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# UNIFORM ESTIMATES FOR THE CAUCHY-RIEMANN EQUATION ON q-CONVEX WEDGES

by C. LAURENT-THIÉBAUT & J. LEITERER

## 0. Introduction.

With this paper we begin a systematic study of the tangential Cauchy-Riemann operator $\overline{\partial}_b$ on real submanifolds of complex manifolds from the viewpoint of uniform estimates and by means of integral formulas. This method was first applied by Henkin and Airapetjan/Henkin to $\overline{\partial}_b$ (see [He1], [He2], [He3], [He4], [AiHe]). In particular, in [AiHe] important ideas are described in greater detail, which are basic for our study.

Concerning other methods in the theory of $\overline{\partial}_b$ we refer to the survey of Henkin [He4] and the recent papers of Nacinovich [N1], [N2] and Trèves [T].

We follow the classical concept first used by Andreotti and Hill (see [AnHi1], [AnHi2]) which consists of two steps :

I. Representation of CR forms as the jump of $\overline{\partial}$-closed forms in certain auxiliary domains (wedges).

II. Solving the $\overline{\partial}$-equation in those domains.

To get uniform estimates, both steps must be done with corresponding estimates. That, under certain strict convexity, resp. concavity conditions, this is possible was first announced by Henkin (see Theorem 2 in [He2] and Theorem 8.15 in [He4]).

*Key words* : Cauchy-Riemann equations – Integral formula with uniform estimates – Piecewise smooth $q$-convex domains – Tangential Cauchy-Riemann equations.
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In the present paper we prove the result which is necessary to do step II in the $q$-convex case. To state this result we use the following

**0.1.** DEFINITION (see (8.7) in [He4]). — *Let $D \subset\subset X$ be a domain in an $n$-dimensional complex manifold $X$. $D$ will be called a* strictly $q$-convex $C^2$ intersection, $0 \leq q \leq n-1$, *if there exists a finite number of real $C^2$ functions $\rho_1, \dots, \rho_N$ in a neighborhood $U_{\overline{D}}$ of $\overline{D}$ such that*

$$D = \{z \in U_{\overline{D}} : \rho_j(z) < 0 \ \text{ for } \ 1 \leq j \leq N\}$$

*and the following condition is fulfilled : if $z \in \partial D$ and $1 \leq k_1 < \dots < k_\ell \leq N$ with $\rho_{k_1}(z) = \dots = \rho_{k_\ell}(z) = 0$, then*

$$d\rho_{k_1}(z) \wedge \dots \wedge d\rho_{k_\ell}(z) \neq 0$$

*and, for all $\lambda_1, \dots, \lambda_\ell \geq 0$ with $\lambda_1 + \dots + \lambda_\ell = 1$, the Levi form at $z$ of the function*

$$\lambda_1 \rho_{k_1} + \dots + \lambda_\ell \rho_{k_\ell}$$

*has at least $q+1$ positive eigenvalues. (See Lemma 2.2 for a weaker formulation of this condition.)*

The main result of the present paper is the following :

**0.2.** THEOREM. — *Let $E$ be a holomorphic vector bundle over an $n$-dimensional complex manifold $X$, and let $D \subset\subset X$ be a strictly $q$-convex $C^2$ intersection, $0 \leq q \leq n-1$. Moreover suppose that $D$ is completely $q$-convex,*

*i.e.* the following condition is fulfilled(*): there exists a real $C^2$ function $\varphi$ on $D$ whose Levi form has at least $(q+1)$ positive eigenvalues at each point in $D$ and such that

$$\{z \in D : \varphi(z) < C\} \subset\subset D \ \text{ for all } \ C > 0.$$

Denote by $B^{\beta}_{n,r}(D, E)$, $\beta \geq 0$, $r = 0, 1, \dots, n$, the Banach space of $E$-valued continuous $(n,r)$-forms $f$ on $D$ such that

$$\sup_{z \in D} \|f(z)\| \left[\operatorname{dist}(z, \partial D)\right]^{\beta} < \infty,$$

and denote by $C^{\alpha}_{n,r}(\overline{D}, E), 0 \leq \alpha \leq 1, r = 0, 1, \dots, n$, the Banach space of $E$-valued $(n,r)$-forms which are Hölder continuous with exponent $\alpha$ on $\overline{D}$

(*) This is automatically the case if $X$ is Stein (*cf.* , e.g., Theorem 5.3 in [HeLe2] and the proof of Lemma 9.2 of the present paper).

(*cf.* sect. 1.16). Further, let $\ker d$ be the space of all closed $E$-valued forms on $D$.

*Then :*

i) *If* $0 \leq \beta < 1/2$, *then there exist linear operators*

$$T_r : B^{\beta}_{n,r}(D,E) \cap \ker d \longrightarrow \bigcap_{0<\varepsilon\leq 1/2-\beta} C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D},E),$$

$n-q \leq r \leq n$, *which are compact as operators from* $B^{\beta}_{n,r}(D,E) \cap \ker d$ *to each* $C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D},E), 0 < \varepsilon \leq 1/2-\beta$, *and such that*

$$dT_r f = f$$

*for all* $n-q \leq r \leq n$ *and* $f \in B^{\beta}_{n,r}(D,E) \cap \ker d$.

ii) *If* $1/2 \leq \beta < 1$, *then there exist linear operators*

$$T_r : B^{\beta}_{n,r}(D,E) \cap \ker d \longrightarrow \bigcap_{\varepsilon>0} B^{\beta+\varepsilon-1/2}_{n,r-1}(D,E),$$

$n-q \leq r \leq n$, *which are compact as operators from* $B^{\beta}_{n,r}(D,E) \cap \ker d$ *to each* $B^{\beta+\varepsilon-1/2}_{n,r-1}(D,E)$, $\varepsilon > 0$, *and such that*

$$dT_r f = f$$

*for all* $n-q \leq r \leq n$ *and* $f \in B^{\beta}_{n,r}(D,E) \cap \ker d$.

For $q = n-1$ (*i.e.* the pseudoconvex case) and $\beta = 0$, this theorem was proved by Range and Siu (see [RS]), and for arbitrary $q$, but smooth $\partial D$ and $\beta = 0$, it was proved by W. Fischer and Lieb (see [FiLi]). For $\beta \neq 0$, but smooth $\partial D$ and $q = n-1$, Theorem 0.2 was obtained in [LiR] (see also [BFi]). Passing from these more special results to the case when $1 \leq q \leq n-2$ and the number $N$ of smooth pieces of $\partial D$ is greater than one, one meets the following new problem : the Leray map (see sect. 1.11) now depends *non-linearly* on $\lambda$, whereas in the piecewise strictly pseudoconvex case considered in [RS] this dependence is linear and therefore can be eliminated by explicit integration over $\lambda$. In the literature this problem was first discussed by Airapetjan and Henkin in [AiHe]. They observe that in the case of non-linear dependence of $\lambda$ "the explicit integration with respect to $\lambda \dots$ becomes a rather difficult problem" (see the beginning of sect. 1.4 in [AiHe]), and then they present a very important idea : if the Leray map depends in a certain special rational form on $\lambda$ (see formula (1.4.1) in [AiHe]), then explicit integration is also possible, using a formula (see Proposition 1.4.1 in [AiHe]) which is called by them *generalized Fantappie-Feynman formula*.

Further, in the survey article [He4] of Henkin, one can find the statement (see Theorem 8.12 d) in [He4]) that from the papers [He2] and [AiHe] follows the following result, which is an important special case of our Theorem 0.2 : *if $D$ is as in Theorem 0.2, then, for any $f \in C^0_{n,r}(\overline{D}, E) \cap \ker d$, the equation $du = f$ has a continuous solution on $\overline{D}$.* From the text following Proposition 6.2.1 in [AiHe] and from personal conversations with Henkin we understood that, writing this, Henkin had the following idea for the proof :

1) Construct a Leray map whose dependence on $\lambda$ is piecewise of the special rational form mentioned above.

2) Explicit integration with respect to $\lambda$ by means of the generalized Fantappie-Feynman formula.

3) Estimation of the integrals over the moduli of the obtained integrands.

After certain attempts to realize this program of Henkin, we understood that this is not so easy. Therefore we modified Henkin's idea as follows : we do not try to choose a Leray map of a special form – we take the first one which one obtains by generalizing the constructions of Range and Siu [RS] and W. Fischer and Lieb [FiLi], and then we prove that, in certain infinitesimal sense, this Leray map is of the mentioned special rational form (Lemma 7.4). Though now explicit integration with respect to $\lambda$ is impossible, this enables us to get a suitable estimate for this integration (Theorem 7.2). The key to this estimate is an auxiliary estimate (Theorem 6.1) which is close to the generalized Fantappie-Feynman formula of Airapetjan and Henkin.

This article is organized as follows : to prove Theorem 0.2 we use an integral operator for certain special domains which we call *local $q$-convex domains* and which will be defined in sect. 2. The construction of this operator is given in sects. 3-4 by now well-known ideas. First, in sect. 3, we construct the Leray map mentioned above. Then in sect. 4, we replace the boundary integrals in the corresponding Cauchy-Fantappie formula (see sect. 1.13) by integrals over some submanifolds of the domain. This is necessary to include the case of unbounded forms ($\beta > 0$) which we need for the intended applications to the $\overline{\partial}_b$-operator. Note that this construction of sect. 4 is similar to the construction of J. Michel in his paper [M] on $C^k$-estimates for the $\overline{\partial}$-equation on piecewise strictly pseudoconvex domains, where the boundary integrals are replaced by integrals over certain submanifolds *outside* the domain. In sects. 5-8 we prove the estimates. After a first description of the singularity of the kernel of our operator in sect. 5, in sects. 6-7 we solve the main problem : we estimate the integration with respect to $\lambda$ (see above). After that it remains to repeat the arguments of Range and Siu [RS], what is done in sect. 8. In sect. 9 we sketch the standard arguments (Fredholm theory and Grauert's "Beulenmethode") which lead from local results with uniform estimates to global results with uniform estimates. Also in sect. 9 we complete the proof of Theorem 0.2, using the Andreotti-Grauert theorem (see [AnG]) on solvability of the Cauchy-Riemann equation on completely $q$-convex manifolds.

## 1. Preliminaries.

**1.1**. For $z \in \mathbb{C}^n$ we denote by $z_1, \ldots, z_n$ the canonical complex coordinates of $z$. We write $\langle z, w\rangle = z_1 w_1 + \cdots + z_n w_n$ and $|z| = \langle z, z\rangle^{1/2}$ for $z, w \in \mathbb{C}^n$.

**1.2**. Let $M$ be a closed real $C^1$ submanifold of a domain $\Omega \subseteq \mathbb{C}^n$, and let $\zeta \in M$. Then we denote by $T_\zeta^{\mathbb{C}}(M)$ the *complex*, and by $T_\zeta^{\mathbb{R}}(M)$ the *real* tangent space of $M$ at $\zeta$. We identify these spaces with subspaces of $\mathbb{C}^n$ as follows : if $\rho_1, \ldots, \rho_N$ are real $C^1$ functions in a neighborhood $U_\zeta$ of $\zeta$ such that $M \cap U = \{\rho_1 = \cdots = \rho_N = 0\}$ and $d\rho_1(\zeta) \wedge \cdots \wedge d\rho_N(\zeta) \neq 0$, then

$$T_\zeta^{\mathbb{C}}(M) = \Big\{t \in \mathbb{C}^n : \sum_{\nu=1}^{n} \frac{\partial \rho_j(\zeta)}{\partial \zeta_\nu} t_\nu = 0 \text{ for } j = 1, \ldots, n\Big\}$$

and

$$T_\zeta^{\mathbb{R}}(M) = \Big\{t \in \mathbb{C}^n : \sum_{\nu=1}^{2n} \frac{\partial \rho_j(\zeta)}{\partial x_\nu} x_\nu(t) = 0 \text{ for } j = 1, \ldots, n\Big\},$$

where $x_1, \ldots, x_{2n}$ are the real coordinates on $\mathbb{C}^n$ with $t_\nu = x_\nu(t) + i x_{\nu+n}(t)$ for $t \in \mathbb{C}^n$ and $\nu = 1, \ldots, n$.

**1.3**. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $\rho$ a real $C^2$ function on $\Omega$. Then we denote by $L_\rho(\zeta)$ the Levi form of $\rho$ at $\zeta \in \Omega$, and by $F_\rho(\cdot, \zeta)$ the Levi polynomial of $\rho$ at $\zeta \in \Omega$, *i.e.*

$$L\rho(\zeta)t = \sum_{j,k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \overline{\zeta}_j \partial \zeta_k} \overline{t}_j t_k$$

$\zeta \in \Omega,\ t \in \mathbb{C}^n$, and

$$F_\rho(z,\zeta) = 2\sum_{j=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_j}(\zeta_j - z_j) - \sum_{j,k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k)$$

$\zeta \in \Omega,\ z \in \mathbb{C}^n$. Recall that by Taylor's theorem (see, e.g., Lemma 1.4.13 in [HeLe1])

$$\operatorname{Re} F_\rho(z,\zeta) = \rho(\zeta) - \rho(z) + L_\rho(\zeta)(\zeta - z) + o(|\zeta - z|^2). \tag{1.1}$$

**1.4**. Let $J = (j_1, \ldots, j_\ell), 1 \leq \ell < \infty$, be an ordered collection of integers. Then we write $|J| = \ell,\ J(\hat{\nu}) = (j_1, \ldots, j_{\nu-1}, j_{\nu+1}, \ldots, j_\ell)$ for $\nu = 1, \ldots, \ell$, and $j \in J$ if $j \in \{j_1, \ldots, j_\ell\}$.

**1.5**. Let $N \geq 1$ be an integer. Then we denote by $P(N)$ the set of all ordered collections $K = (k_1, \ldots, k_\ell), \ell \geq 1$, of integers with $1 \leq k_1, \ldots, k_\ell \leq N$, and we denote by $P'(N)$ the subset of all $K = (k_1, \ldots, k_\ell) \in P(N)$ with $k_1 < \cdots < k_\ell$.

**1.6**. Let $J = (j_1, \ldots, j_\ell),\ 1 \leq \ell < \infty$, be an ordered collection of integers with $0 \leq j_1 < \cdots < j_\ell$. Then we denote by $\Delta_J$ (or $\Delta_{j_1 \cdots j_\ell}$) the simplex of all sequences $\{\lambda_j\}_{j=0}^{\infty}$ of numbers $0 \leq \lambda_j \leq 1$ such that $\lambda_j = 0$ if $j \notin J$ and $\Sigma \lambda_j = 1$. We orient $\Delta_J$ by the form $d\lambda_{j_2} \wedge \cdots \wedge d\lambda_{j_\ell}$ if $\ell \geq 2$, and by $+1$ if $\ell = 1$.

**1.7**. We denote by $\overset{\circ}{\chi}$ a fixed $C^\infty$ function

$$\overset{\circ}{\chi} : [0,1] \longrightarrow [0,1]$$

with $\overset{\circ}{\chi}(\lambda) = 0$ if $0 \leq \lambda \leq 1/4$ and $\overset{\circ}{\chi}(\lambda) = 1$ if $1/2 \leq \lambda \leq 1$.

**1.8**. Let $N \geq 1$ be an integer and $K = (k_1, \ldots, k_\ell) \in P'(N)$. Then, for $\lambda \in \Delta_{OK}$ with $\lambda_0 \neq 1$, we denote by $\overset{\circ}{\lambda}$ the point in $\Delta_K$ defined by

$$\overset{\circ}{\lambda}_{k_\nu} = \frac{\lambda_{k_\nu}}{1 - \lambda_0} \quad (\nu = 1, \ldots, \ell).$$

**1.9**. Let $D \subset\subset \mathbb{C}^n$ be a domain. $D$ will be called a $C^k$ *intersection*, $k = 1, 2, \ldots, \infty$, if there exist a neighborhood $U_{\overline{D}}$ of $\overline{D}$ and a finite number of real $C^k$ functions $\rho_1, \ldots, \rho_N$ in a neighborhood of $\overline{U_{\overline{D}}}$ such that

$$D = \{z \in U_{\overline{D}} : \rho_j(z) < 0 \ \text{ for } \ j = 1, \ldots, N\}$$

and

$$d\rho_{k_1}(z) \wedge \cdots \wedge d\rho_{k_\ell}(z) \neq 0$$

for all $(k_1, \ldots, k_\ell) \in P'(N)$ and $z \in \partial D$ with $\rho_{k_1}(z) = \cdots = \rho_{k_\ell}(z) = 0$. In this case, the collection $(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ will be called a $C^k$ *frame* (or a *frame*) for $D$.

**1.10**. Let $D \subset\subset \mathbb{C}^n$ be a $C^1$ intersection and $(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ a frame for $D$. Then, for $K = (k_1, \ldots, k_\ell) \in P(N)$, we set

$$S_K = \{z \in \partial D : \rho_{k_1}(z) = \cdots = \rho_{k_\ell}(z) = 0\}$$

if $k_1, \ldots, k_\ell$ are different in pairs, and

$$S_K = \emptyset$$

otherwise. We orient the manifolds $S_K$ so that the orientation is skew symmetric in $k_1, \ldots, k_\ell$, and

$$\partial D = \sum_{j=1}^{N} S_j \tag{1.2}$$

and

$$\partial S_K = \sum_{j=1}^{N} S_{Kj} \tag{1.3}$$

for all $K \in P(N)$.

