)}{\zeta(-\alpha)} \int (\varphi(x_0 + x) - \varphi(x_0)) \frac{dx}{|x|^{1+\alpha}}.
\end{equation}

E.g., generalizing Levy’s representation, $k^{-\alpha} * \psi_y(x) = |y|^\alpha \cdot \psi_y(x)$, i.e. \(\psi_y(x) = \psi(y \cdot x)\) is a $k^{-\alpha}$-eigenvector with eigenvalue $|y|^\alpha$. For \(\varphi \in S\), $k^{\alpha} * \varphi$ is again locally constant (with the same modulus as that of \(\varphi\)), and moreover, $k^{\alpha} * \varphi(x) = O(|x|^{|\text{Re}\alpha|-1})$, hence we can form $k^\beta * (k^{\alpha} * \varphi)$ for $\text{Re}(\alpha + \beta) < 1$. For $\alpha, \beta$ such that $0 < \alpha, \beta, \alpha + \beta < 1$, the equation $k^{\alpha} * k^\beta = k^{\alpha + \beta}$ is immediate upon taking Fourier transforms, and since everything is holomorphic in $\alpha, \beta$ we get

3.6. M. Riesz Reproduction Formula. — For $\varphi \in S$, and $\alpha, \beta \neq 1 \mod \frac{2\pi i}{\log p}$ with $\text{Re}(\alpha + \beta) < 1$,
\[k^{\alpha} * (k^\beta * \varphi) = k^{\alpha + \beta} * \varphi.\]

3.7. Corollary. — For $\varphi \in S$ and $\alpha \neq \pm 1 \mod \frac{2\pi i}{\log p}$, $k^{-\alpha} \varphi(y) = O(|y|^{-(1 + |\text{Re}\alpha|)})$ as $|y| \to \infty$, and we have $k^{\alpha} * (k^{-\alpha} * \varphi) = \varphi$.

4. Basic principles for potentials.

We let:
\[\mathcal{M}^+ = \{\text{positive measures on } \mathbb{Q}_p\},\]
\[\mathcal{M} = \{\text{signed measures on } \mathbb{Q}_p\},\]
\[\mathcal{M}_c^+, \mathcal{M}_c = \{\text{(positive) measures with compact support on } \mathbb{Q}_p\},\]
\[\mathcal{M}^+(K), \mathcal{M}(K) = \{\text{(positive) measures with compact support on } K\}.$

We shall consider next $k^{\alpha}$ as an operator on $\mathcal{M}$ via convolution, restricting ourselves to the case $\alpha \in (0, 1)$.

For $\nu \in \mathcal{M}_c^+$, we have
\[k^{\alpha} * \nu(x) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int |x-y|^\alpha \, d\nu(y)\]
\[= \lim_{N \to \infty} \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_{|y| \geq N} |x-y|^\alpha \, d\nu(y)\]
an increasing limit of continuous functions, hence is a lower-semicontinuous function of \( x \), i.e. \( k^\alpha \ast \nu(x_0) \leq \lim_{x \to x_0} k^\alpha \ast \nu(x) \); moreover \( k^\alpha \ast \nu \in L^1_{\text{loc}} \) as follows from Fubini's theorem:

\[
\int_{|x| \leq p^N} k^\alpha \ast \nu(x) \, dx = \int_{\sup(\nu)} dv(y) \int_{|x| \leq p^N} k^\alpha(x-y) \, dx < \infty.
\]

For \( 0 \neq \nu \in \mathcal{M}^+ \), \( k^\alpha \ast \nu(x) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int |x-y|^{\alpha-1} \, d\nu(y) \) is a well-defined positive function of \( x \), although it may assume the value \( \infty \), \( 0 < k^\alpha \ast \nu(x) \leq \infty \).

4.1. Definition. — For \( \nu \in \mathcal{M}^+ \), we write \( \nu \in \mathcal{M}_\alpha^+ \) if it satisfies one of the following equivalent conditions:

(i) \( \{ x \mid k^\alpha \ast \nu(x) = \infty \} \) has measure 0 w.r.t. \( dx \).

(ii) \( \int_{|y| > p^N} |y|^{\alpha-1} \, d\nu(y) < \infty \) for some \( N \).

(iii) \( \int_{|y| > p^N} |y|^{\alpha-1} \, d\nu(y) < \infty \) for all \( N \).

Indeed, if (iii) is not satisfied, then for \( |x| \leq p^N \)

\[
\frac{\zeta(\alpha)}{\zeta(1-\alpha)} \cdot k^\alpha \ast \nu(x) = \int |x-y|^{\alpha-1} \, d\nu(y) \geq \int_{|y| > |x|} |y|^{\alpha-1} \, d\nu(y) = \infty,
\]

and \( k^\alpha \ast \nu(x) \equiv \infty \), showing (i) \( \Rightarrow \) (iii); while since \( \int_{|x| < p^N} |x-y|^{\alpha-1} \, dx = O(|y|^{\alpha-1}) \) as \( |y| \to \infty \), we have assuming (ii),

\[
\int_{|x| < p^N} k^\alpha \ast \nu(x) \, dx = \int d\nu(y) \int_{|x| < p^N} k^\alpha(x-y) \, dx
\]

\[
= O\left( \int_{|y| > p^N} |y|^{\alpha-1} \, d\nu(y) \right) < \infty,
\]

showing (ii) \( \Rightarrow \) (i).

We say \( \nu \in \mathcal{M}_\alpha \) if \( \nu = \nu^+ - \nu^- \) with \( \nu^+, \nu^- \in \mathcal{M}_\alpha^+ \). We have \( \mathcal{M}_c \subseteq \mathcal{M}_\alpha^+ \), \( \mathcal{M}_c \subseteq \mathcal{M}_\alpha \).

We denote by \( B(p^N) = \{ x \mid |x| \leq p^N \} \), \( \bar{B}_N = \{ x \mid |x| \geq p^{-N} \} \), the ball of radius \( p^N \) and its exterior.

We note that the Harnack inequalities (cf. [8] IV, §5, n°20, p. 266) become in our non-archimedean setting actually equalities. For \( \nu \in \)
\[ M(B(p^N)) \text{ and any } y \notin B(p^N) \text{ we have } k^\alpha \ast \nu(y) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \nu(1) \cdot |y|^{\alpha-1}, \]

and so for \( y_1, y_2 \notin B(p^N) : k^\alpha \ast \nu(y_1) = \frac{|y_1/y_2|^{\alpha-1} \cdot k^\alpha \ast \nu(y_2)}. \] Similarly, for \( \nu \in M(\tilde{B}_N) \text{ and any } y \notin \tilde{B}_N, k^\alpha \ast \nu(y) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int |x|^{\alpha-1} d\nu(x), \]

and so for \( y_1, y_2 \notin \tilde{B}_N : k^\alpha \ast \nu(y_1) = k^\alpha \ast \nu(y_2). \) For a general \( \nu \in M_\alpha, \) we have that \( k^\alpha \ast \nu \) is locally constant away from \( \text{supp } \nu, \)

\[ \int_{|x|>1} |k^\alpha \ast \nu(x)| \frac{dx}{|x|^1+\alpha} = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int d\nu(y) \int_{|x|>1} |x-y|^{\alpha-1}|x|^{-1-\alpha} dx \]

\[ \leq \text{const} \cdot \int |y|^{\alpha-1} d\nu(y) < \infty. \]

4.2. PRINCIPLE OF DESCENT. — Let \( \nu_n \in M_\alpha^+, \nu_n \rightsquigarrow \nu_\infty, x_n \to x_\infty, \) then

\[ k^\alpha \ast \nu_\infty(x_\infty) \leq \lim_{n \to \infty} k^\alpha \ast \nu_n(x_n). \]

4.3. COROLLARY. — For \( \nu \in M_\alpha^+, k^\alpha \ast \nu \) is lower-semicontinuous.

4.4. COROLLARY. — For \( \nu_n \in M_\alpha^+, \nu_n \rightsquigarrow \nu, k^\alpha \ast \nu = \lim k^\alpha \ast \nu_n. \)

4.5. COROLLARY. — For \( \nu_n \in M_\alpha^+, \nu_n \rightsquigarrow \nu, \nu_n \leq \nu_{n+1}, k^\alpha \ast \nu = \lim k^\alpha \ast \nu_n. \)

Proof of 4.2. — Assume first \( \text{supp}(\nu_n) \subseteq \{ x \ | \ |x| \leq p^N \} \) for all \( n. \) Let \( k_\varepsilon^\alpha \ast \nu_n(x) = \int_{|x-y|>\varepsilon} k^\alpha(x-y) d\nu_n(y), \) it’s uniformly continuous and \( k_\varepsilon^\alpha \ast \nu(x) = \lim k_\varepsilon^\alpha \ast \nu_n(x), \) hence we have \( k_\varepsilon^\alpha \ast \nu(x_\infty) = \lim k_\varepsilon^\alpha \ast \nu_n(x_n) \leq \lim k_\varepsilon^\alpha \ast \nu_n(x_n) \) and taking \( \varepsilon \to 0 \) we get the theorem in this special case. In the general case, let \( \nu_n^{(N)} \), \( \nu_\infty^{(N)} \) denote the restrictions of \( \nu_n, \nu_\infty \) to \( \{ x \ | \ |x| \leq p^N \}. \) Then \( \nu_n^{(N)} \rightsquigarrow \nu_\infty^{(N)}, \) and by the above \( k^\alpha \ast \nu_\infty^{(N)}(x_\infty) \leq \lim k^\alpha \ast \nu_n^{(N)}(x_n). \) Taking the limit \( N \to \infty, \) first on the right-hand side and then on the left we conclude the theorem.

4.6. DICHOTOMY PRINCIPLE. — Let \( \nu_n \in M^+ \) such that \( k^\alpha \ast \nu_n \leq k^\alpha \ast \nu_{n+1}. \) Then either \( \lim \to \infty k^\alpha \ast \nu_n(x) \) for all \( x, \) or \( \lim \to \infty k^\alpha \ast \nu_n(x) = k^\alpha \ast \nu(x) + c, \) for some \( \nu \in M_\alpha^+, c \geq 0, \) and almost all \( x. \)

Proof. — Suppose \( \lim \to \infty k^\alpha \ast \nu_n(x) \neq \infty. \)
(i) For any compact $K$, $\nu_n(K)$ is bounded. For if $\nu_n(K) \to \infty$ then
\[
\lim_{i \to \infty} k^\alpha * \nu_n(x) \geq \lim_{i \to \infty} \int_K |x-y|^{\alpha-1} d\nu_n(y) \equiv \infty.
\]
(ii) For any $N$, $\int_{|y|>pN} |y|^{\alpha-1} d\nu_n(y)$ is bounded. For if $\int_{|y|>pN} |y|^{\alpha-1} d\nu_n(y) \to \infty$ then
\[
\lim_{i \to \infty} k^\alpha * \nu_n(x) \geq \lim_{i \to \infty} \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_{|y|>|x|} |y|^{\alpha-1} d\nu_n(y) \equiv \infty.
\]
From (i), (ii) we get $\nu_n \sim \nu$, for some $\nu \in \mathcal{M}^+$. Let $F(x) = \lim_{n \to \infty} k^\alpha * \nu_n(x)$. $F \in L^1_{\text{loc}}$ since
\[
\int_{|x|<pN} F(x) \, dx \leq \lim_{n \to \infty} \int_{|x|<pN} k^\alpha * \nu_n(x) \, dx \leq \lim_{n \to \infty} \int d\nu_n(y) \int_{|x|<pN} k^\alpha(x-y) \, dx < \infty.
\]
Let $\varphi \in \mathcal{S}$, $\int \varphi(x) \, dx = 0$, then $|x|^{-\alpha} \mathcal{F} \varphi(x) \in \mathcal{S}$, and $k^\alpha * \varphi = \mathcal{F}^{-1} |x|^{-\alpha} \mathcal{F} \varphi \in \mathcal{S}$. We have:
\[
\int F(x) \varphi(x) \, dx = \lim_{n \to \infty} \int k^\alpha * \nu_n(x) \varphi(x) \, dx = \lim_{n \to \infty} \int k^\alpha * \varphi(x) \, d\nu_n(x) = \int k^\alpha * \varphi(x) \, d\nu(x) = \int k^\alpha * \nu(x) \varphi(x) \, dx.
\]
Thus, for any $\varphi \in \mathcal{S}$ orthogonal to 1 : $\int \lim_{n \to \infty} k^\alpha * \nu_n(x) \varphi(x) \, dx = k^\alpha * \nu(x) \varphi(x) \, dx$ and hence $\lim_{n \to \infty} k^\alpha * \nu_n(x) - k^\alpha * \nu(x) = c$ almost everywhere, where $c$ is a constant, by the principle of descent $c \geq 0$.

4.7. 1st MAXIMUM PRINCIPLE. — Let $\nu \in \mathcal{M}^+$ be such that $k^\alpha * \nu(x) \leq M$, $\nu$-almost everywhere. Then $k^\alpha * \nu(x) \leq M$ for all $x$.

Proof. — Assume first $x \in \text{supp}(\nu)$, then the inequality follows by the lower semi-continuity of $k^\alpha * \nu$, hence $k^\alpha * \nu(x) \leq M$ for $x \in \text{supp}(\nu)$. Let $x \notin \text{supp}(\nu)$, and let $x' \in \text{supp}(\nu)$ be such that $|x-x'|$ is minimal. Then for $y \in \text{supp}(\nu)$, $|y-x'| \leq \max(|y-x|, |x-x'|) \leq |y-x|$, hence $|y-x'|^{\alpha-1} \geq |y-x|^{\alpha-1}$, and so $k^\alpha * \nu(x) \leq k^\alpha * \nu(x') \leq M$. 

4.8. COROLLARY. — If \( k^\alpha \ast \nu \) is continuous on \( \text{supp}(\nu) \), then it's continuous everywhere.

Proof. — On \( \mathbb{Q}_p \setminus \text{supp}(\nu) \), \( k^\alpha \ast \nu \) is locally constant, so it's enough to check continuity at the points of \( \text{supp}(\nu) \). Let \( x_0 \in \text{supp}(\nu) \), \( \nu_\epsilon \) the restriction of \( \nu \) to \( \{ x \mid |x-x_0| < \epsilon \} \), \( \nu'_\epsilon = \nu - \nu_\epsilon \), so that \( k^\alpha \ast \nu(x) = k^\alpha \ast \nu_\epsilon(x) + k^\alpha \ast \nu'_\epsilon(x) \), hence \( |k^\alpha \ast \nu(x) - k^\alpha \ast \nu(x_0)| \leq k^\alpha \ast \nu_\epsilon(x) + k^\alpha \ast \nu'_\epsilon(x_0) + |k^\alpha \ast \nu'_\epsilon(x) - k^\alpha \ast \nu'_\epsilon(x_0)| \). Since \( k^\alpha \ast \nu'_\epsilon \) is continuous at \( x_0 \), it suffices to show that \( k^\alpha \ast \nu_\epsilon(x) \xrightarrow{\epsilon \to 0} 0 \) uniformly in \( x \). Let \( \eta > 0 \). Since \( k^\alpha \ast \nu(x_0) < \infty \) we can find \( \epsilon_1 \) such that \( k^\alpha \ast \nu_\epsilon_1(x_0) \leq \frac{1}{2} \eta \); and since \( k^\alpha \ast \nu_\epsilon_1 \) is continuous on \( \text{supp}(\nu) \) we can find \( \epsilon < \epsilon_1 \) such that \( |k^\alpha \ast \nu_\epsilon_1(x) - k^\alpha \ast \nu_\epsilon_1(x_0)| \leq \frac{1}{2} \eta \) for all \( x \in \text{supp}(\nu) \), \( |x-x_0| < \epsilon \); hence \( k^\alpha \ast \nu_\epsilon(x) \leq k^\alpha \ast \nu_\epsilon_1(x) \leq \eta \) for \( x \in \text{supp}(\nu_\epsilon) \). By the maximum principle, \( k^\alpha \ast \nu_\epsilon(x) \leq \eta \) everywhere.

