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ESTIMATES ON THE NUMBER OF SCATTERING POLES NEAR THE REAL AXIS FOR STRICTLY CONVEX OBSTACLES

by J. SJÖSTRAND & M. ZWORSKI

0. Introduction.

In this paper we consider scattering by strictly convex obstacles and study the angular density of scattering poles near the real axis. We obtain optimal estimates on the rate at which this density decays as the angle goes to zero. This answers a question suggested by our work on the angular density of poles conducted in a more abstract setting ([SZ2], Corollary 1.3). The presentation in this paper is essentially self contained but the method of proof follows some ideas first developed in [S].

Let $\mathcal{O} \subset \mathbf{R}^n$ with n odd be an open bounded set with a smooth strictly convex boundary. Let $-P$ denote the Dirichlet Laplacian on $\mathbf{R}^n \backslash \mathcal{O}$. In other words, P is the Friedrichs extension of $-\Delta = -\Sigma \partial_{x_j}^2$ defined on $C_0^\infty(\mathbf{R}^n \backslash \overline{\mathcal{O}})$. Following the standard terminology, we then define the scattering poles as the poles of the meromorphic extension of $(P-k^2)^{-1}$: $C_0^\infty(\mathbf{R}^n \backslash \mathcal{O}) \longrightarrow C^\infty(\mathbf{R}^n \backslash \mathcal{O})$ from $\{k \in \mathbf{C}; \operatorname{Im} k > 0\}$ to $\mathbf{C}$. Global bounds on the number of poles in the case of odd n were first obtained by R. Melrose [M1], [M2]. More general results (for general compactly supported perturbations of the Laplacian) were obtained by the present authors [SZ1] and later by Vodev [V].

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If for $\delta \in]0, \delta_0]$ with $\delta_0 > 0$ small and fixed, we let $N_\delta(\lambda)$ denote the number of poles in $\{k \in \mathbf{C};\ 0 \leq |k| < \lambda, 0 \leq -\arg k \leq \delta\}$, then it follows from the general result in [SZ2] that

$$N_\delta(\lambda) \leq \varepsilon(\delta)\lambda^n,\ \lambda \geq \lambda(\delta), \tag{0.1}$$

for some positive function $\lambda = \lambda(\delta)$ and with a constant $\varepsilon(\delta)$ that tends to zero when δ tends to zero, and we were able to estimate $\varepsilon(\delta)$ by $\delta^{2/5}$. On the other hand, in the case when $\mathcal{O}$ is a ball, the poles can be expressed as zeros of special functions, in which case (see [O]) we can take $\varepsilon(\delta) = \mathcal{O}(\delta^{3/2})$ and in fact this estimate is optimal. In the present paper, we shall prove :

THEOREM. — *There exist a constant $C_0 > 0$ and a positive continuous function $\lambda = \lambda(\delta)$ on $]0, \delta_0]$, such that*

$$N_\delta(\lambda) \leq C_0\delta^{3/2}\lambda^n,\ \text{for } 0 < \delta \leq \delta_0,\ \lambda > \lambda(\delta). \tag{0.2}$$

Remark. — In the above result we assumed that n is odd. It will be clear from the proof (and the techniques of [SZ1], [SZ2]) that the same result holds for even n if $\delta_0 < \pi$ and we replace $N_\delta(\lambda)$ by $\widetilde{N}_\delta(\lambda)$, the number of scattering poles in $\{k \in \mathbf{C}; 1 \leq |k| \leq \lambda,\ 0 \leq -\arg k \leq \delta\}$.

We remark that if suffices to prove the theorem for some sufficiently small δ_0, since the estimate (0.2) for δ bounded from below by some fixed constant follows from the result of Melrose (and from the method of proof of [SZ1] in the case of even dimensions).

As in [SZ2], the proof is based on the method of complex scaling, but contrary to the method of that paper, where a more abstract situation was considered, we now scale all the way up to the boundary, and work explicitly with the corresponding (scaled) elliptic boundary value problem. We were led to the particular scaling by applying the point of view of escape functions in [S].

Many open problems remain. One would be to understand the geometric meaning of the best constant C_0 in (0.2), another problem is to get also some estimates in certain parabolic neighborhoods of the real axis. The methods of this work could certainly give some results about the second problem at the expense of some technical complications (*).

The plan of the paper is the following :

(*) *Added in proof :* Progress on this problem has now been achieved and will appear elsewhere.

In section 1 we review complex scaling outside a strictly convex set and add some considerations which become necessary by the fact that we scale all the way up to the boundary. In section 2 we study ellipticity properties of certain scaled boundary problems. In section 3, we construct asymptotic parametrices of these boundary value problems. In section 4, we study the trace of functions of a certain self-adjoint operator, and finally in section 5, we end the proof, in the same spirit as in [S], [SZ1], [SZ2].

1. Complex scaling up to the boundary.

The convexity of the obstacle allows us to scale all the way to the boundary. When studying characteristic values of the scaled operator this will avoid additional contributions present when the more abstract situation was considered in [SZ2].

The function $d(x) = \mathrm{dist}(x, \mathcal{O})$ belongs to $C^\infty(\mathbf{R}^n \backslash \mathcal{O})$ and $d''(x) \geq 0$, $\mathrm{Ker}\, d''(x)$ is generated by the normal of $\mathcal{O}$ which passes through x. For large x we also know that $d''(x)$ restricted to the orthogonal space of the normal direction is bounded from below by $1/(C|x|)$ and from above by $C/|x|$. For x in some large ball shaped neighborhood of $\mathcal{O}$, we put $f(x) = d(x)^2/2$. Then $f'(x) = d(x)d'(x)$, $f''(x) = d(x)d''(x) + d'(x) \otimes d'(x)$, so outside any fixed neighborhood of $\overline{\mathcal{O}}$, we know that $f'' \geq 1/C$. We may extend f to a function in $C^\infty(\mathbf{R}^n \backslash \mathcal{O})$ in such a way that the above properties remain valid and so that $f(x) = |x|^2/2$ for large $|x|$. As in [SZ2], we let $\theta > 0$ be small but fixed, and put :

$$\Gamma_\theta = \{z = x + i\theta f'(x);\ x \in \mathbf{R}^n \backslash \mathcal{O}\}. \tag{1.1}$$

We let P_θ denote $-\Delta_{|\Gamma_\theta}$, equipped with the domain $(H_0^1 \cap H^2)(\Gamma_\theta)$, where Γ_θ is identified with $\mathbf{R}^n \backslash \mathcal{O}$ by means of (1.1). The principal symbol p_θ of P_θ is then of the form :

$$\begin{aligned} p_\theta(x,\xi) &= \langle (1+i\theta f''(x))^{-1}\xi, (1+i\theta f''(x))^{-1}\xi \rangle \\ &= \langle (1-(\theta f''(x))^2 \tilde{\xi}, \tilde{\xi}\rangle - 2i\theta \langle f''(x)\tilde{\xi}, \tilde{\xi}\rangle, \end{aligned} \tag{1.2}$$

with $\tilde{\xi} = (1+(\theta f''(x))^2)^{-1}\xi$, and we may notice that $p_{-\theta} = \bar{p}_\theta$. Here we assume that $\theta \geq 0$ is sufficiently small so that :

$$\|\theta f''(x)\| \leq 2^{-1/2}. \tag{1.3}$$

Writing

$$p_\theta = a_\theta(x,\xi) - i b_\theta(x,\xi), \tag{1.4}$$

we conclude that $a_\theta(x,\xi) \sim \xi^2$, and that $b_\theta \sim \xi^2$ outside any neighborhood of $\overline{\mathcal{O}}$. Near the boundary we also see that $b_\theta \geq \widetilde{C}^{-1}(d(x)\xi^2+\tilde{\xi}_n^2) \geq C^{-1}(d(x)\xi^2+\xi_n^2)$, with $\tilde{\xi}_n = \langle d'(x), \tilde{\xi}\rangle$, $\xi_n = \langle d'(x), \xi\rangle$. Near any boundary point, we can choose local (normal geodesic) coordinates $y = (y', y_n)$ such that $y_n = d(y)$ and such that $\langle \nabla y_n, \nabla y_j\rangle = 0$, $1 \leq j \leq n-1$. Then the principal symbol of $-\Delta = P_0$ becomes $p_0 = \eta_n^2+r(y,\eta') = \langle {}^t(\partial y/\partial x)\eta, {}^t(\partial y/\partial x)\eta\rangle$. We also notice that $(\partial y/\partial x \circ {}^t\partial y/\partial x)_{j,k}$ vanishes for $j < k = n$ and for $k < j = n$ and is equal to one for $j = k = n$. For $y_n = 0$, we have

$$f''(x) = {}^t(\partial y/\partial x)\pi_n \partial y/\partial x = (f''(x))^2,$$

where π_n is the orthogonal projection on the $n{:}th$ canonical basis vector. We also notice that $\partial y/\partial x f''{}^t\partial y/\partial x = \pi_n$. Then $(1+i\theta f''(x))^{-1} = (1-(i\theta/(1+i\theta))f'')$ and a simple calculation gives :

$$(1.5)\quad \begin{aligned} p_\theta(x',0,\xi) &= \langle (1+i\theta f'')^{-1}\, {}^t(\partial y/\partial x)\eta, (1+i\theta f'')^{-1}\, {}^t(\partial y/\partial x)\eta\rangle \\ &= (1+i\theta)^{-2}\eta_n^2+r(y',0,\eta'). \end{aligned}$$

PROPOSITION 1.1. — *Choosing first $\theta>0$ and then $\delta_0>0$ sufficiently small, the spectrum of the operator p_θ in the sector $\{k \in \mathbf{C};\, 0 \leq -\arg k \leq \delta_0\}$, consists only of isolated eigenvalues of finite algebraic multiplicity, and moreover these eigenvalues coincide with the squares of the scattering poles in the same sector.*

In what follows, the squares of the scattering poles will be called resonances.