**1.11**. Let $D \subset\subset \mathbb{C}^n$ be a $C^1$ intersection, $(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ a frame for $D$, and let $S_K$ be the corresponding manifolds introduced in sect. 1.10.

A *Leray map* for $D$ or, more precisely, for the frame $(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ is, by definition, a map $\psi$ which attaches to each $K = (k_1, \ldots, k_\ell) \in P'(N)$ a $\mathbb{C}^n$-valued map

$$\psi_K(z, \zeta, \lambda) = \big(\psi_K^1(z, \zeta, \lambda), \ldots, \psi_K^n(z, \zeta, \lambda)\big)$$

defined for $(z, \zeta, \lambda) \in D \times S_K \times \Delta_K$ such that

$$\langle \psi_K(z, \zeta, \lambda), \zeta - z \rangle = 1$$

for all $(z, \zeta, \lambda) \in D \times S_K \times \Delta_K$, and, for $\nu = 1, \ldots, \ell$,

$$\psi_{K(\hat{\nu})}(z, \zeta, \lambda) = \psi_K(z, \zeta, \lambda)$$

if $(z, \zeta, \lambda) \in D \times S_{K(\hat{\nu})} \times \Delta_{K(\hat{\nu})}$.

**1.12**. We denote by $\widehat{B}(z,\zeta)$ the Martinelli-Bochner kernel for $(n,r)$-forms, *i.e.*

$$\widehat{B}(z,\zeta) = \frac{1}{(2\pi i)^n} \det\left(\overbrace{\frac{\overline{\zeta}-\overline{z}}{|\zeta-z|^2}}^{1}, \overbrace{d\frac{\overline{\zeta}-\overline{z}}{|\zeta-z|^2}}^{n-1}\right) \wedge dz_1 \wedge \cdots \wedge dz_n$$

for all $z,\zeta \in \mathbb{C}^n$ with $z \neq \zeta$ (for the definition of determinants of matrices of differential forms, see, e.g., sect. 0.7 in [HeLe2]). If $D \subset\subset \mathbb{C}^n$ is a domain and $f$ is a continuous differential form with integrable coefficients on $D$, then we set

$$B_D f(z) = \int_{\zeta\in D} f(\zeta) \wedge \widehat{B}(z,\zeta), \quad z \in D$$

(for the definition of integration with respect to a part of the variables, see, e.g., sect. 0.2 in [HeLe2]).

**1.13**. Let $D \subset\subset \mathbb{C}^n$ be a $C^1$ intersection, $(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ a frame for $D$, and let $S_K$ be the corresponding manifolds introduced in sect. 1.10.

Further, let $\psi$ be a Leray map for the frame $(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$. Then we set

$$\psi_{OK}(z,\zeta,\lambda) = \mathring{\chi}(\lambda_0)\frac{\overline{\zeta}-\overline{z}}{|\zeta-z|^2} + \big(1-\mathring{\chi}(\lambda_0)\big)\psi_K(z,\zeta,\mathring{\lambda}) \tag{1.4}$$

for $K \in P'(N)$ and $(z,\zeta,\lambda) \in D \times S_K \times \Delta_{OK}$. Note that $1 - \mathring{\chi}(\lambda_0) = 0$ for $\lambda$ in the neighborhood $\Delta_{OK} \setminus \mathring{\Delta}_{OK}$ of $\Delta_0$ and therefore $\psi_{OK}$ is of class $C^1$. For $K \in P'(N)$ we introduce the differential form

$$\widehat{R}^{\psi}_K(z,\zeta,\lambda) = \frac{(-1)^{|K|}}{(2\pi i)^n} \det\Big(\overbrace{\psi_{OK}(z,\zeta,\lambda)}^{1}, \overbrace{d\psi_{OK}(z,\zeta,\lambda)}^{n-1}\Big) \wedge dz_1 \wedge \cdots \wedge dz_n$$

defined for $(z,\zeta,\lambda) \in D \times S_K \times \Delta_{OK}$, and the differential form

$$\widehat{L}^{\psi}_K(z,\zeta,\lambda) = \frac{1}{(2\pi i)^n} \det\Big(\overbrace{\psi_K(z,\zeta,\lambda)}^{1}, \overbrace{d\psi_K(z,\zeta,\lambda)}^{n-1}\Big) \wedge dz_1 \wedge \cdots \wedge dz_n$$

defined for $(z,\zeta,\lambda) \in D \times S_K \times \Delta_K$ (here $d$ denotes the exterior differential operator with respect to all variables $z,\zeta,\lambda$). If $f$ is a continuous differential form on $\overline{D}$, then, for all $K \in P'(N)$, we set

$$R^{\psi}_K f(z) = \int_{(\zeta,\lambda)\in S_K\times\Delta_{OK}} f(\zeta) \wedge \widehat{R}^{\psi}_K(z,\zeta,\lambda), \quad z \in D,$$

and

$$L^{\psi}_K f(z) = \int_{(\zeta,\lambda)\in S_K\times\Delta_K} f(\zeta) \wedge \widehat{L}^{\psi}_K(z,\zeta,\lambda), \quad z \in D.$$

Then, for each continuous $(n,r)$-form $f$ on $D$, $0 \le r \le n$, such that $df$ is also continuous on $\overline{D}$, one has the representation

$$(1.5)\quad (-1)^{r+n} f = dB_D f - B_D df + \sum_{K \in P'(N)} \left( L_K^{\psi} f + dR_K^{\psi} f - R_K^{\psi} df \right) \text{ on } D.$$

This formula is basic for the present paper. It has different names and a long history (see, e.g., the notes at the end of ch. 4 in [HeLe1]), we call it *Cauchy-Fantappie formula*. For more special Leray maps it was proved by Range and Siu (see [RS]). In the case considered here, this formula was obtained by Airapetjan and Henkin (see Proposition 1.3.1 in [AiHe]). As mentioned by Airapetjan and Henkin, the proof of Range and Siu can be used also in this more general case (see sect. 3.12 in [HeLe2], where this is carried out).

**1.14.** Let $f$ be a differential form on a domain $D \subseteq \mathbb{C}^N$. Then we denote by $\|f(z)\|, z \in D$, the Riemannian norm of $f$ at $z$ (see, e.g., sect. 0.4 in [HeLe2]).

**1.15.** If $M$ is an oriented real $C^1$ manifold and $f$ is a differential form of maximal degree, then we denote by $|f|$ the absolute value of $f$ (see, e.g., sect. 0.3 in [HeLe2]).

**1.16.** Let $D \subset\subset \mathbb{C}^n$ be a domain. Then we shall use the following spaces and norms of differential forms :

$C_*^0(D)$ is the set of continuous forms on $D$. Set

$$(1.6)\qquad \|f\|_0 = \|f\|_{O,D} = \sup_{z \in D} \|f(z)\|$$

for $f \in C_*^0(D)$.

$C_*^{\alpha}(\overline{D}), 0 \le \alpha < 1$, is the space of forms $f \in C_*^0(D)$ whose coefficients admit a continuous extension to $\overline{D}$ which are, if $\alpha > 0$, even Hölder continuous with exponent $\alpha$ on $\overline{D}$. Set

$$(1.7)\qquad \|f\|_{\alpha} = \|f\|_{\alpha,D} = \|f\|_{O,D} + \sup_{\substack{z,\zeta \in D \\ z \neq \zeta}} \frac{\|f(z) - f(\zeta)\|}{|\zeta - z|^{\alpha}}$$

for $0 < \alpha < 1$ and $f \in C_*^{\alpha}(\overline{D})$.

$B_*^{\beta}(D), \beta \ge 0$, is the space of forms $f \in C_*^0(D)$ such that, for some constant $C > 0$,

$$\|f(z)\| \le C[\operatorname{dist}(z, \partial D)]^{-\beta}, \quad z \in D,$$

where $\operatorname{dist}(z, \partial D)$ is the Euclidean distance between $z$ and $\partial D$. Set

$$\|f\|_{-\beta} = \|f\|_{-\beta,D} = \sup_{z\in D} \|f(z)\| [\operatorname{dist}(z, \partial D)]^\beta \tag{1.8}$$

for $\beta \geq 0$ and $f \in B_*^\beta(D)$.

If $\Lambda_{p,r}(D)$ is the space of forms of bidegree $(p, r)$ on $D$, then we set

$$C^0_{p,r}(D) = C^0_*(D) \cap \Lambda_{p,r}(D),$$
$$C^\alpha_{p,r}(\overline{D}) = C^\alpha_*(\overline{D}) \cap \Lambda_{p,r}(D),$$
$$B^\beta_{p,r}(D) = B^\beta_*(D) \cap \Lambda_{p,r}(D),$$

and

$$C^0_{p,*}(D) = \cup_{0\leq r\leq n} C^0_{p,r}(D),$$
$$C^\alpha_{p,*}(\overline{D}) = \cup_{0\leq r\leq n} C^\alpha_{p,r}(\overline{D}),$$
$$B^\beta_{p,*}(D) = \cup_{0\leq r\leq n} B^\beta_{p,r}(D).$$

## 2. Local $q$-convex domains.

In this section $n$ and $q$ are fixed integers with $0 \leq q \leq n-1$. Denote by $G(n, q)$ the complex Grassmann manifold of $q$-dimensional subspaces of $\mathbb{C}^n$, and by $MO(n, q)$ the complex manifold of all complex $n \times n$-matrices which define an orthogonal projection from $\mathbb{C}^n$ onto some $q$-dimensional subspace of $\mathbb{C}^n$. Sometimes we shall identify the projection $P \in MO(n, q)$ with its image $\operatorname{Im} P \in G(n, q)$. Observe that this identification is only of class $C^\infty$ but not holomorphic.

**2.1.** DEFINITION. — *A collection $(U, \rho_1, \ldots, \rho_N)$ will be called a* $q$-configuration *in $\mathbb{C}^n$ if $U \subseteq \mathbb{C}^n$ is a convex domain, and $\rho_1, \ldots, \rho_N$ are real $C^2$ functions on $U$ satisfying the following conditions :*

(i) $\{z \in U : \rho_1(z) = \cdots = \rho_N(z) = 0\} \neq \emptyset$.

(ii) $d\rho_1(z) \wedge \cdots \wedge d\rho_N(z) \neq 0$ *for all* $z \in U$.

(iii) *If* $\lambda \in \Delta_{1\ldots N}$ *(see sect. 1.6) and*

$$\rho_\lambda := \lambda_1\rho_1 + \cdots + \lambda_N\rho_N,$$

*then the Levi form $L_{\rho_\lambda}(z)$ (see sect. 1.3) has at least $q+1$ positive eigenvalues.*

**2.2.** LEMMA. — *Let $\xi \in \mathbb{C}^n$ and let $\varphi_1, \ldots, \varphi_N$ be real $C^2$ functions in a neighborhood $V$ of $\xi$ such that the following conditions are fulfilled :*

(i) $d\varphi_1(\xi) \wedge \cdots \wedge d\varphi_N(\xi) \neq 0$.

(ii) $\varphi_1(\xi) = \cdots = \varphi_N(\xi) = 0$.

(iii) *Set* $Y_j = \{z \in V : \varphi_j(z) = 0\}$ *for* $j = 1, \ldots, N$, *and* $\varphi_\lambda = \lambda_1\varphi_1 + \cdots + \lambda_N\varphi_N$ *for* $\lambda \in \Delta_{1\cdots N}$. *Then, for all* $K = (k_1, \ldots, k_\ell) \in P'(N)$ *and* $\lambda \in \Delta_K$ *(see sects. 1.5 and 1.6), the Levi form* $L_{\varphi_\lambda}(\xi)$ *restricted to* $T_\xi^{\mathbb{C}}(Y_{k_1} \cap \cdots \cap Y_{k_\ell})$ *(see sect. 1.2) has at least*

$$\dim_{\mathbb{C}} T_\xi^{\mathbb{C}}(Y_{k_1} \cap \cdots \cap Y_{k_\ell}) - n + q + 1$$

*positive eigenvalues.*

*Then there exist a convex neighborhood* $U \subseteq V$ *of* $\xi$ *and a constant* $C_0 > 0$ *such that, for all* $C \geq C_0$,

$$(U, e^{C\varphi_1} - 1, \ldots, e^{C\varphi_N} - 1)$$

*is a q-configuration in* $\mathbb{C}^n$.

*Proof.* — For $\lambda \in \Delta_{1\cdots N}, C > 0$ and $z \in V$, we denote by $L_\lambda^C(z)$ the Levi form at $z$ of the function

$$\lambda_1 e^{C\varphi_1} + \cdots + \lambda_N e^{C\varphi_N}.$$

It is sufficient to prove that for all $\lambda \in \Delta_{1\cdots N}$ there exist a constant $C_\lambda > 0$, a space $T_\lambda \in G(n, q+1)$ and neighborhoods $U_\lambda \subseteq V$ of $\xi$ and $\Gamma_\lambda \subseteq \Delta_{1\cdots N}$ of $\lambda$ such that, for all $C \geq C_\lambda, z \in U_\lambda$ and $\mu \in \Gamma_\lambda$, the Levi form $L_\mu^C(z)$ is positive definite on $T_\lambda$.

Let $\lambda \in \Delta_{1\cdots N}$ be fixed, and let $K = (k_1, \ldots, k_\ell) \in P'(N)$ be the collection of indices with $\lambda_{k_\nu} \neq 0$ for $\nu = 1, \ldots, \ell$ and $\lambda_j = 0$ if $j \notin K$. Then by condition (iii) we can find a subspace $\widetilde{T}_\lambda$ of $\mathbb{C}^n$ such that $L_{\varphi_\lambda}(\xi)$ is positive definite on $\widetilde{T}_\lambda$,

$$\widetilde{T}_\lambda \subseteq T_\xi^{\mathbb{C}}(Y_{k_1} \cap \cdots \cap Y_{k_\ell})$$

and

$$\dim_{\mathbb{C}} \widetilde{T}_\lambda = \dim_{\mathbb{C}} T_\xi^{\mathbb{C}}(Y_{k_1} \cap \cdots \cap Y_{k_\ell}) - n + q + 1.$$

We choose a subspace $T_\lambda \in G(n, q+1)$ so that

$$\widetilde{T}_\lambda = T_\lambda \cap T_\xi^{\mathbb{C}}(Y_{k_1} \cap \cdots \cap Y_{k_\ell}).$$

Set $M = \{t \in T_\lambda : |t| = 1\}$. Since $L_{\varphi_\lambda}(\xi)$ is positive definite on $\widetilde{T}_\lambda$ and $L_{\varphi_\mu}(z)$ depends continuously on $\mu$ and $z$, then we can find neighborhoods $U_\lambda^0 \subseteq V$ of $\xi$ and $\Gamma_\lambda^0 \subseteq \Delta_{1\cdots N}$ of $\lambda$ such that

$$\gamma := \inf_{z \in U_\lambda^0, \mu \in \Gamma_\lambda^0, t \in M \cap \widetilde{T}_\lambda} L_{\varphi_\mu}(z)t > 0.$$

Set

$$t(\varphi_j, z) = \sum_{k=1}^{n} \frac{\partial \varphi_j(z)}{\partial z_k} t_k$$

for $z \in V, t \in \mathbb{C}^n, j = 1, \ldots, N$. Then

$$L_\mu^C(z)t = C\Big(C \sum_{j=1}^{N} \mu_j |t(\varphi_j, z)|^2 + L_{\varphi_\mu}(z)t\Big) \tag{2.1}$$

for all $C > 0, \mu \in \Delta_{1\cdots N}, z \in V$. Set

$$M' = \Big\{ t \in M : \inf_{z \in U_\lambda^0, \mu \in \Gamma_\lambda^0} L_{\varphi_\mu}(z)t \leq \frac{\gamma}{2} \Big\}.$$

Then it follows from (2.1) that

$$L_\mu^C(z)t \geq C\frac{\gamma}{2} \tag{2.2}$$

if $t \in M \setminus M', z \in U_\lambda^0, \mu \in \Gamma_\lambda^0$. Further, since

$$T_\xi^{\mathbb{C}}(Y_{k_1} \cap \cdots \cap Y_{k_\ell}) = \{t \in \mathbb{C}^n : t(\varphi_{k_\nu}, \xi) = 0 \ \text{ for } \ \nu = 1, \ldots, \ell\}$$

and by definition of the number $\gamma$,

$$M' \cap T_\xi^{\mathbb{C}}(Y_{k_1} \cap \cdots \cap Y_{k_\ell}) = \emptyset,$$

we have the inequality

$$\gamma' := \min_{t \in M'} \sum_{j=1}^{N} \lambda_j |t(\varphi_j, \xi)|^2 > 0.$$

Choose neighborhoods $U_\lambda \subseteq U_\lambda^0$ of $\xi$ and $\Gamma_\lambda \subseteq \Gamma_\lambda^0$ of $\lambda$ so small that

$$\sum_{j=1}^{N} \mu_j |t(\varphi_j, z)|^2 \geq \frac{\gamma'}{2}$$

for $t \in M', z \in U_\lambda, \mu \in \Gamma_\lambda$. Moreover, we choose $C_\lambda > 0$ with

$$|L_{\varphi_\mu}(z)t| \leq \frac{C_\lambda \gamma'}{4}$$

for $z \in U_\lambda, \mu \in \Gamma_\lambda, t \in M'$. Then it follows from (2.1) that

$$|L_\mu^C(z)t| \geq \frac{C^2 \gamma'}{4}$$

for $C \geq C_\lambda, z \in U_\lambda, \mu \in \Gamma_\lambda, t \in M'$. Together with (2.2) this implies that, for all $C \geq C_\lambda, z \in U_\lambda$ and $\mu \in \Gamma_\lambda$ the form $L_\mu^C(z)$ is positive definite on $T_\lambda$. □