Let \( \phi^{(N)}(x) = p^N \phi(p^{-N} x) \) denote the uniform distribution on \( \{ x \mid |x| \leq p^{-N} \} \). For any measure \( \nu \), let \( \nu^{(N)} = \nu \ast \phi^{(N)} \) its \( N \)th regularization, and let \( \nu^{(N)}(x) = (\nu^{(N)})_x \) denote the restriction of \( \nu^{(N)} \) to \( \{ x \mid |x| \leq p^{-N} \} \).

4.9. REGULARIZATION PRINCIPLE. — Let \( \nu \in \mathcal{M}_1^+ \), then \( k^\alpha \ast \nu^{(N)}(x) \xrightarrow{N \to \infty} k^\alpha \ast \nu(x) \) for all \( x \).

Proof.

\[
k^\alpha \ast \nu^{(N)}(x) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} p^N \int_{|y| \leq p^{-N}} \int_{\text{supp}(\nu)} |z-y|^{\alpha-1} \, d\nu(y) \int_{|z-x| \leq p^{-N}} |x-y|^{1-\alpha} \, dz
\]

where \( C = \sup_{N,y} \left\{ p^N \int_{|z-x| \leq p^{-N}} |x-y|^{1-\alpha} \, dz \right\} < \infty \).

If \( k^\alpha \ast \nu(x) = \infty \) then \( k^\alpha \ast \nu^{(N)}(x), k^\alpha \ast \nu^{(N)}(x) \xrightarrow{N \to \infty} \infty \) by lower-semicontinuity. So assume \( k^\alpha \ast \nu(x) < \infty \). Given \( \eta > 0 \), we can find \( \epsilon > 0 \) such that writing \( \nu = \nu_\epsilon + \nu'_\epsilon \), where \( \nu_\epsilon \) is the restriction of \( \nu \) to \( \{ z \mid |z-x| < \epsilon \} \), we have \( k^\alpha \ast \nu_\epsilon(x) < \eta \). Hence by the above, \( k^\alpha \ast (\nu_\epsilon^{(N)}) < C \cdot \eta \). Now
$k^\alpha * \nu'_\epsilon$ is continuous at $x$, so $k^\alpha * (\nu'_\epsilon)^{(N)}(x) \underset{N \to \infty}{\longrightarrow} k^\alpha * \nu'_\epsilon(x)$, and we get:

$$k^\alpha * \nu(x) - (C + 1) \eta \leq \lim_{N \to \infty} k^\alpha * \nu^{(N)}(x) \leq \lim_{N \to \infty} k^\alpha * \nu^{(N)}(x) \leq k^\alpha * \nu(x) + (C + 1) \eta.$$ 

Letting $\eta \to 0$ we obtain the theorem for $\nu^{(N)}$.

Next, considering $\nu^{(N)}$, we have:

$$\frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_{|y| > p_N} |x-y|^{\alpha-1} \, d\nu^{(N)}(y) \leq \frac{\zeta(1-\alpha)}{\zeta(\alpha)} p^N \int_{|z| \leq p_N} \int_{|y| > p_N} |x-z-y|^{\alpha-1} \, d\nu(y).$$

But since $\nu \in \mathcal{M}_\alpha^+$, we know that $\lim_{N \to \infty} \int_{|y| > p_N} |x-z-y|^{\alpha-1} \, d\nu(y) = 0$ uniformly with respect to bounded $z$'s, which gives the theorem for $\nu^{(N)}$.

4.10. UNIQUENESS PRINCIPLE. — Let $\nu \in \mathcal{M}_\alpha$, if $k^\alpha * \nu(x) = 0$ for a.a. $x$'s, then $\nu = 0$ (and $k^\alpha * \nu \equiv 0$).

Proof. — Let $\varphi \in \mathcal{S}$, $\varphi(x) = \varphi(-x)$, and apply Riesz formula to write $\varphi = k^\alpha * (k^{-\alpha} * \varphi)$. Since $|k^{-\alpha} * \varphi|(x) = O(|x|^{-(1+\alpha)})$ as $|x| \to \infty$, we easily obtain $k^\alpha * |k^{-\alpha} * \varphi|(x) = O(|x|^{\alpha-1})$, and since by assumption, $\int_{|y| > p_N} |y|^{\alpha-1} |\nu(y)| < \infty$, we see that the convolution $|\nu| * (k^\alpha * |k^{-\alpha} * \varphi|)$ is defined, and hence $\nu * (k^\alpha * (k^{-\alpha} * \varphi)) = (k^\alpha * \nu) * (k^{-\alpha} * \varphi)$. Thus, $\nu(\varphi) = \nu * \varphi(0) = \nu * (k^\alpha * (k^{-\alpha} * \varphi))(0) = (k^\alpha * \nu) * (k^{-\alpha} * \varphi)(0) = 0$ since $k^\alpha * \nu(x) = 0$ for a.a. $x$'s. Since $\varphi \in \mathcal{S}$ was arbitrary, we get $\nu \equiv 0$.

5. Measures with finite energy, $0 < \alpha < 1$.

We shall consider next signed measures $\nu = \nu^+ - \nu^- \in \mathcal{M}_\alpha$ such that

$$\int \int k^\alpha(x-y) \, d\nu^+(x) \, d\nu^-(y) < \infty.$$  

The “mutual $\alpha$-energy” of two such measures $\nu_1$ and $\nu_2$ is

$$\langle \nu_1, \nu_2 \rangle_{\alpha} = \int \int k^\alpha(x-y) \, d\nu_1(x) \, d\nu_2(y)$$

which is well defined under the condition that $\int \int k^\alpha(x-y) d\nu_1^\pm(x) d\nu_2^\pm(y) < \infty$. Since $k^\alpha(x-y) = k^\alpha(y-x)$, we get the
5.1. Symmetry principle. — \( \langle \nu_1, \nu_2 \rangle_\alpha = \langle \nu_2, \nu_1 \rangle_\alpha \).

5.2. Positive definiteness. — \( \langle \nu, \nu \rangle_\alpha \geq 0 \) and \( \langle \nu, \nu \rangle_\alpha = 0 \iff \nu = 0 \).

**Proof.** Write \( \nu = \nu^+ - \nu^- \), \( |\nu| = \nu^+ + \nu^- \); we are assuming \( \langle \nu^+, \nu^- \rangle_\alpha < \infty \), and we may assume \( \langle \nu, \nu \rangle_\alpha < \infty \), which gives us \( \langle |\nu|, |\nu| \rangle_\alpha < \infty \). Hence we can apply Fubini's theorem and the M. Riesz Formula to obtain:

\[
\langle \nu, \nu \rangle_\alpha = \int \int \delta(x) \delta(y) k^\alpha(x-y)
\]
\[
= \int \int \delta(x) \delta(y) \int k^{\alpha/2}(x-y-z) k^{\alpha/2}(z) \, dz
\]
\[
= \int \int \delta(x) \delta(y) \int k^{\alpha/2}(x-z) k^{\alpha/2}(z-y) \, dz
\]
\[
= \int k^{\alpha/2}(x-z) \, dz \int k^{\alpha/2}(z-y) \, d\nu(y)
\]
\[
= \int |k^{\alpha/2} \star \nu(z)|^2 \, dz \geq 0.
\]
I.e. \( \langle \nu, \nu \rangle_\alpha = \|k^{\alpha/2} \star \nu\|_{L^2}^2 \).

Moreover, if \( \langle \nu, \nu \rangle_\alpha = 0 \) then \( k^{\alpha/2} \star \nu \equiv 0 \) for \( \nu \)-a.a. \( x \)'s, and by the 1\(^{st} \) uniqueness principle \( \nu = 0 \).

As a corollary we get,

5.3. 2\(^{nd} \) uniqueness principle. — Let \( \nu \in \mathcal{M}_\alpha \) satisfy \( \langle \nu^+, \nu^- \rangle_\alpha < \infty \). If \( k^\alpha \star \nu(x) = 0 \) for \( \nu \)-a.a. \( x \)'s, then \( \nu = 0 \).

We can now define the "\( \alpha \)-energy" of \( \nu \), by \( \|\nu\|_\alpha = \langle \nu, \nu \rangle_\alpha^{1/2} \). We let \( \mathcal{E}_\alpha = \{ \nu \in \mathcal{M}_\alpha \text{ such that } \langle \nu^+, \nu^- \rangle_\alpha < \infty \text{ and } \|\nu\|_\alpha < \infty \} \) denote the space of signed measures with finite \( \alpha \)-energy, and \( \mathcal{E}_\alpha^+ = \mathcal{E}_\alpha \cap \mathcal{M}_\alpha^+ \):

- e.g. \( \text{supp}(\nu) \) compact and \( k^\alpha \star \nu \) bounded on \( \text{supp}(\nu) \Rightarrow \nu \in \mathcal{E}_\alpha \);
- e.g. \( k^\alpha \star \nu \) bounded of compact support \( \Rightarrow \nu \in \mathcal{E}_\alpha \).

\( \mathcal{E}_\alpha \) with the inner product \( \langle \, , \, \rangle_\alpha \) is a pre-Hilbert-space. In particular we have Schwartz inequality : \( |\langle \nu_1, \nu_2 \rangle_\alpha| \leq \|\nu_1\|_\alpha \cdot \|\nu_2\|_\alpha \). We write \( \nu_n \Rightarrow_\alpha \nu \) for strong convergence (i.e. \( \|\nu - \nu_n\|_\alpha \to 0 \)), and \( \nu_n \to_\alpha \nu \) for weak convergence (i.e. \( \langle \nu_n, \mu \rangle_\alpha \to \langle \nu, \mu \rangle_\alpha \), \( \forall \mu \in \mathcal{E}_\alpha \)). By the Schwartz inequality, \( \nu_n \Rightarrow_\alpha \nu \) implies \( \nu_n \to_\alpha \nu \).
We note also the "lower-semi-continuity" of $(\langle , \rangle)_\alpha$, namely: if $\nu_n, \nu'_n \in \mathcal{E}_\alpha$, $\nu_n \to \nu, \nu'_n \to \nu'$ then $\langle \nu, \nu' \rangle_\alpha \leq \lim_{N \to \infty} \langle \nu_n, \nu'_n \rangle_\alpha$. In particular, $\|\nu\|_\alpha \leq \lim_{N \to \infty} \|\nu_n\|_\alpha$.

5.4. REGULARIZATION LEMMA. — If $\nu \in \mathcal{E}_\alpha$ then $\nu^{(N)} \Rightarrow \nu$ and $\nu^{((N))} \Rightarrow \nu$.

**Proof.** — It's enough to consider $\nu \in \mathcal{E}_\alpha^+$. 

$$\|\nu - \nu^{(N)}\|_\alpha^2 = \int |k^{\alpha/2} \ast \nu(x) - k^{\alpha/2} \ast \nu^{(N)}(x)|^2 \, dx.$$ 

But 

$$|k^{\alpha/2} \ast \nu(x) - k^{\alpha/2} \ast \nu^{(N)}(x)|^2 \leq 2|k^{\alpha/2} \ast \nu(x)|^2 + 2|k^{\alpha/2} \ast \nu^{(N)}(x)|^2,$$

Hence, 

$$\lim_{N \to \infty} \|\nu - \nu^{(N)}\|_\alpha^2 = \int \lim_{N \to \infty} |k^{\alpha/2} \ast \nu(x) - k^{\alpha/2} \ast \nu^{(N)}(x)| \, dx = 0$$

by the regularization principle; similarly for $\nu^{((N))}$.

5.5. COROLLARY. — $\mathcal{S} \cap \mathcal{E}_\alpha$ is dense in $\mathcal{E}_\alpha$.

**Proof.** — Indeed $\nu^{((N))} \in \mathcal{S}$.

5.6. COROLLARY. — $\{\nu \in \mathcal{E}_\alpha$ such that $k^\alpha \ast \nu \in \mathcal{S}\}$ is dense in $\mathcal{E}_\alpha$.

**Proof.** — By the above it's enough to show that for any $\nu \in \mathcal{S} \cap \mathcal{E}_\alpha$, $\exists \varphi_n \in \mathcal{E}_\alpha$ such that $k^\alpha \ast \varphi_n \in \mathcal{S}$, $\varphi_n \Rightarrow \nu$. Let $f_n(x) = k^\alpha \ast \nu(x) \cdot \phi(p^n x)$ where $n \gg 0$ so that $\nu = \nu^{(N)}$. We have $f_n \in \mathcal{S}$ hence $f_n = k^\alpha \ast \varphi_n$, $\varphi_n = k^{-\alpha} \ast f_n$. Note that $k^\alpha \ast \varphi_n(x) = k^\alpha \ast \nu(x)$ for $|x| \leq p^n$, and $\|\varphi_n\|_\alpha^2 = \int_{|x| \leq p^n} f_n(x) \varphi_n(x) \, dx < \infty$, so $\varphi_n \in \mathcal{E}_\alpha$. Recall that $\varphi_n(x) = k^{-\alpha} \ast f_n(x)$ is given by

$$\int \frac{1-p^{-1}}{1-p^{-(1+\alpha)}} f_n(x) + \frac{1-p^{-\alpha}}{1-p^{-(1+\alpha)}} \left[ \int_{|y| > 1} f_n(x+y) \frac{dy}{|y|^{1+\alpha}} + \int_{|y| \leq 1} (f_n(x+y) - f_n(x)) \frac{dy}{|y|^{1+\alpha}} \right]$$

which for $|x| > p^n$ reduces to

$$\int \frac{1-p^{-\alpha}}{1-p^{-(1+\alpha)}} f_n(x+y) \frac{dy}{|y|^{1+\alpha}} = \frac{1-p^{-\alpha}}{1-p^{-(1+\alpha)}} \int_{|z| \leq p^n} f_n(z) \frac{dz}{|z-x|^{1+\alpha}} = \frac{1-p^{-\alpha}}{1-p^{-(1+\alpha)}} |x|^{-(1+\alpha)} \int_{|z| \leq p^n} k^\alpha \ast \nu(z) \, dz.$$
But for $|z| > p^{n_0}$, $k^\alpha \ast \nu(z) = \frac{1-p^{-\alpha}}{1-p^{-\alpha-1}} \nu(1)|z|^{\alpha-1}$, hence we get for $n \geq n_0$

$$|\varphi_n(x)| = O(|x|^{-(1+\alpha)p^n}), \quad |x| > p^n.$$  

Now,

$$\|\varphi_n - \nu\|_\alpha^2 = \int_{|x| > p^n} k^\alpha \ast (\varphi_n - \nu)(x)(\varphi_n - \nu)(x) \, dx$$

since $k^\alpha \ast (\varphi_n - \nu)(x) = 0$, $|x| \leq p^n$,

$$= -\int_{|x| > p^n} k^\alpha \ast \nu(x)\varphi_n(x) \, dx$$

since $k^\alpha \ast \varphi_n(x) = \nu(x) = 0$, $|x| > p^n$,

$$= O\left(\int_{|x| > p^n} (|x|^{\alpha-1} \cdot |x|^{-(1+\alpha)p^n} \, dx\right) = O(p^{n(\alpha-1)}) \to 0$$

since $\alpha < 1$.