Proof. — It suffices to link our new scaling to the scaling away from the obstacles in [SZ1], [SZ2] by means of the following local result :

LEMMA 1.2. — *There exists an $\theta_0 > 0$ such that for all $x_0 \in \partial\mathcal{O}$ and all $\lambda \in \mathbf{C}$, we have the following : Let $u \in C^\infty(\mathbf{R}^n\backslash\mathcal{O})$ satisfy $(-\Delta-\lambda)^{k_0}u = 0$, $\partial^\alpha u_{|\partial\Omega} = \bar{u}_\alpha \in C^\infty(\partial\Omega)$ in a neighborhood of x_0. Then there exists a complex neighborhood W of x_0 such that u extends holomorphically to an open neighborhood of $W \cap \cup_{|\theta|\leq\theta_0}$ (Interior of Γ_θ) and such that $u_\theta = u_{|\Gamma_\theta}$ is smooth up to $\partial\mathcal{O}$ and satisfies $(-\Delta-\lambda)^{k_0}u_\theta = 0$, $\partial^\alpha u_{\theta|\partial\Gamma_\theta} = \bar{u}_\alpha$ in $\Gamma_\theta \cap W$. Moreover, in the above result, we may replace $\mathbf{R}^n\backslash\mathcal{O}$ by any fixed Γ_η with $|\eta| < \theta_0$.*

Proof of the lemma. — Let $x_0 \in \partial\mathcal{O}$, and assume that u is of class $C^\infty(\mathbf{R}^n\backslash\mathcal{O})$ in a neighborhood of x_0, and satisfies :

$$(1.6)\qquad (-\Delta-\lambda)^{k_0}u = 0,$$

for some $k_0 \in \{1, 2, 3, \dots\}$. We have to show that u extends holomorphically sufficiently far so that $u_\theta = u_{|\Gamma_\theta}$ can be defined, that this restriction is of class $C^\infty(\Gamma_\theta)$ (near x_0, where we work), and finally that the boundary values of u_θ (*i.e.* the restriction to $\partial\mathcal{O}$) and of its derivatives, coincide with the corresponding boundary values of u.

In order to prove this, we take some boundary point x_1 close to x_0 and make the change of variables :

$$x \longmapsto y, \quad x = x_1 + \varepsilon y. \tag{1.7}$$

Here $\varepsilon > 0$ is a parameter that will tend to 0, so that we are "blowing-up" a neighborhood of x_0. We now restrict the attention to the region $|y| < 2$. The equation (1.7) becomes :

$$(-\Delta_y - \varepsilon^2\lambda)^{k_0} u = 0. \tag{1.8}$$

Let y_0 with $|y_0| = 1$ be close to the normal to the boundary through $y = 0$. (We now express everything in the y-variables.) Let $\Gamma_{\theta,x_1,\varepsilon}$ be the image in (the complexified) y-space of Γ_θ, parametrized by $z = y + i\theta\varepsilon^{-1} f'(x_1+\varepsilon y) = y + i\theta\partial_y(f_{\varepsilon,x_1}(y))$, $f_{\varepsilon,x_1}(y) = \varepsilon^{-2} f(x_1+\varepsilon y)$. Notice that $\partial_y^\alpha f_{\varepsilon,x_1} = \mathcal{O}(|y|^{2-|\alpha|})$, for $|\alpha| \le 2$ and $= \mathcal{O}(\varepsilon^{|\alpha|-2})$, for $|\alpha| \ge 2$. Let $\chi \in C_0^\infty(B(y_0, 1/2))$ be equal to 1 on $B(y_0, 1/4)$ and consider the intermediate contour $\widetilde{\Gamma}_{\theta,x_1,\varepsilon}$, defined by $z = y + i\theta\partial_y(\chi f_{\varepsilon,x_1}(y))$. We let $\Omega_0, \Omega_1, \Omega_2 \subset \widetilde{\Gamma}_{\theta,x_1,\varepsilon}$ be the images of the balls $B(y_0, 1/4)$, $B(y_0, 1/2)$ and $B(y_0, 3/4)$ respectively, so that $\Omega_0 \subset \Gamma_{\theta,x_1,\varepsilon}$ and $\Omega_2 \backslash \Omega_1 \subset \mathbf{R}^n$.

If θ is not too large, then we have uniformly with respect to x_1 and ε :

$$\|v\|_{\Omega_1} \le C(\|(-\Delta_y - \varepsilon^2\lambda)^{k_0} v\|_{\Omega_2} + \|v\|_{\Omega_2\backslash\Omega_1}), \tag{1.9}$$

for smooth functions, defined on Ω_2. Here Δ_y indicates the restriction of the Laplace operator to the intermediate contour and $\|\cdot\|_{\Omega_j}$ is the L^2-norm over Ω_j. The inequality (1.9) is a consequence of the ellipticity of the restriction of the Laplace operator and of the strong uniqueness property of $(-\Delta_y - \varepsilon^2\lambda)^{k_0}$, see the appendix for more details.

By a non-characteristic deformation, u extends to a holomorphic solution of (1.8) over a family of contours which are intermediate between $B(y_0, 3/4) \cap \mathbf{R}^n$ and $\Gamma_{\theta,x_1,\varepsilon}$, and in particular, u satisfies (1.8) on Ω_2.

Let $\bar{u}(x_1)$ be the boundary value of u at x_1 (when u is considered as a function on the original real domain). From (1.8), we get :

$$(-\Delta_y - \varepsilon^2\lambda)^{k_0}(u - \bar{u}(x_1)) = -(-\varepsilon^2\lambda)^{k_0}\bar{u}(x_1) \tag{1.10}$$

and applying (1.9), we get :

$$\|u - \bar{u}(x_1)\|_{\Omega_1} \le C((\varepsilon^2\lambda)^{k_0}|\bar{u}(x_1)| + \|u - \bar{u}(x_1)\|_{\Omega_2\backslash\Omega_1}). \tag{1.11}$$

We can also combine (1.8), (1.9) directly and get :

$$\|u\|_{\Omega_1} \leq C\|u\|_{\Omega_2\backslash\Omega_1}. \tag{1.12}$$

If we use (1.12) with suitable x_1 and ε and go back to the original coordinates, we see that $u_{|\Gamma_\theta}$ is well defined in L^2 (we still only discuss the situation locally near x_0). Since $u_{|\Gamma_\theta}$ satisfies a non-characteristic equation (deduced from (1.8)), we see that $u_{|\Gamma_\theta}$ has a boundary value $w \in \mathcal{D}'(\partial\mathcal{O})$ and we shall show that w coincides with $\bar{u}$, the "real" boundary value of u. By (1.11), we have

$$(\mathrm{vol}\,\Omega_0)^{-1}\|u-\bar{u}(x_1)\|^2_{\Omega_2} = o(1), \quad \varepsilon \to 0, \tag{1.13}$$

uniformly with respect to x_1, when x_1 varies in a compact set of the boundary and where we now write u instead of $u_{|\Gamma_\theta}$. Notice that the statement (1.13) is independent of whether we take L^2-norms with respect to the x or the y-coordinates. It is the y-coordinate version which most clearly follows from (1.11). Choosing y_0 and the coordinates $x = (x', x_n)$ in Γ_θ conveniently, we may assume that y_0 corresponds to (x', ε), $x_1 = (x', 0)$. Using the continuity of $\bar{u}(x')$ we then deduce from (1.13) that

$$\varepsilon^{-n}\|u(x)-\bar{u}(x')\|^2_{B((y',\varepsilon);\varepsilon/C)} \longrightarrow 0, \quad \varepsilon \to 0, \tag{1.14}$$

uniformly with respect to y'. From this we deduce that the boundary value w coincides with $\bar{u}(x')$ in the following way : We know that $u \in C([0,\varepsilon_0[;\mathcal{D}'(\mathbf{R}^{n-1}))$ and we want to show that $u(0,x') = \bar{u}(x')$. We replace u by $u(x)-\bar{u}(x')$ so that (1.14) becomes

$$\varepsilon^{-n}\|u\|^2_{B((y',\varepsilon);\varepsilon/C)} \longrightarrow 0. \tag{$*$}$$