**2.3.** DEFINITION. — *A local $q$-convex domain, $0 \le q \le n-1$, is a $C^2$ intersection $D \subset\subset \mathbb{C}^n$ (see sect. 1.9) for which one can find a $C^2$ frame $(U_{\overline{D}}, \rho_1, \dots, \rho_N)$ satisfying the following two conditions :*

(i) *If $K = \{k_1, \dots, k_\ell) \in P'(N)$ and*

$$U_{\overline{D}}^K := \{z \in U_{\overline{D}} : \rho_{k_1}(z) = \cdots = \rho_{k_\ell}(z)\},$$

*then $(d\rho_{k_1}(z) - d\rho_{k_2}(z)) \wedge \cdots \wedge (d\rho_{k_1}(z) - d\rho_{k_\ell}(z)) \neq 0$ for all $z \in U_{\overline{D}}^K$.*

(ii) *There exist a $C^\infty$ map*

$$Q : \Delta_{1\cdots N} \longrightarrow MO(n, n-q-1)$$

*and constants $\alpha, A > 0$ such that*

$$\operatorname{Re} F_{\rho_\lambda}(z,\zeta) \ge \rho_\lambda(\zeta) - \rho_\lambda(z) + \alpha|\zeta - z|^2 - A|Q(\lambda)(\zeta - z)|^2$$

*for all $\lambda \in \Delta_{1\cdots N}$ and $z, \zeta \in U_{\overline{D}}$ (for the definition of the Levi polynomial $F_{\rho_\lambda}(z,\zeta)$, see sect. 1.3).*

**2.4.** LEMMA. — *Let $(U, \rho_1, \dots, \rho_N)$ be a $q$-configuration in $\mathbb{C}^n$, $0 \le q \le n-1$. Then for each point $\xi \in U$ with $\rho_1(\xi) = \cdots = \rho_N(\xi) = 0$ there exists a number $R_\xi > 0$ such that, for all $R$ with $0 < R \le R_\xi$,*

$$D := \{z \in U : \rho_j(z) < 0 \ \text{ for } \ j = 1, \dots, N\} \cap \{z \in \mathbb{C}^n : |z - \xi| < R\}$$

*is a local $q$-convex domain.*

*Proof.* — Set for $R > 0$,

$$\rho_{N+1}^R(z) = |z - \xi|^2 - R^2,$$
$$D_R = \{z \in U : \rho_j(z) < 0 \ \text{ for } \ j = 1, \dots, N \ \text{ and } \ \rho_{N+1}^R(z) < 0\},$$
$$U_{\overline{D}_R} = \{z \in \mathbb{C}^n : |z - \xi| < 2R\}.$$

We have to prove that, for sufficiently small $R$, $D_R$ is a local $q$-convex intersection. First note the following : it is clear that there is $R'_\xi > 0$ such that $D_R$ is a $C^2$ intersection and $(U_{\overline{D}_R}, \rho_1, \dots, \rho_N, \rho_{N+1}^R)$ is a frame for $D_R$ satisfying condition (i) in Definition 2.3 if $0 < R \le R'_\xi$.

Therefore it remains to find constants $\alpha, A, R_\xi > 0$ with $R_\xi \le R'_\xi$, as well as a $C^\infty$ map

$$Q : \Delta_{1\cdots N+1} \longrightarrow MO(n, n-q-1)$$

such that, for $0 < R \le R_\xi$, $z, \zeta \in U_{\overline{D}_R}$ and $\lambda \in \Delta_{1\cdots N+1}$,

$$\operatorname{Re} F_{\rho_\lambda^R}(z,\zeta) \ge \rho_\lambda^R(\zeta) - \rho_\lambda^R(z) + \alpha|\zeta - z|^2 - A|Q(\lambda)(\zeta - z)|^2, \tag{2.3}$$

where $\rho_\lambda^R := \lambda_1\rho_1 + \cdots + \lambda_N\rho_N + \lambda_{N+1}\rho_{N+1}^R$.

Note that $L_{\rho_\lambda^R}(\xi)$ is independent of $R > 0$. Denote by $G_+(\lambda), \lambda \in \Delta_{1\cdots N+1}$, the set of all spaces $E \in G(n, q+1)$ such that $L_{\rho_\lambda^R}(\xi)$ is positive definite on $E$. Since $L_{\rho_{N+1}^R}(\xi)$ is positive definite on $\mathbb{C}^n$ and by condition (iii) in Definition 2.1,

$$G_+(\lambda) \neq \emptyset \quad \text{for all } \lambda \in \Delta_{1\cdots N+1}.$$

Moreover, it is easy to see that, for all $\lambda \in \Delta_{1\cdots N+1}, G_+(\lambda)$ is open and connected. Therefore, by elementary topological arguments ($\Delta_{1\cdots N+1}$ is contractible), one obtains a $C^\infty$ map

$$T : \Delta_{1\cdots N+1} \longrightarrow G(n, q+1)$$

such that $L_{\rho_\lambda^R}(\xi)$ is positive definite on $T(\lambda)$ for all $\lambda \in \Delta_{1\cdots N+1}$.

Denote by $P(\lambda), \lambda \in \Delta_{1\cdots N+1}$, the orthogonal projection from $\mathbb{C}^n$ onto $T(\lambda)$, and set $Q(\lambda) = I - P(\lambda)$. Choose $\alpha > 0$ with

$$L_{\rho_\lambda^R}(\xi)P(\lambda)t \geq 4\alpha|P(\lambda)t|^2$$

for all $\lambda \in \Delta_{1\cdots N+1}$ and $t \in \mathbb{C}^n$. Further, choose $R''_\xi > 0$ with $R''_\xi \leq R'_\xi$ so small that

$$L_{\rho_\lambda^R}(\zeta)P(\lambda)t \geq 3\alpha|P(\lambda)t|^2$$

if $|\zeta - \xi| \leq R''_\xi, \lambda \in \Delta_{1\cdots N+1}, t \in \mathbb{C}^n$. Finally, we choose constants $A, A' > 0$ such that

$$\begin{aligned} |L_{\rho_\lambda^R}(\zeta)t - L_{\rho_\lambda^R}(\zeta)P(\lambda)t| &\leq A'\big(|P(\lambda)t||Q(\lambda)t| + |Q(\lambda)t|^2\big) \\ &\leq \alpha|P(\lambda)t|^2 + \Big(\frac{A}{2} - 2\alpha\Big)|Q(\lambda)t|^2 \end{aligned}$$

and therefore

$$\begin{aligned} L_{\rho_\lambda^R}(\zeta)t &\geq 2\alpha|P(\lambda)t|^2 + 2\alpha|Q(\lambda)t|^2 - \frac{A}{2}|Q(\lambda)t|^2 \\ &= 2\alpha|t|^2 - \frac{A}{2}|Q(\lambda)t|^2 \end{aligned}$$

for $|\zeta - \xi| \leq R''_\xi, \lambda \in \Delta_{1\cdots N+1}, t \in \mathbb{C}^n$. In view of relation (1.1) in sect. 1.3, this implies that there exists a constant $R_\xi$ with $0 < R_\xi \leq R''_\xi$ such that

$$\operatorname{Re} F_{\rho_\lambda^R}(z,\zeta) \geq \rho_\lambda^R(\zeta) - \rho_\lambda^R(z) + \alpha|\zeta - z|^2 - A|Q(\lambda)(\zeta - z)|^2$$

for $|\zeta - \xi|, |z - \xi| \leq 2R_\xi$ and $\lambda \in \Delta_{1\cdots N+1}$. □

*Notes.* — The results of this section are closely related to §3 in [AiHe] (cp. Lemma 2.2 with Lemma 3.1.1 in [AiHe], and the proof of Lemma 2.4 with the proof of Proposition 3.3.1 in [AiHe].

## 3. A Leray map for local $q$-convex domains.

In this section $D \subset\subset \mathbb{C}^n$ is a local $q$-convex domain, $0 \leq q \leq n-1$, and $(U_{\overline{D}}, \rho_1, \ldots, \rho_N), \alpha, A, Q$ are just as in Definition 2.3.

**3.1. Construction of the Leray map $\psi$.** — Since $\rho_1, \ldots, \rho_N$ are defined and of class $C^2$ in a neighborhood of $\overline{U}_{\overline{D}}$, we can find $C^\infty$ functions $a_\nu^{kj} (\nu = 1, \ldots, N; k, j = 1, \ldots, n)$ on $U_{\overline{D}}$ such that

$$\left| a_\nu^{kj}(\zeta) - \frac{\partial^2 \rho_\nu(\zeta)}{\partial \zeta_k \partial \zeta_j} \right| < \frac{\alpha}{2n^2}$$

for all $\zeta \in U_{\overline{D}}$. Set

and
$$\rho_\lambda = \lambda_1 \rho_1 + \cdots + \lambda_N \rho_N$$
$$a_\lambda^{kj} = \lambda_1 a_1^{kj} + \cdots + \lambda_N a_N^{kj}$$

for $\lambda \in \Delta_{1\cdots N}$. Then

$$\left| \sum_{k,j=1}^{n} \left( a_\lambda^{kj}(\zeta) - \frac{\partial^2 \rho_\lambda(\zeta)}{\partial \zeta_k \partial \zeta_j} \right) t_k t_j \right| \leq \frac{\alpha}{2} |t|^2 \tag{3.1}$$

for all $\zeta \in U_{\overline{D}}, t \in \mathbb{C}^n$ and $\lambda \in \Delta_{1\cdots N}$. Set

$$\widetilde{F}_{\rho_\lambda}(z, \zeta) = 2 \sum_{j=1}^{n} \frac{\partial \rho_\lambda(\zeta)}{\partial \zeta_j} (\zeta_j - z_j) - \sum_{k,j=1}^{n} a_\lambda^{kj}(\zeta)(\zeta_k - z_k)(\zeta_j - z_j)$$

for $(z, \zeta, \lambda) \in \mathbb{C}^n \times U_{\overline{D}} \times \Delta_{1\cdots N}$. Then it follows from (3.1) and condition (ii) in Definition 2.3 that

$$\operatorname{Re} \widetilde{F}_{\rho_\lambda}(z, \zeta) \geq \rho_\lambda(\zeta) - \rho_\lambda(z) + \frac{\alpha}{2} |\zeta - z|^2 - A |Q(\lambda)(\zeta - z)|^2 \tag{3.2}$$

for all $(z, \zeta, \lambda) \in U_{\overline{D}} \times U_{\overline{D}} \times \Delta_{1\cdots N}$. Denote by $Q_{kj}(\lambda)$ the entries of the matrix $Q(\lambda)$, *i.e.*

$$Q(\lambda) = \big(Q_{kj}(\lambda)\big)_{k,j=1}^{n} \quad (k = \text{ column index}).$$

If $(z, \zeta, \lambda) \in \mathbb{C}^n \times U_{\overline{D}} \times \Delta_{1\cdots N}$, then we set

$$\begin{cases} w^j(z, \zeta, \lambda) = 2 \dfrac{\partial \rho_\lambda(\zeta)}{\partial \zeta_j} - \displaystyle\sum_{k=1}^{n} a_\lambda^{kj}(\zeta)(\zeta_k - z_k) + A \sum_{k=1}^{n} \overline{Q_{kj}(\lambda)(\zeta_k - z_k)}, \\ w(z, \zeta, \lambda) = \big(w^1(z, \zeta, \lambda), \ldots, w^n(z, \zeta, \lambda)\big), \\ \Psi(z, \zeta, \lambda) = \langle w(z, \zeta, \lambda), \zeta - z \rangle. \end{cases} \tag{3.3}$$

Since $Q(\lambda)$ is an *orthogonal* projection, then we have

$$\Psi(z, \zeta, \lambda) = \widetilde{F}_{\rho_\lambda}(z, \zeta) + A |Q(\lambda)(\zeta - z)|^2 \tag{3.4}$$

for all $(z, \zeta, \lambda) \in \mathbb{C}^n \times U_{\overline{D}} \times \Delta_{1\cdots N}$, and it follows from estimate (3.2) that

$$\operatorname{Re} \Psi(z, \zeta, \lambda) \geq \rho_\lambda(\zeta) - \rho_\lambda(z) + \frac{\alpha}{2}|\zeta - z|^2 \tag{3.5}$$

for all $(z, \zeta, \lambda) \in U_{\overline{D}} \times U_{\overline{D}} \times \Delta_{1\cdots N}$. In particular $\Psi(z, \zeta, \lambda) \neq 0$ if $(z, \zeta, \lambda) \in D \times S_K \times \Delta_K$ for some $K \in P'(N)$. Therefore, by setting

$$\psi_K(z, \zeta, \lambda) = \frac{w(z, \zeta, \lambda)}{\Psi(z, \zeta, \lambda)} \tag{3.6}$$

for $(z, \zeta, \lambda) \in D \times S_K \times \Delta_K, K \in P'(N)$, we obtain a family $\psi = \{\psi_K\}_{K \in P'(N)}$ of $\mathbb{C}^n$-valued $C^1$ maps. Obviously, $\psi$ is a Leray map for the frame $(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ (see sect. 1.11).

**3.2.** DEFINITION. — *A map $f$ defined on some complex manifold $X$ will be called* $k$-holomorphic *if, for each point $\xi \in X$, there exist holomorphic coordinates $h_1, \ldots, h_n$ in a neighborhood of $\xi$ such that $f$ is holomorphic with respect to $h_1, \ldots, h_k$.*

**3.3.** LEMMA.

(i) *For every fixed $(\zeta, \lambda) \in U_{\overline{D}} \times \Delta_{1\cdots N}$, the map $w(z, \zeta, \lambda)$ and the function $\Psi(z, \zeta, \lambda)$ (see (3.3)) are $(q + 1)$-holomorphic in $z \in \mathbb{C}^n$.*

(ii) *For each $K \in P'(N)$ and all fixed $(\zeta, \lambda) \in S_K \times \Delta_K$, the map $\psi_K(z, \zeta, \lambda)$ (see (3.6)) is $(q + 1)$-holomorphic in $z \in D$.*

*Proof.* — Assertion (ii) follows from (i). Therefore we must prove only assertion (i). Let $(\zeta, \lambda) \in U_{\overline{D}} \times \Delta_{1\cdots N}$ be fixed.

Choose complex linear coordinates $h_1, \ldots, h_n$ on $\mathbb{C}^n$ with

$$\{z \in \mathbb{C}^n : Q(\lambda)z = 0\} = \{z \in \mathbb{C}^n : h_{q+2}(z) = \cdots = h_n(z) = 0\}.$$

Then the map $\mathbb{C}^n \ni z \to \overline{Q(\lambda)(\zeta - z)}$ is independent of $h_1, \ldots, h_{q+1}$. This implies that $w(\cdot, \zeta, \lambda)$ is complex linear with respect to $h_1, \ldots, h_{q+1}$, and $\Psi(\cdot, \zeta, \lambda)$ is a quadratic complex polynomial with respect to $h_1, \ldots, h_{q+1}$. □

*Notes.* — For $N = 1$ such a Leray map was first constructed by W. Fischer and Lieb (see [FiLi]). For the general case, a similar map was constructed by Airapetjan and Henkin (see Proposition 3.3.1 in [AiHe]).

## 4. Homotopy formulas on local $q$-convex domains.

Throughout this section we assume :

$D \subset\subset \mathbb{C}^n$ is a local $q$-convex domain, $0 \leq q \leq n-1$ (see Definition 2.3);

$(U_{\overline{D}}, \rho_1, \dots, \rho_N)$ is a frame for $D$ satisfying conditions (i) and (ii) in Definition 2.3;

$S_K, K \in P(N)$, are the submanifolds of $\partial D$ which belong to the frame $(U_{\overline{D}}, \rho_1, \dots, \rho_N)$ according to sect. 1.10;

$\psi$ is the Leray map constructed in sect. 3.1 for the frame $(U_{\overline{D}}, \rho_1, \dots, \rho_N)$.

We set

$$T^\psi = B_D + \sum_{K \in P'(N)} R_K^\psi$$

and

$$L^\psi = \sum_{K \in P'(N)} L_K^\psi$$

(for the definition of the operators $B_D, R_K^\psi$ and $L_K^\psi$, see sects. 1.12 and 1.13).

**4.1.** THEOREM. — *If $n-q \leq r \leq n$, then, for each continuous $(n,r)$-form $f$ on $\overline{D}$ such that $df$ is also continuous on $\overline{D}$,*

$$(-1)^{r+n} f = dT^\psi f - T^\psi df \quad \text{on } D. \tag{4.1}$$

*Proof.* — In view of the Cauchy-Fantappie formula (1.5), we must prove that $L_K^\psi f = 0$ for all $K \in P'(N)$.

Fix $K \in P'(N)$ and denote by $\psi_K^1, \dots, \psi_K^n$ the components of the map $\psi_K$. Since, by Lemma 3.2 (ii), the map $\psi_K(z,\zeta,\lambda)$ is $(q+1)$-holomorphic in $z$, and since $r \geq n-q$, this implies that

$$d_z\psi_K^{j_1}(z,\zeta,\lambda) \wedge \cdots \wedge d_z\psi_K^{j_r}(z,\zeta,\lambda) \wedge dz_1 \wedge \cdots \wedge dz_n = 0$$

for all $1 \leq j_1, \dots, j_r \leq n$. Looking at the definition of $L_K^\psi f$ now it is easy to see that $L_K^\psi f = 0$. □

Now we are going to replace the integrals over the manifolds $S_K$ in the homotopy formula (4.1) by integrals over certain submanifolds $\Gamma_K$ of $D$.