So $\varphi_n \Rightarrow \nu$ and the theorem is proved.

5.7. COROLLARY. — Let $\nu_n \in \mathcal{E}_\alpha^+$, $\nu_n \to \nu$, then $\nu_n \rightharpoonup \nu$.

Proof. — Let $f \in \mathcal{S}$ and write $f = k^\alpha \ast \varphi$, $\varphi = k^{-\alpha} \ast f$, then $\varphi \in \mathcal{E}_\alpha$ and so $\lim_{n \to \infty} \langle \varphi, \nu_n \rangle_\alpha = \langle \varphi, \nu \rangle_\alpha$, i.e. $\lim_{n \to \infty} \int f(x) \, d\nu_n(x) = \int f(x) \, d\nu(x)$, hence $\nu_n \rightharpoonup \nu$.

5.8. COROLLARY. — Let $\nu_n \in \mathcal{E}_\alpha^+$, $\nu_n \rightharpoonup \nu$, $\|\nu_n\|_\alpha < C$. Then $\nu \in \mathcal{E}_\alpha^+$ and $\nu_n \to \nu$.

Proof. — By lower-semi-continuity $\|\nu\|_\alpha \leq \lim_{n \to \infty} \|\nu_n\|_\alpha < C$, so $\nu \in \mathcal{E}_\alpha^+$.

Let $\mu \in \mathcal{E}_\alpha$, $k^\alpha \ast \mu \in \mathcal{S}$, then

$$\langle \nu_n, \mu \rangle_\alpha = \int k^\alpha \ast \mu(x) \, d\nu_n(x) \to_n \int k^\alpha \ast \mu(x) \, d\nu(x) = \langle \nu, \mu \rangle_\alpha.$$  

Given any $\xi \in \mathcal{E}_\alpha$, and $\varepsilon > 0$, we can find by Corollary 5.6 such a $\mu$ with $\|\xi - \mu\|_\alpha < \varepsilon$, and then for $n > n_0(\varepsilon)$:

$$|\langle \nu_n, \xi \rangle_\alpha - \langle \nu, \xi \rangle_\alpha| \leq |\langle \nu_n, \mu \rangle_\alpha - \langle \nu, \mu \rangle_\alpha| + |\langle \nu_n, \xi - \mu \rangle_\alpha| + |\langle \nu, \xi - \mu \rangle_\alpha|$$

$$\leq |\langle \nu_n, \mu \rangle_\alpha - \langle \nu, \mu \rangle_\alpha| + 2C \cdot \varepsilon < (1 + 2C) \cdot \varepsilon.$$  

Thus $\langle \nu_n, \xi \rangle_\alpha \to_n \langle \nu, \xi \rangle_\alpha$ and $\nu_n \to \nu$.

5.9. LEMMA. — Let $\nu_n \in \mathcal{E}_\alpha^+$ be a Cauchy sequence such that $\nu_n \rightharpoonup \nu$. Then $\nu \in \mathcal{E}_\alpha^+$ and $\nu_n \Rightarrow \nu$.  

Proof. — There exists a bound \( \|\nu_n\|_\alpha < C \), so by the above corollary, \( \nu \in \mathcal{E}_\alpha^+ \) and \( \nu_n \to_\alpha \nu \). Hence we get

\[
\|\nu - \nu_n\|_\alpha^2 = \langle \nu - \nu_n, \nu - \nu_n \rangle_\alpha = \lim_{m \to \infty} \langle \nu - \nu_n, \nu_m - \nu_n \rangle_\alpha \\
\leq \lim_{m \to \infty} \|\nu - \nu_n\|_\alpha \cdot \|\nu_m - \nu_n\|_\alpha,
\]

so \( \|\nu - \nu_n\|_\alpha \leq \lim_{m \to \infty} \|\nu_m - \nu_n\|_\alpha \) and

\[
\lim_{n \to \infty} \|\nu - \nu_n\|_\alpha \leq \lim_{n \to \infty} \lim_{m \to \infty} \|\nu_m - \nu_n\|_\alpha = 0;
\]

or \( \nu_n \to_\alpha \nu \).

5.10. Completeness of the positive cone. — \( \mathcal{E}_\alpha^+ \) is a complete metric space.

Proof. — Let \( \nu_n \in \mathcal{E}_\alpha^+ \) be a Cauchy sequence. By Lemma 5.9 it suffices to show \( \nu_n \to \nu \) for some \( \nu \). Given any \( \mu \in \mathcal{E}_\alpha \), we have

\[
|\langle \nu_n - \nu_m, \mu \rangle_\alpha| \leq \|\nu_n - \nu_m\|_\alpha \cdot \|\mu\|_\alpha
\]

hence the sequence \( \langle \nu_n, \mu \rangle_\alpha \) converges. Since any \( \varphi \in \mathcal{S} \) can be written as \( \varphi = k^\alpha \ast \mu \) with \( \mu \in \mathcal{E}_\alpha \), and

\[
\int \varphi(x) d\nu_n(x) = \langle \nu_n, \mu \rangle_\alpha
\]

converges, we conclude the theorem.

6. Capacity and equilibrium measure.

\( K \subseteq \mathbb{Q}_p \) will denote a compact set.

\( \mathcal{M}^+(K) \), resp. \( \mathcal{M}^{(1)}(K) \), will be the positive, resp. probability, measures supported in \( K \).

\( \mathcal{E}_\alpha^+(K) = \mathcal{M}^+(K) \cap \mathcal{E}_\alpha \), \( \mathcal{E}_\alpha^{(1)}(K) = \mathcal{M}^{(1)}(K) \cap \mathcal{E}_\alpha \).

6.1. Lemma. — \( \mathcal{M}^{(1)}(K) \) is convex and vaguely compact.

Proof. — That \( \mathcal{M}^{(1)}(K) \) is convex is immediate; we prove compactness. Let \( \mathcal{S}_\mathbb{Q} \) denote the set of locally constant compactly supported functions \( \varphi : \mathbb{Q}_p \to \mathbb{Q} \). \( \mathcal{S}_\mathbb{Q} \) is denumerable; write \( \mathcal{S}_\mathbb{Q} = \{ \varphi_m \}_{m=1}^\infty \). Let \( \mu_n \in \mathcal{M}^{(1)}(K) \) be an arbitrary sequence. Since \( |\mu_n(\varphi)| \leq \sup \varphi \), we can find a subsequence \( \mu_{n_1} \) such that \( \{ \mu_{n_1}(\varphi_1) \} \) converges. Proceeding by induction we find a subsequence \( \mu_{n_{i-1}}(\varphi_1) \), ..., \( \{ \mu_{n_i}(\varphi_m) \} \) all converges. Looking at the diagonal, we have a subsequence \( \{ \mu_{n_{i_{m}}}(\varphi_{m}) \} = \{ \mu_{n_{i}} \} \) of our original sequence \( \{ \mu_n \} \) such that \( \{ \mu_{n_{i}}(\varphi_{m}) \} \) converges for
all $m$'s. Define $\mu(\varphi_m) = \lim \mu_{n_j}(\varphi_m)$, and extend $\mu$ by linearity to all of $S = S_\mathbb{Q} \otimes \mathbb{C}$, to give a positive distribution, hence a positive measure $\mu$. Moreover, $\mu_{n_j} \rightharpoonup \mu$, hence $\mu \in \mathcal{M}^{(1)}(K)$.

Let $W_\alpha(K) = \inf \{\|\mu\|_\alpha^2 \mid \mu \in \mathcal{M}^{(1)}(K)\} \in (0, \infty]$. We have $W_\alpha(K) \geq \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \text{diam}(K)^{\alpha-1} > 0$, and $W_\alpha(K) = \infty$ iff $\mathcal{E}_\alpha^{(1)}(K)$ is empty. We let $\text{cap}_\alpha(K) = W_\alpha(K)^{-1} \in [0, \infty)$, the $\alpha$-capacity of $K$. We have

6.2. Theorem. — The following conditions are equivalent:

(i) $\text{cap}_\alpha(K) = 0$;
(ii) $\mathcal{E}_\alpha^{(1)}(K)$ is empty;
(iii) $\nu(K) = 0$ for all $\nu \in \mathcal{E}_\alpha^+$;
(iv) $\nu(K) = 0$ for all $\nu \in \mathcal{E}_\alpha$.

Proof. — Easy.

In particular, if $\text{cap}_\alpha(K) = 0$ then $K$ has measure zero with respect to Haar measure. For an arbitrary set $X \subseteq \mathbb{Q}_p$ we define the inner and outer capacity respectively by

$$\text{cap}_\alpha^-(X) = \sup\{\text{cap}_\alpha(K) \mid K \subseteq X \text{ compact}\}$$
$$\text{cap}_\alpha^+(X) = \inf\{\text{cap}_\alpha(U) \mid X \subseteq U \text{ open}\}.$$  

We have $\text{cap}_\alpha^-(X) \leq \text{cap}_\alpha^+(X)$; if equality holds we say $X$ is capacitable and denote the common value by $\text{cap}_\alpha(X)$.

We say that a property holds for $\alpha$-almost-all $x$'s ($\alpha - \text{a.a.} x$'s) if the set of $x$ where it doesn’t hold has inner capacity 0.

6.3. Theorem. — Assume $\text{cap}_\alpha(K) > 0$. There exists a unique $\lambda_K \in \mathcal{E}_\alpha^{(1)}(K)$ such that $\|\lambda_K\|_\alpha^2 = W_\alpha(K)$; moreover, if $\mu_n \in \mathcal{E}_\alpha^{(1)}(K)$ is any sequence such that $\|\mu_n\|_\alpha^2 \to W_\alpha(K)$ then $\mu_n \rightharpoonup \alpha \lambda_K$.

Proof. — Let $\mu_n$ be as above; by vague compactness there is a subsequence $\mu_{n_j}$ such that $\mu_{n_j} \rightharpoonup \lambda \in \mathcal{M}^{(1)}(K)$, and by the principle of descent $\|\lambda\|_\alpha^2 \leq \lim\|\mu_{n_j}\|_\alpha^2 = W_\alpha(K)$, so $\|\lambda\|_\alpha^2 = W_\alpha(K)$. Moreover, since $W_\alpha(K)^{1/2} \leq \|\frac{1}{2}(\mu_n + \mu_m)\|_\alpha \leq \frac{1}{2}\|\mu_n\|_\alpha + \frac{1}{2}\|\mu_m\|_\alpha$ and $\|\mu_n\|_\alpha \rightarrow$
We get \( \|\frac{1}{2}(\mu_n + \mu_m)\|^2_{\alpha, n,m} \longrightarrow W_\alpha(K) \); and since we have
\[
\|\mu_n - \mu_m\|^2_{\alpha, n,m} = 2\|\mu_n\|^2_{\alpha, n,m} + 2\|\mu_m\|^2_{\alpha, n,m} - 4\|\frac{1}{2}(\mu_n + \mu_m)\|^2_{\alpha, n,m}
\]
we see that \( \|\mu_n - \mu_m\|_{\alpha, n,m} \longrightarrow 0 \), i.e. \( \mu_n \) is a Cauchy sequence. By Lemma 5.9 \( \mu_{n_j} \Rightarrow \lambda \), hence also \( \mu_n \Rightarrow \lambda \). Suppose next \( \lambda, \lambda' \in E^{(1)}_\alpha(K) \) are such that \( \|\lambda\|^2_{\alpha, n} = \|\lambda'\|^2_{\alpha, n} = W_\alpha(K) \), then
\[
W_\alpha(K)^{1/2} \leq \|\frac{1}{2}(\lambda + \lambda')\|_{\alpha, n} \leq \frac{1}{2}\|\lambda\|_{\alpha, n} + \frac{1}{2}\|\lambda'\|_{\alpha, n} = W_\alpha(K)^{1/2}
\]
so
\[
\frac{1}{2}(\lambda + \lambda') \in W_\alpha(K)
\]
and since \( \|\lambda - \lambda'\|^2_{\alpha, n} = 2\|\lambda\|^2_{\alpha, n} + 2\|\lambda'\|^2_{\alpha, n} - 4\|\frac{1}{2}(\lambda + \lambda')\|^2_{\alpha, n} = 0 \) we get \( \lambda = \lambda' \).

6.4. — We let \( \gamma_K = \text{cap}_\alpha(K) \cdot \lambda_K \in E^+_\alpha(K) \) be the equilibrium measure of \( K \). Thus \( \|\gamma_K\|^2_{\alpha} = \gamma_K(1) = \text{cap}_\alpha(K) \).

6.5. Theorem. — \( \gamma_K \) is characterized among all \( \gamma \in \mathcal{M}^+_\alpha(K) \) by any of the following equivalent conditions:

(i) \( k^\alpha \ast \gamma(x) = 1 \) for \( \alpha - a.a.x \in K \), and \( k^\alpha \ast \gamma(x) \leq 1 \) for all \( x \);

(ii) \( k^\alpha \ast \gamma(x) \leq 1 \) for all \( x \), and \( \gamma(1) \) is maximal;

(iii) \( \gamma(1) = \text{cap}_\alpha(K) \), and \( \|k^\alpha \ast \gamma\|_{L^\infty} \) is minimal;

(iv) \( \|\gamma\|^2_{\alpha} - 2\gamma(1) \) is minimal.

Proof. — We first show \( \gamma_K \) satisfies (i), by showing

(a) \( k^\alpha \ast \gamma_K(x) \geq 1 \) for \( \alpha - a.a.x \in K \),

(b) \( k^\alpha \ast \gamma_K(x) \leq 1 \) for \( x \in \text{supp} \gamma_K \).