Let $\chi_n \in C_0^\infty(\mathbf{R})$, $\chi'(x') \in C_0^\infty(\mathbf{R}^{n-1})$ be non-negative functions with support close to 1 and 0 respectively, and with $\int \chi_n(x_n)\, dx_n = 1$, $\int \chi'(x')\, dx' = 1$. For $\varphi \in C_0^\infty(\mathbf{R}^{n-1})$, we have

$$\begin{aligned}\langle u(x',0), \varphi\rangle_{\mathcal{D}'(\mathbf{R}^n)} &= \lim_{\varepsilon\to 0} \int u(x)\varepsilon^{-1}\chi_n(x_n/\varepsilon)\varphi(x')\, dx \\ &= \lim_{\varepsilon\to 0} \int \varepsilon^{-n} \int u(x)\chi_n(x_n/\varepsilon)\chi'((x'-y')/\varepsilon)\varphi(x')\, dx\, dy' \\ &= 0,\end{aligned}$$

since $\varepsilon^{-n} \int u(x)\chi_n(x_n/\varepsilon)\chi'((x'-y')/\varepsilon)\varphi(x')\, dx$ has a uniformly compact support with respect to y' and tends to zero (uniformly in y') by Cauchy-Schwarz and $(*)$.

In the same way we can show that the boundary values of $\partial_x^\alpha u$ (with x being the original coordinates) along $\mathbf{R}^n\backslash\mathcal{O}$ and along Γ_θ coincide. In fact, we just have to repeat the above arguments for the differentiated equation

$$(-\Delta-\lambda)^{k_0}(D^\alpha u) = 0. \tag{1.15}$$

Finally, we notice that in the above proof, we can replace $\mathbf{R}^n \backslash \mathcal{O}$ by Γ_η for any fixed η with $|\eta| \leq \theta_0$. This concludes the proof of the lemma and of the proposition. □

As in [S], [SZ1], [SZ2] the strategy now is to choose $\omega_0 \in \mathbf{C}$ with $\operatorname{Re}\omega_0$, $\operatorname{Im}\omega_0 > 0$, and to study the following two problems when $h \longrightarrow 0$:

1) get a lower bound for $(h^2 P_\theta - \omega_0)^*(h^2 P_\theta - \omega_0)$,

2) estimate the number of eigenvalues smaller than $(\operatorname{Im}\omega_0 + \delta)^2$ of the same operator.

The final step in the proof will be to apply some inequalities of H. Weyl.

2. Ellipticity properties of Q.

We put

$$Q = Q_\theta = (h^2 P_\theta - \omega_0)^*(h^2 P_\theta - \omega_0). \tag{2.1}$$

The domain of $(h^2 P_\theta - \omega_0)$ is $H_0^1 \cap H^2$ (using the global parametrization (1.1)) and the same is true for $(h^2 P_\theta - \omega_0)^*$. The natural domain of Q is then $\{u \in H^4(\Gamma_\theta); u_{|\partial\mathcal{O}} = ((h^2 P_\theta - \omega_0)u)_{|\partial\mathcal{O}} = 0\}$ and we notice that the domain does not change if we replace the second boundary condition by $h^2 P_\theta u_{|\partial\mathcal{O}} = 0$. Using the next result, we can verify that Q is selfadjoint with the domain just indicated.

PROPOSITION 2.1. — *If $\theta \geq 0$ is small enough, then the problem*

$$Q_\theta u = v, \quad u_{|\partial\mathcal{O}} = v_0, \quad ((h^2 P_\theta - \omega_0)u)_{|\partial\mathcal{O}} = v_1 \tag{2.2}$$

is an elliptic boundary value problem in the classical sense.

Proof. — This property only concerns the principal symbols in the classical sense and we may assume that $h = 1$. The classical principal symbol of Q is $q_\theta = \overline{p_\theta(x,\xi)} p_\theta(x,\xi) = a_\theta(x,\xi)^2 + b_\theta(x,\xi)^2$, which is elliptic since a_θ is. (This also follows more easily from the fact that θ is small.) Recall also that $\bar{p}_\theta = p_{-\theta}$.

We now work near a boundary point and choose local coordinates in Γ_θ, so that Γ_θ becomes the half-space $x_n \geq 0$ (locally) and so that (1.5) holds. The symbol $p_\theta(x', 0; \xi', \xi_n)$ has the roots $\xi_n = \lambda_{\pm,\theta}(x', \xi')$ with $\pm \operatorname{Im} \lambda_{\pm,\theta}(x', \xi') > 0$, when $\xi' \neq 0$ is real. Notice that $\lambda_{-,\theta} \neq \bar{\lambda}_{+,\theta}$ when $\theta \neq 0$ so in this case q_θ has the four distinct roots $\lambda_{\pm,\theta}$ and $\bar{\lambda}_{\pm,\theta} = \lambda_{\mp,-\theta}$.

In order to show that (2.2) is elliptic, it suffices to show that for all real x', ξ' with $\xi' \neq 0$, there are no bounded non-trivial solutions of the problem :

$$(2.3)\quad q_\theta(x',0;\xi',D_{x_n})u = 0,\ x_n \geq 0,\ u(0) = 0,\ p_\theta(x',0;\xi',D_{x_n})u(0) = 0.$$

We also recall that the bounded solutions of the first equation in (2.3) are exponential solutions which are decaying near infinity. We first notice that if v is a bounded solution of $p_{-\theta}(x',0,\xi',D_{x_n})v = 0$, $x_n \geq 0$, with $v(0) = 0$, then $v = 0$. Since $q_\theta(x',0;\xi',D_{x_n}) = p_{-\theta}(x',0,\xi',D_{x_n})p_\theta(x',0,\xi',D_{x_n})$, we conclude then from (2.3) that $p_\theta(x',0,\xi',D_{x_n})u = 0$, $u(0) = 0$, and hence that $u = 0$. □

The next problem is to determine when (2.3) is an elliptic boundary value problem in the natural semi-classical sense.

The semi-classical principal symbol of $Q-z$ is $(\bar{p}_\theta-\bar{\omega}_0)(p_\theta-\omega_0)-z = |p_\theta-\omega_0|^2-z$. Since p_θ takes its values in the lower half plane, we have $|p_\theta-\omega_0|^2 \geq (\operatorname{Im}\omega_0)^2$. It follows that $Q-z$ is elliptic in the semi-classical sense, when $z \in \mathbf{C}\backslash[(\operatorname{Im}\omega_0)^2,\infty[$.

In the following discussion we may choose the coordinates so that (1.5) holds and we assume that $z \in \mathbf{C}\backslash[(\operatorname{Im}\omega_0)^2,\infty[$. Let

$$(p_{-\theta}(x',0;\xi',\xi_n)-\bar{\omega}_0)(p_\theta(x',0;\xi',\xi_n)-\omega_0)-z = 0$$

have the roots $\lambda_{+,1}$, $\lambda_{+,2}$ in the upper half plane and the roots $\lambda_{-,1}$, $\lambda_{-,2}$ in the lower half plane. We are interested in knowing whether

$$(2.4)\qquad ((p_{-\theta}-\bar{\omega}_0)(p_\theta-\omega_0)(x',0;\xi',D_{x_n})-z)u = v,\ x_n \geq 0$$

$$u(0) = v_0$$

$$p_\theta(x',0;x',D_{x_n})u(0) = v_1$$

is elliptic for all real ξ' in the sense that the corresponding homogenous problem ($v = 0$, $v_j = 0$) admits no non-trivial bounded solutions. In the case $\lambda_{+,1} \neq \lambda_{+,2}$ ellipticity is equivalent to the fact that $p_\theta(x',0;\xi',\lambda_{+,2})-p_\theta(x',0;\xi',\lambda_{+,1}) \neq 0$ which is clear from (1.5) since we obviously cannot have $\lambda_{+,2} = -\lambda_{+,1}$. In the case $\lambda_{+,1} = \lambda_{+,2}$ the ellipticity of (2.4) amounts to $(\partial_{\xi_n}p_\theta)(x',0;\xi',\lambda_{+,1}) \neq 0$, which holds by (1.5), since $\lambda_{+,1} \neq 0$.

Summing up, we have verified that for $z \in \mathbf{C}\backslash[(\operatorname{Im}\omega_0)^2,\infty[$,

$$(2.5)\qquad (Q_\theta-z)u = v,\quad u_{|\partial\mathcal{O}} = v_0,\quad (h^2P_\theta-\omega_0)u_{|\partial\mathcal{O}} = v_1,$$

is an elliptic boundary value problem in the sense that $|p_\theta-\omega_0|^2-z \neq 0$ everywhere and that for every $(x',\xi') \in T^*\partial\mathcal{O}$, the problem (2.4) with v, v_0, v_1 all equal to zero, has non non-trivial bounded solutions on $x_n \geq 0$.