**4.2. The manifolds $\Gamma_K$.** — For $K = (k_1, \dots, k_\ell) \in P(N)$ we set

$$U_{\overline{D}}^K = \{\zeta \in U_{\overline{D}} : \rho_{k_1}(\zeta) = \cdots = \rho_{k_\ell}(\zeta)\}$$

if $k_1, \dots, k_\ell$ are different in pairs, and $U_{\overline{D}}^K = \emptyset$ otherwise. By condition (i) in Definition 2.3 each $U_{\overline{D}}^K$ is a closed $C^2$ submanifold of $U_{\overline{D}}$. We denote by $\rho_K, K \in P(N)$, the function on $U_{\overline{D}}^K$ which is defined by

$$\rho_K(\zeta) = \rho_{k_\nu}(\zeta) \quad (\zeta \in U_{\overline{D}}^K; \nu = 1, \dots, \ell). \tag{4.2}$$

Now, for all $K \in P(N)$, we define

$$\Gamma_K = \{\zeta \in U_{\overline{D}}^K : \rho_j(\zeta) \le \rho_K(\zeta) \le 0 \ \text{ for } \ j = 1, \dots, N\}. \tag{4.3}$$

Then it is easy to see that all $\Gamma_K$ are $C^2$ submanifolds of $\overline{D}$ with piecewise $C^2$ boundary, and that

$$\overline{D} = \Gamma_1 \cup \cdots \cup \Gamma_N \tag{4.4}$$

and

$$\partial \Gamma_K = S_K \cup \Gamma_{K1} \cup \cdots \cup \Gamma_{KN}, \quad K \in P(N). \tag{4.5}$$

We choose the orientation on $\Gamma_K$ such that the orientation is skew symmetric in the components of $K$, and the following conditions hold :

$$\left\{ \begin{array}{l} \Gamma_1, \dots, \Gamma_N \text{ carry the orientation of } \mathbb{C}^n, \text{ and} \\ \text{if } K \in P(N) \text{ and } 1 \le j \le N \text{ with } j \notin K, \text{ then} \\ \Gamma_{Kj} \text{ is oriented just as } -\partial\Gamma_K. \end{array} \right. \tag{4.6}$$

**4.3.** LEMMA. — *If $\Gamma_K$ are the manifolds defined in sect. 4.2, then*

$$\partial \Gamma_K = S_K - \sum_{j=1}^N \Gamma_{Kj}$$

*for all $K \in P(N)$.*

*Proof.* — Denote by $\widetilde{S}_K, K \in P(N)$, the manifold which is equal to $S_K$ as a set and which carries the orientation of $\partial\Gamma_K$. Then it follows from (4.5) and (4.6) that

$$\partial \Gamma_K = \widetilde{S}_K - \sum_{i=1}^N \Gamma_{Ki} \tag{4.7}$$

for all $K \in P(N)$. Therefore we must prove that $\widetilde{S}_K = S_K$ for all $K \in P(N)$. We do this by induction over $|K|$. Since $\partial D = S_1 + \cdots + S_N$ (see sect. 1.10), it is clear that $\widetilde{S}_K = S_K$ if $|K| = 1$.

Now let $\ell \ge 1$ and assume that the relation $\widetilde{S}_K = S_K$ is already proved for all $K \in P(N)$ with $|K| = \ell$.

Consider $K \in P(N)$ with $|K| = \ell + 1$. Set $K' = K(\widehat{\ell+1})$ (see sect. 1.4). Then by hypothesis and (4.7)

$$\partial\Gamma_{K'} = S_{K'} - \sum_{j=1}^{N} \Gamma_{K'j}.$$

This implies that

$$\sum_{j=1}^{N} \partial\Gamma_{K'j} = \partial S_{K'}$$

and therefore, by (1.3),

$$\sum_{j=1}^{N} \partial\Gamma_{K'j} = \sum_{j=1}^{N} S_{K'j}. \tag{4.8}$$

Moreover, since

$$\sum_{j,i=1}^{N} \Gamma_{K'ji} = 0,$$

it follows from (4.7) that

$$\sum_{j=1}^{N} \partial\Gamma_{K'j} = \sum_{j=1}^{N} \widetilde{S}_{K'j}.$$

Comparing this with (4.8) we see that $\widetilde{S}_{K'j} = S_{K'j}$ for all $1 \le j \le N$. Hence, in particular, $\widetilde{S}_K = S_K$. □

**4.4. Lemma.** — *If $\Gamma_K$ are the manifolds defined in sect. 4.2 and $\Delta_K, \Delta_{OK}$ are oriented simplices introduced in sect. 1.6, then*

$$\sum_{K\in P'(N)} (-1)^{|K|}\partial(\Gamma_K \times \Delta_{OK})$$
$$= \overline{D} \times \Delta_O + \sum_{K\in P'(N)} (-1)^{|K|} S_K \times \Delta_{OK} - \sum_{K\in P'(N)} \Gamma_K \times \Delta_K.$$

*Proof.* — If $K = (k_1, \dots, k_\ell) \in P'(N)$, then (in addition to sect. 1.4) we introduce the notations

$$j(K,\nu) = k_\nu \quad (\nu = 1, \dots, \ell)$$

and

$$CK = \{1, \dots, N\} \setminus \{k_1, \dots, k_\ell\}.$$

Then we obtain from Lemma 4.3 that

$$\partial(\Gamma_K \times \Delta_{OK}) = S_K \times \Delta_{OK} - \sum_{j\in CK} \Gamma_{Kj} \times \Delta_{OK}$$
$$+ \sum_{\nu=1}^{|K|} (-1)^{|K|+\nu+1} \Gamma_K \times \Delta_{OK(\hat{\nu})} + (-1)^{|K|+1} \Gamma_K \times \Delta_K$$

for all $K \in P'(N)$. Since

$$\Gamma_K = (-1)^{|K|+\nu}\Gamma_{K(\hat{\nu})j(K,\nu)},$$

this implies that

$$\begin{aligned}\sum_{K\in P'(N)} &(-1)^{|K|}\partial(\Gamma_K \times \Delta_{OK})\\ &= \sum_{K\in P'(N)}(-1)^{|K|}S_K \times \Delta_{OK} - \sum_{K\in P'(N)}\Gamma_K \times \Delta_K + \sum_{s=1}^{N} M_s,\end{aligned}$$

where

$$M_1 := \sum_{j=1}^{N}\Gamma_j \times \Delta_0$$

and, for $2 \leq s \leq N$,

$$M_s := \sum_{\substack{K\in P'(N)\\ |K|=s\\ 1\leq\nu\leq s}}(-1)^{|K|}\Gamma_{K(\hat{\nu})j(K,\nu)} \times \Delta_{OK(\hat{\nu})} - \sum_{\substack{K\in P'(N)\\ |K|=s-1\\ j\in CK}}(-1)^{|K|}\Gamma_{Kj} \times \Delta_{OK}.$$

Since $\Gamma_1 + \cdots + \Gamma_N = \overline{D}$ and the sets of pairs

$$\{(K,j) : K \in P'(N), |K| = \ell - 1,\ \ j \in CK\}$$

and

$$\{\big(K(\hat{\nu}), j(K,\nu)\big) : K \in P'(N), |K| = \ell,\ \ 1 \leq \nu \leq \ell\}$$

are equal, this completes the proof. □

**4.5. The function $\Phi(z,\zeta,\lambda)$ and the map $\eta(z,\zeta,\lambda)$.** — Set

$$\rho_\lambda = \lambda_1\rho_1 + \cdots + \lambda_N\rho_N \quad \text{for} \quad \lambda \in \Delta_{1\cdots N},$$

and let $\Psi(z,\zeta,\lambda)$ and $w(z,\zeta,\lambda)$ be the maps defined by (3.3). We set

$$\Phi(z,\zeta,\lambda) = \Psi(z,\zeta,\lambda) - 2\rho_\lambda(\zeta) \tag{4.9}$$

for all $(z,\zeta,\lambda) \in \mathbb{C}^n \times U_{\overline{D}} \times \Delta_{1\cdots N}$. Then it follows from (3.5) that

$$\operatorname{Re}\Phi(z,\zeta,\lambda) \geq -\rho_\lambda(\zeta) - \rho_\lambda(z) + \frac{\alpha}{2}|\zeta - z|^2, \tag{4.10}$$

for all $(z,\zeta,\lambda) \in \mathbb{C}^n \times U_{\overline{D}} \times \Delta_{1\cdots N}$, where $\alpha > 0$ is the constant from condition (ii) in Definition 2.3. In particular, $\Phi(z,\zeta,\lambda) \neq 0$ if $(z,\zeta,\lambda) \in D \times \overline{D} \times \Delta_{1\cdots N}$, and we can define the $C^1$ map

$$\eta(z,\zeta,\lambda) = \mathring{\chi}(\lambda_0)\frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^2} + \big(1 - \mathring{\chi}(\lambda_0)\big)\frac{w(z,\zeta,\mathring{\lambda})}{\Phi(z,\zeta,\mathring{\lambda})} \tag{4.11}$$

for all $(z,\zeta,\lambda) \in D \times \overline{D} \times \Delta_{O1\cdots N}$ with $z \neq \zeta$ (for the definitions of $\overset{\circ}{\chi}$ and $\overset{\circ}{\lambda}$, see sects. 1.7 and 1.8). Note that

$$\eta(z,\zeta,\lambda) = \frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^2} l \text{ if } 1/2 \leq \lambda_0 \leq 1, \tag{4.12}$$

$$\eta(z,\zeta,\lambda) = \frac{w(z,\zeta,\overset{\circ}{\lambda})}{\Phi(z,\zeta,\overset{\circ}{\lambda})} \text{ if } 0 \leq \lambda_0 \leq 1/4, \tag{4.13}$$

$$\eta(z,\zeta,\lambda) = \frac{w(z,\zeta,\lambda)}{\Phi(z,\zeta,\lambda)} \text{ if } \lambda_0 = 0. \tag{4.14}$$

Further we notice that, by (4.11), (4.9), (3.6) and (1.4), for all $K \in P'(N)$ we have the relation

$$\eta(z,\zeta,\lambda) = \psi_{OK}(z,\zeta,\lambda) \text{ if } (\zeta,\lambda) \in S_K \times \Delta_{OK}. \tag{4.15}$$

From Lemma 3.3 one immediately obtains the following

**4.6. Lemma.** — *For fixed $(\zeta,\lambda) \in U_{\overline{D}} \times \Delta_{1\cdots N}$, the function $\Phi(z,\zeta,\lambda)$ is $(q+1)$-holomorphic in $z \in \mathbb{C}^n$, and the map $\eta(z,\zeta,\lambda)$ is $(q+1)$-holomorphic in $z \in D$.*

**4.7. The kernels $\widehat{G}(z,\zeta,\lambda)$ and $\widehat{H}(z,\zeta,\lambda)$.** — Let $\eta(z,\zeta,\lambda)$ be the map defined by (4.11). Then, for all $(z,\zeta,\lambda) \in D \times \overline{D} \times \Delta_{01\cdots N}$ with $z \neq \zeta$ we introduce the continuous differential forms

$$\widehat{G}(z,\zeta,\lambda) = \frac{1}{(2\pi i)^n} \det\Big(\overbrace{\eta(z,\zeta,\lambda)}^{1}, \overbrace{d\eta(z,\zeta,\lambda)}^{n-1}\Big) \wedge dz_1 \wedge \cdots \wedge dz_n \tag{4.16}$$

and

$$\widehat{H}(z,\zeta,\lambda) = \frac{1}{(2\pi i)^n} \det\Big(\overbrace{d\eta(z,\zeta,\lambda)}^{n}\Big) \wedge dz_1 \wedge \cdots \wedge dz_n, \tag{4.17}$$

where $d$ is the exterior differential operator with respect to all variables $z,\zeta,\lambda$ (for the definition of the determinants, see, e.g., sect. 0.7 in [HeLe2]).

Then it is easy to see that

$$d\widehat{G} = \widehat{H}. \tag{4.18}$$

Further, it follows from (4.12) and the definition of the Martinelli-Bochner kernel $\widehat{B}$ (see sect. 1.12) that

$$\widehat{G}|_{D\times\overline{D}\times\Delta_0} = \widehat{B}, \tag{4.19}$$

and it follows from (4.15) and the definition of the Cauchy-Fantappie kernels $\widehat{R}_K$ (see sect. 1.13) that, for all $K \in P'(N)$,

$$\widehat{G}|_{D\times S_K\times\Delta_{OK}} = (-1)^{|K|}\widehat{R}_K. \tag{4.20}$$

We omit the simple proof of the following

**4.8. Lemma.** — *Denote by $[\widehat{G}(z,\zeta,\lambda)]_{\deg\lambda=k}$ and $[\widehat{H}(z,\zeta,\lambda)]_{\deg\lambda=k}$ the parts of the forms $\widehat{G}(z,\zeta,\lambda)$ and $\widehat{H}(z,\zeta,\lambda)$, respectively, which are of degree $k$ in $\lambda$. Then the following statements hold :*

(i) *The singularity at $z=\zeta$ of the form $[\widehat{G}(z,\zeta,\lambda)]_{\deg\lambda=k}$ is of order $\leq 2n-2k-1$.*

(ii) *The singularities at $z=\zeta$ of the first-order derivatives with respect to $z$ of the coefficients of $[\widehat{G}(z,\zeta,\lambda)]_{\deg\lambda=k}$ are of order $\leq 2n-2k$.*

(iii) *The singularity at $z=\zeta$ of the form $[\widehat{H}(z,\zeta,\lambda)]_{\deg\lambda=k}$ is of order $\leq 2n-2k+1$.*

**4.9. Lemma.** — *Let $\widehat{G}(z,\zeta,\lambda)$ be the form defined by (4.16). Then the following two statements hold :*

(i) *If $f \in C^0_{n,r}(\overline{D})$ with $n-q+1 \leq r \leq n$, then*

$$\int_{(\zeta,\lambda)\in\Gamma_K\times\Delta_K} f(\zeta)\wedge\widehat{G}(z,\zeta,\lambda) = 0$$

*for all $K \in P'(N)$ and $z \in D$.*

(ii) *If $f \in C^0_{n,n-q}(\overline{D})$, then*

$$d_z\int_{(\zeta,\lambda)\in\Gamma_K\times\Delta_K} f(\zeta)\wedge\widehat{G}(z,\zeta,\lambda) = 0$$

*for all $z \in D$ and $K \in P'(N)$, where $d_z$ is the exterior differential operator with respect to $z \in D$.*

*Proof.* — Denote by $[\widehat{G}(z,\zeta,\lambda)]_j$ the part of $\widehat{G}(z,\zeta,\lambda)$ which is of bidegree $(n,j)$ in $z$, and let $K \in P'(N)$. Then

$$\int_{\Gamma_K\times\Delta_K} f(\zeta)\wedge\widehat{G}(z,\zeta,\lambda) = \int_{\Gamma_K\times\Delta_K} f(\zeta)\wedge[\widehat{G}(z,\zeta,\lambda)]_{r-1},\ z\in D,$$

if $f \in C^0_{n,r}(\overline{D})$. On the other hand, since, by Lemma 4.6, $\eta(z,\zeta,\lambda)$ is $(q+1)$-holomorphic in $z$ if $\lambda_0 = 0$, we see that

$$[\widehat{G}(z,\zeta,\lambda)]_{r-1} = 0 \ \text{ on } \ D\times\Gamma_K\times\Delta_K$$

if $r \geq n - q + 1$, and

$$d_z[\widehat{G}(z,\zeta,\lambda)]_{n-q-1} = 0 \quad \text{on} \quad D \times \Gamma_K \times \Delta_K.$$

Together this implies assertions (i) and (ii) of the lemma. □

**4.10. The operator** $H$. — Let $f \in B^{\beta}_{n,*}(D), 0 \leq \beta < 1$ (see sect. 1.16). Then, for all $K \in P'(N)$, we define

$$H_K f(z) = \int_{(\zeta,\lambda)\in\Gamma_K\times\Delta_{OK}} f(\zeta) \wedge \widehat{H}(z,\zeta,\lambda), \quad z \in D. \tag{4.21}$$

It follows from Lemma 4.8 (iii) that these integrals converge and the so defined differential forms $H_K f$ are continuous on $D$. We set

$$Hf = \sum_{K\in P'(N)} (-1)^{|K|} H_K f \tag{4.22}$$

for $f \in B^{\beta}_{n,*}(D), 0 \leq \beta < 1$.

Now let $f \in B^{\beta}_{n,r}(D), 0 \leq \beta < 1, 0 \leq r \leq n$. Since $\widehat{H}(z,\zeta,\lambda)$ is of degree $2n$ and contains the factor $dz_1 \wedge \cdots \wedge dz_n$ and since $\dim_{\mathbb{R}} \Gamma_K \times \Delta_{OK} = 2n+1$, then only such monomials of $\widehat{H}(z,\zeta,\lambda)$ contribute to the integral in (4.21) which are of degree $(n+1-r)$ in $(\zeta,\lambda)$ and hence of bidegree $(n, r-1)$ in $z$. This implies that $H_K f = 0$ if $r = 0$ or $n + 1 - r < |K| = \dim_{\mathbb{R}} \Delta_{OK}$.