Indeed, by the maximum principle we get \( k^\alpha \ast \gamma_K(x) \leq 1 \) for all \( x \), hence (i). Assume that (a) does not hold, then we find a compact \( K_0 \subseteq K \), \( \text{cap}_\alpha(K_0) > 0 \), such that \( k^\alpha \ast \gamma_K(x) < 1 \) for \( x \in K_0 \). If \( \nu \in E^{(1)}_\alpha(K_0) \) then, on the one hand,
\[
\langle \nu, \gamma_K \rangle_{\alpha} = \int k^\alpha \ast \gamma_K(x) dv(x) < 1 \quad \text{and so} \quad \langle \nu, \lambda_K \rangle_{\alpha} < \|\lambda_K\|^2_{\alpha}.
\]

On the other hand, for any \( t \in [0, 1] \), \( t \cdot \nu + (1-t)\lambda_K \in E^{(1)}_\alpha(K) \), so that
\[
t^2\|\nu\|^2_{\alpha} + 2t(1-t)\langle \nu, \lambda_K \rangle_{\alpha} + (1-t)^2\|\lambda_K\|^2_{\alpha} = \|t \cdot \nu + (1-t)\lambda_K\|^2_{\alpha} \geq \|\lambda_K\|^2_{\alpha}
\]
from which we get $\langle \nu, \lambda_K \rangle_0 \geq \|\lambda_K\|_0^2$, a contradiction. Assume that (b) does not hold, say $k_0 \ast \gamma_K(x_0) > 1$, $x_0 \in \text{supp } \gamma_K$; then by lower semicontinuity $k_0 \ast \gamma_K(x) > 1$ for $x \in U$, a neighbourhood of $x_0$, and moreover $\gamma_K(U) > 0$. But then we get a contradiction, using (a), namely

$$\text{cap}_\alpha(K) = \|\gamma_K\|_0^2 = \int_U k_0 \ast \gamma_K(x) d\gamma_K(x) + \int_{\supp \gamma_K \setminus U} k_0 \ast \gamma_K(x) d\gamma_K(x) > \gamma_K(U) + \gamma_K(\supp \gamma_K \setminus U) = \gamma_K(1) = \text{cap}_\alpha(K).$$

Next, let $\gamma \in \mathcal{M}^+(K)$ satisfy (i). We have $\|\varphi\|^2_\alpha = \int k_0 \ast \varphi(x) d\gamma(x) = \gamma(1)$, but $\gamma(1)^{-1} \cdot \gamma \in \mathcal{M}^{(1)}(K)$, so

$$\text{cap}_\alpha(K)^{-1} \leq \|\gamma(1)^{-1} \cdot \gamma\|^2_\alpha = \gamma(1)^{-2} \cdot \|\gamma\|^2_\alpha = \gamma(1)^{-1},$$

or $\gamma(1) \leq \text{cap}_\alpha(K)$. Hence

$$\|\gamma - \gamma_K\|^2_\alpha = \|\gamma\|^2_\alpha + \|\gamma_K\|^2_\alpha - 2\langle \gamma, \gamma_K \rangle_0 \leq \gamma(1)^{-1}. \gamma(1) = \text{cap}_\alpha(K) - 2 \int k_0 \ast \gamma(x) d\gamma_K(x) \leq 2 \cdot \text{cap}_\alpha(K) - 2\gamma(1) = 0$$

and we get that $\gamma = \gamma_K$.

To prove (ii), let $\gamma \in \mathcal{M}^+(K)$ be such that $k_0 \ast \gamma(x) \leq 1$. Then

$$\|\gamma\|^2_\alpha = \int k_0 \ast \gamma(x) d\gamma(x) \leq \gamma(1),$$

and since $\gamma(1)^{-1} \cdot \gamma \in \mathcal{M}^{(1)}(K)$, we get $\gamma_K(1)^{-1} = W_\alpha(K) \leq \|\gamma(1)^{-1} \cdot \gamma\|^2_\alpha = \gamma(1)^{-2} \cdot \|\gamma\|^2_\alpha \leq \gamma(1)^{-1}$, so $\gamma(1) \leq \gamma_K(1)$. Moreover, if equality holds, then $\|\gamma(1)^{-1} \cdot \gamma\|^2_\alpha = W_\alpha(K)$, so $\gamma(1)^{-1} \cdot \gamma = \lambda_K$ and $\gamma = \gamma_K$.

To prove (iii), assume $\gamma \in \mathcal{M}^+(K)$ is such that $\gamma(1) = \text{cap}_\alpha(K)$ and $\|k_0 \ast \gamma\|_{L^\infty} \leq \|k_0 \ast \gamma_K\|_{L^\infty} = 1$. Then $\|\gamma\|^2_\alpha = \int k_0 \ast \gamma(x) d\gamma(x) \leq \gamma(1) = \text{cap}_\alpha(K) = \gamma_K(1)$, and again since $\gamma(1)^{-1} \cdot \gamma \in \mathcal{M}^{(1)}(K)$ we have $W_\alpha(K) \leq \|\gamma(1)^{-1} \cdot \gamma\|^2_\alpha = \gamma(1)^{-2} \cdot \|\gamma\|^2_\alpha \leq \gamma(1)^{-1} = W_\alpha(K)$, $\gamma(1)^{-1} \cdot \gamma = \lambda_K$, $\gamma = \gamma_K$. To prove (iv), we note that for any $\gamma \in \mathcal{M}^+(K)$,

$$\|\gamma - \gamma_K\|^2_\alpha = \|\gamma\|^2_\alpha + \|\gamma_K\|^2_\alpha - 2 \int k_0 \ast \gamma_K(x) d\gamma(x) \leq \|\gamma\|^2_\alpha + \text{cap}_\alpha(K) - 2\gamma(1)$$

hence $\|\gamma\|^2_\alpha - 2 \cdot \gamma(1) \geq \|\gamma - \gamma_K\|^2_\alpha - \text{cap}_\alpha(K)$ and the minimum is obtained for $\gamma = \gamma_K$ where $\|\gamma_K\|^2_\alpha - 2 \cdot \gamma_K(1) = - \text{cap}_\alpha(K)$.

Since the maximum in (ii) above is equal $\gamma_K(1) = \text{cap}_\alpha(K)$ we get

$$\text{cap}_\alpha(K) = \max\{\gamma(1) \mid \gamma \in \mathcal{M}^+(K), k_0 \ast \gamma(x) \leq 1 \text{ for all } x\}.$$
Let $\phi(N)(x) = p^N \phi(p^{-N} x)$, the uniform distribution on $B(p^{-N}) = \{x \mid |x| \leq p^{-N}\}$. We have

$$k^\alpha \ast \phi(N)(y) = \frac{\zeta(1 - \alpha)}{\zeta(\alpha)} p^N \int_{|x| \leq p^{-N}} |y - x|^{\alpha - 1} dx$$

$$= \begin{cases} \frac{\zeta(1 - \alpha)}{\zeta(1)} p^{N(1 - \alpha)} & \text{for } y \in B(p^{-N}) \\ \frac{\zeta(1 - \alpha)}{\zeta(\alpha)} |y|^{\alpha - 1} & \text{for } |y| > p^{-N}. \end{cases}$$

Thus, $\gamma_{B(p^{-N})}(x) = \frac{\zeta(1)}{\zeta(1 - \alpha)} p^{N(\alpha - 1)} \cdot \phi(N)(x) = \frac{1 - p^{-1}}{1 - p^{-1}} \cdot p^{N\alpha} \phi(p^{-N} x)$ is the equilibrium measure on $B(p^{-N})$:

$$k^\alpha \ast \gamma_{B(p^{-N})}(y) = \begin{cases} 1 & \text{for } y \in B(p^{-N}) \\ \frac{\zeta(1)}{\zeta(\alpha)} |p^{-N} y|^{\alpha - 1} < \frac{\zeta(1)}{\zeta(\alpha)} < 1 & \text{for } |y| > p^{-N}. \end{cases}$$

and $\text{cap}_\alpha(B(p^{-N})) = \frac{\zeta(1)}{\zeta(1 - \alpha)} p^{N(\alpha - 1)} = \frac{1 - p^{-1}}{1 - p^{-1}} p^{N(\alpha - 1)}$ (cf. [8], Appendix A.1, A.2 on p. 401).

6.7. Theorem. — $\text{cap}_\alpha$ satisfies the following properties:

(i) monotonicity: $K_1 \subseteq K_2 \Rightarrow \text{cap}_\alpha(K_1) \leq \text{cap}_\alpha(K_2)$;

(ii) subadditivity: $K = \bigcup_{i=1}^\infty K_i \Rightarrow \text{cap}_\alpha(K) \leq \sum_{i=1}^\infty \text{cap}_\alpha(K_i)$;

(iii) continuity: given $K$, $\varepsilon > 0$, we can find an open $U \supseteq K$ such that for all $K'$,

$$K \subseteq K' \subseteq U \Rightarrow \text{cap}_\alpha(K') \leq \text{cap}_\alpha(K) + \varepsilon;$$

(iv) translation invariance: $\text{cap}_\alpha(a + K) = \text{cap}_\alpha(K)$, $\text{cap}_\alpha(aK) = |a|^{1-\alpha} \text{cap}_\alpha(K)$.

Proof.

(i) follows since $\mathcal{M}^+(K_1) \subseteq \mathcal{M}^+(K_2)$, hence $W_\alpha(K_1) \geq W_\alpha(K_2)$.

(ii) follows since if $\gamma_i = \gamma_{K_i|K_i}$ is the restriction of $\gamma_K$ to $K_i$, then $\gamma_i(1) \leq \text{cap}_\alpha(K_i)$, and so $\text{cap}_\alpha(K) = \gamma_K(1) \leq \sum_{i=1}^\infty \gamma_i(1) \leq \sum_{i=1}^\infty \text{cap}_\alpha(K_i)$.

Assume (iii) does not hold, so that we have $K$, $\varepsilon > 0$, and a sequence of compact sets $K_i \supseteq K_{i+1} \supseteq \cdots \supseteq K$, $\bigcap_{i=0}^\infty K_i = K$, $\text{cap}_\alpha(K_i) \geq$
cap_\alpha(K) + \varepsilon$. The equilibrium measures \(\gamma_{K_i}\) are weakly bounded \((\gamma_{K_i}(1) = cap_\alpha(K_i) \leq cap_\alpha(K_0))\), hence vaguely compact, so after passage to a subsequence we may assume \(\gamma_{K_i} \rightharpoonup \mu \in \mathcal{M}^+(K)\). By the principle of descent \(k^\alpha * \mu \leq \lim k^\alpha * \gamma_{K_i} \leq 1\) so \(\mu(1) \leq cap_\alpha(K)\). On the other hand, \(\mu(1) = \lim \gamma_{K_i}(1) = \lim cap_\alpha(K_i) \geq cap_\alpha(K) + \varepsilon\), a contradiction.

(iv) follows easily from the definition.

By Choquet's Theorem, we obtain

6.8. COROLLARY. — Every analytic set is capactable.

For \(x_1 \cdots x_n \in K\), put \(d^\alpha(x_1 \cdots x_n) = \binom{n}{2}^{-1} \sum_{i < j} |x_i - x_j|^\alpha^{-1}\). This function obtains its minimal value on the compact set \(K\) at certain points \(x_i = \xi_i^{(n)}\), and we define

\[
\text{diam}_\alpha^{(n)}(K) = \left[ \min_{x_1 \cdots x_n \in K} d^\alpha(x_1 \cdots x_n) \right]^{-1} = \binom{n}{2}^{-1} \left[ \sum_{i < j} |\xi_i^{(n)} - \xi_j^{(n)}|^\alpha^{-1} \right]^{-1}.
\]

We have

\[
[diam^{(n)}_{\alpha}(K)]^{-1} = \binom{n}{2}^{-1} \frac{1}{n - 2} \sum_{k=1}^{n} \sum_{i \neq j} |\xi_i^{(n)} - \xi_j^{(n)}|^\alpha^{-1}
\]

\[
\geq \binom{n}{2}^{-1} \frac{1}{n - 2} \binom{n - 1}{2} [diam^{(n-1)}_{\alpha}(K)]^{-1}
\]

\[
= [diam^{(n-1)}_{\alpha}(K)]^{-1}.
\]

Thus \(diam^{(n-1)}_{\alpha}(K) \geq diam^{(n)}_{\alpha}(K)\), and we have a well defined limit \(diam_{\alpha}(K) = \lim_{n \to \infty} diam^{(n)}_{\alpha}(K)\).

6.9. THEOREM. — \(cap_\alpha(K) = \frac{\zeta(\alpha)}{\zeta(1 - \alpha)} diam_{\alpha}(K)\).

Proof. — Taking the inequality \(\binom{n}{2}[diam^{(n)}_{\alpha}(K)]^{-1} \leq \sum_{i < j} |x_i - x_j|^{\alpha^{-1}}\), multiplying by \(d\mu(x_i)d\mu(x_j)\), where \(\mu \in \mathcal{M}^{(1)}(K)\) and integrating \(\binom{n}{2}\) times with respect to \((x_i, x_j) \in K \times K\), we get

\[
[diam^{(n)}_{\alpha}(K)]^{-1} \leq \int_{K \times K} |x - y|^{\alpha^{-1}} d\mu(x)d\mu(y) = \frac{\zeta(\alpha)}{\zeta(1 - \alpha)} \|\mu\|_{\alpha}^2.
\]

Letting \(n \to \infty\), then taking the inf over all \(\mu \in \mathcal{M}^{(1)}(K)\), we get

\[
[diam_{\alpha}(K)]^{-1} \leq \frac{\zeta(\alpha)}{\zeta(1 - \alpha)} [cap_\alpha(K)]^{-1}.
\]

On the other hand, let
\[ \mu^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\xi_i^{(n)}} \in \mathcal{M}^{(1)}(K), \text{ and consider the truncated kernel} \]
\[ k_\varepsilon^{(n)}(x) = \begin{cases} 
\kappa_\alpha(x), & |x| \geq |\varepsilon| \\
\kappa_\alpha(\varepsilon), & |x| \leq |\varepsilon| 
\end{cases}, \text{ where } \varepsilon \in \mathbb{Q}_p \text{ is small. We have} \]
\[ \int \int_{K \times K} k_\varepsilon^{(n)}(x-y) d\mu^{(n)}(x) d\mu^{(n)}(y) \]
\[ \leq \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \left[ \frac{1}{n^2} \sum_{i \neq j} |\xi_i^{(n)} - \xi_j^{(n)}|^\alpha - \frac{1}{n} + \frac{|\varepsilon|^{\alpha-1}}{n} \right] \]
\[ = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \left[ \frac{2}{n^2} \left( \frac{n}{2} \right) (\text{diam}_\alpha^{(n)}(K))^{-1} + \frac{|\varepsilon|^{\alpha-1}}{n} \right]. \]

By vague compactness, after passage to a subsequence, we may assume \( \mu^{(n)} \sim \mu \in \mathcal{M}^{(1)}(K) \). Since \( k_\varepsilon^{(n)} \) is continuous, we can pass to the limit \( n \to \infty \), and obtain
\[ \int \int_{K \times K} k_\varepsilon^{(n)}(x-y) d\mu(x) d\mu(y) \leq \frac{\zeta(1-\alpha)}{\zeta(\alpha)} [\text{diam}_\alpha(K)]^{-1}. \]

Now letting \( \varepsilon \to 0 \), we get \( \|\mu\|_\alpha^2 \leq \frac{\zeta(1-\alpha)}{\zeta(\alpha)} [\text{diam}_\alpha(K)]^{-1} \), hence
\[ [\text{cap}_\alpha(K)]^{-1} \leq \frac{\zeta(1-\alpha)}{\zeta(\alpha)} [\text{diam}_\alpha(K)]^{-1}. \]

Note. — By the uniqueness of the measure \( \lambda_K \), we also get \( \lambda_K = \mu \).