In the next two sections we will also need to consider slightly degenerate cases, and we will then have to verify the ellipticity of our boundary value problems differently.

3. A semi-classical parametrix.

The main work will be local near some fixed point of the boundary and we shall concentrate on that case. Our constructions will be "elliptic" and in the absence of propagation phenomena we shall simplify the arguments by never writing explicitly the partitions of unity required to build a global parametrix. Choose the coordinates near a boundary point of Γ_θ, so that Γ_θ is given by $x_n \geq 0$. We let z vary in a compact set in $\mathbf{C}\backslash[(\operatorname{Im}\omega_0)^2,\infty[$. To start with, we let this set be independent of h and later we will also have to consider the slightly degenerate case when the distance from z to $\mathbf{R}$ may tend to zero, but remains $\geq h^\varepsilon$ for some fixed sufficiently small $\varepsilon > 0$.

We recall that the semi-classical principal symbol of $Q-z$ is :

$$q-z = |p_\theta-\omega_0|^2-z, \tag{3.1}$$

and that $q(x;\xi',\xi_n)-z = 0$ has the roots $\xi_n = \lambda_{+,j}$, $j = 1,2$ in the open upper half-plane and the roots $\xi_n = \lambda_{-,j}$, $j = 1,2$ in the open lower half-plane (when ξ' is real and when x_n is sufficiently small). For large $|\xi'|$ we have $|\operatorname{Im}\lambda_{\pm,j}| \sim |\xi'|$, $|\lambda_{\pm,j}| \sim |\xi'|$ and to the leading order (in ξ') $\lambda_{\pm,j}$ are homogeneous of degree 1 and independent of ω_0 and z.

If $V \subset \mathbf{R}^n_x \times \mathbf{R}^n_\xi$ is open and conic with respect to ξ for large ξ, we let $S^{m,k}(V) = S^{m,k}_1(V)$ be the space of functions $a(x,\xi;h)$ on $V\times]0,h_0]$ for some $h_0 > 0$ which are C^∞ with respect to $(x,\xi) \in V$, and such that for all $\alpha,\beta \in \mathbf{N}^n$ and all closed sets $V' \subset V$ with compact projection in $\mathbf{R}^n_x \times \mathbf{S}^{n-1}_\xi$ and with $(x,\xi) \in V' \Longrightarrow \operatorname{dist}((x,\xi),\partial V) \geq \langle\xi\rangle/\operatorname{Const}$, there is a constant $C = C_{\alpha,\beta,V'}$ such that

$$|\partial^\alpha_x\partial^\beta_\xi a| \leq C_{\alpha,\beta}h^{-m}\langle\xi\rangle^k \ \text{ for all } \ (x,\xi) \in V' . \tag{3.2}$$

Our symbols may also depend on z or some other parameters, and it is then understood that the constants in (3.2) should be independent of those additional parameters. If $a_j \in S^{m-j,k-j}$ we say that $a \sim \Sigma a_j$ (in $S^{m,k}$) if $a-\sum\limits_0^{N-1} a_j \in S^{m-N,k-N}$, for every $N \in \mathbf{N}$.

The standard elliptic construction gives us a symbol $R(x,\xi,z;h)$ of class $S^{0,-4}$ defined for x in a neighborhood of 0, and for $\xi \in \mathbf{R}^n$, such that

$$(Q-z)\#_h R \sim 1, \quad R\#_h(Q-z) \sim 1 \quad \text{in} \quad S^{0,0}, \tag{3.3}$$

where $\#_h$ indicates compositions of symbols of h-pseudodifferential operators (using the classical and not the Weyl quantization). For the corresponding operators (that we shall denote by the same letters), this means that

$$(Q-z)\circ R = I+K_1, \quad R\circ(Q-z) = I+K_2, \tag{3.4}$$

where the distribution kernels $K_j(x,y)$ of K_j satisfy :

$$|\partial_x^\alpha \partial_y^\beta K_j(x,y,z;h)| \leq C_{\alpha,\beta,N} h^N \tag{3.5}$$

for all α, β, N.

In order to treat boundary value problems, we notice that we can choose R with a holomorphic extension in the ξ_n-variable, still of class $S^{0,-4}$ in the obvious sense, to the domain $\{(x,\xi)\ ;\ x \in \operatorname{neigh}(0),\ \xi' \in \mathbf{R}^{n-1},$ $\xi_n \in \Omega_{x,\xi',z}\}$ with $\Omega_{x,\xi',z} = \{\xi_n \in \mathbf{C}\ ;\ |\xi_n| < \varepsilon^{-1}\langle\xi'\rangle,\ |\xi_n - \lambda_{j,k}| > \varepsilon\langle\xi'\rangle,$ $j = +,-,\ k = 1,2\}$, for an arbitrary but fixed $\varepsilon > 0$. Let $\gamma = \gamma(x',\xi') \subset$ $\Omega_{x,\xi',z} \cap \{\xi_n \in \mathbf{C};\ \operatorname{Im}\xi_n \geq \varepsilon\langle\xi'\rangle\}$ be a suitable simple loop, which encircles $\lambda_{+,1}$ and $\lambda_{\lambda,2}$ in the positive sense. We can then define (locally) the operators $\Pi_j : C_0^\infty(\mathbf{R}^{n-1}) \longrightarrow C^\infty(\overline{\mathbf{R}}_+^n)$, $j = 0,1$ (with $\mathbf{R}_+^n$ denoting the half-space $x_n > 0$) by :

$$\Pi_j u(x) = \int_{\mathbf{R}^{n-1}} \int_\gamma e^{ix\xi/h} R(x,\xi,z;h) \xi_n^{2j} \hat{u}(\xi') \, (d\xi'/(2\pi h)^{n-1})\, d\xi_n/(2\pi i), \tag{3.6}$$

where $\hat{u}(\xi') = \int e^{-ix'\xi'/h} u(x')\, dx'$ is the natural semi-classical Fourier transform. Notice that

$$\Pi_j u = (i/h) R(u \otimes (hD_{x_n})^{2j} \delta_{x_n=0}) + K_j u,$$

where K_j has a kernel $K_j(x,y';h)$ satisfying $\partial_x^\alpha \partial_{y'}^{\beta'} K = \mathcal{O}(h^\infty)$. (See the explanation following (3.19).)

It is easy to see that the distribution kernel $\Pi_j(x,y',z;h)$ satisfies

$$\partial_x^\alpha \partial_{y'}^\beta \Pi_j = \mathcal{O}(h^N) \tag{3.7}$$

for all α,β,N, uniformly in any compact set disjoint from $\{(x,y');\ x_n = 0,$ $x' = y'\}$. Further,

$$(Q-z)\Pi_j u = \iint_\gamma e^{ix\xi/h}((Q-z)\#_h R)\xi_n^{2j} \hat{u}(\xi')\, d\xi_n\, d\xi' (2\pi h)^{-(n-1)} (2\pi i)^{-1}, \tag{3.8}$$

and using that (3.3) extends to complex ξ_n, we get :

(3.9) $(Q-z)\Pi_j \equiv 0$ modulo an operator with kernel satisfying :
$$\partial_x^\alpha \partial_{y'}^\beta K = \mathcal{O}(h^N) \text{ for all } \alpha, \beta, N.$$

Let $\gamma_0 u(x',0) = u(x',0)$ (boundary value from $x_n \geq 0$). Let $B(x',\xi;h)$ be of class $S^{m,k}$ and polynomial in ξ. Then,

$$\begin{aligned}\gamma_0 B(x',hD)\Pi_j u(x') &= (2\pi h)^{-(n-1)}(2\pi i)^{-1} \iint_{\xi_n \in \gamma} e^{ix'\xi'/h} \\ &\quad \times (B(x',\xi)\#_h R(x,\xi,z;h))_{x_n=0}\xi_n^{2j}\, d\xi_n \hat{u}(\xi')\, d\xi' \\ &= C(x',hD_{x'};h)u(x'),\end{aligned} \tag{3.10}$$

where

$$C(x',\xi';h) = (2\pi i)^{-1} \int_{\xi_n \in \gamma} (B(x',\xi)\#_h R(x,\xi,z;h))_{x_n=0}\xi_n^{2j}\, d\xi_n \in S^{m,k+2j-3}. \tag{3.11}$$

We have $R \equiv (q-z)^{-1} \bmod S^{-1,-5}$, and if we assume that $B \equiv b \bmod S^{m-1,k-1}$, then

$$C \equiv c \bmod S^{m-1,k+2j-4}, \tag{3.12}$$

where

$$c(x',\xi') = (2\pi i)^{-1} \int_\gamma b(x',\xi)\xi_n^{2j}(q(x',0,\xi)-z)^{-1}\, d\xi_n. \tag{3.13}$$

If $\lambda_{+,1} \neq \lambda_{+,2}$, we get

$$c(x',\xi') = \sum_{\nu=1,2} b(x',\xi',\lambda_{+,\nu})(\lambda_{+,\nu})^{2j}/\partial_{\xi_n} q(x',0,\xi',\lambda_{+,\nu}). \tag{3.14}$$