Hence, for $f \in B^{\beta}_{n,r}(D), 0 \leq \beta < 1, 0 \leq r \leq n$, we have

$$\begin{cases} Hf = \displaystyle\sum_{\substack{K\in P'(N)\\ |K|\leq n+1-r}} (-1)^{|K|} H_K f, \\ Hf = 0 \text{ if } r = 0, \text{ and } Hf \in C^0_{n,r-1}(D) \text{ if } 1 \leq r \leq n. \end{cases} \tag{4.23}$$

**4.11. Theorem.** — *Let $n - q \leq r \leq n$ and $0 \leq \beta < 1$. Then*

$$f = dHf + Hdf \quad \text{on} \quad D \tag{4.24}$$

*for all $f \in B^{\beta}_{n,r}(D)$ such that also $df \in B^{\beta}_{*}(D)$.*

*Proof.* — First consider a form $g \in C^0_{n,j}(\overline{D})$. Then by (4.18)

$$d_{\zeta,\lambda}(g \wedge \widehat{G}) = dg \wedge \widehat{G} - d_z(g \wedge \widehat{G}) + (-1)^{n+j} g \wedge \widehat{H}$$

and it follows from Stokes'formula (which can be applied in view of Lemma 4.8) that

$$\int_{\partial(\Gamma_K\times\Delta_{OK})} g \wedge \widehat{G} = \int_{\Gamma_K\times\Delta_{OK}} dg \wedge \widehat{G} + d\int_{\Gamma_K\times\Delta_{OK}} g \wedge \widehat{G} + (-1)^{n+j} H_K g$$

for all $K \in P'(N)$. In view of Lemma 4.4 this implies that

$$\int_{D\times\Delta_0} g\wedge\widehat{G} + \sum_{K\in P'(N)} (-1)^{|K|} \int_{S_K\times\Delta_{OK}} g\wedge\widehat{G} - \sum_{K\in P'(N)} \int_{\Gamma_K\times\Delta_K} g\wedge\widehat{G}$$

$$= \sum_{K\in P'(N)} (-1)^{|K|} \left( \int_{\Gamma_K\times\Delta_{OK}} dg\wedge\widehat{G} + d\int_{\Gamma_K\times\Delta_{OK}} g\wedge\widehat{G} + (-1)^{n+j} H_K g \right).$$

Taking into account (4.19) and (4.20) as well as the definitions of $T^\psi$ and $H$, this can be written

$$\begin{aligned} T^\psi g - \sum_{K\in P'(N)} \int_{\Gamma_K\times\Delta_K} g\wedge\widehat{G} = \sum_{K\in P'(N)} (-1)^{|K|} \Bigg( \int_{\Gamma_K\times\Delta_{OK}} dg\wedge\widehat{G} \\ + d\int_{\Gamma_K\times\Delta_{OK}} g\wedge\widehat{G} \Bigg) + (-1)^{n+j} Hg. \end{aligned} \tag{4.25}$$

Now we consider a form $f \in C^0_{n,r}(\overline{D})$ with $n-q\le r\le n$ such that $df$ is also continuous on $\overline{D}$. Setting $g = df$ in (4.25) and taking into account Lemma 4.9 (i), we obtain that

$$T^\psi df = \sum_{K\in P'(N)} (-1)^{|K|} d\int_{\Gamma_K\times\Delta_{OK}} df\wedge\widehat{G} + (-1)^{n+r+1} Hdf.$$

Setting $g = f$ in (4.25), applying $d$ to the resulting relation and taking into account Lemma 4.9 (ii), we obtain that

$$dT^\psi f = \sum_{K\in P'(N)} (-1)^{|K|} d\int_{\Gamma_K\times\Delta_{OK}} df\wedge\widehat{G} + (-1)^{n+r} dHf.$$

Together this implies that

$$dT^\psi f - T^\psi df = (-1)^{n+r}(dHf + Hdf),$$

and hence, by Theorem 4.1,

$$f = dHf + Hdf. \tag{4.26}$$

Finally we consider the general case. Let $f \in B^\beta_{n,r}(D), 0\le\beta<1, n-q\le r\le n$, such that also $df \in B^\beta_*(D)$. Choose $\varepsilon>0$ with $\beta+\varepsilon<1$. Then, by local shifts of $f$ and a partition of unity argument, we can find a sequence of forms $f_\nu \in C^0_{n,r}(\overline{D})$ such that also the forms $df_\nu$ are continuous on $\overline{D}$ and

$$f_\nu \longrightarrow f \quad\text{and}\quad df_\nu \longrightarrow df$$

in the space $B^{\beta+\varepsilon}_*(D)$. By Lemma 4.8 (iii), then

$$Hf_\nu \longrightarrow Hf \quad\text{and}\quad Hdf_\nu \longrightarrow Hdf$$

uniformly on the compact subsets of $D$. Since, by (4.26),

$$f_\nu = dHf_\nu + Hdf_\nu,$$

this implies that

$$f = dHf + Hdf. \qquad \square$$

**4.12. THEOREM.**

(i) *Let $0 \leq \beta < 1/2, 0 < \varepsilon \leq 1/2 - \beta$, and $1 \leq r \leq n$. Then*

$$H\big(B^{\beta}_{n,r}(D)\big) \subseteq C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D})$$

*and the operator $H$ is compact as operator between the Banach spaces $B^{\beta}_{n,r}(D)$ and $C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D})$.*

(ii) *Let $1/2 \leq \beta < 1, 0 < \varepsilon \leq 1 - \beta$, and $1 \leq r \leq n$. Then*

$$H\big(B^{\beta}_{n,r}(D)\big) \subseteq B^{\beta+\varepsilon-1/2}_{n,r-1}(D)$$

*and the operator $H$ is compact as operator between the Banach spaces $B^{\beta}_{n,r}(D)$ and $B^{\beta+\varepsilon-1/2}_{n,r-1}(D)$.*

The following sects. 5-8 are devoted to the proof of this theorem.

## 5. A first description of the singularity of the kernel of $H$.

In this section we assume :

$D \subset\subset \mathbb{C}^n$ is a local $q$-convex domain, $0 \leq q \leq n-1$ (see Definition 2.3);

$(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ is a frame for $D$ satisfying conditions (i) and (ii) in Definition 2.3;

$\Gamma_K, K \in P(N)$, are the submanifolds of $\overline{D}$ which belong to the frame $(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ according to sect. 4.2;

$\Phi(z, \zeta, \lambda)$ is the function defined for $(z, \zeta, \lambda) \in \mathbb{C}^n \times U_{\overline{D}} \times \Delta_{1\ldots N}$ by (4.9) in sect. 4.5.

**5.1. DEFINITION.** — *Let $K \in P'(N)$ and let $s$ be an integer.*

*A form of type $O_s$* (or of type $O_s(z, \zeta, \lambda)$) *on $D \times \Gamma_K \times \Delta_{OK}$ is, by definition, a continuous differential form $f(z, \zeta, \lambda)$ defined for all $(z, \zeta, \lambda) \in D \times \Gamma_K \times \Delta_{OK}$ with $z \neq \zeta$ such that the following conditions are fulfilled :*

(i) *All derivatives of the coefficients of* $f(z,\zeta,\lambda)$ *which are of order* 0 *in* $\zeta$, *of order* $\leq 1$ *in* $z$, *and of arbitrary order in* $\lambda$ *are continuous for all* $(z,\zeta,\lambda)\in D\times\Gamma_K\times\Delta_{OK}$ *with* $z\neq\zeta$.

(ii) *Let* $\nabla_z^\kappa, \kappa=0,1$, *be a differential operator with constant coefficients which is of order 0 in* $\zeta$, *of order* $\kappa$ *in* $z$, *and of arbitrary order in* $\lambda$. *Then there is a constant* $C>0$ *such that, for each coefficient* $\varphi(z,\zeta,\lambda)$ *of the form* $f(z,\zeta,\lambda)$,

$$|\nabla_z^\kappa\varphi(z,\zeta,\lambda)|\leq C|\zeta-z|^{s-\kappa}$$

*for all* $(z,\zeta,\lambda)\in D\times\Gamma_K\times\Delta_{OK}$ *with* $z\neq\zeta$.

(iii) *There exist neighborhoods* $U_0,U_K\subseteq\Delta_{OK}$ *of* $\Delta_0$ *and* $\Delta_K$, *respectively, such that* $f(z,\zeta,\lambda)=0$ *for all* $(z,\zeta,\lambda)\in D\times\Gamma_K\times(U_0\cup U_K)$.

The symbols $O_s(z,\zeta,\lambda)$ and $O_s$ will be used also to denote forms of this type, also in formulas. For example :

$f=O_s$ means : $f$ is a form of type $O_s$.

$O_s\wedge f=O_k\wedge g+O_m$ means : for each form $h$ of type $O_s$ there exist a form $u$ of type $O_k$ and a form $v$ of type $O_m$ such that $h\wedge f=u\wedge g+v$.

The equation

$$Ef(z)=\int_{(\zeta,\lambda)\in S_K\times\Delta_{OK}}O_s(z,\zeta,\lambda)\wedge f(z,\zeta,\lambda)$$

means : there exists a form $\widehat{E}$ of type $O_s$ such that

$$Ef(z)=\int_{(\zeta,\lambda)\in S_K\times\Delta_{OK}}\widehat{E}(z,\zeta,\lambda)\wedge f(z,\zeta,\lambda)$$

for all $f$.

**5.2. Definition.** — *Let* $m\geq 0$ *be an integer.* An operator of type $m$ *is, by definition, a map*

$$E:\bigcup_{0\leq\beta<1}B^\beta_{n,*}(D)\longrightarrow C^0_{n,*}(D)$$

*such that there exist*

*– an integer* $k\geq 0$,

*–* $K\in P'(N)$,

*– a form* $\widehat{E}(z,\zeta,\lambda)$ *of type* $O_{|K|-2n+2k+m}$ *on* $D\times\Gamma_K\times\Delta_{OK}$ *such that, for all* $f\in B^\beta_{n,*}(D), 0\leq\beta<1$,

$$Ef(z)=\int_{(\zeta,\lambda)\in\Gamma_K\times\Delta_{OK}}\tilde{f}(\zeta)\wedge\frac{\widehat{E}(z,\zeta,\lambda)\wedge\Theta(\zeta)}{\Phi^{k+m}(z,\zeta,\overset{\circ}{\lambda})}$$

*where $\tilde{f} \in B^{\beta}_{0,*}(D)$ is the form with*

$$f(\zeta) = \tilde{f}(\zeta) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

*and for $\Theta$ holds the following :*

*if $m = 0$, then $\Theta = 1$;*

*if $m \geq 1$, then there exist indices $i_1, \ldots, i_m \in K$ such that either*

$$\Theta = \partial\rho_{i_1} \wedge \cdots \wedge \partial\rho_{i_m} \quad \text{or} \quad \Theta = \overline{\partial}\rho_{i_1} \wedge \partial\rho_{i_2} \wedge \cdots \wedge \partial\rho_{i_m}$$

*(for the definition of $\overset{\circ}{\lambda}$, see sect. 1.8).*

**5.3.** LEMMA. — *Let $E$ be an operator of type 0, let $0 \leq \beta < 1$ and $0 < \varepsilon < 1 - \beta$. Then $E\big(B^{\beta}_{n,*}(D)\big) \subseteq C^{1-\beta-\varepsilon}_{n,*}(\overline{D})$ and there exists a constant $C > 0$ such that*

$$\|Ef\|_{1-\beta-\varepsilon,D} \leq C\|f\|_{-\beta,D} \tag{5.1}$$

*for all $f \in B^{\beta}_{n,*}(D)$.*

*Proof.* — In this proof we denote all positive constants by the same letter $C$, and we use the abbreviations

$$\rho_\lambda = \lambda_1\rho_1 + \cdots + \lambda_N\rho_N, \quad \lambda \in \Delta_{1\cdots N},$$

and

$$d(z) = \operatorname{dist}(z, \partial D), \quad z \in D.$$

Further, let $\rho_K$ be the function on $\Gamma_K$ defined by $\rho_K(\zeta) = \rho_j(\zeta)$ for $j \in K$ and $\zeta \in \Gamma_K$ (see (4.2)). Note that

$$d(\zeta) \geq C|\rho_K(\zeta)| \quad \text{for} \quad \zeta \in \Gamma_K, \tag{5.2}$$

$$|\rho_\lambda(z)| \geq Cd(z) \quad \text{for} \quad (z, \lambda) \in D \times \Delta_{1\cdots N}, \tag{5.3}$$

and

$$\rho_K(\zeta) = \rho_\lambda(\zeta) \quad \text{for} \quad (\zeta, \lambda) \in \Gamma_K \times \Delta_K. \tag{5.4}$$

In view of (5.3) and (5.4) it follows from (4.10) that

$$|\Phi(z, \zeta, \overset{\circ}{\lambda})| \geq C\big(|\rho_K(\zeta)| + d(z) + |\zeta - z|^2\big) \tag{5.5}$$

for all $(z, \zeta, \lambda) \in D \times \Gamma_K \times (\Delta_{OK} \setminus \Delta_0)$.

Now we first consider the case that the integer $k$ in Definition 5.2 is zero. Then it follows from (5.2) that

$$\|f(\zeta) \wedge \widehat{E}(z, \zeta, \lambda)\| \leq \frac{C\|f\|_{-\beta,D}}{|\rho_K(\zeta)|^{\beta}|\zeta - z|^{2n-|K|}} \tag{5.6}$$

and, if $\nabla_z$ is one of the operators $\partial/\partial z_j, \partial/\partial \overline{z}_j$,

$$\|\nabla_z f(\zeta) \wedge \widehat{E}(z,\zeta,\lambda)\| \leq \frac{C\|f\|_{-\beta,D}}{|\rho_K(\zeta)|^\beta |\zeta - z|^{2n-|K|+1}} \tag{5.7}$$

for all $f \in B^\beta_{n,*}(D)$ and $(z,\zeta,\lambda) \in D \times \Gamma_K \times \Delta_{OK}$. Set

$$I_1(f,z,w) = \int_{\substack{(\zeta,\lambda)\in\Gamma_K\times\Delta_{OK} \\ |\zeta-z|<|z-w|}} f(\zeta) \wedge \widehat{E}(z,\zeta,\lambda) - \int_{\substack{(\zeta,\lambda)\in\Gamma_K\times\Delta_{OK} \\ |\zeta-w|<|z-w|}} f(\zeta) \wedge \widehat{E}(w,\zeta,\lambda)$$

and

$$I_2(f,z,w) = \int_{\substack{(\zeta,\lambda)\in\Gamma_K\times\Delta_{OK} \\ |\zeta-z|,|\zeta-w|>|z-w|}} f(\zeta) \wedge \left(\widehat{E}(z,\zeta,\lambda) - \widehat{E}(w,\zeta,\lambda)\right)$$

for all $f \in B^\beta_{n,*}(D)$ and $z, w \in D$. Then

$$Ef(z) - Ef(w) = I_1(f,z,w) + I_2(f,z,w) \tag{5.8}$$

for all $f \in B^\beta_{n,*}(D)$ and $z, w \in D$. Since $\rho_K$ is a local coordinate on $\Gamma_K$, it follows from (5.6) that

$$\begin{aligned} \|I_1(f,z,w)\| &\leq C\|f\|_{-\beta,D} \int_{\substack{t\in\mathbf{R}^{2n-|K|+1} \\ |t|<|z-w|}} \frac{dt_1 \wedge \cdots \wedge dt_{2n-|K|+1}}{|t_1|^\beta |t|^{2n-|K|}} \\ &\leq C|z-w|^{1-\beta}\|f\|_{-\beta,D} \end{aligned} \tag{5.9}$$

for all $f \in B^\beta_{n,*}(D)$ and $z, w \in D$. Further, it follows from (5.6) that

$$\begin{aligned} \|Ef(z)\| &\leq C\|f\|_{-\beta,D} \int_{\substack{t\in\mathbf{R}^{2n-|K|+1} \\ |t|<C}} \frac{dt_1 \wedge \cdots \wedge dt_{2n-|K|+1}}{|t_1|^\beta |t|^{2n-|K|}} \\ &\leq C\|f\|_{-\beta,D} \end{aligned} \tag{5.10}$$

for all $f \in B^\beta_{n,*}(D)$ and $z \in D$. From (5.7) it follows that

$$\begin{aligned} \|f(\zeta) \wedge \left(\widehat{E}(z,\zeta,\lambda) - \widehat{E}(w,\zeta,\lambda)\right)\| &\leq \frac{C|z-w|\|f\|_{-\beta,D}}{|\rho_K(\zeta)|^\beta|\zeta - z|^{2n-|K|+1}} \\ &\leq \frac{C|z-w|^{1-\beta-\varepsilon}\|f\|_{-\beta,D}}{|\rho_K(\zeta)|^\beta|\zeta - z|^{2n-|K|+1-\beta-\varepsilon}} \end{aligned}$$

for all $f \in B^\beta_{n,*}(D), z, w \in D$ and $(\zeta,\lambda) \in \Gamma_K \times \Delta_{OK}$ with $|\zeta - z|, |\zeta - w| > |z-w|$. Hence

$$\begin{aligned} \|I_2(f,z,w)\| &\leq C|z-w|^{1-\beta-\varepsilon}\|f\|_{-\beta,D} \int_{\substack{t\in\mathbf{R}^{2n-|K|+1} \\ |t|<C}} \frac{dt_1 \wedge \cdots \wedge dt_{2n-|K|+1}}{|t_1|^\beta |t|^{2n-|K|+1-\beta-\varepsilon}} \\ &\leq C|z-w|^{1-\beta-\varepsilon}\|f\|_{-\beta,D} \end{aligned} \tag{5.11}$$

for all $f \in B^{\beta}_{n,*}(D)$ and $z, w \in D$. Now it follows from (5.8), (5.9) and (5.11) that

$$\|Ef(z) - Ef(w)\| \leq C|z-w|^{1-\beta-\varepsilon}\|f\|_{-\beta,D}$$

for all $f \in B^{\beta}_{n,*}(D)$ and $z, w \in D$. Together with (5.10) this implies (5.1).