6.10. Corollary. — If \( f : K \to \mathbb{Q}_p \) satisfies \( |f(x) - f(y)| \leq |x-y| \), then \( \text{cap}_\alpha(f(K)) \leq \text{cap}_\alpha(K) \).

Proof. — This property is immediate if \( \text{cap}_\alpha \) is replaced by \( \text{diam}_\alpha \), hence it follows from the above theorem.

6.11. Limit Theorem. — Let \( \nu_n \in \mathcal{M}_\alpha^+, \nu_n \sim \nu \), and assume that we have
\[ \lim_{R \to \infty} \int_{|x|>R} |x|^\alpha d\nu_n(x) = 0 \text{ uniformly with respect to } n. \]
Then \( k_\alpha \ast \nu(x) = \lim_{n \to \infty} k_\alpha \ast \nu_n(x) \) for \( \alpha - a.a.x's. \) Moreover, if we set
\[ f(x) = \lim_{n \to \infty} k_\alpha \ast \nu_n(x) \] and introduce its lower semicontinuous regularization \( \tilde{f}(x) = \lim_{y \to x} f(y) \), then \( k_\alpha \ast \nu(x) = \tilde{f}(x) \) for all \( x's. \)

Proof. — By the descent principle \( k_\alpha \ast \nu(x) \leq \lim_{n \to \infty} k_\alpha \ast \nu_n(x) \), so assume we have strict inequality on a set \( K \) of a positive \( \alpha \)-capacity, and
without loss of generality $K$ compact. Then
\[
\int_K k^\alpha \ast \nu(x) d\gamma_K(x) < \int_K \lim_{n \to \infty} k^\alpha \ast \nu_n(x) d\gamma_K(x) \\
\leq \lim_{n \to \infty} \int_K k^\alpha \ast \nu_n(x) d\gamma_K(x).
\]
On the other hand,
\[
\int_K k^\alpha \ast (\nu_n - \nu)(x) d\gamma_K(x) \\
= \int_K k^\alpha \ast \gamma_K(x) d(\nu_n - \nu)(x) \\
\leq \int_{|x| \leq R} d(\nu_n - \nu)(x) + O\left( \int_{|x| > R} |x|^{-1} d(\nu_n - \nu)(x) \right).
\]
Since $\nu_n \sim \nu$, the first integral approaches zero $n \to \infty$, while the second can be made arbitrary small by our uniformity assumption. This gives the desired contradiction and establishes the first part of our theorem. For the second part, note that since $k^\alpha \ast \nu$ is lower semicontinuous and everywhere $\leq f$, we have
\[
k^\alpha \ast \nu(x) \leq \lim_{y \to x} k^\alpha \ast \nu(y) \leq \lim_{y \to x} f(y) = \tilde{f}(x).
\]
On the other hand, using the above and the regularization principle, we have
\[
\tilde{f}(x) = \lim_{N \to \infty} \min_{|y - x| \leq 2^{-N}} f(y) \leq \lim_{N \to \infty} f^{(N)}(x) \\
= \lim_{N \to \infty} k^\alpha \ast \nu^{(N)}(x) = k^\alpha \ast \nu(x).
\]

7. The Green measure.

The group $\text{PGL}_2(\mathbb{Q}_p)$ acts on $\mathbb{P}^1(\mathbb{Q}_p)$ by fractional linear transformations, and we get an induced action on $C(\mathbb{P}^1(\mathbb{Q}_p))$, the $\mathbb{C}$-valued continuous functions on $\mathbb{P}^1(\mathbb{Q}_p)$, and an adjoint action on the (signed) measures on $\mathbb{P}^1(\mathbb{Q}_p)$. We write $\mathbb{P}^1(\mathbb{Q}_p) = \mathbb{Q}_p \cup \{\infty\}$, $C(\mathbb{P}^1(\mathbb{Q}_p)) = C_0(\mathbb{Q}_p) \oplus \mathbb{C}$, $\mathcal{M}(\mathbb{P}^1(\mathbb{Q}_p)) = \mathcal{M}(\mathbb{Q}_p) \oplus \mathbb{R} \delta_\infty$. For $g \in \text{PGL}_2(\mathbb{Q}_p)$ and $\nu \in \mathcal{M}(\mathbb{P}^1(\mathbb{Q}_p))$, we have $g\nu \in \mathcal{M}(\mathbb{P}^1(\mathbb{Q}_p))$ defined by the formula
\[
\int \varphi(x) d\nu(x) = \int \varphi(gx) d\nu(x) \text{ for } \varphi \in C(\mathbb{P}^1(\mathbb{Q}_p)).
\]
We shall be concerned only with the action of the “inversion” $I_{x_0} = \begin{bmatrix} x_0 & 1 - x_0^2 \\ 1 & -x_0 \end{bmatrix}$, $x_0 \in \mathbb{Q}_p$, where $I_{x_0}(x) = (x - x_0)^{-1} + x_0$ for $x \neq x_0$, $\infty$ and
\[ I_{x_0}(x_0) = \infty, \quad I_{x_0}(\infty) = x_0. \] We have: \[ I_{x_0} \circ I_{x_0} = \text{id}, \quad |I_{x_0}(x) - I_{x_0}(y)| = \frac{|x-y|}{|x-x_0| \cdot |y-x_0|}. \]

If \( K \subseteq \mathbb{Q}_p \) is compact and \( x_0 \not\in K \), there are positive constants \( \Gamma_1 = p^{n_1}, \Gamma_2 = p^{n_2} \) such that for all \( x, y \in K : \Gamma_1 \cdot |x-y| \leq |I_{x_0}(x) - I_{x_0}(y)| \leq \Gamma_2 \cdot |x-y| \). We obtain \( \Gamma_1^{1-\alpha} \cdot \text{cap}_\alpha(K) \leq \text{cap}_\alpha(I_{x_0}(K)) \leq \Gamma_2^{1-\alpha} \cdot \text{cap}_\alpha(K) \); in particular, \( \text{cap}_\alpha(K) = 0 \) if and only if \( \text{cap}_\alpha(I_{x_0}(K)) = 0 \), from which we get that \( I_{x_0} \) preserves the collection of sets of inner capacity zero.

In the following, let \( K \subseteq \mathbb{P}^1(\mathbb{Q}_p) \) be an arbitrary closed set such that \( x_0 \not\in K \). Since \( x_0 \not\in K, \infty \not\in I_{x_0}(K) \), the set \( I_{x_0}(K) \) is compact, so let \( \gamma_{I_{x_0}(K)} \) denote its equilibrium measure. We have a well defined measure \( k^{\alpha} \ast \delta_{x_0} \in \mathcal{M}^+(I_{x_0}(K)) \) given by
\[
(k^{\alpha} \ast \delta_{x_0}) \cdot \gamma_{I_{x_0}(K)}(y) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} |y-x_0|^{\alpha-1} \cdot \gamma_{I_{x_0}(K)}(y).
\]

We define \( P_K^{\alpha} \delta_{x_0} = I_{x_0}(k^{\alpha} \ast \delta_{x_0} \cdot \gamma_{I_{x_0}(K)}) \in \mathcal{M}^+(K) \), explicitly:
\[
\int \varphi(y) dP_K^{\alpha} \delta_{x_0}(y) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int K \varphi(y) |y-x_0|^{1-\alpha} d\gamma_{I_{x_0}(K)}((y-x_0)^{-1} + x_0).
\]

Let us estimate the potential of the measure \( P_K^{\alpha} \delta_{x_0} \). We have
\[
k^{\alpha} \ast P_K^{\alpha} \delta_{x_0}(x) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_K |x-y|^{\alpha-1} dP_K^{\alpha} \delta_{x_0}(y) = \left( \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \right)^2 \int_K |x-y|^{\alpha-1} |y-x_0|^{1-\alpha} d\gamma_{I_{x_0}(K)}(I_{x_0}(y))
\]
and upon substituting \( y := I_{x_0}(y) \) we get
\[
= \left( \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \right)^2 \int_{I_{x_0}(K)} \left( \frac{|I_{x_0}(x) - y|}{|y-x_0|} \right)^{\alpha-1}.
\]
\[
= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} |x-x_0|^{\alpha-1} \cdot \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_{I_{x_0}(K)} |I_{x_0}(x) - y|^{\alpha-1} d\gamma_{I_{x_0}(K)}(y)
\]
\[
l = k^{\alpha} \ast \delta_{x_0}(x) \cdot k^{\alpha} \ast \gamma_{I_{x_0}(K)}(I_{x_0}(x)).
\]

Since \( k^{\alpha} \ast \gamma_{I_{x_0}(K)}(x^*) \leq 1 \), with equality for \( \alpha - a.a.x^* \in I_{x_0}(K) \), and since \( I_{x_0} \) preserves sets of capacity zero, we get:
\[
k^{\alpha} \ast P_K^{\alpha} \delta_{x_0}(x) \leq k^{\alpha} \ast \delta_{x_0}(x), \quad \text{with equality for } \alpha - a.a.x \in K.
\]

We will say that a measure \( \nu \) on \( \mathbb{P}^1(\mathbb{Q}_p) \) is \( \alpha \)-absolutely continuous if for any set \( E, \text{cap}_\alpha(E) = 0 \) implies \( \nu(E) = 0 \). Thus every measure of finite
α-energy is α-absolutely continuous, but not conversely. We summarize the above discussion of \( P^\alpha_K \delta_{x_0} \) in the following

7.1. THEOREM. — Let \( K \subseteq \mathbb{P}^1(Q_p) \) be a closed set, and fix \( x_0 \notin K \). Then there exists a (unique) α-absolutely continuous measure \( P^\alpha_K \delta_{x_0} \in \mathcal{M}^+(K) \) such that \( k^\alpha \ast P^\alpha_K \delta_{x_0}(x) = k^\alpha \ast \delta_{x_0}(x) \) for α – a.a. \( x \in K \). Furthermore \( k^\alpha \ast P^\alpha_K \delta_{x_0}(x) \leq k^\alpha \ast \delta_{x_0}(x) \) for all \( x \), and \( P^\alpha_K \delta_{x_0}(1) = k^\alpha \ast \gamma_{I_{x_0}(K)}(x_0) \leq 1. \)

The formula for \( P^\alpha_K \delta_{x_0}(1) \) is immediate from the definition. The uniqueness of \( P^\alpha_K \delta_{x_0} \) follows from:

7.2. 3rd UNIQUENESS PRINCIPLE. — Let \( K \) be a closed set, \( \nu_1, \nu_2 \in \mathcal{M}_\alpha^+(K) \), \( \nu_1 \) α-absolute continuous. If \( k^\alpha \ast \nu_1(x) = k^\alpha \ast \nu_2(x) \) for \( \alpha – a.a.x \in K \), then \( \nu_1 = \nu_2. \)

Proof. — Setting \( \nu = \nu_1 – \nu_2 \), we claim that \( |k^\alpha \ast \nu_1(x)| < \infty \) for \( \alpha – a.a.x \)‘s, hence a posteriori for \( \nu – a.a.x \)‘s. Indeed, otherwise we could find a compact \( K' \subseteq K \), \( \cap \nu_1(K') \) \( > 0 \), with \( k^\alpha \ast \nu_1(x) = \infty \) for all \( x \in K' \). But then we would get a contradiction:

\[
\infty = \int_{K'} k^\alpha \ast \nu_1(x) \gamma_{K'}(x) \, dx = \int_{K'} k^\alpha \ast \gamma_{K'}(x) d\nu_1(x) \\
\leq \int_{|x| \leq R} d\nu_1(x) + O \left( \int_{|x| > R} |x|^{\alpha-1} d\nu_1(x) \right) < \infty.
\]

We have, moreover, \( k^\alpha \ast \nu(x) = 0 \) for \( \alpha – a.a.x \in K \), hence again for \( \nu – a.a.x \)‘s, so that \( \|\nu\|_\alpha^2 = \int_K k^\alpha \ast \nu(x) d\nu(x) = 0 \), and we conclude that \( \nu = 0. \)

In the following we will let \( K \subseteq \mathbb{P}(Q_p) \) denote a closed set and \( U = \mathbb{P}(Q_p) \setminus K \) its open complement, with \( y \)‘s denoting points of \( U \), \( x \)‘s points of \( K \). We have the following

7.3. SYMMETRY PROPERTY. — \( k^\alpha \ast P^\alpha_K \delta_{y_1}(y_2) = k^\alpha \ast P^\alpha_K \delta_{y_2}(y_1), \) \( y_1, y_2 \in U. \)
Proof.
\[ k^\alpha * P_K \delta_{y_1}(y_2) = \int_K k^\alpha(y_2 - x) dP^\alpha_K \delta_{y_1}(x) = \int_K k^\alpha * P^\alpha_K \delta_{y_2}(x) dP^\alpha_K \delta_{y_1}(x) \]
\[ = \int_K k^\alpha * P^\alpha_K \delta_{y_1}(x) dP^\alpha_K \delta_{y_2}(x) \]
\[ = \int_K k^\alpha(y_1 - x) * P^\alpha_K \delta_{y_2}(x) = k^\alpha * dP^\alpha_K \delta_{y_2}(y_1). \]

7.4. DEFINITION.
\[ K_{\text{reg}} = \{ x \in K | k^\alpha * \delta_y(x) = k^\alpha * P^\alpha_K \delta_y(x) \text{ for all } y \in U \} \]
\[ K_{\text{irr}} = \{ x \in K | k^\alpha * \delta_y(x) > k^\alpha * P^\alpha_K \delta_y(x) \text{ for some } y \in U \}. \]

The points of \( K_{\text{reg}} \) (resp. \( K_{\text{irr}} \)) are called regular (resp. irregular) points, and \( K = K_{\text{reg}} \cup K_{\text{irr}} \). By the regularization principle and since \( k^\alpha * \delta_y(x) = k^\alpha * P^\alpha_K \delta_y(x) \) for \( \alpha - a.a.x \in K \), it follows that \( k^\alpha * \delta_y(x) = k^\alpha * P^\alpha_K \delta_y(x) \) for an interior point \( x \in \mathring{K} = K \setminus \partial K \), so \( K_{\text{irr}} \subseteq \partial K \). In particular, if \( \partial K = \emptyset \), \( K = K_{\text{reg}} \). If \( k^\alpha * \delta_y(x) > k^\alpha * P^\alpha_K \delta_y(x) \), then also \( k^\alpha * \delta_y'(x) > k^\alpha * P^\alpha_K \delta_y'(x) \) for \( y' \) near \( y \). From this we get \( \text{cap}_\alpha(K_{\text{irr}}) = 0 \).