If $\lambda_{+,1}(x',0,\xi',z) = \lambda_{+,2}(x',0,\xi',z)$ we write $q(x',0,\xi)-z = a(\xi_n)(\xi_n-\lambda_{+,1})^2$ (with a defined only for certain values of x',ξ',z), and we get :

$$c(x',\xi') = \partial_{\xi_n}(b(x',\xi)\xi_n^{2j}/a)_{\xi_n=\lambda_{+,1}(x',0,\xi',z)}. \tag{3.15}$$

We shall now take $B_0 = \mathrm{id}$, $B_1 = h^2 P_\theta$ for which we know that the problem $Qu = v$, $\gamma_0 B_0 u = v_0$, $\gamma_0 B_1 u = v_1$ is an elliptic boundary value problem in the semi-classical sense. (See the discussion of (2.4).) We want to find tangential h-pseudodifferential operators $A_{j,k}$ with symbols of class $S^{0,3-2j-2k}$, such that

$$(\gamma_0 B_j \Pi_k)(A_{j,k}) \equiv I \tag{3.16}$$

in the sense of 2×2-matrices of operators and modulo operators with distribution kernels K satisfying $\partial_{x'}^\alpha \partial_{y'}^\beta K = \mathcal{O}(h^N)$ for all α, β, N. Of course there will be a unique (up to " $\equiv$") such choice, if we check that $(\gamma_0 B_j \Pi_k)_{0\le j,k\le 1}$ is an elliptic matrix of h-pseudodifferential operators.

1) In the case when $\lambda_{+,1}(x',0,\xi') \neq \lambda_{+,2}(x',0,\xi')$, the principal symbol of this matrix (in the natural semi-classical sense) becomes

$$\Big(\sum_{\nu=1,2} b_j(\lambda_{+,\nu})(\lambda_{+,\nu})^{2k}/(\partial_{\xi_n} q(\lambda_{+,\nu}))\Big)_{0\le j,k\le 1}$$
$$= \begin{pmatrix} b_0(\lambda_{+,1}) & b_0(\lambda_{+,2}) \\ b_1(\lambda_{+,1}) & b_1(\lambda_{+,2}) \end{pmatrix} \begin{pmatrix} 1/\partial_{\xi_n} q(\lambda_{+,1}) & 0 \\ 0 & 1/\partial_{\xi_n} q(\lambda_{+,2}) \end{pmatrix} \begin{pmatrix} 1 & \lambda_{+,1}^2 \\ 1 & \lambda_{+,2}^2 \end{pmatrix}.$$

Here the first factor is invertible since our boundary value problem is elliptic, and clearly so are the other two.

2) We now assume that $\lambda_{+,1} = \lambda_{+,2} \underset{\text{def}}{=} \lambda$. The principal symbol of $(\gamma_0 B_j \Pi_k)_{0\le j,k\le 1}$ becomes

$$\begin{pmatrix} \partial_{\xi_n}(b_0/a)_{\xi_n=\lambda} & \partial_{\xi_n}(\xi_n^2 b_0/a)_{\xi_n=\lambda} \\ \partial_{\xi_n}(b_1/a)_{\xi_n=\lambda} & \partial_{\xi_n}(\xi_n^2 b_1/a)_{\xi_n=\lambda} \end{pmatrix}$$

which has the same determinant as

$$a^{-1}\begin{pmatrix} \partial_{\xi_n} b_0 & 2\lambda b_0 \\ \partial_{\xi_n} b_1 & 2\lambda b_1 \end{pmatrix}.$$

Since $b_0 = 1$, $\partial_{\xi_n} b_1(\lambda) \neq 0$, this determinant is non-vanishing, and this concludes the verification that we do have unique $A_{j,k}$ satisfying (3.16).

Put

$$G_0 = \Pi_0 A_{0,0} + \Pi_1 A_{1,0}$$
$$G_1 = \Pi_0 A_{0,1} + \Pi_1 A_{1,1}$$

so that

$$(Q-z)G_j \equiv 0,\ \gamma_0 B_j G_k \equiv \delta_{j,k} I,\ 0\le j,k\le 1. \tag{3.17}$$

If $u \in C_0^\infty(\overline{\mathbf{R}}^n_+)$ (with support in a neighborhood of the origin, since our whole discussion is local), we let $\tilde u$ denote the zero extension of u to $\mathbf{R}^n$, and we define $Ru = R\tilde u$. Put

$$E = R - G_0\gamma_0 B_0 R - G_1 \gamma_0 B_1 R. \tag{3.18}$$

We have $(Q-z)R \equiv I$ modulo an operator with kernel $K(x,y;h)$ in $C^\infty(\overline{\mathbf{R}}^{n2}_+)$, satisfying $\partial_x^\alpha \partial_y^\beta K = \mathcal{O}(h^N)$ for all α,β,N. Moreover, $(Q-z)G_j\gamma_0 B_j R = [\equiv 0]\circ \gamma_0 B_j R$. The kernel of this operator is of the form

$$\iint k(x,z';h) e^{i(z'-y)\xi/h} \sigma(B_j R)(z',0,\xi;h)\, d\xi'\, dz', \tag{3.19}$$

where $\partial_x^\alpha \partial_{z'}^\beta k = \mathcal{O}(h^\infty)$. For any $N \in \mathbf{N}$, we can write the symbol of $B_j R$ as $p(z',\xi;h)+r_N(z',\xi;h)$, where $r_N \in S^{-N,2j-4-N}$ and where p is holomorphic in ξ_n and of class $S^{0,2j-4}$ in the domain given by $z',\xi' \in \mathbf{R}^{n-1}$, $|\xi_n - \lambda_{\nu,\pm}(z',\xi')| \geq \varepsilon\langle\xi'\rangle$. The contribution from r_N in (3.19) is of the form $K(x,y;h)$ with $\partial_x^\alpha \partial_y^\beta K = \mathcal{O}(h^M)$ for $|\alpha|+|\beta| \leq M$, where $M = M(N)$ tends to infinity when N tends to infinity. In the contribution from p, we may change the contour of integration to the negatively oriented boundary of $\{\xi_n \in \mathbf{C}\ ;\ \operatorname{Im}\xi_n \leq 0,\ |\xi_n| \leq \varepsilon^{-1}\langle\xi'\rangle\}$ for some sufficiently small $\varepsilon > 0$. Applying repeated integrations by parts, we then see that this contribution is of the form $K(x,y;h)$, with $\partial_x^\alpha \partial_y^\beta K = \mathcal{O}(h^M)$ for all α,β,M, and finally we conclude that the kernel (3.19) has the same property. Summing up we have proved

$$(Q-z)E(z) \equiv I. \tag{3.20}$$

Using (3.17), we also prove (as above) that

$$\gamma_0 B_j E(z) \equiv 0,\ j=0,1. \tag{3.21}$$

Modifying $E(z)$ by an operator $\equiv 0$, we may even achieve that

$$\gamma_0 B_j E(z) = 0. \tag{3.21$'$}$$

We shall next extend the above constructions to the case when z varies in some compact set in $\mathbf{C}$ and satisfies the additional restriction

$$|\operatorname{Im} z| \geq h^\varepsilon, \tag{3.22}$$

for some sufficiently small but fixed $\varepsilon > 0$. To do that, we reexamine the preceding constructions. If $V \subset \mathbf{R}_x^n \times \mathbf{R}_\xi^n$ has the same properties as in the definition of $S^{m,k}(V)$, and if $0 \leq \delta < 1$, we let $S_{1-\delta}^{m,m',k}(V)$ be the space of $a(x,\xi;h) \in C^\infty(V)$ which are of class $S^{m,k}$ for ξ outside some fixed bounded set, and which satisfy :

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta,K} h^{-m'-\delta(|\alpha|+|\beta|)}, \tag{3.23}$$

for $(x,\xi) \in K$, for all $\alpha,\beta \in \mathbf{N}^n$ and all compact subsets K of V. We may remark that these spaces are semi-classical analogues of the Hörmander spaces $S_{1-\delta,\delta}^m$, and we have a corresponding analogy on the quantized level.

Under the assumption (3.22) with $0 \leq \varepsilon < 1/2$, we can still construct R as in (3.2), now of class $S_{1-\varepsilon}^{0,\varepsilon,-4}$. Again we have (3.3), (3.4).