Now we consider the case that $k \geq 1$.

In view of (5.2) and (5.5) then we obtain that

$$\left\| f(\zeta) \wedge \frac{\widehat{E}(z,\zeta,\lambda)}{\Phi^k(z,\zeta,\overset{\circ}{\lambda})} \right\| \leq \frac{C\|f\|_{-\beta,D}}{|\rho_K(\zeta)|^{\beta}|\zeta - z|^{2n-|K|}}$$

for all $(z,\zeta,\lambda) \in D \times \Gamma_K \times \Delta_{OK}$. By the same arguments as in the case $k = 0$ this leads to estimate (5.10). Further we see that now, since $k \geq 1, Ef$ is of class $C^1$ and, if $\nabla_z$ is one of the operators $\partial/\partial z_j$ and $\partial/\partial \overline{z_j}$, then

$$\|\nabla_z Ef(z)\| \leq C\|f\|_{-\beta,D} \int_{(\zeta,\lambda)\in\Gamma_K\times\Delta_{OK}} \frac{d\sigma_{K\times\Delta}}{|\rho_K(\zeta)|^{\beta}|\Phi^{k+1}(z,\zeta,\overset{\circ}{\lambda})||\zeta - z|^{2n-|K|-2k}}$$
$$+ C\|f\|_{-\beta,D} \int_{(\zeta,\lambda)\in\Gamma_K\times\Delta_{OK}} \frac{d\sigma_{K\times\Delta}}{|\rho_K(\zeta)|^{\beta}|\Phi^{k}(z,\zeta,\overset{\circ}{\lambda})||\zeta - z|^{2n-|K|-2k+1}}$$

for all $f \in B^{\beta}_{n,*}(D)$ and $z \in D$, where $d\sigma_{K\times\Delta}$ is the Euclidean volume form on $\Gamma_K \times \Delta_{OK}$. In view of (5.5) this implies that

$$\|\nabla_z Ef(z)\| \leq C\|f\|_{-\beta,D} \int_{\zeta\in\Gamma_K} \frac{d\sigma_K}{|\rho_K(\zeta)|^{\beta}\big(|\rho_K(\zeta)| + d(z) + |\zeta - z|^2\big)^2|\zeta - z|^{2n-|K|-2}}$$
$$+ C\|f\|_{-\beta,D} \int_{\zeta\in\Gamma_K} \frac{d\sigma_K}{|\rho_K(\zeta)|^{\beta}\big(|\rho_K(\zeta)| + d(z) + |\zeta - z|^2\big)|\zeta - z|^{2n-|K|-1}}$$

for all $f \in B^{\beta}_{n,*}(D)$ and $z \in D$. Since $\rho_K$ can be used as local coordinate on $\Gamma_K$, this implies that

$$\|\nabla_z Ef(z)\| \leq C\|f\|_{-\beta,D} \int_{t\in\mathbf{R}^{2n-|K|+1}} \frac{dt_1 \wedge \cdots \wedge dt_{2n-|K|+1}}{|t_1|^{\beta}\big(|t_1| + d(z) + |t|^2\big)^2|t|^{2n-|K|-2}}$$
$$+ C\|f\|_{-\beta,D} \int_{t\in\mathbf{R}^{2n-|K|+1}} \frac{dt_1 \wedge \cdots \wedge dt_{2n-|K|+1}}{|t_1|^{\beta}\big(|t_1| + d(z) + |t|^2\big)|t|^{2n-|K|-1}}$$

for all $f \in B^{\beta}_{n,*}(D)$ and $z \in D$. After integrating over $t_1$, one obtains

$$\|\nabla_z Ef(z)\| \leq C\|f\|_{-\beta,D} \int_{t\in\mathbf{R}^{2n-|K|}} \frac{dt_1 \wedge \cdots \wedge dt_{2n-|K|}}{\big(d(z) + |t|^2\big)^{1+\beta}|t|^{2n-|K|-2}}$$
$$+ C\|f\|_{-\beta,D} \int_{t\in\mathbf{R}^{2n-|K|}} \frac{dt_1 \wedge \cdots \wedge dt_{2n-|K|}}{\big(d(z) + |t|^2\big)^{\beta+1/2}|t|^{2n-|K|-1}}$$

for all $f \in B^{\beta}_{n,*}(D)$ and $z \in D$. Hence

$$\begin{aligned} \|\nabla_z Ef(z)\| &\leq C\|f\|_{-\beta,D}\left[\int_0^\infty \frac{rdr}{(d(z)+r^2)^{1+\beta}} + \int_0^\infty \frac{dr}{(d(z)+r^2)^{\beta+1/2}}\right] \\ &\leq C\|f\|_{-\beta,D}[d(z)]^{-\beta} \end{aligned} \tag{5.12}$$

for all $f \in B^{\beta}_{n,*}(D)$ and $z \in D$. (5.10) and (5.12) together imply estimate (5.1) (cp., e.g., Proposition 2 in [HeLe1]). □

*Remark.* — This proof shows that in the case $k \geq 1$ estimate (5.1) holds even with $\varepsilon = 0$.

**5.4. THEOREM.** — *The operator $H$ defined in sect. 4.10 is a finite sum of operators of type $m$ (for certain integers $m \geq 0$ - see Definition 5.2).*

*Proof.* — It is sufficient to prove that each of the operators $H_k, K \in P'(N)$, is a finite sum of operators of type $m$. Let $K \in P'(N)$ be fixed. By (4.23) we may assume that $\ell \leq n$.

We use the same notations as in sects. 3 and 4. Set

$$W = W(z,\zeta,\mathring{\lambda}) = \langle w(z,\zeta,\mathring{\lambda}), d\zeta\rangle$$

and

$$M = M(z,\zeta) = \frac{\langle \overline{\zeta}-\overline{z}, d\zeta\rangle}{|\zeta - z|^2}$$

for $(z,\zeta,\lambda) \in D \times \Gamma_K \times \Delta_{OK} \smallsetminus \Delta_0$, where

$$\langle w(z,\zeta,\mathring{\lambda}), d\zeta\rangle = \sum_{j=1}^n w^j(z,\zeta,\mathring{\lambda})d\zeta_j$$

and

$$\langle \overline{\zeta}-\overline{z}, d\zeta\rangle = \sum_{j=1}^n (\overline{\zeta_j}-\overline{z_j})d\zeta_j.$$

Further, we use the abbreviations $w = w(z,\zeta,\lambda), \Phi = \Phi(z,\zeta,\mathring{\lambda})$, $\eta = \eta(z,\zeta,\lambda)$, $\mathring{\chi} = \mathring{\chi}(\lambda_0)$. Then, by (4.11),

$$d\eta = \Big(\frac{\overline{\zeta}-\overline{z}}{|\zeta-z|^2} - \frac{w}{\Phi}\Big)d\mathring{\chi} + \mathring{\chi}\, d\frac{\overline{\zeta}-\overline{z}}{|\zeta-z|^2} + (1-\mathring{\chi})\frac{dw}{\Phi} + (\mathring{\chi}-1)\frac{w}{\Phi^2}\wedge d\Phi$$

and therefore

$$\langle d\eta, d\zeta\rangle = \Big(\frac{W}{\Phi} - M\Big)\wedge d\mathring{\chi} + \mathring{\chi}\, dM + (1-\mathring{\chi})\frac{dW}{\Phi} + (1-\mathring{\chi})\frac{W}{\Phi^2}\wedge d\Phi, \tag{5.13}$$

for all $(z,\zeta,\lambda) \in D \times \overline{D} \times \Delta_{01\cdots N}$ with $z \neq \zeta$, where $\langle d\eta, d\zeta\rangle = d\eta^1 \wedge d\zeta_1 + \cdots + d\eta^n \wedge d\zeta_n$ and $\eta^1, \ldots, \eta^n$ are the components of $\eta$.

In the following all differential forms which are defined on $D \times \overline{D} \times \Delta_{01\cdots N}$ will be regarded as forms restricted to $D \times \Gamma_K \times \Delta_{OK}$. If $g$ is such a form, then we denote by $[g]_{\deg \lambda = r}$ the part of $g$ which is of degree $r$ in $\lambda$. Then the forms $d\Phi$ and $dW$ are obtained by lifting from $D \times \Gamma_K \times \overset{\circ}{\Delta}_K$ with respect to the map $(z, \zeta, \lambda) \to (z, \zeta, \overset{\circ}{\lambda})$. Since $\dim_{\mathbb{R}} \overset{\circ}{\Delta}_K = \ell - 1$, this implies that

$$[(dW)^s]_{\deg \lambda = \ell} = [(dW)^s \wedge d\Phi]_{\deg \lambda = \ell} = 0 \quad \text{for } s = 1, 2, \cdots.$$

Therefore it follows from (5.13) that

$$\left[\langle d\eta, d\zeta\rangle^n \wedge dz_1 \wedge \cdots \wedge dz_n\right]_{\deg \lambda = \ell} = n\Big(\frac{W}{\Phi} - M\Big) \wedge d\overset{\circ}{\chi} \wedge \Big[\Big(\overset{\circ}{\chi}\, dM + (1 - \overset{\circ}{\chi})\frac{dW}{\Phi} + (1 - \overset{\circ}{\chi})\frac{W \wedge d\Phi}{\Phi^2}\Big)^{n-1}\Big]_{\deg \lambda = \ell - 1} \wedge dz_1 \wedge \cdots \wedge dz_n$$

on $D \times \Gamma_K \times \Delta_{OK}$. Since $\langle d\eta, d\zeta\rangle^n \wedge dz_1 \wedge \cdots \wedge dz_n$ contains the factor $d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge dz_1 \wedge \cdots \wedge dz_n$, in this relation $dW$ and $d\Phi$ may be replaced by $d_\lambda W + \overline{\partial}_{z,\zeta} W$ and $d_\lambda \Phi + \overline{\partial}_{z,\zeta}\Phi$. Hence

$$\begin{aligned}
&\left[\langle d\eta, d\zeta\rangle^n \wedge dz_1 \wedge \cdots \wedge dz_n\right]_{\deg \lambda = \ell} \\
&= n\binom{n-1}{\ell-1}\Big(\frac{W}{\Phi} - M\Big) \wedge d\overset{\circ}{\chi} \wedge \Big(\overset{\circ}{\chi}\, dM + (1 - \overset{\circ}{\chi})\frac{\overline{\partial}_{z,\zeta} W}{\Phi} + (1 - \overset{\circ}{\chi})\frac{W \wedge \overline{\partial}_{z,\zeta}\Phi}{\Phi^2}\Big)^{n-\ell} \\
&\qquad \wedge \Big((1 - \overset{\circ}{\chi})\frac{d_\lambda W}{\Phi} + (1 - \overset{\circ}{\chi})\frac{W \wedge d_\lambda \Phi}{\Phi^2}\Big)^{\ell - 1} \wedge dz_1 \wedge \cdots \wedge dz_n
\end{aligned}$$

on $D \times \Gamma_K \times \Delta_{OK}$. Since $d\overset{\circ}{\chi} = O_0$, $O_0 \wedge M = O_{-1}$, $O_0 \wedge dM = O_{-2}$ and $O_0 \wedge \overline{\partial}_{z,\zeta} W = O_0$ on $D \times \Gamma_K \times \Delta_{OK}$, this implies that

$$\begin{aligned}
&\left[\langle d\eta, d\zeta\rangle^n \wedge dz_1 \wedge \cdots \wedge dz_n\right]_{\deg \lambda = \ell} \\
&= O_0 \wedge \Big(\frac{W}{\Phi} + O_{-1}\Big) \wedge \Big(O_{-2} + \frac{O_0}{\Phi} + \frac{O_0}{\Phi^2} \wedge W \wedge \overline{\partial}_{z,\zeta}\Phi\Big)^{n-\ell} \\
&\qquad \wedge \Big(\frac{O_0}{\Phi} \wedge d_\lambda W + \frac{O_0}{\Phi^2} \wedge W \wedge d_\lambda \Phi\Big)^{\ell-1}
\end{aligned}$$

on $D \times \Gamma_K \times \Delta_{OK}$. Taking into account that $W \wedge W = 0$, it follows that

$$\left[\langle d\eta, d\zeta\rangle^n \wedge dz_1 \wedge \cdots \wedge dz_n\right]_{\deg \lambda = \ell} = \widehat{E}_1 + \cdots + \widehat{E}_4, \tag{5.14}$$

where $\widehat{E}_1, \ldots, \widehat{E}_4$ are forms on $D \times \Gamma_K \times \Delta_{OK}$ with

$$\widehat{E}_1 = \frac{O_{-1}}{\Phi^{\ell-1}} \wedge \Big(O_{-2} + \frac{O_0}{\Phi}\Big)^{n-\ell} \wedge (d_\lambda W)^{\ell-1}$$

$$\widehat{E}_2 = \frac{O_0 \wedge W}{\Phi^{\ell}} \wedge \Big(O_{-2} + \frac{O_0}{\Phi}\Big)^{n-\ell} \wedge (d_\lambda W)^{\ell-1}$$

$$\widehat{E}_3 = \frac{O_{-1}}{\Phi^{\ell+1}} \wedge \Big(O_{-2} + \frac{O_0}{\Phi}\Big)^{n-\ell-1} \wedge (d_\lambda W)^{\ell-1} \wedge W \wedge \overline{\partial}_{z,\zeta}\Phi$$

$$\widehat{E}_4 = \begin{cases} \dfrac{O_{-1}}{\Phi^{\ell}} \wedge \Big(O_{-2} + \dfrac{O_0}{\Phi}\Big)^{n-\ell} \wedge (d_\lambda W)^{\ell-2} \wedge W \wedge d_\lambda \Phi & \text{if } \ell \geq 2 \\ 0 & \text{if } \ell = 1. \end{cases}$$

Now for $f \in B^\beta_{n,*}(D), 0 \le \beta < 1$, we denote by $\tilde{f}$ the form in $B^\beta_{0,*}(D)$ with $f(\zeta) = \tilde{f}(\zeta) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$. Then

$$f(\zeta) \wedge \widehat{H}(z,\zeta,\lambda) = \tilde{f}(\zeta) \wedge \frac{n(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \langle d\eta, d\zeta \rangle^n \wedge dz_1 \wedge \cdots \wedge dz_n. \tag{5.15}$$

Since $\dim_{\mathbf{R}} \Delta_{OK} = \ell$, it follows from (5.15) and (5.14) that $H_K = E_1 + \cdots + E_4$, where

$$E_j f(z) = \int_{(\zeta,\lambda) \in \Gamma_K \times \Delta_{OK}} f(\zeta) \wedge \widehat{E}_j(z,\zeta,\lambda), \quad z \in D,$$

for all $f \in B^\beta_{n,*}(D), 0 \le \beta < 1, (j = 1, \ldots, 4)$.

From the definition of $w$ and $\Phi$ (see (3.3) and (4.9)) it follows that

$$O_0 \wedge W = \sum_{j \in K} O_0 \wedge \partial\rho_j(\zeta) + O_1, \tag{5.16}$$

$$O_0 \wedge d_\lambda W = \sum_{j \in K} O_0 \wedge \partial\rho_j(\zeta) + O_1, \tag{5.17}$$

$$O_0 \wedge \overline{\partial}_{z,\zeta} \Phi = \sum_{j \in K} O_0 \wedge \overline{\partial}\rho_j(\zeta) + O_1$$

on $D \times \Gamma_K \times \Delta_{OK}$ and therefore

$$O_0 \wedge (d_\lambda W)^{\ell-1} = \sum_{\substack{0 \le m \le \ell-1 \\ i_1,\ldots,i_m \in K}} O_{\ell-1-m} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta),$$

$$O_0 \wedge (d_\lambda W)^{\ell-1} \wedge W = \sum_{\substack{0 \le m \le \ell \\ i_1,\ldots,i_m \in K}} O_{\ell-m} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta),$$

$$\begin{aligned}
&O_0 \wedge (d_\lambda W)^{\ell-1} \wedge W \wedge \overline{\partial}_{z,\zeta}\Phi \\
&= \sum_{\substack{0 \le m \le \ell \\ i_1,\ldots,i_{m+1} \in K}} O_{\ell-m} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta) \wedge \overline{\partial}\rho_{i_{m+1}}(\zeta) \\
&\quad + \sum_{\substack{0 \le m \le \ell \\ i_1,\ldots,i_m \in K}} O_{\ell-m+1} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta) \\
&= \sum_{\substack{1 \le m \le \ell+1 \\ i_1,\ldots,i_m \in K}} O_{\ell-m+1} \wedge \overline{\partial}\rho_{i_1}(\zeta) \wedge \partial\rho_{i_2}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta) \\
&\quad + \sum_{\substack{0 \le m \le \ell \\ i_1,\ldots,i_m \in K}} O_{\ell-m+1} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta)
\end{aligned}$$

on $D \times \Gamma_K \times \Delta_{OK}$. This implies that

$$\widehat{E}_1 = \sum_{\substack{0 \le s \le n-\ell \\ 0 \le m \le \ell-1 \\ i_1,\dots,i_m \in K}} \frac{O_{\ell-2n+2(\ell-1+s-m)+m}}{\Phi^{(\ell-1+s-m)+m}} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta),$$

$$\widehat{E}_2 = \sum_{\substack{0 \le s \le n-\ell \\ 0 \le m \le \ell \\ i_1,\dots,i_m \in K}} \frac{O_{\ell-2n+2(\ell+s-m)+m}}{\Phi^{(\ell+s-m)+m}} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta),$$

$$\widehat{E}_3 = \sum_{\substack{0 \le s \le n-\ell-1 \\ 1 \le m \le \ell+1 \\ i_1,\dots,i_m \in K}} \frac{O_{\ell-2n+2(\ell+1+s-m)+m}}{\Phi^{(\ell+1+s-m)+m}} \wedge \overline{\partial}\rho_{i_1}(\zeta) \wedge \partial\rho_{i_2}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta)$$

$$+ \sum_{\substack{0 \le s \le n-\ell-1 \\ 0 \le m \le \ell \\ i_1,\dots,i_m \in K}} \frac{O_{\ell-2n+2(\ell+1+s-m)+m}}{\Phi^{(\ell+1+s-m)+m}} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta)$$

on $D \times \Gamma_K \times \Delta_{OK}$. Hence each of the operators $E_1, E_2, E_3$ is a finite sum of operators of type $m$ (with $0 \le m \le \ell + 1$). It remains to prove that this is true also for $E_4$.