Regular points of the boundary \( \partial K \) are characterized by the following property of “concentration” of the Green measure,

7.5. THEOREM. — Let \( x_0 \in \partial K; x_0 \in K_{\text{reg}} \iff P^\alpha_K \delta_y \rightharpoonup \delta_{x_0} \) as \( y \to x_0, y \in U \).

Proof. — To prove “\( \iff \)” we use the regularization principle and the fact that \( k^\alpha * \delta_y(x) = k^\alpha * P^\alpha_K \delta_y(x) \) for \( \alpha - a.a.x \in K \), hence for \( a.a.x \in K \), to write :
\[ k^\alpha*\delta_{y_0}(x_0) - k^\alpha*P^\alpha_K \delta_{y_0}(x_0) = \lim_{N \to \infty} p^N \int_{|x-x_0| \leq p^{-N}} k^\alpha((\delta_{y_0} - P^\alpha_K \delta_{y_0}))(x) dx \]
\[ = \lim_{N \to \infty} p^N \int_{|y-y_0| \leq p^{-N}} k^\alpha((\delta_{y_0} - P^\alpha_K \delta_{y_0}))(y) dy. \]
Thus if \( x_0 \notin K_{\text{reg}} \), we can find \( y_0 \in U \) such that the left hand side is \( > 0 \), hence we can find a sequence of points \( y_n \to x_0, y_n \in U \), such that \( \lim_{n \to \infty} k^\alpha((\delta_{y_0} - P^\alpha_K \delta_{y_0}))(y_n) > 0 \). But assuming \( P^\alpha_K \delta_{y_n} \rightharpoonup \delta_{x_0} \), we get using the descent principle, and the symmetry property,
\[ \lim_{n \to \infty} k^\alpha * \delta_{y_0}(y_n) = k^\alpha * \delta_{y_0}(x_0) = k^\alpha * \delta_{x_0}(y_0) \]
\[ \leq \lim_{n \to \infty} k^\alpha * P^\alpha_K \delta_{y_n}(y_0) = \lim_{n \to \infty} k^\alpha * P^\alpha_K \delta_{y_0}(y_n). \]
To prove \( \Rightarrow \) we take \( 0 < \varphi \in \mathcal{S}(\mathbb{Q}_p) \), and show \( P^\alpha_K \delta_y(\varphi) \to \delta_{x_0}(\varphi) = \varphi(x_0) \). We write \( \varphi = k^\alpha \ast \varphi' \), where \( \varphi' = k^{-\alpha} \ast \varphi \) is locally constant and \( O(|x|^{-(1+\alpha)}) \) as \( |x| \to \infty \). We have

\[
\varphi(x_0) - P^\alpha_K \delta_y(\varphi) = k^\alpha \ast \varphi'(x_0) - \int k^\alpha \ast \varphi'(x) dP^\alpha_K \delta_y(x) \\
= \int (k^\alpha(x_0 - x) - k^\alpha \ast P^\alpha_K \delta_y(x)) \cdot \varphi'(x) dx \\
= \int (k^\alpha(y - x) - k^\alpha \ast P^\alpha_K \delta_y(x)) \cdot \varphi'(x) dx \\
+ \int (k^\alpha(x_0 - x) - k^\alpha(y - x)) \cdot \varphi'(x) dx.
\]

Letting \( y \to x_0 \), the second integral tends to zero, and the first integral may be rewritten

\[
\left| \int_U k^\alpha \ast (\delta_y - P^\alpha_K \delta_y)(\tilde{y}) \varphi'(\tilde{y}) d\tilde{y} \right| \leq \max |\varphi'| \cdot \int_U k^\alpha \ast (\delta_y - P^\alpha_K \delta_y)(\tilde{y}) d\tilde{y} \\
\leq \max |\varphi'| \cdot \left[ \int_{|\tilde{y} - x_0| > \delta} k^\alpha \ast (\delta_y - P^\alpha_K \delta_y)(\tilde{y}) d\tilde{y} + \int_{|\tilde{y} - x_0| \leq \delta} k^\alpha \ast (\delta_y - P^\alpha_K \delta_y)(\tilde{y}) d\tilde{y} \right].
\]

Taking \( \delta \) sufficiently small the second integral inside the brackets can be made arbitrarily small, while for the first we have by the symmetry property and our assumption that \( x_0 \in K_{\text{reg}} \),

\[
\lim_{y \to x_0} \int_{|\tilde{y} - x_0| > \delta} k^\alpha \ast (\delta_y - P^\alpha_K \delta_y)(\tilde{y}) d\tilde{y} = \int_{|\tilde{y} - x_0| > \delta} k^\alpha \ast (\delta_y - P^\alpha_K \delta_y)(x_0) d\tilde{y} = 0.
\]

We shall next extend the Green measure \( P^\alpha_K \delta_y \) to points of \( K \). For \( x \in K_{\text{reg}} \) we set \( P^\alpha_K \delta_x = \delta_x \).

Assuming \( x \in K_{\text{irr}} \), by the above theorem we can find some sequence \( y_n \to x, y_n \in U \), such that \( P^\alpha_K \delta_{y_n} \) does not converge to \( \delta_x \). By the vague compactness of \( \{P^\alpha_K \delta_{y_n}\} \) we may assume that \( P^\alpha_K \delta_{y_n} \to \nu \neq \delta_x \). Set \( m = \nu(\{x\}) \in [0, 1) \) and define

\[
P^\alpha_K \delta_x = \frac{1}{1 - m} (\nu - m \cdot \delta_x) \in \mathcal{M}^+(K).
\]

The independence of this definition from the choice of \( \{y_n\} \) follows from the 3rd uniqueness principle and the following property which completely characterizes \( P^\alpha_K \delta_x \):

7.6. THEOREM. — \( P^\alpha_K \delta_x \) is \( \alpha \)-absolutely continuous and

\[
k^\alpha \ast P^\alpha_K \delta_x(x') = k^\alpha \ast \delta_x(x') \text{ for } \alpha - \text{a.a.} x' \in K.
\]
Proof. — Using the limit theorem, we have for $\alpha - a.a.x' \in K$,

$$k^\alpha * \nu(x') = \lim_{n \to \infty} k^\alpha * P_K \delta_{y_n}(x') = \lim_{n \to \infty} k^\alpha * \delta_{y_n}(x') = k^\alpha * \delta_{\nu}(x')$$

and by the descent principle $k^\alpha * \nu(x') \leq k^\alpha * \delta_{\nu}(x')$ for all $x'$. Since $k^\alpha * \nu$ is bounded away from $x$, it follows that $P^\alpha_K \delta_{\nu} = \frac{1}{1-m} \nu |_{K \setminus (x)}$ is $\alpha$-absolutely continuous. Moreover, for $\alpha - a.a.x' \in K$,

$$k^\alpha * P^\alpha_K \delta_{\nu}(x') = \frac{1}{1-m} k^\alpha * (\nu - m \cdot \delta_x)(x') = \frac{1}{1-m} k^\alpha * (\delta_x - m \delta_x)(x') = k^\alpha * \delta_{\nu}(x').$$

7.7. DEFINITION. — $G^\alpha_U(x,y) = k^\alpha * \delta_y(x) - k^\alpha * P_K^\alpha \delta_y(x)$ will be called the Green's function of $U$. We have $G^\alpha_U(x,y) \geq 0$ for all $x, y$; $G^\alpha_U(x,y) = 0$ for all $x \in K_{\text{reg}}, y \in U$ and all $y \in K_{\text{reg}}$. $G^\alpha_U(x,y) = G^\alpha_U(y,x)$ for all $x, y \notin K_{\text{irr}}$.

We consider next the problem of reconstructing the potential $k^\alpha * \nu, \nu \in \mathcal{M}_\alpha(K)$, from its values on $K$: for all $y \in U$ we have

$$\int_{K_{\text{irr}}} G^\alpha_U(x,y) d\nu(x) = \int_K G^\alpha_U(x,y) d\nu(x)$$

$$= \int_K k^\alpha * \delta_y(x) d\nu(x) - \int_K k^\alpha * P_K^\alpha \delta_y(x) d\nu(x)$$

$$= k^\alpha * \nu(y) - \int_K k^\alpha * \nu(x) dP_K^\alpha \delta_y(x).$$

Thus, we obtain

7.8. M. Riesz RECONSTRUCTION FORMULA. — For $\nu \in \mathcal{M}_\alpha(K)$, $y \in U = \mathbb{Q}_p \setminus K$,

$$k^\alpha * \nu(y) = \int_K k^\alpha * \nu(x) dP_K^\alpha \delta_y(x) + \int_{K_{\text{irr}}} G^\alpha_U(x,y) d\nu(x).$$

In particular, if $\nu |_{K_{\text{irr}}} = 0$ then

$$k^\alpha * \nu(y) = \int_K k^\alpha * \nu(x) dP_K^\alpha \delta_y(x).$$

As a corollary we obtain

7.9. 4th UNIQUENESS THEOREM. — Let $\nu_1, \nu_2 \in \mathcal{M}(K), \nu_1 |_{K_{\text{irr}}} = 0$ (or just $\nu_1 |_{K_{\text{irr}}} = \nu_2 |_{K_{\text{irr}}}$). If $k^\alpha * \nu_1(x) = k^\alpha * \nu_2(x)$ for $\alpha - a.a.x \in K$, then $\nu_1 = \nu_2$. 
Proof. — Setting \( \nu = \nu_1 - \nu_2 \), \( \nu|_{K_{\text{irr}}} = 0 \), so \( k^{\alpha} \ast \nu(y) = \int_{K} k^{\alpha} \ast \nu(x) dP_{K}^{\alpha} \delta_{y}(x) = 0 \) since \( P_{K}^{\alpha} \delta_{y} \) is \( \alpha \)-absolutely continuous. Thus \( k^{\alpha} \ast \nu(z) = 0 \) for \( \alpha - \text{a.a.} z \)'s hence for \( \text{a.a.} z \)'s and our theorem follows from the 1st uniqueness principle.

The family of measures \( \{P_{K}^{\alpha} \delta_{z}\} \) is Borel, i.e. for any \( \varphi \in S(\mathbb{Q}_{p}) \), \( P_{K}^{\alpha} \delta_{z}(\varphi) \) is a Borel measurable function of \( z \). For any measure \( \nu \) we define \( P_{K}^{\alpha} \nu = \int P_{K}^{\alpha} \delta_{z} d\nu(z) \), i.e. \( P_{K}^{\alpha} \nu(\varphi) = \int P_{K}^{\alpha} \delta_{z}(\varphi) d\nu(z) \). \( P_{K}^{\alpha} \nu \) is a positive distribution, hence a measure : \( P_{K}^{\alpha} \nu \in M^{+}(K) \), and is such that \( P_{K}^{\alpha} \nu|_{K_{\text{irr}}} = 0 \). Similarly, if \( \nu \) is any signed measure, \( P_{K}^{\alpha} \nu \) is again a signed measure. Moreover, \( P_{K}^{\alpha} \nu \) solves the problem of "Balayage" or sweeping out \( \nu \) onto \( K \) while preserving the potential.

7.10. BALAYAGE PRINCIPLE. — For any (signed) measure \( \nu \) there exists a unique (signed) measure \( P_{K}^{\alpha} \nu \), such that:

(i) \( \text{supp} \, P_{K}^{\alpha} \nu \subseteq K \);

(ii) \( P_{K}^{\alpha} \nu|_{K_{\text{irr}}} = 0 \);

(iii) \( k^{\alpha} \ast P_{K}^{\alpha} \nu(x) = k^{\alpha} \ast \nu(x) \) for \( \alpha - \text{a.a.} x \in K \).

Moreover, if \( \nu \) is a measure, then we have also

(iv) \( k^{\alpha} \ast P_{K}^{\alpha} \nu(x) \leq k^{\alpha} \ast \nu(x) \) for all \( x \).

Proof. — Properties (iii) and (iv) follow immediately from the corresponding properties of \( P_{K}^{\alpha} \delta_{z} \). Uniqueness follows from the 4th uniqueness theorem.

From the uniqueness of \( P_{K}^{\alpha} \nu \) we get the following transitivity property of balayage:

7.11. COROLLARY. — Let \( K_{1} \supset K_{2} \) be closed sets, \( \nu \) a signed measure, then

\[
P_{K_{2}}^{\alpha} \nu = P_{K_{2}}^{\alpha} (P_{K_{1}}^{\alpha} \nu).
\]
Note also that we have, for $K$ compact,

$$P_K^\alpha \nu(1) = \int_K 1 dP_K^\alpha \nu(x) = \int_K k^\alpha * \gamma_K(x) dP_K^\alpha \nu(x)$$

$$= \int_K k^\alpha * P_K^\alpha \nu(x) d\gamma_K(x) = \int_K k^\alpha * \nu(x) d\gamma_K(x)$$

$$= \int k^\alpha * \gamma_K(x) d\nu(x) \leq \int d\nu = \nu(1).$$

7.12. Example. — For $B(p^N) = \{x \mid |x| \leq p^N\}$, $x_0 \notin B(p^N)$, we have

$$P_{B(p^N)}^\alpha \delta_{x_0} = \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} |x_0|^{\alpha-1} \phi(p^N x) dx.$$  

Indeed, the potential of this measure is equal to

$$\frac{\zeta(1-\alpha)}{\zeta(\alpha)} p^{-N\alpha} |x_0|^{\alpha-1} \int_{|x| \leq p^N} |y-x|^{\alpha-1} dx;$$

for $y \in B(p^N)$ we can put $x := x+y$ and we get $\frac{\zeta(1-\alpha)}{\zeta(\alpha)} |x_0|^{\alpha-1} = k^\alpha * \delta_{x_0}(y)$. For $y \notin B(p^N)$ we obtain $\frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1)}{\zeta(\alpha)} p^{N(1-\alpha)} |x_0|^{\alpha-1} |y|^{\alpha-1}$

which is $< k^\alpha * \delta_{x_0}(y)$, since $\zeta(1) \zeta(\alpha) < 1$, and since for $|y| > p^N$ and $|x_0| > p^N$, $|x_0 - y|^{\alpha-1} < p^{-N}$, hence $p^{N(1-\alpha)} |x_0|^{\alpha-1} |y|^{\alpha-1} < |x_0 - y|^{\alpha-1}$ (cf. [8], Appendix, A.6, p. 402).