We next look at the holomorphic extension of R with respect to the ξ_n-variable. When $|\xi'| \geq \text{Const}$, this works as before. When $|\xi'| \leq \text{Const}$, we first observe that (3.22) implies

$$|\operatorname{Im} \lambda_{\pm,j}(x,\xi',z)| \geq h^\varepsilon/C, \tag{3.24}$$

and that R has a holomorphic extension with respect to ξ_n which is of class $S^{0,\varepsilon,-4}_{1-\varepsilon}$. It is defined for ξ_n as after (3.5) when $|\xi'| \geq$ Const and for $|\xi'| \leq$ Const in a region :

$$\{|\xi_n| < C,\ \mathrm{Im}\,\xi_n \geq 0\}$$
$$\cap(\{\mathrm{Im}\,\xi_n < h^\varepsilon/C\} \cup \{\mathrm{Im}\,\xi_n \geq h^\varepsilon/C \text{ and } |\xi_n - \lambda_{+,j}| > C^{-1},\ j=1,2\})$$

for an arbitrarily large but fixed C. Choosing γ suitably for $|\xi'| \leq$ Const (and as before when $|\xi'| \geq$ Const), we can still define Π_j by (3.6), and we still have (3.7--3.11), now with C in (3.11) of class $S^{m,m+\varepsilon,k+2j-3}_{1-\varepsilon}$. Instead of (3.12), we have :

$$C \equiv c \operatorname{mod} S^{m-1,m-1+2\varepsilon,k+2j-4}_{1-\varepsilon}, \tag{3.12'}$$

with c given by (3.13).

For the study of the ellipticity at the boundary, we shall follow a method which is slightly different from the one of section 2. Let $p(x',\xi',D_{x_n}) = p_\theta(x',0,\xi',D_{x_n}) - \omega_0$. We first give another proof of the absence of L^2-solutions u to the homogeneous version of (2.4), when $\mathrm{Im}\, z \neq 0$. Recall that $p(D_{x_n}) = \alpha D^2_{x_n} + \beta$, $\alpha, \beta \in \mathbf{C}$, $\alpha \neq 0$ (by (1.5)). In general for $u_1, u_2 \in C^\infty_0([0,+\infty[)$:

$$(p(D)u_1|u_2) = \alpha(\partial u_1 \bar{u}_2 - u_1 \partial \bar{u}_2)(0) + (u_1|p(D)^* u_2), \tag{3.25}$$

where $(\cdot|\cdot)$ is the standard L^2 inner product on the half line. If $u(0) = 0$, $p(D)u(0) = 0$, we have :

$$((p^*p - z)u|u) = -z\|u\|^2 + \|pu\|^2, \tag{3.26}$$

and taking imaginary parts, we get :

$$|\mathrm{Im}\, z|\ \|u\| \leq \|(p^*p - z)u\|, \tag{3.27}$$

which easily implies :

$$\|u\|_4 \leq (C/\mathrm{Im}\, z|)\ \|(p^*p - z)u\| \quad (\text{when } u(0) = pu(0) = 0), \tag{3.28}$$

where $\|\cdot\|_4$ is the norm in the standard Sobolev space $H^4([0,+\infty[)$.

Now consider $u \in C^\infty_0([0,+\infty[)$ satisfying :

$$(p^*p - z)u = v, \quad u(0) = v_0, \quad pu(0) = v_1, \tag{3.29}$$

which essentially is the same as (2.4). Let $H_0, H_1 \in C^\infty_0([0,+\infty[)$ be fixed functions with $H_0(0) = 1$, $pH_0(0) = 0$, $H_1(0) = 0$, $pH_1(0) = 1$. Put $\tilde{u} = u - v_0 H_0 - v_1 H_1$, so that $(p^*p - z)\tilde{u} = v - (p^*p - z)(v_0 H_0 + v_1 H_1) \underset{\text{def}}{=} \tilde{v}$, with $\|\tilde{v}\| \leq \|v\| + C(|v_0| + |v_1|)$. Applying (3.28), we get

$$\|u\|_4 \leq (C/\mathrm{Im}\, z|)(\|v\| + |v_0| + |v_1|). \tag{3.30}$$

Let $(p^*p-z)^{-1}$ denote the inverse for temperate distributions on the whole x_n-axis. For $j=0,1,2$, the distribution $u_j=(p^*p-z)^{-1}(D^j(\delta))$, satisfies :

$$u_{j\,|x_n>0}=(2\pi)^{-1}\int_\gamma e^{ix_n\xi_n}(q-z)^{-1}\xi_n^j\,d\xi_n, \tag{3.31}$$

with the contour γ as in the definition of Π_j. We denote this restriction by u_j. Put

$$\begin{aligned} b_{0,0}&=u_0(0), & b_{0,1}&=u_2(0)\\ b_{1,0}&=pu_0(0), & b_{1,1}&=pu_2(0)\end{aligned} \tag{3.32}$$

(with the natural boundary value from the right). Then, $B=(b_{j,k})=\mathcal{O}(|\operatorname{Im}z|^{-1})$ (when x',ξ' are fixed), and we want to estimate its inverse. Let $a_0,a_1\in\mathbf{C}$, and put $u=a_0u_0+a_1u_2$, so that

$$(p^*p-z)u=0,\ \ u(0)=b_{0,0}a_0+b_{0,1}a_1,\ \ pu(0)=b_{1,0}a_0+b_{1,1}a_1. \tag{3.33}$$

From (3.30), we get :

$$\|u\|_4\le(C/\operatorname{Im}z|)\|Ba\|_{\mathbf{C}^2},\quad\text{with } a=(a_1,a_2). \tag{3.34}$$

On the other hand, $u=\tilde u_{|x_n>0}$, where $\tilde u=(p^*p-z)^{-1}(a_0\delta+a_1D_{x_n}^2\delta)$, so $(p^*p-z)\tilde u=a_0\delta+a_1D_{x_n}^2\delta$, and using that $\tilde u$ is even :

$$\|a_0\delta+a_1D_{x_n}^2\delta\|_{-4}\le C\|\tilde u\|\le 2^{1/2}C\|u\|\le 2^{1/2}C\|u\|_4. \tag{3.35}$$

However, $\|a_0\delta+a_1D_{x_n}^2\delta\|_{-4}$ is of the same order of magnitude as $\|a\|_{\mathbf{C}^2}$, so combining this with (3.34), we get with a new constant C :

$$\|B^{-1}\|\le C/|\operatorname{Im}z|. \tag{3.36}$$

We also recall that,

$$\|B\|\le C/|\operatorname{Im}z|. \tag{3.37}$$

The symbol of $\gamma_0B_j\Pi_k$ is in $S_{1-\varepsilon}^{0,\varepsilon,-3+2j+2k}$ and is congruent to $b_{j,k}\bmod S_{1-\varepsilon}^{-1,2\varepsilon-1,-4+2j+2k}$, where the new $b_{j,k}$ is given by

$$b_{j,k}=(2\pi i)^{-1}\int_\gamma(p_\theta(x',0,\xi)-\omega_0)^j\xi_n^{2k}/(q(x',0,\xi)-z)\,d\xi_n, \tag{3.38}$$

and differs from the earlier one by a constant factor (independent of z,j,k,x',ξ'). Using (3.36), (3.37), we see that if $(a_{j,k})=(b_{j,k})^{-1}$, then $a_{j,k}\in S_{1-\varepsilon}^{0,\varepsilon,-3-2j-2k}$. We then get $A_{j,k}$ satisfying (3.16) with

$$A_{j,k}\in S_{1-\varepsilon}^{0,\varepsilon,-3-2j-2k},\quad A_{j,k}-a_{j,k}\in S_{1-3\varepsilon}^{0,7\varepsilon-1,3-2j-2k-1}, \tag{3.39}$$

provided that $\varepsilon<1/6$. We define G_0, G_1 as before. Then (3.20), (3.21) hold, and again we may modify E by an operator $\equiv 0$ in order to have (3.21′).

4. The trace of $f(Q)$.

The function f which will be applied to the operator Q will belong to $C_0^\infty(]-\infty, (\operatorname{Im}\omega_0+\delta)^2[)$ for some sufficiently small $\delta > 0$. Before starting the actual computations, we shall localize the problem in such a way that certain operators will be of trace class.