Since $E_4 = 0$ if $\ell = 1$, we may assume that $\ell \ge 2$. For $j \in K$, we denote by $\partial/\partial\lambda_j$ the partial derivative on $\Delta_{OK}$ with respect to $\lambda_j$ as a member of the system of coordinates $\lambda_i, i \in K$. Then it follows from (5.16) that

$$O_0 \wedge W \wedge d_\lambda\Phi = \sum_{i,j \in K} \frac{\partial\Phi}{\partial\lambda_j} O_0 \wedge \partial\rho_i + \sum_{j \in K} \frac{\partial\Phi}{\partial\lambda_j} O_1$$

on $D \times S_K \times \Delta_{OK}$. Together with (5.17) this implies that

$$O_0 \wedge (d_\lambda W)^{\ell-2} \wedge W \wedge d_\lambda\Phi = \sum_{\substack{0 \le m \le \ell-1 \\ i_1,\dots,i_m,j \in K}} \frac{\partial\Phi}{\partial\lambda_j} O_{\ell-1-m} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta)$$

on $D \times \Gamma_K \times \Delta_{OK}$. Hence

$$\widehat{E}_4 = \sum_{\substack{0 \le s \le n-\ell \\ 0 \le m \le \ell-1 \\ i_1,\dots,i_m,j \in K}} \frac{\partial\Phi}{\partial\lambda_j} \frac{O_{\ell-2n+2(\ell+s-1-m)+m}}{\Phi^{\ell+s}} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta) \,. \tag{5.18}$$

Now let $s, m, i_1, \dots, i_m, j$ be as in (5.8). Then $\ell + s \ge 2$ and therefore

$$\frac{\partial\Phi}{\partial\lambda_j} \frac{1}{\Phi^{\ell+s}} = (1 - \ell - s) \frac{\partial}{\partial\lambda_j} \Big( \frac{1}{\Phi^{\ell+s-1}} \Big).$$

Moreover, then

$$\{\lambda \in \Delta_{OK} : \lambda_j = 0\} = \Delta_{OK(\hat{j})} \text{ and } \Big\{\lambda \in \Delta_{OK} : \lambda_j = 1 - \sum_{i \notin K} \lambda_i \Big\} = \Delta_K.$$

By partial integration with respect to $\lambda_j$ and taking into account that $\partial O_k/\partial\lambda_j = O_k$ for all integers $k$, and that forms of type $O_k$ vanish for $\lambda$ in a neighborhood of $\Delta_K$, this implies that

$$\begin{aligned}
&\int_{\Gamma_K\times\Delta_{OK}} \tilde{f}(\zeta)\wedge\frac{\partial\Phi}{\partial\lambda_j}\frac{O_{\ell-2n+2(\ell+s-1-m)+m}}{\Phi^{\ell+s}}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)\\
&=\int_{\Gamma_K\times\Delta_{OK}} \tilde{f}(\zeta)\wedge\frac{O_{\ell-2n+2(\ell+s-1-m)+m}}{\Phi^{\ell+s-1}}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)\\
&\quad+\int_{\Gamma_K\times\Delta_{OK(\hat{j})}} \tilde{f}(\zeta)\wedge\frac{O_{\ell-2n+2(\ell+s-1-m)+m}}{\Phi^{\ell+s-1}}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)
\end{aligned}$$

for all $f\in B^{\beta}_{n,*}(D), 0\le\beta<1$. In view of (5.18) this implies that $E_4$ is a finite sum of operators of type $m$ (with $0\le m\le\ell-1$). □

## 6. An auxiliary estimate.

In this section we assume :

$\ell\ge 2$ is an integer;

$\Delta_K, K\in P'(\ell)$, are the simplices introduced in sect. 1.6, and we set $d\lambda_K = d\lambda_{k_2}\wedge\cdots\wedge d\lambda_{k_r}$ for $K=(k_1,\ldots,k_r)\in P'(\ell)$ and $\lambda\in\Delta_K$;

$C_*,\delta,\varepsilon$ are positive numbers;

$\Phi_1,\ldots,\Phi_\ell$ are complex numbers with

$$\mathrm{Re}\,\Phi_j\ge\delta+\varepsilon\quad(j=1,\ldots,\ell). \tag{6.1}$$

If $i,j\in\{1,\ldots,\ell\}$ with $i\ne j$, then $\nabla^i_j$ denotes the partial derivative $\partial/\partial\lambda_j$ with respect to $\lambda_j$ as a member of the system of coordinates $\lambda_1,\cdots\underset{\hat{i}}{}\cdots,\lambda_\ell$ on $\Delta_{1\ldots\ell}$; and we write

$$\nabla^{i_1\cdots i_s}_{j_1\cdots j_s}=\nabla^{i_1}_{j_1}\cdots\nabla^{i_s}_{j_s}$$

for $s=2,3,\cdots$ and $1\le i_\nu, j_\nu\le s$ with $i_\nu\ne j_\nu (\nu=1,\ldots,s)$.

$\gamma$ and $\Gamma$ are complex $C^\infty$ functions on $\Delta_{1\ldots\ell}$ such that

$$|\gamma(\lambda)|\le\frac{\varepsilon}{4}, \tag{6.2}$$

$$|\nabla^{i_1\cdots i_s}_{j_1\cdots j_s}\gamma(\lambda)|\le\frac{\varepsilon}{4}, \tag{6.3}$$

$$|\Gamma(\lambda)|\le C_*, \tag{6.4}$$

$$|\nabla^{i_1\cdots i_s}_{j_1\cdots j_s}\Gamma(\lambda)|\le C_* \tag{6.5}$$

for all $\lambda \in \Delta_{1\dots\ell}$ and $1 \le s \le \ell+2, 1 \le i_\nu, j_\nu \le \ell$ with $i_\nu \ne j_\nu (\nu = 1, \dots, s)$.

**6.1. THEOREM.** — *Set $C_p = (3p)! 2^{7p}$ for $p = 0, 1, \cdots$. Then*

$$\left| \int_{\Delta_{1\dots\ell}} \frac{\Gamma(\lambda) d\lambda_{1\dots\ell}}{\left( \sum\limits_{j=1}^{\ell} \lambda_j \Phi_j + \gamma(\lambda) \right)^p} \right| \le \frac{C_p C_*}{(\delta + \varepsilon)^p} \tag{6.6}$$

*for all integers $p \ge 0$, and*

$$\left| \int_{\Delta_{1\dots\ell}} \frac{\Gamma(\lambda) d\lambda_{1\dots\ell}}{\left( \sum\limits_{j=1}^{\ell} \lambda_j \Phi_j + \gamma(\lambda) \right)^p} \right| \le \frac{C_p C_*}{\prod\limits_{j \in K} |\Phi_j| (\delta + \varepsilon)^{p - |K|}} \tag{6.7}$$

*for all $K \in P'(\ell)$ and all integers $p \ge |K| + 1$.*

*Proof.* — Estimate (6.6) follows immediately from (6.1), (6.2) and (6.4). To prove (6.7) we may assume that

$$|\Phi_1| \ge \cdots \ge |\Phi_\ell| \tag{6.8}$$

and $K = (1, \dots, r)$ for some fixed $r \le \ell$. Let also $p \ge r + 1$ be fixed.

We introduce the following notations :

$(\gamma)_s$ and $(\Gamma)_s, s = 1, \dots, \ell + 2$, are the sets of all functions of the form

$$\nabla^{i_1 \cdots i_m}_{j_1 \cdots j_m} \gamma \quad \text{resp.} \quad \nabla^{i_1 \cdots i_m}_{j_1 \cdots j_m} \Gamma,$$

where $0 \le m \le s$ and $1 \le i_\nu, j_\nu \le \ell$ with $i_\nu \ne j_\nu (\nu = 1, \dots, m)$;

$X_s, s = 0, \dots, \ell$, is the set of functions defined as follows : $X_0 = \{\Gamma\}$ and, for $0 \le s \le \ell - 1$, $X_{s+1}$ is the set of all functions which are of one of the forms

$$\frac{b}{b + \nabla^i_j \gamma} \varphi, \ \frac{b \nabla^{ii}_{jj} \gamma}{(b + \nabla^i_j \gamma)^2} \varphi, \ \text{or} \ \frac{b}{b + \nabla^i_j \gamma} \nabla^i_j \varphi$$

where $\varphi \in X_s, 1 \le i, j \le \ell$ with $i \ne j$, and $b$ is a complex number with

$$|b| \ge \frac{\varepsilon}{2}. \tag{6.9}$$

It is easy to see that each function in $X_s, 0 \le s \le \ell$, is the sum of not more than $(3s)!$ functions of the form

$$\frac{b_1 \cdots b_\mu \psi_1 \cdots \psi_\nu \Psi}{(b_1 + \varphi_1)^{1+\alpha_1} \cdots (b_\mu + \varphi_\mu)^{1+\alpha_\mu}},$$

where $\mu, \nu, \alpha_1, \dots, \alpha_\mu \ge 0$ are integers with $0 \le \mu + \nu \le 2s$ and $\alpha_1 + \cdots + \alpha_\mu = \nu, b_1, \dots, b_\mu$ are complex numbers with

$$|b_i| \ge \frac{\varepsilon}{2} \quad (i = 1, \dots, \mu),$$

$\psi_i \in (\gamma)_{s+1}(1 \leq i \leq \nu), \Psi \in (\Gamma)_s$, and $\varphi_i \in (\gamma)_1(1 \leq i \leq \mu)$. In view of (6.3)-(6.5), this implies that

$$|\varphi(\lambda)| \leq (3s)!2^{2s}C_* \tag{6.10}$$

for all $\varphi \in X_s, 0 \leq s \leq \ell$, and $\lambda \in \Delta_{1\ldots\ell}$. Now, for $0 \leq s \leq r$ we formulate the following

STATEMENT $(s)$. — *If $K = (k_1, \ldots, k_m) \in P'(\ell)$ with $m \geq r - s$ and if $\varphi \in X_s$, then*

$$\left| \int_{\Delta_K} \frac{\varphi(\lambda) d\lambda_K}{\left( \sum\limits_{j \in K} \lambda_j \Phi_j + \gamma(\lambda) \right)^{p-s}} \right| \leq \frac{(3p)!2^{7p-3s}C_*}{\prod\limits_{j \in K_{r-s}} |\Phi_j| (\delta + \varepsilon)^{p-r}}, \tag{6.11}$$

*where $K_{r-s} := (k_1, \ldots, k_{r-s})$.*

In view of (6.8), statement (0) (setting $K = (1, \ldots, \ell)$ and $\varphi = \Gamma$) implies estimate (6.7). Statement $(r)$ is true by (6.6). To complete the proof of the theorem, it is therefore sufficient to prove the implications

$$\text{Statement } (s+1) \Rightarrow \text{Statement } (s), \quad 0 \leq s \leq r-1.$$

Assume that $0 \leq s \leq r - 1$ such that statement $(s + 1)$ is true. Further, let $K = (k_1, \ldots, k_m) \in P'(\ell)$ with $m \geq r - s$ be given. To prove (6.11) we distinguish two cases.

First case. — $|\Phi_{k_1} - \Phi_j| \leq 1/2 |\Phi_{k_1}|$ for all $j \in K$. Since $\sum\limits_{j \in K} \lambda_j = 1$ for $\lambda \in \Delta_K$, then

$$\left| \sum_{j \in K} \lambda_j \Phi_j \right| = \left| \Phi_{k_1} + \sum_{j \in K(\hat{\imath})} \lambda_j (\Phi_j - \Phi_{k_1}) \right| \geq \frac{1}{2} |\Phi_{k_1}|$$

for all $\lambda \in \Delta_K$. By (6.1) and (6.2), this implies that

$$\left| \sum_{j \in K} \lambda_j \Phi_j + \gamma(\lambda) \right| \geq \frac{1}{4} |\Phi_{k_1}|$$

for all $\lambda \in \Delta_K$. Together with (6.10), (6.8), (6.1) this implies (6.11).

Second case. — There exists $\tau \in \{1, \ldots, m\}$ with

$$\left| \Phi_{k_1} - \Phi_{k_\tau} \right| > \frac{1}{2} |\Phi_{k_1}|. \tag{6.12}$$

Set $b = \Phi_{k_1} - \Phi_{k_\tau}$,

$$\varphi_1 = \frac{b}{b + \nabla_{k_1}^{k_\tau} \gamma} \varphi, \varphi_2 = \frac{b \nabla_{k_1 k_1}^{k_\tau k_\tau} \gamma}{(b + \nabla_{k_1}^{k_\tau} \gamma)^2} \varphi, \varphi_3 = \frac{b}{b + \nabla_{k_1}^{k_\tau} \gamma} - \nabla_{k_1}^{k_\tau} \varphi.$$

Then, by (6.12) and (6.1), $\varphi_1, \varphi_2, \varphi_3 \in X_{s+1}$. Further, since

$$\sum_{j\in K} \lambda_j \Phi_j = \varphi_1 = \Phi_{K_\tau} + \sum_{j\in K(\hat{\tau})} \lambda_j(\Phi_j - \Phi_{k_\tau})$$

for $\lambda \in \Delta_K$, then we have the relation

$$\nabla_{k_1}^{k_\tau} \sum_{j\in K} \lambda_j \Phi_j = b$$

for $\lambda \in \Delta_K$, whence

$$\frac{b(1-p+s)\varphi(\lambda)}{\Big(\sum\limits_{j\in K}\lambda_j\Phi_j+\gamma(\lambda)\Big)^{p-s}} = \nabla_{k_1}^{k_\tau} \frac{\varphi_1(\lambda)}{\Big(\sum\limits_{j\in K}\lambda_j\Phi_j+\gamma(\lambda)\Big)^{p-s-1}}$$
$$+ \frac{\varphi_2(\lambda)}{\Big(\sum\limits_{j\in K}\lambda_j\Phi_j+\gamma(\lambda)\Big)^{p-s-1}} - \frac{\varphi_3(\lambda)}{\Big(\sum\limits_{j\in K}\lambda_j\Phi_j+\gamma(\lambda)\Big)^{p-s-1}}$$

for all $\lambda \in \Delta_K$. Since, for each $C^1$ function $f$ on $\Delta_K$,

$$\int_{\Delta_K} \nabla_{k_1}^{k_\tau} f(\lambda) d\lambda_K = \pm \int_{\Delta_{K(\hat{1})}} f(\lambda) d\lambda_{K(\hat{1})} \pm \int_{\Delta_{K(\hat{\tau})}} f(\lambda) d\lambda_{K(\hat{\tau})},$$

this implies, in view of Statement $(s+1)$, that

$$|b| \left| \int_{\Delta_K} \frac{\varphi(\lambda) d\lambda_K}{\Big(\sum\limits_{j\in K}\lambda_j\Phi_j+\gamma(\lambda)\Big)^{p-s}} \right|$$
$$\leq \left( \frac{1}{\prod\limits_{j\in J_1} |\Phi_j|} + \frac{1}{\prod\limits_{j\in J_2} |\Phi_j|} + \frac{2}{\prod\limits_{j\in K_{r-s-1}} |\Phi_j|} \right) \frac{(3p)! 2^{7p-3s-3} C_*}{(\delta+\varepsilon)^{p-r}},$$

where $J_1 = (k_2, \ldots, k_{r-s})$, $J_2 = (k_1, \cdots \underset{\hat{\tau}}{} \cdots, k_{r-s})$ if $\tau \leq r-s$, and $J_2 = (k_1, \ldots, k_{r-s-1})$ if $\tau > r-s$. Since, by (6.12) and (6.8),

$$|b| \geq \frac{1}{2} |\Phi_j|$$

for all $j \in K$, this implies estimate (6.11). □

## 7. Estimation of operators of type $m \geq 1$ by $\lambda$-free bounds.

In this section we assume :

$D \subset\subset \mathbb{C}^n$ is a local $q$-convex domain, $0 \leq q \leq n-1$ (see Definition 2.3);

$(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ is a frame for $D$ satisfying conditions (i) and (ii) in Definition 2.3;

$\Gamma_K, K \in P'(N)$, are the submanifolds of $\overline{D}$ which belong to the frame $(U_{\overline{D}}, \rho_1, \ldots, \rho_N)$ according to sect 4.2;

if $i, j \in \{0, \ldots, N\}$ with $i \neq j$, then $\nabla^i_j$ denotes the partial derivative $\partial/\partial\lambda_j$ with respect to $\lambda_j$ as a member of the system of coordinates $\lambda_0, \cdots \underset{\hat{i}}{} \cdots, \lambda_N$ on $\Delta_{0\cdots N}$, and we write

$$\nabla^{i_1\cdots i_s}_{j_1\cdots j_s} = \nabla^{i_1}_{j_1} \cdots \nabla^{i_s}_{j_s},$$

for $s = 2, 3, \cdots$ and $0 \leq i_\nu, j_\nu \leq N$ with $i_\nu \neq j_\nu (\nu = 1, \ldots, s)$ ;

$\Phi(z, \zeta, \lambda)$ is the function defined by (4.9) in sect. 4.5;

$C^0_+(D)$ is the space of continuous non-negative functions on $D$;

$\rho_K(\zeta) := \rho_j(\zeta)$ for $K \in P'(N), \zeta \in \Gamma_K$ and $j \in K$ (see sect. 4.2) $d(z) := \mathrm{dist}(z, \partial D)$ for $z \in D$.