7.13. Example. — Let $\tilde{B}_N = \{x \mid |x| \geq p^{-N}\}$, and set

$$\delta_{(N)}^\alpha \overset{\text{def}}{=} \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} (1-\phi(p^{-1-N} x)) \frac{dx}{|x|^{1+\alpha}} \in M^+(\tilde{B}_N).$$

We have for all $x_0 \notin \tilde{B}_N$, $P_{\tilde{B}_N}^\alpha \delta_{x_0} = \delta_{(N)}^\alpha$. Indeed, for $y \in \tilde{B}_N$
\[
\begin{align*}
 k^\alpha \ast \tilde{\delta}_{(N)}(y) &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \geq p^{-N}} \frac{|y-x|^{\alpha-1}}{|x|^{1+\alpha}} \, dx \\
 &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \geq p^{-N}} |y-x|^{\alpha-1} \, dx \\
 &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} |y|^{-1} \int_{|x| \geq p^{-N} |y|} |x|^{\alpha-1} \, dx \\
 &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} |y|^{-1} \int_{|x| \geq p^{-N} |y|} \frac{\alpha}{|x|} \, dx \\
 &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} |y|^{-1} \int_{|x| \geq p^{-N} |y|} \frac{\alpha}{|x|} \, dx \\
 &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} |y|^{\alpha-1} = k^\alpha \ast \delta_{x_0}(y).
\end{align*}
\]

On the other hand for \( y \notin \tilde{B}_N \),
\[
k^\alpha \ast \tilde{\delta}_{(N)}(y) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \geq p^{-N}} \frac{dx}{|x|^2} = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1)}{\zeta(\alpha)} p^{N(1-\alpha)}.
\]

Since for any \( x_0 \notin \tilde{B}_N \), \(|y-x_0| < p^{-N}\) so \(|y-x_0|^{\alpha-1} > p^{N(1-\alpha)}\); and since \(\frac{\zeta(1)}{\zeta(\alpha)} < 1\), we get \(k^\alpha \ast \tilde{\delta}_{(N)}(y) < \frac{\zeta(1-\alpha)}{\zeta(\alpha)} |y-x_0|^{\alpha-1} = k^\alpha \ast \delta_{x_0}(y)\) (cf. [8], Appendix, A.3, A4, p. 401).

Note that \(\tilde{\delta}_{(N)}(1) = \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \geq p^{-N}} \frac{dx}{|x|^{1+\alpha}} = 1\), i.e. \(\tilde{\delta}_{(N)}\) is a probability measure. We note also that we have, as is easily verified,
\[
\tilde{\delta}_{(N)} \sim \delta_0 \text{ as } N \to \infty,
\]
\[
\tilde{\delta}_{(N)} \sim \tilde{\delta}_\beta \text{ as } \alpha \to \beta.
\]

The measure \(\tilde{\delta}_{(N)}\) defines an operator (via convolution), applicable to any function \(f \in L^1_{\text{loc}}\) such that \(\int_{|x| > 1} |f(x)| \frac{dx}{|x|^{1+\alpha}} < \infty\); in particular, it's applicable to any potential \(f(x) = k^\alpha \ast \mu(x)\), \(\mu \in \mathcal{M}_\alpha\).

8. \(\alpha\)-super-harmonic-functions.

The following lemma establishes a relation between \(\tilde{\delta}_{(N)}\) and \(k^{-\alpha}\) that will motivate much of the development of this section.
8.1. LEMMA. — Let \( f \in L^1_{\text{loc}}, \int_{|x|>1} |f(x)| \frac{dx}{|x|^{1+\alpha}} < \infty \), and assume \( f \) is locally constant near \( x_0 \). Then

\[
k^{-\alpha} \ast f(x_0) = \frac{\zeta(1+\alpha)}{\zeta(1)} \lim_{N \to \infty} p^{(1+N)\alpha}(\delta_0 - \delta^\alpha_{(N)}) \ast f(x_0),
\]

indeed, equality holds for \( N \geq N_0 \).

**Proof.**

\[
\frac{\zeta(1+\alpha)}{\zeta(1)} p^{(1+N)\alpha}(\delta_0 - \delta^\alpha_{(N)}) \ast f(x_0)
\]

\[
= \frac{\zeta(1+\alpha)}{\zeta(\alpha)} p^\alpha \int_{|x|\geq p^{-N}} (f(x_0) - f(x_0 + x)) \frac{dx}{|x|^{1+\alpha}}
\]

\[
= \frac{\zeta(1+\alpha)}{\zeta(-\alpha)} \int_{|x|\geq p^{-N}} (f(x_0 + x) - f(x_0)) \frac{dx}{|x|^{1+\alpha}}
\]

since \( \frac{1}{\zeta(\alpha)} p^\alpha = p^\alpha - 1 = -\frac{1}{\zeta(-\alpha)} \), and the lemma is obtained upon letting \( N \to \infty \).

8.2. DEFINITION. — A function \( f : \mathbb{Q}_p \to [0, \infty], f \neq \infty \), is called \( \alpha \)-superharmonic (\( \alpha \)-s.h.) if

(i) \( f \) is lower semicontinuous;

(ii) \( \int_{|x|>1} f(x) \frac{dx}{|x|^{1+\alpha}} < \infty \);

(iii) For any \( x_0 \in \mathbb{Q}_p \), there exists \( N_0 \) such that for all \( N \geq N_0 \):

\[
f(x_0) \geq \delta^\alpha_{(N)} \ast f(x_0).
\]

8.3. DEFINITION. — A function \( f : \mathbb{Q}_p \to \mathbb{C} \cup \{\infty\}, f \neq \infty \), is called \( \alpha \)-harmonic (\( \alpha \)-h.) at \( x_0 \), if :

(i) \( f \) is locally constant near \( x_0 \);

(ii) \( \int_{|x|>1} |f(x)| \frac{dx}{|x|^{1+\alpha}} < \infty \);

(iii) \( k^{-\alpha} \ast f(x_0) = 0 \).

By Lemma 8.1 conditions (i) and (iii) are equivalent to \( f(x_0) = \delta^\alpha_{(N)} \ast f(x_0) \) for all \( N \geq N_0 \).

\( f \) will be called \( \alpha \)-harmonic in an open set \( U \), if it is \( \alpha \)-h. at each \( x_0 \in U \). Explicitly, \( f(x_0) = (1-p^{-\alpha}) \int_{|x|\geq 1} f(x_0 + p^N x) |x|^{-\alpha} d^* x \), for all \( N \geq N(x_0) \).
8.4. Example. — The constant function \( f(x) \equiv c \) is everywhere harmonic, since \( \tilde{\delta}_{(N)}(1) = 1 \); for \( c \geq 0 \) it is \( \alpha \)-s.h.

8.5. Example. — A potential \( f(x) = k^{\alpha} \ast \nu(x), \nu \in \mathcal{M}_{\alpha}^{+} \), is \( \alpha \)-s.h. Indeed, we have

\[
\tilde{\delta}_{(N)} \ast f(x_0) = \int k^{\alpha} \ast \nu(x_0 - x) d\tilde{\delta}_{(N)}(x) = \int k^{\alpha} \ast \tilde{\delta}_{(N)}(x_0 - x) d\nu(x) \\
\leq \int k^{\alpha}(x_0 - x) d\nu(x) = k^{\alpha} \ast \nu(x_0) = f(x_0).
\]

Moreover, in the complement of \( \text{supp}(\nu) \) it’s \( \alpha \)-h. Indeed, if \( x_0 + B(p^{-N_0}) \) is disjoint from \( \text{supp}(\nu) \), then for \( x \in \text{supp}(\nu), x_0 - x \in \tilde{B}_N \) for all \( N \geq N_0 \), and we obtain

\[
\tilde{\delta}_{(N)} \ast f(x_0) = \int k^{\alpha} \ast \nu(x_0 - x) d\tilde{\delta}_{(N)}(x) = \int k^{\alpha} \ast \tilde{\delta}_{(N)}(x_0 - x) d\nu(x) \\
= \int k^{\alpha}(x_0 - x) d\nu(x) = f(x_0).
\]

We shall establish below a converse to these examples.


(i) Let \( f \in L^1(B(p^N), dx) \) be extended to all of \( \mathbb{Q}_p \) by putting, for \( |x_0| > p^N \), \( f(x_0) = \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \leq p^N} f(x) dx \cdot |x_0|^{\alpha - 1} \); then \( f \) is \( \alpha \)-h. in \( \mathbb{Q}_p \setminus B(p^N) \).

(ii) Similarly, let \( f \in L^1(\tilde{B}_N, \frac{dx}{|x|^{1+\alpha}}) \) be extended to all of \( \mathbb{Q}_p \) by giving it the constant value \( \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \geq p^{-N}} f(x) \frac{dx}{|x|^{1+\alpha}} \) on \( \mathbb{Q}_p \setminus \tilde{B}_N \); then \( f \) is \( \alpha \)-h. in \( \mathbb{Q}_p \setminus \tilde{B}_N \).

Proof. — An easy calculation gives for \( |x_0| > p^N > p^{-M} \):

\[
\frac{\zeta(1)}{\zeta(\alpha)} \int_{|x| \geq p^{-M}} |x - x_0|^{\alpha - 1} |x|^{-(1+\alpha)} dx = p^{M\alpha} |x_0|^{\alpha - 1} - p^{N\alpha} |x_0|^{\alpha - 1}.
\]
To prove (i) we calculate for $|x_0| > p^N$, $M > -N$,
\[
\delta_q^\alpha \ast f(x_0) = \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \int_{|x| \geq p^{-M}} f(x_0-x) \frac{dx}{|x|^{1+\alpha}}
\]
\[
= \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \left[ \int_{|y| \leq p^N} f(y)dy \cdot |x_0|^{-(1+\alpha)} \right]
\]
\[
+ \int_{|x_0| > p^N} \int_{|x| > p^M} f(y)dy \cdot \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|y| \leq p^N} f(y)dy
\]
\[
= \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|y| \leq p^N} f(y)dy \cdot |x_0|^{\alpha-1} \cdot \left[ |x_0|^{-2\alpha p(N-M)\alpha} \right]
\]
\[
+ |x_0|^{1-\alpha} p^{-M\alpha} \frac{\zeta(1)}{\zeta(\alpha)} \int_{|x_0| > p^N} |x-x_0|^{\alpha-1} |x|^{-(1+\alpha)} dx
\]
\[
= f(x_0)
\]
by the definition of $f(x_0)$ and the above claim.

The calculation for (ii) is even easier : for $|x_0| < p^N$, $M > N$,
\[
\delta_q^\alpha \ast f(x_0) = \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \left[ \int_{|x| \geq p^N} f(x) dx \right]
\]
\[
+ \int_{p^M \leq |y| < p^N} \frac{dy}{|y|^{1+\alpha}} \cdot \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \geq p^N} f(x) \frac{dx}{|x|^{1+\alpha}}
\]
\[
= \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \geq p^N} f(x) \frac{dx}{|x|^{1+\alpha}} \cdot \left[ p^{(N-M)\alpha} \right]
\]
\[
+ \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \int_{p^M \leq |y| < p^N} \frac{dy}{|y|^{1+\alpha}}
\]
\[
= f(x_0). \]

We note that if $f$ is $\alpha$-s.h., then since $\delta_q^\alpha \rightarrow \delta_0$ and $f$ is semicontinuous, we get $\lim_{N \to \infty} \delta_q^\alpha \ast f(x) = f(x)$. In particular, if $f_1$, $f_2$ are $\alpha$-s.h., and $f_1(x) = f_2(x)$ for a.a.$x$'s then $f_1 \equiv f_2$ everywhere.

Let $U \subseteq \mathbb{P}(Q_p)$ be an open bounded set and put $K = \mathbb{P}(Q_p) \setminus U$. Let $f$ be a continuous function in $K$ such that $\int_{|x| > 1} f(x) dx < \infty$. Define :
\[
h_f(y) = \int_K f(x) dP^K_{x} \delta_y(x). \]
8.7. Solution to the Dirichlet problem. — \( h_f \) is \( \alpha \)-h. in \( U \),

\[
h_f|_{K_{\text{reg}}} = f|_{K_{\text{reg}}}.\]

Proof. — For \( x \in K_{\text{reg}} \), \( P^K_\alpha \delta_x = \delta_x \) by definition, so indeed \( h_f(x) = f(x) \). Fix \( y_0 \in U \). The integral defining \( h_f(y_0) \) is finite; indeed, take \( M \) such that \( \bar{B}_M \subseteq K \), then as is easily seen \( P^K_\alpha \delta_{y_0}|_{\bar{B}_M} \leq P^K_\alpha \delta_{y_0} \), and by our assumption on \( f \), we get

\[
\int_{\bar{B}_M} |f(x)|dP^K_\alpha \delta_{y_0}(x) \leq \int_{\bar{B}_M} |f(x)|dP^K_{\bar{B}_M} \delta_{y_0}(x)
= \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \int_{|x| \geq p^{-M}} |f(x)| \frac{dx}{|x|^{1+\alpha}} < \infty.
\]

Next, let \( N_0 \) be such that \( y_0 + B(p^{-N_0}) \subseteq U \), then for any \( N \geq N_0 \), \( K \subseteq y_0 + \bar{B}_N \), and we have

\[
\delta_{\alpha}^{(N)} \ast h_f(y_0) = \int_{y_0 + \bar{B}_N} h_f(y)dP^K_\alpha \delta_{y_0 + \bar{B}_N}(y)
= \int_{y_0 + \bar{B}_N} \int_K f(x)dP^K_\alpha \delta_y dP^K_{y_0 + \bar{B}_N} \delta_{y_0}(y) \quad \text{(by definition of } h_f) \]

\[
= \int_K f(x) d \int_{y_0 + \bar{B}_N} dP^K_\alpha \delta_y(x) dP^K_{y_0 + \bar{B}_N} \delta_{y_0}(y)
= \int_K f(x) dP^K_\alpha \delta_{y_0}(x) \quad \text{(by transitivity of Balayage)}
= h_f(y_0).
\]

Since for any \( y'_0 \in y_0 + B(p^{-N_0}) \), \( y'_0 + \bar{B}_N = y_0 + \bar{B}_N \), we also see that \( h_f(y'_0) = h_f(y_0) \), i.e. \( h_f \) is locally constant in \( U \), concluding the proof that it is \( \alpha \)-h. in \( U \). We note that, since for \( U \ni y \to x_0 \in K_{\text{reg}} \cap \partial K \), \( P^K_\alpha \delta_{x_0} \sim \delta_{x_0} \), we have \( h_f(y) \rightarrow h_f(x_0) \), i.e. \( h_f \) is also continuous in \( K_{\text{reg}} \), and its only discontinuities can appear in \( K_{\text{irr}} \).

8.8. Principle of the harmonic minorant. — Let \( f \) be \( \alpha \)-s.h., and let \( g \) be \( \alpha \)-h. in an open set \( U \), continuous in \( \bar{U} \), and assume \( \infty \not\in U \) (possibly \( \infty \in \bar{U} \)).

If \( f \geq g \) on \( \mathbb{Q}_p \setminus U \), then \( f \geq g \) everywhere; if \( f(x_0) = g(x_0) \) for some \( x_0 \in U \), then \( f = g \) almost everywhere.