Let $\operatorname{Re} z \leq (\operatorname{Im}\omega_0+\delta)^2$, and z in a fixed compact set. If $\chi \in C_0^\infty(\Gamma_\theta)$ is equal to 1 in a neighborhood of the boundary, and $\delta > 0$ is sufficiently small, then $Q-z$ is a semiclassical elliptic operator in a neighborhood of the support of $1-\chi$, in view of (1.4) and the properties of a_θ, b_θ. We can therefore find $E_0(z)$ depending holomorphically on z and such that

$$(Q-z)E_0(z) = I-\chi+K_0(z), \quad \gamma_0 B_j E_0(z) = 0, \quad j = 1,2, \tag{4.1}$$

when h is sufficiently small, and where the trace class norm of K_0 is $\mathcal{O}(h^\infty)$. Near the boundary we can further construct a symbol R of class $S^{0,-4}$ as in the preceding section, provided that we let $|\xi'| \geq \text{Const} > 0$. We then have the corresponding $E_1(z)$, depending holomorphically on z, such that :

$$(Q-z)E_1(z) = I-\chi_1(x', hD_{x'})+K_1(z), \quad \gamma_0 B_j E_1(z) = 0, \tag{4.2}$$

where $\chi_1(x',\xi')$ vanishes for large ξ' and the trace class norm $[K_1(z)]$ of $K_1(z)$ is $\mathcal{O}(h^N)$ for every N. Patching E_0 and E_1 together by means of a partition of unity, we get $E_2(z)$ depending holomorphically on z, such that :

$$(Q-z)E_2(z) = I-\chi_2(x, hD_{x'})+K_2(z), \quad \gamma_0 B_j E_2(z) = 0, \tag{4.3}$$

where χ_2 is supported in a region with x_n small and with ξ' bounded. Moreover, $[K_2] = \mathcal{O}(h^\infty)$. If we add the assumption that $\operatorname{Im} z \neq 0$, we get :

$$(z-Q)^{-1} = (z-Q)^{-1}\chi_2(x, hD_{x'}) - (z-Q)^{-1}K_2(z) - E_2(z). \tag{4.4}$$

Let $f \in C_0^\infty(]-\infty, (\operatorname{Im}\omega_0+\delta)^2[)$, and let $\tilde{f} \in C_0^\infty$ $(\{z \in \mathbf{C};\ \operatorname{Re} z < (\operatorname{Im}\omega_0+\delta)^2\})$ be an almost analytic extension. Using the formula

$$f(Q) = -\pi^{-1} \int \bar{\partial}\tilde{f}(z)(z-Q)^{-1}L(dz), \tag{4.5}$$

with $L(dz) = d(\operatorname{Re} z)d(\operatorname{Im} z)$, and the fact that E_2 is holomorphic, we get :
(4.6)

$$f(Q) = -\pi^{-1} \int \bar{\partial}\tilde{f}(z)(z-Q)^{-1}\chi_2(x, hD_{x'})L(dz) + K_3, \quad [K_3] = \mathcal{O}(h^\infty).$$

We shall see that $(z-Q)^{-1}\chi_2(x, hD_{x'})$ is of trace class and give some (weak) estimate on its trace class norm. Let $z_0 \in \mathbf{C}$ satisfy

$\operatorname{Im} z_0 \neq 0$, $\operatorname{Re} z_0 < (\operatorname{Im}\omega_0+\delta)^2$, and consider first $(z_0-Q)^{-1}\chi_2(x, hD_{x'})$. Let $E = E(z_0)$ be the corresponding parametrix, defined by (3.18), so that $(Q-z_0)E = I+K_4$ where the trace class norm of K_4 is $\mathcal{O}(h^\infty)$. Then $(z_0-Q)^{-1} = -E+K_5$, where the trace class norm of $K_5 = (Q-z_0)^{-1}K_4$ is $\mathcal{O}(h^\infty)$. If follows that $(z_0-Q)^{-1}\chi_2(x, hD_{x'}) = -E(z_0)\chi_2(x, hD_{x'})+K_6$ with $[K_6] = \mathcal{O}(h^\infty)$. In order to study $E(z_0)\chi_2$, we use (3.18). The first contribution is then $R(x, hD_x; h)\chi_2(x, hD_{x'})$. This composition is an h-pseudodifferential operator with the leading symbol $(q-z_0)^{-1}\chi_2(x, \xi')$, where we now give up gains in powers of $\langle\xi\rangle$ in the calculus and consider symbols $a(x,\xi)$ satisfying $\partial_x^\alpha\partial_\xi^\beta a(x,\xi) = \mathcal{O}(\langle\xi\rangle^m\langle\xi'\rangle^k)$ for suitable m, k independent of α, β. Using a criterion of D. Robert [R] (Théorème II-49), we see that $R\chi_2$ is of trace class and that the corresponding norm is $\mathcal{O}(h^{-n})$. The other two contributions to $-E(z_0)\chi_2(x, hD_{x'})$ are of the form, $G_j\gamma_0 B_j R\chi_2 = (\Pi_0\alpha_{0,j}+\Pi_1 A_{1,j})\gamma_0 B_j R\chi_2$. Here we recall that $A_{0,1}$, $A_{1,j}$ are h-pseudodifferential operators on the boundary. From (3.6) it follows that Π_j has a distribution kernel of the form :

$$\Pi_j(x, y') = (2\pi h)^{-(n-1)} \int e^{i(x'-y')\xi'/h} r_j(x, y', \xi')\, d\xi', \tag{4.7}$$

with

$$|\partial_{x'}^{\alpha'}\partial_{y'}^{\beta'}\partial_{\xi'}^{\gamma'}(hD_{x_n})^k r_j(x, y', \xi')| \leq C_{\alpha',\beta',\gamma',k}\langle\xi'\rangle^{-3+2j+k-|\gamma'|}e^{-x_n\langle\xi'\rangle/Ch}. \tag{4.8}$$

Similarly, $\gamma_0 B_j R$ has a distribution kernel of the form

$$k_j(x', y) = (2\pi h)^{-n} \int e^{i(x'-y')\xi'/h} b_j(x', y, \xi')\, d\xi', \tag{4.9}$$

up to an error which can be estimated as (3.19), with

$$|\partial_{x'}^{\alpha'}\partial_{y'}^{\beta'}\partial_{\xi'}^{\gamma'}(hD_{y_n})^k b_j(x', y, \xi')| \leq C_{\alpha',\beta',\gamma',k}\langle\xi'\rangle^{-3+2j+k-|\gamma'|}e^{-y_n\langle\xi'\rangle/Ch}. \tag{4.10}$$

If we view all our operators as h-pseudodifferential operators in x', we conclude that $G_j\gamma_0 B_j R\chi_2$ has a distribution kernel of the form

$$\ell_j(x, y; h) = (2\pi h)^{-n} \int e^{i(x'-y')\xi'/h} c_j(x, y, \xi')\, d\xi', \tag{4.11}$$

where

$$|\partial_{x'}^{\alpha'}\partial_{y'}^{\beta'}\partial_{\xi'}^{\gamma'}(hD_{x_n})^k(hD_{y_n})^\ell c_j(x, y, \xi')| \leq C_{\alpha',\beta',\gamma',k,\ell,N}\langle\xi'\rangle^{-N}\exp(-(x_n+y_n)/Ch)+\mathcal{O}(h^\infty). \tag{4.12}$$

It follows that ℓ_j is smooth with

$$|\partial_{x'}^{\alpha'}\partial_{y'}^{\beta'}(hD_{x_n})^k(hD_{y_n})^\ell \ell_j(x,y;h)| \leq C_{\alpha',k,\beta',\ell}h^{-n}e^{-(x_n+y_n)/Ch}+\mathcal{O}(h^\infty). \tag{4.13}$$

We conclude that $G_j\gamma_0 B_j R\chi_2$ is of trace class and that

$$[G_j\gamma_0 B_j R\chi_2] = \mathcal{O}(h^{-N_0}), \tag{4.14}$$

for some fixed value of N_0.

We have then proved that $(z_0-Q)^{-1}\chi_0$ is of trace class and that the corresponding norm is $\mathcal{O}(h^{-N_0})$. Returning to (4.6), we write

$$(z-Q)^{-1}\chi_2(x,hD_{x'})=(I+(z_0-z)(z-Q)^{-1})(z_0-Q)^{-1}\chi_2(x,hD_{x'}), \tag{4.15}$$

so for z in some fixed compact set in $\mathbf{C}$ and with $\operatorname{Im} z \neq 0$, we conclude that $(z-Q)^{-1}\chi_2$ is of trace class and that

$$[(z-Q)^{-1}\chi_2(x,hD_{x'})] \leq C|\operatorname{Im} z|^{-1}h^{-N_0}. \tag{4.16}$$

Since $\bar\partial\tilde f(z) = \mathcal{O}(|\operatorname{Im} z|^N)$ for every N it follows from this estimate and (4.6) that for every $\varepsilon > 0$:

$$f(Q) = -\pi^{-1}\int_{|\operatorname{Im} z|\geq h^\varepsilon} \bar\partial\tilde f(z)(z-Q)^{-1}\chi_2(x,hD_{x'})L(dz)+K_\varepsilon, \tag{4.17}$$

with $[K_\varepsilon] = \mathcal{O}(h^\infty)$. We now recall that $f\in C_0^\infty(]-\infty,(\operatorname{Im}\omega_0+\delta)^2[)$, so we may further restrict z to $\operatorname{Re} z \leq (\operatorname{Im}\omega_0+\delta)^2$ in (4.17). We can then apply the constructions of the end of section 3. Let $E(z)$ be the parametrix of $(z-Q)$. As after (4.6), we see that $(z-Q)^{-1} = E(z)+K$ with $[K] = \mathcal{O}(h^\infty)$, when $|\operatorname{Im} z| \geq h^\varepsilon$, $\operatorname{Re} z \leq (\operatorname{Im}\omega_0+\delta)^2$, $z\in$ compact, and using this in (4.17), we get :

$$f(Q)=-\pi^{-1}\int_{|\operatorname{Im} z|\geq h^\varepsilon} \bar\partial\tilde f(z)E(z)\chi_2(x,hD_{x'})L(dz)+K, [K]=\mathcal{O}(h^\infty). \tag{4.18}$$

Here we substitute (3.18), where now $R(x,\xi,z;h) \equiv (q(x,\xi)-z)^{-1} \bmod S^{-1,2\varepsilon-1,-4}_{1-\varepsilon}$. It follows that, $\pi^{-1}\int_{|\operatorname{Im} z|\geq h^\varepsilon}\bar\partial\tilde f(z)R(z)L(dz)$ is an h-pseudo-differential operator with symbol $f(q(x,\xi)) \bmod S^{-1,2\varepsilon-1,-4}_{1-\varepsilon}$, so the composition of this operator with $\chi_2(x,hD_{x'})$ to the right is of trace class and the trace of the composition is

$$(2\pi h)^{-n}\iint f(q(x,\xi))\,dx\,d\xi+\mathcal{O}(h^{1-2\varepsilon-n}). \tag{4.19}$$

Here we also use that $\chi_2(x,\xi') = 1$ on the support of $f(q(x,\xi))$.

The slightly degenerate calculus in the second half of section 3 now shows that (4.14) remains valid with z_0 replaced by z and that the distribution kernel $\ell_j(x,y;h)$ of $G_j\gamma_0 B_j R\chi_2$ satisfies

$$|\ell_j(x,y;h)| \leq Ce^{-|\operatorname{Im} z|(x_n+y_n)/Ch} h^{-n-3\varepsilon} + \mathcal{O}(h^\infty). \tag{4.20}$$

In particular,

$$\operatorname{tr} G_j\gamma_0 B_j R\chi_2 = \mathcal{O}(h^{1-n-4\varepsilon}). \tag{4.21}$$

Returning to (4.18) we get :

$$\operatorname{tr} f(Q) = (2\pi h)^{-n} \iint f(q(x,\xi))\, dx\, d\xi + \mathcal{O}(h^{-n+1-4\varepsilon}). \tag{4.22}$$

A minor additional effort would certainly give a complete asymptotic expansion for $\operatorname{tr} f(Q)$ in powers of h, and in particular, one would be able to improve the error in (4.22) to $\mathcal{O}(h^{-n+1})$.

5. End of the proof.

For $\delta > 0$ small enough, let $f \in C_0^\infty(]-\infty, (\operatorname{Im}\omega_0+2\delta)^2[;[0,1])$ be equal to 1 on $[0,(\operatorname{Im}\omega_0+\delta)^2]$, and consider $f(q(x,\xi)) = f(|p_\theta(x,\xi)-\omega_0|^2)$. On the support of $f\circ q$, we have $|p_\theta(x,\xi)-\omega_0| \leq \operatorname{Im}\omega_0+2\delta$, which implies that with $p_\theta = a_\theta(x,\xi)-ib_\theta(x,\xi)$:

$$b_\theta(x,\xi) \leq C\delta,\ |a_\theta - \operatorname{Re}\omega_0| \leq ((\operatorname{Im}\omega_0+2\delta)^2-(\operatorname{Im}\omega_0)^2)^{1/2} \sim \delta^{1/2}. \tag{5.1}$$

In section 1, we showed that $a_\theta \sim \xi^2$, $b_\theta \geq C^{-1}(\xi_n^2+d(x)\xi^2)$, so we see that the volume of this set is $\mathcal{O}(\delta^2)$. From (4.22), it then follows that the number $M(\delta;h)$ of eigenvalues of $|h^2P_\theta-\omega_0|$ smaller than $\operatorname{Im}\omega_0+\delta$ satisfies :

$$M(\delta;h) \leq C\delta^2 h^{-n},\ h \leq h(\delta), \tag{5.2}$$

for some $h(\delta) > 0$. Let $0 \leq \mu_1 \leq \mu_2 \leq \cdots$ be an enumeration of these values (followed by an infinite repetition of $\inf \sigma(|h^2P_\theta-\omega_0|)$ in the case there are only finitely many eigenvalues).

From the fact that $(h^2P_\theta-\omega_0)^*(h^2P_\theta-\omega_0)-z$ is given by an elliptic boundary value problem when $z < (\operatorname{Im}\omega_0)^2$, it follows that

$$\mu_1 \geq \operatorname{Im}\omega_0 - o(1),\ h \to 0. \tag{5.3}$$

Let $\lambda_1,\ldots,\lambda_N$, $N = N(\delta;h)$ be the eigenvalues of $h^2P_\theta-\omega_0$ with $0 \leq |\lambda_1| \leq \cdots \leq |\lambda_N| \leq \operatorname{Im}\omega_0+\delta$. We claim that

$$N(\delta;h) \leq C\delta^2 h^{-n},\ h \leq h(\delta), \tag{5.4}$$

when $h(\delta) > 0$ is sufficiently small.

If $N(\delta; h) \leq M(2\delta; h)$, there is nothing to prove. If not, we use the Weyl inequality,

$$\mu_1 \cdots \mu_N \leq |\lambda_1| \cdots |\lambda_N|, \tag{5.5}$$

with $N = N(\delta; h)$, which implies (with $M = M(2\delta; h)$) :

$$\mu_1^M (\operatorname{Im} \omega_0 + 2\delta)^{N-M} \leq (\operatorname{Im} \omega_0 + \delta)^N, \tag{5.6}$$

or equivalently,

$$[(\operatorname{Im} \omega_0 + 2\delta)/(\operatorname{Im} \omega_0 + \delta_)]^N \leq [(\operatorname{Im} \omega_0 + 2\delta)/\mu_1]^M. \tag{5.7}$$

For some constant $C > 0$ we have

$$(\operatorname{Im} \omega_0 + 2\delta)/(\operatorname{Im} \omega_0 + \delta) \geq 1 + \delta/C, \ \ (\operatorname{Im} \omega_0 + 2\delta)/\mu_1 \leq 1 + C\delta,$$

when δ, h are sufficiently small, and hence $(1+(\delta/C))^N \leq (1+C\delta)^M$, so $N \leq M(\log(1+C\delta))/(\log(1+\delta/C)) \leq \widetilde{C}M$, and again we get (5.4).

By a scaling argument, it is then clear how to complete the proof as in [SZ1], [SZ2].

Appendix.

We outline here the proof of (1.9). Let P be an $m{:}th$ order differential operator with holomorphic coefficients on some open set in $\mathbf{C}^n$. Let Γ be a totally real connected smooth submanifold of maximal dimension, such that P is defined in some neighborhood of Γ. Let P_Γ be the natural restriction of P to Γ (see [SZ1]), and assume that P_Γ is elliptic. We then have the following strong uniqueness property :

(A.1) If $u \in \mathcal{D}'(\Gamma)$, $P_\Gamma u = 0$ on Γ and $u(x)$ is 0 in a neighborhood of some point $x_0 \in \Gamma$, then u is identically 0.

In fact, if u is as above, then by Lemma 3.1 of [SZ1], u is the restriction of a function $\tilde{u}$ which is holomorphic in some neighborhood of Γ. On the other hand $\tilde{u}$ and all its derivatives vanish at x_0, and hence be the unique continuation property for holomorphic functions, we have $\tilde{u} = 0$ everywhere in a neighborhood of Γ.

Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Gamma$ be open sets. Then

(A.2) There exists a constant $C > 0$ such that for all $u \in H^m(\Omega_2)$:

$$\|u\|_{H^m(\Omega_1)} \leq C(\|Pu\|_{H^0(\Omega_2)} + \|u\|_{H^0(\Omega_2 \setminus \Omega_0)}).$$

In order to prove this we assume the contrary. Then there is a sequence $u_\nu \in H^m(\Omega_2)$ with

$$\|Pu_\nu\|_{H^0(\Omega_2)} + \|u_\nu\|_{H^0(\Omega_2\setminus\Omega_0)} \longrightarrow 0, \ \nu \to \infty,$$

$\|u_\nu\|_{H^m(\Omega_1)} = 1$. Let E be a properly supported parametrix of P in Ω_2. Then $u_{\nu|\Omega_1} = EPu_{\nu|\Omega_1} + Ku_{\nu|\Omega_1}$, where K has C^∞ kernel and is properly supported. Since $\{u_\nu\}$ is bounded in $L^2(\Omega_2)$ and $L^2(\Omega_2) \ni u \mapsto Ku_{|\Omega_1} \in H^m(\Omega_1)$ is compact, and since $EPu_{\nu|\Omega_1} \to 0$ in $H^m(\Omega_1)$, we obtain after passing to a subsequence, that $u_\nu \to u_0$ in $H^m(\Omega_1)$, where $\|u_0\|_{H^m(\Omega_1)} = 1$. Also u_0 vanishes in $\Omega_1\setminus\Omega_0$ and $Pu_0 = 0$. This contradicts the uniqueness property (A.1).

Our final remark is that if P, Γ, Ω_0, Ω_1, Ω_2 depend continuously on an additional parameter μ which varies in some compact set, then we still have (A.2) with a constant C which is independent of μ. This is proved in the same way be taking also sequences in the set of domains and operators. The estimate (1.9) follows from this observation.

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