Proof. — Let \( d = f - g \); \( d \) is lower semicontinuous on \( \bar{U} \). At any point \( b \in \bar{U} \setminus U \), we have \( \lim_{x \to b} d(x) = \lim_{x \to b} f(x) - \lim_{x \to b} g(x) \geq f(b) - g(b) \geq 0 \).
Consequently, if \( d \) assumes negative values, then there is a point \( x_0 \in U \),
where \( d(x_0) \) achieves its minimum, and we have a contradiction: for \( N \gg 0 \),
\[
d(x_0) < \delta^{\alpha}_{(N)} * d(x_0) = \delta^{\alpha}_{(N)} * f(x_0) - \delta^{\alpha}_{(N)} * g(x_0) \leq f(x_0) - g(x_0) = d(x_0).
\]

If \( f(x_0) = g(x_0) \) for some \( x_0 \in U \), then again \( \delta^{\alpha}_{(N)} * d(x_0) = 0 \), hence
\( d(x) = 0 \) for a.a. \( x \)’s.

8.9. Corollary. — Let \( f \) be \( \alpha \)-s.h. If \( f(x_0) = \inf f(x) \) for some
\( x_0 \in U \), then \( f(x) \equiv f(x_0) \).

E.g., we have \( G_{(N)}^{\alpha}(y_1, y_2) > 0 \) for all \( y_1, y_2 \in U \); indeed, \( G_{(N)}^{\alpha}(y_1, y_2) \) is
\( \alpha \)-s.h., and \( \alpha \)-h. for \( y_1 \neq y_2 \).

We have the following converse to the above principle:

8.10. Lemma. — Let \( f : \mathbb{Q}_p \rightarrow [0, \infty] \) (\( f \neq \infty \)) be lower semicon-
tinuous, \[
\int_{|x| > 1} f(x) \frac{dx}{|x|^{1+\alpha}} < \infty,
\]
and assume \( f \) satisfies the principle of the harmonic minorant. Then \( f \) is \( \alpha \)-s.h.

Proof. — Fix \( x_0 \in \mathbb{Q}, N \in \mathbb{Z} \) and let \( g \) be the function constructed
in lemma (ii) with respect to \( f \mid_{x_0 + B_N} \). Since \( g \) is \( \alpha \)-h. and continuous (even
constant) in the open (and closed) set \( \{ x \mid |x-x_0| < p^{-N} \} \), and since on
the complement we have \( f \geq g \) (even \( f = g \)), we obtain by applying the
principle of the harmonic minorant \( \delta^{\alpha}_{(N)} * f(x_0) = g(x_0) \leq f(x_0) \).

Noting that in the above proof \( N \) was arbitrary, we get

8.11. Corollary. — If \( f \) is \( \alpha \)-s.h., then \( \delta^{\alpha}_{(N)} * f \leq f \) for all \( N \).

8.12. Theorem. — The class of \( \alpha \)-s.h. functions is closed under :

(i) addition and multiplication by a positive constant;

(ii) passage to the limit of a sequence converging uniformly;

(iii) passage to the limit of a monotone increasing sequence (if the
limit is \( \neq \infty \));

(iv) the operation \( \inf \), applied to a finite number of elements;

(v) convolution with a measure (if the result is \( \neq \infty \)).

Proof. — (i), (ii) and (iv) are trivial. To prove (iii), assume \( f_n \) are \( \alpha \-
s.h., f_n \leq f_{n+1}, f(x) = \lim f_n(x) \neq \infty \). Then \( f(x) \geq f_n(x) \geq \delta^{\alpha}_{(N)} * f_n(x) \),
and by monotonicity we can let \( n \to \infty \) to obtain \( f(x) \geq \delta^{(N)}_0 * f(x) \). To prove (v), let \( f \) be \( \alpha \)-s.h., \( \mu \) a positive measure such that \( \mu * f \neq \infty \). Then we have \( \mu * f = \lim_{n \to -\infty} (\mu|_{B(p^n)}) * f \) a monotone increasing limit, so by (iii) we are reduced to the case that \( \mu \) has compact support, but then we can use Fubini theorem to obtain
\[
\mu * f(x_0) \geq \mu * (\delta^{(N)}_0 * f)(x_0) = (\mu * f) * \delta^{(N)}_0(x_0).
\]

8.13. Corollary. — If \( f \) is \( \alpha \)-s.h., then \( k^{-\alpha} * f \) is a positive measure.

Proof. — Here \( k^{-\alpha} * f(\varphi) \) \( \overset{\Delta}{=} k^{-\alpha}(f * \varphi) \) for \( \varphi \in S(\mathbb{Q}_p) \), and we need to show that, if \( \varphi \geq 0 \), then \( k^{-\alpha} * f(\varphi) \geq 0 \). By (v), \( f * \varphi \) is again \( \alpha \)-s.h., and noting that it satisfies the conditions of Lemma 8.1, we get
\[
k^{-\alpha}(f * \varphi) = \zeta(1+\alpha) \lim_{N \to -\infty} p^{(1+N)\alpha}(\delta_0 - \delta^{(N)}_0) * (f * \varphi) \geq 0.
\]

We now arrive at the following important result:

8.14. Riesz Representation Theorem. — A function \( f \) is \( \alpha \)-s.h. if and only if \( f = k^\alpha * \nu + c \), \( \nu \in \mathcal{M}_\alpha^+ \), \( c \geq 0 \). In this representation, the measure \( \nu \) and the constant \( c \) are unique. Moreover, \( f \) is \( \alpha \)-h. at \( x_0 \) if and only if \( x_0 \notin \text{supp}(\nu) \).

Proof. — The "if" part of the first statement has been established above, so let \( f \) be \( \alpha \)-s.h.. Consider the function \( f_N \) constructed in Lemma 8.6, (i) for \( f|_{B(p^N)} \), i.e.
\[
f_N(x) = \begin{cases} f(x) & |x| \leq p^N \\ a_N \cdot |x|^{\alpha-1} & |x| > p^N \end{cases} \text{ where } a_N = \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \leq p^N} f(x) dx.
\]
Since \( f_N \) is \( \alpha \)-h. outside \( B(p^N) \), and since \( f_N(x) \leq f(x) \) for \( x \in B(p^N) \) and \( x = \infty \), we get by applying the principle of the harmonic minorant : \( f_N \leq f \) everywhere. Consequently, for \( x_0 \in B(p^N) : \delta^{(N)}_0 * f_N(x_0) \leq \delta^{(N)}_0 * f(x_0) = f_N(x_0) \), i.e. \( f_N \) is again \( \alpha \)-s.h.. Moreover, since \( f_{N-1} \) is \( \alpha \)-h. outside \( B(p^{N-1}) \), and since \( f_{N-1}(x) \leq f_N(x) \), for \( x \in B(p^{N-1}) \) and \( x = \infty \), we obtain by another application of the principle of the harmonic minorant : \( f_{N-1} \leq f_N \) everywhere.

Since \( f_N(x) - a_N|x|^{\alpha-1} \) has compact support, we easily get
\[
f_N(x_0) - a_N|x_0|^{\alpha-1} = k^\alpha * (k^{-\alpha} * (f_N(x) - a_N|x|^\alpha))(x_0) \text{ for a.a. } x_0 \text{'s.}
\]
But the right hand side equals $k^\alpha \ast (k^{-\alpha} \ast f_N)(x_0) - a_N|x_0|^\alpha - 1$, hence

$$f_N(x) = k^\alpha \ast (k^{-\alpha} \ast f_N)(x) \text{ for } a.a.x's.$$ 

Since $f_N$ is $\alpha$-s.h., $\nu_N = k^{-\alpha} \ast f_N$ is a positive measure and $k^\alpha \ast \nu_N$ is again $\alpha$-s.h., hence

$$f_N(x) = k^\alpha \ast \nu_N(x) \text{ for all } x.$$ 

Since the $f_N$ monotonically increase to $f$, we obtain by the dichotomy principle (4.6)

$$f(x) = \lim_{N \to \infty} f_N(x) = \lim_{N \to \infty} k^\alpha \ast \nu_N(x) = k^\alpha \ast \nu(x) + c \text{ for } a.a.x's$$

for some $\nu \in \mathcal{M}^+_\alpha$ and $c \geq 0$. Again, since both sides are $\alpha$-s.h., we get

$$f(x) = k^\alpha \ast \nu(x) + c \text{ for all } x.$$ 

Furthermore, since $\int_{|x|>1} k^\alpha \ast \nu(x) \frac{dx}{|x|^{1+\alpha}} < \infty$, we have

$$\delta^\alpha_{(-N)}(k^\alpha \ast \nu) = \frac{\zeta(1)}{\zeta(\alpha)} \int_{|x| \geq 1} k^\alpha \ast \nu(p^{-N}x) \frac{dx}{|x|^{1+\alpha}} \overset{N \to \infty}{\longrightarrow} 0$$

hence $\lim_{N \to -\infty} \delta^\alpha_{(N)}(f) = c$, which proves the uniqueness of $c$, and hence of $\nu$. The harmonicity statement is clear.

8.15. COROLLARY. — A function $f$ is a potential of a measure if and only if $f$ is $\alpha$-s.h. and $\lim_{N \to -\infty} \delta^\alpha_{(N)}(f) = 0$.

8.16. COROLLARY. — Let $f$ be $\alpha$-s.h. and $\nu \in \mathcal{M}^+_\alpha$. Then $\inf \{f, k^\alpha \ast \nu\}$ is a potential of a measure.

Proof. — Since both $f$ and $k^\alpha \ast \nu$ are $\alpha$-s.h. so is $\inf \{f, k^\alpha \ast \nu\}$, but we also have

$$\lim_{N \to -\infty} \delta^\alpha_{(N)}(\inf \{f, k^\alpha \ast \nu\}) \leq \lim_{N \to -\infty} \delta^\alpha_{(N)}(k^\alpha \ast \nu) = 0.$$ 

As another application we have,

8.17. PRINCIPLE OF DOMINATION. — Let $f$ be $\alpha$-s.h., $\nu \in \mathcal{E}^+_\alpha$; then if $k^\alpha \ast \nu(x) \leq f(x)$ for $\nu - a.a.x's$, then $k^\alpha \ast \nu \leq f$ everywhere.

Proof. — By the above corollary $\inf \{f, k^\alpha \ast \nu\} = k^\alpha \ast \nu'$, for some $\nu' \in \mathcal{M}^+_\alpha$. But $\|\nu\|_\alpha = \int k^\alpha \ast \nu'(x)\nu'(x) dx \leq \int k^\alpha \ast \nu(x)\nu'(x) = \int k^\alpha \ast \nu'(x)d\nu(x) \leq \int k^\alpha \ast \nu(x)d\nu(x) = \|\nu\|_\alpha < \infty$, i.e. $\nu' \in \mathcal{E}^+_\alpha$. Moreover,

$$\|\nu - \nu'\|^2_\alpha = \int (k^\alpha \ast \nu(x) - k^\alpha \ast \nu'(x)) (d\nu(x) - d\nu'(x))$$
and we can restrict the integral to \( \{ x \mid k^\alpha \ast \nu(x) > k^\alpha \ast \nu'(x) \} \) which by assumption has \( \nu \)-measure 0, hence we obtain 

\[
\| \nu - \nu' \|_\alpha^2 = -\int (k^\alpha \ast \nu(x) - k^\alpha \ast \nu'(x)) d\nu'(x) \leq 0 \]

hence \( \| \nu - \nu' \|_\alpha = 0 \), and by the 2nd uniqueness principle \( \nu = \nu' \), so \( k^\alpha \ast \nu = k^\alpha \ast \nu' = \inf \{ f, k^\alpha \ast \nu \} \)

i.e. \( k^\alpha \ast \nu \leq f \).

8.18. Remark. — The condition \( \nu \in \mathcal{E}^+_\alpha \) can be weakened to :

\( k^\alpha \ast \nu(x) \leq \infty \) for \( \nu - a.a.x \)'s (cf. [8], Theorem 1.29, p. 115).

Recall that for a compact set \( K \), we have \( \gamma_K \) its equilibrium measure, and denote by \( f_K = k^\alpha \ast \gamma_K \) its potential, so \( f_K(x) \leq 1 \), \( f_K(x) = 1 \) for \( \alpha - a.a.\, x \in K \).

8.19. Corollary. — If \( K' \subseteq K \), then \( f_{K'} \leq f_K \).

Proof. — \( f_{K'}(x) = 1 = f_K(x) \) for \( \alpha - a.a.\, x \in K' \), so in particular \( f_{K'}(x) \leq f_K(x) \) for \( \gamma_{K'} - a.a.\, x \)'s, and by the principle of domination \( f_{K'} \leq f_K \) everywhere.

8.20. Corollary. — \( f_{K_1 \cap K_2} \leq f_{K_1} + f_{K_2} \).

Proof. — Since \( f_{K_1 \cap K_2} \leq f_{K_2} \), and since for \( \alpha - a.a.\, x \in K_1 \), \( f_{K_1 \cup K_2}(x) = 1 = f_{K_1}(x) \), we obtain \( k^\alpha \ast (\gamma_{K_1 \cup K_2} + \gamma_{K_1 \cap K_2})(x) = f_{K_1 \cup K_2}(x) + f_{K_1 \cap K_2}(x) \leq f_{K_1}(x) + f_{K_2}(x) = k^\alpha \ast (\gamma_{K_1} + \gamma_{K_2})(x) \) for \( \alpha - a.a.\, x \in K_1 \). By symmetry this inequality holds also for \( \alpha - a.a.\, x \in K_2 \), hence it holds for \( (\gamma_{K_1 \cup K_2} + \gamma_{K_1 \cap K_2}) - a.a.\, x \)'s and by the principle of domination it holds everywhere.

This corollary gives us the following convexity property of \( \text{cap}_\alpha \).

8.21. Corollary. —

\[
\text{cap}_\alpha(K_1 \cup K_2) + \text{cap}_\alpha(K_1 \cap K_2) \leq \text{cap}_\alpha(K_1) + \text{cap}_\alpha(K_2).
\]

Proof. — Set \( \nu = \gamma_{K_1} + \gamma_{K_2} - \gamma_{K_1 \cup K_2} - \gamma_{K_1 \cap K_2} \), so that \( k^\alpha \ast \nu(x) \geq 0 \) for all \( x \). We have

\[
\text{cap}_\alpha(K_1) + \text{cap}_\alpha(K_2) - \text{cap}_\alpha(K_1 \cup K_2) - \text{cap}_\alpha(K_1 \cap K_2)
\]

\[
= \gamma_{K_1}(1) + \gamma_{K_2}(1) - \gamma_{K_1 \cup K_2}(1) - \gamma_{K_1 \cap K_2}(1)
\]

\[
= \nu(1) = \int k^\alpha \ast \gamma_{K_1 \cup K_2}(x) d\nu(x)
\]

\[
= \int k^\alpha \ast \nu(x) d\gamma_{K_1 \cup K_2}(x) \geq 0.
\]
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