Further, we use the following conventions : the letter $d$ stands for $d(z), \rho_j$ and $\rho_K$ stand for $\rho_j(\zeta)$ and $\rho_K(\zeta)$, and $f$ stands for $f(\zeta)$.

**7.1.** DEFINITION. — *A* $\lambda$-free bound *(of first or second kind) is, by definition, a map*

$$M : \bigcup_{0 \leq \beta < 1} B^\beta_*(D) \longrightarrow C^0_+(D)$$

*such that : there exist a number $C > 0$, a monomial $\sigma$ in $d\zeta_1, \ldots, d\zeta_n$, $d\bar{\zeta}_1, \ldots, d\bar{\zeta}_n$, a multiindex $K \in P'(N)$, an integer $0 \leq s \leq |K|$, and (if $s \geq 1$) points $\lambda^1, \ldots, \lambda^s \in \Delta_K$ such that if we use the abbreviations*

$$t_\nu = \mathrm{Im}\, \Phi(z, \zeta, \lambda^\nu)$$

*and*

$$dt_\nu = d_\zeta \,\mathrm{Im}\, \Phi(z, \zeta, \lambda^\nu),$$

*then $M$ is defined by one of the following equations : If $s = 0$, then*

$$Mf(z) = C \int_{\zeta \in \Gamma_K} \frac{\|f\| |\sigma \wedge d\rho_K|}{\big(|\rho_K| + d + |\zeta - z|^2\big)|\zeta - z|^{2n-|K|-1}}, \tag{7.1}$$

$$Mf(z) = C \int_{\zeta \in \Gamma_K} \frac{\|f\| |\sigma \wedge d\rho_K|}{\big(|\rho_K| + d + |\zeta - z|^2\big)|\zeta - z|^{2n-|K|}}, \tag{7.2}$$

*or*

$$Mf(z) = C \int_{\zeta \in \Gamma_K} \frac{\|f\| \; |\sigma \wedge d\rho_K|}{\big(|\rho_K| + d + |\zeta - z|^2\big)^2 |\zeta - z|^{2n-|K|-1}}, \tag{7.3}$$

*for* $f \in B_*^\beta(D), 0 \le \beta < 1$, *and* $z \in D$. *If* $s \ge 1$, *then*

$$(7.4)\qquad Mf(z) = C \int_{\zeta \in \Gamma_K} \frac{\|f\| \left| \sigma \wedge d\rho_K \wedge \bigwedge_{\nu=1}^{s} dt_\nu \right|}{\left(|\rho_K| + d + |\zeta - z|^2\right) \prod_{\nu=1}^{s} \left(|t_\nu| + d + |\zeta - z|^2\right) |\zeta - z|^{2n-|K|-s-1}},$$

*or*

$$(7.5)\qquad Mf(z) = C \int_{\zeta \in \Gamma_K} \frac{\|f\| \left| \sigma \wedge d\rho_K \wedge \bigwedge_{\nu=1}^{s} dt_\nu \right|}{\left(|\rho_K| + |d| + |\zeta - z|^2\right)^2 \prod_{\nu=1}^{s} \left(|t_\nu| + d + |\zeta - z|^2\right) |\zeta - z|^{2n-|K|-s-1}},$$

*for* $f \in B_*^\beta(D), 0 \le \beta < 1$, *and* $z \in D$.

*In the cases (7.1) and (7.4), M will be called a* $\lambda$*-free bound of* first kind, *and in the cases (7.2), (7.3) and (7.5), M will be called a* $\lambda$*-free bound of* second kind.

**7.2.** THEOREM. — *Let* $E$ *be an operator of type* $m$ *with* $m \ge 1$ *(see Definition 5.2). Denote by* $\nabla_z$ *one of the operators* $\partial/\partial z_1, \dots, \partial/\partial z_n$, $\partial/\partial \overline{z}_1, \dots, \partial/\partial \overline{z}_n$. *Then there exist a finite number of* $\lambda$*-free bounds of first kind* $M_1, \dots, M_\kappa$, *and a finite number of* $\lambda$*-free bounds of second kind* $M'_1, \dots, M'_{\kappa'}$ *such that*

$$\|Ef(z)\| \le M_1(z) + \cdots + M_\kappa(z)$$

*and*

$$\|\nabla_z Ef(z)\| \le M'_1(z) + \cdots + M'_{\kappa'}(z)$$

*for all* $f \in B_*^\beta(D), 0 \le \beta < 1$, *and* $z \in D$.

For the proof of this theorem we need some preparations.

**7.3.** DEFINITION. — *Let* $\alpha$ *be the positive constant from condition* (ii) *in Definition 2.3. An* admissible collection of corners *is, by definition, an ordered collection* $(\lambda^1, \dots, \lambda^\ell)$ *of points* $\lambda^1, \dots, \lambda^\ell \in \Delta_{1 \cdots N}$ *such that the following conditions are fulfilled :*

(i) $\lambda^1, \dots, \lambda^\ell$ *are linearly independent as vectors in* $\mathbb{R}^\ell$.

(ii) *There exists* $K = (k_1, \dots, k_\ell) \in P'(N)$ *with* $\lambda^1, \dots, \lambda^\ell \in \Delta_K$.

(iii) *For all* $(z, \zeta, \mu) \in \mathbb{C}^n \times U_{\overline{D}} \times \Delta_{1 \cdots \ell}$, *the function*

$$(7.6)\qquad \gamma(z, \zeta, \mu) := \Phi\Big(z, \zeta, \sum_{\nu=1}^{\ell} \mu_\nu \lambda^\nu\Big) - \sum_{\nu=1}^{\ell} \mu_\nu \Phi(z, \zeta, \lambda^\nu)$$

*satisfies the estimates*

$$|\gamma(z,\zeta,\mu)| \le \frac{\alpha}{8}|\zeta - z|^2 \tag{7.7}$$

*and*

$$\left|\nabla^{i_1\cdots i_s}_{j_1\cdots j_s}\gamma(z,\zeta,\mu)\right| \le \frac{\alpha}{8}|\zeta - z|^2 \tag{7.8}$$

*for all* $1 \le s \le \ell + 2$ *and* $1 \le i_\nu, j_\nu \le \ell$ *with* $i_\nu \neq j_\nu (1 \le \nu \le \ell)$.

*If* $(\lambda^1,\dots,\lambda^\ell)$ *is an admissible collection of corners, then we denote by* $\Delta(\lambda^1,\dots,\lambda^\ell)$ *the simplex spanned by* $\lambda^1,\dots,\lambda^\ell$, *i.e.*

$$\Delta(\lambda^1,\dots,\lambda^\ell) := \Big\{\sum_{\nu=1}^{\ell}\mu_\nu\lambda^\nu : \mu \in \Delta_{1\cdots\ell}\Big\}.$$

*An* admissible simplex *is, by definition, a simplex* $\Delta$ *such that, for certain admissible collection of corners* $(\lambda^1,\dots,\lambda^\ell), \Delta = \Delta(\lambda^1,\dots,\lambda^\ell)$.

**7.4. Lemma.** — *There exists* $\varepsilon>0$ *such that: if* $K = (k_1,\dots,k_\ell) \in P'(N)$ *and* $\lambda^1,\dots,\lambda^\ell \in \Delta_K$ *are linearly independent (as vectors in* $\mathbb{R}^\ell$*) points with*

$$|\lambda^\nu - \lambda^\kappa| < \varepsilon \quad (1 \le \nu, \kappa \le \ell), \tag{7.9}$$

*then* $(\lambda^1,\dots,\lambda^\ell)$ *is an admissible collection of corners.*

*Proof.* — Let $\lambda^1,\dots,\lambda^\ell \in \Delta_K$. Then it follows from the definition of the function $\Phi$ (see (4.9) and (3.4)) that

$$\Phi(z,\zeta,\lambda) = \widetilde{F}_{\rho_\lambda}(z,\zeta) + A|Q(\lambda)(\zeta - z)|^2$$

and therefore, since, for each $\lambda, Q(\lambda)$ is an orthogonal projection in $\mathbb{C}^n$,

$$\Phi(z,\zeta,\lambda) = \widetilde{F}_{\rho_\lambda}(z,\zeta) + A\langle Q(\lambda)(\zeta - z), \overline{\zeta} - \overline{z}\rangle$$

for all $(z,\zeta,\lambda) \in \mathbb{C}^n \times U_{\overline{D}} \times \Delta_{1\cdots N}$. Since $\widetilde{F}_{\rho_\lambda}(z,\zeta)$ depends linearly on $\lambda$, this implies that if $\gamma(z,\zeta,\mu)$ is the function defined by (7.6) in Definition 7.3, then

$$\gamma(z,\zeta,\mu) = A\Big\langle\Big[Q\Big(\sum_{\nu=1}^{\ell}\mu_\nu\lambda^\nu\Big) - \sum_{\nu=1}^{\ell}\mu_\nu Q(\lambda^\nu)\Big](\zeta - z), \overline{\zeta} - \overline{z}\Big\rangle \tag{7.10}$$

for all $(z,\zeta,\mu) \in \mathbb{C}^n \times U_{\overline{D}} \times \Delta_{1\cdots\ell}$. Since

$$\nabla^i_j \sum_{\nu=1}^{\ell}\mu_\nu\lambda^\nu = \lambda^j - \lambda^i$$

for $\mu \in \Delta_{1\cdots\ell}$ and $1 \le i,j \le \ell$ with $i \ne j$, and since $Q$ is of class $C^\infty$, we can find $C > 0$ (independent of $\lambda^1, \ldots, \lambda^\ell$) such that

$$\left|\nabla^{i_1\cdots i_s}_{j_1\cdots j_s} Q\Big(\sum_{\nu=1}^{\ell} \mu_\nu \lambda^\nu\Big)\right| \le C \max_{1\le i,j\le \ell} |\lambda^j - \lambda^i| \tag{7.11}$$

for all $\mu \in \Delta_{1\cdots\ell}, 1 \le s \le \ell+2$ and $1 \le i_\nu, j_\nu \le \ell$ with $i_\nu \ne j_\nu (1 \le \nu \le s)$. Let $C$ be chosen so that moreover

$$|Q(\lambda) - Q(\lambda')| \le C|\lambda - \lambda'| \tag{7.12}$$

for all $\lambda, \lambda' \in \Delta_{1\cdots N}$. Set

$$\varepsilon = \frac{\alpha}{16CA}$$

and assume that condition (7.9) is fulfilled. Then (7.10) and (7.12) imply (7.7), and from (7.11) it follows that

$$\left|\nabla^{i_1\cdots i_s}_{j_1\cdots j_s} Q\Big(\sum_{\nu=1}^{\ell} \mu_\nu \lambda^\nu\Big)\right| \le \frac{\alpha}{16A} \tag{7.13}$$

for $\mu \in \Delta_{1\cdots\ell}, 1 \le s \le \ell+2$ and $1 \le i_\nu, j_\nu \le \ell$ with $i_\nu \ne j_\nu (1 \le \nu \le s)$. Moreover, since

$$\nabla^{i_1\cdots i_s}_{j_1\cdots j_s} \sum_{\nu=1}^{\ell} \mu_\nu Q(\lambda^\nu) = \begin{cases} Q(\lambda^{j_1}) - Q(\lambda^{i_1}) & \text{if } s = 1 \\ 0 & \text{if } s \ge 2, \end{cases}$$

it follows from (7.12) and (7.9) that also

$$\left|\nabla^{i_1\cdots i_s}_{j_1\cdots j_s} \sum_{\nu=1}^{\ell} \mu_\nu Q(\lambda^\nu)\right| \le \frac{\alpha}{16A}$$

for all $\mu \in \Delta_{1\cdots\ell}, 1 \le s \le \ell+2$ and $1 \le i_\nu, j_\nu \le \ell$ with $i_\nu \ne j_\nu (1 \le \nu \le s)$. Together with (7.10) and (7.13) this implies (7.8). □

**7.5. Lemma.** — *Let $K = (k_1, \ldots, k_\ell) \in P'(N)$, and let $\lambda_1, \ldots, \lambda_\ell \in \Delta_K$ such that $\lambda^1, \ldots, \lambda^\ell$ are linearly independent as vectors in $\mathbf{R}^\ell$. Further let $\sigma$ be a monomial in $d\zeta_1, \ldots, d\zeta_n, d\bar\zeta_1, \ldots, d\bar\zeta_n$. We set*

$$t_j = \operatorname{Im} \Phi(z, \zeta, \lambda^j)$$

*and*

$$dt_j = d_\zeta \operatorname{Im} \Phi(z, \zeta, \lambda^j),$$

*and we use the following definition : if $f$ is a differential form on $\Gamma_K$ and $\tilde f$ is the part of $f$ which is of degree $\dim_{\mathbf{R}} \Gamma_K$, then by $|f|$ we denote the absolute value of $\tilde f$. Then there exists a constant $C > 0$ such that, for all $i_1, \ldots, i_m \in K$,*

$$\begin{aligned} \Big|\sigma \wedge \bigwedge_{\nu=1}^{m} \partial\rho_{i_\nu}\Big| + \Big|\sigma \wedge \bar\partial\rho_{i_1} \wedge \bigwedge_{\nu=2}^{m} \partial\rho_{i_\nu}\Big| &\le C|\sigma_0 \wedge d\rho_K||\zeta - z|^{m-1} \\ &+ C \sum_{\substack{J\in P'(\ell)\\ |J|\le m-1}} \Big|\sigma_J \wedge d\rho_K \wedge \bigwedge_{j\in J} dt_j\Big| |\zeta - z|^{m-1-|J|} \end{aligned} \tag{7.14}$$

*for all $z \in D$ and $\zeta \in \Gamma_K$, where $\sigma_0$ and $\sigma_J$ are some monomials in $d\zeta_1, \dots, d\zeta_n, d\overline{\zeta}_1, \dots, d\overline{\zeta}_n$.*

*Proof.* — It follows from the definition of $\Phi$ (see (4.9) and (4.3)) that

$$d_\zeta \operatorname{Im} \Phi(z, \zeta, \lambda^j) = i\big(\overline{\partial}\rho_{\lambda^j}(\zeta) - \partial\rho_{\lambda^j}(\zeta)\big) + O(|\zeta - z|)$$

for $(z, \zeta) \in \mathbb{C}^n \times U_{\overline{D}}$ and $|\zeta - z| \to 0$ $(1 \leq j \leq n)$. Since $\rho_{\lambda^j}(\zeta) = \rho_K$ for $\zeta \in \Gamma_K$, this implies that

$$\partial\rho_{\lambda^j}|_{\Gamma_K} = \frac{1}{2}d\rho_k + \frac{i}{2}dt_j + O(|\zeta - z|) \tag{7.15}$$

and

$$\overline{\partial}\rho_{\lambda^j}|_{\Gamma_K} = \frac{1}{2}d\rho_k - \frac{i}{2}dt_j + O(|\zeta - z|) \tag{7.16}$$

for $(z, \zeta) \in \mathbb{C}^n \times \Gamma_K$ and $|\zeta - z| \to 0$ $(1 \leq j \leq n)$. Since the points $\lambda^1, \dots, \lambda^\ell$ belong to $\Delta_K$ and are linearly independent as vectors in $\mathbb{R}^n$, we can find numbers $\beta_j^\nu$ with

$$\Delta_{k_\nu} = \sum_{j=1}^{\ell} \beta_j^\nu \lambda^j \quad (1 \leq \nu \leq \ell).$$

Since $\rho_\lambda$ depends linearly on $\lambda$, then

$$\rho_{k_\nu} = \sum_{j=1}^{\ell} \beta_j^\nu \rho_{\lambda^j} \quad (1 \leq \nu \leq \ell)$$

and it follows from (7.15) and (7.16) that

$$\partial\rho_{k_\nu}|_{\Gamma_K} = \sum_{j=1}^{\ell} \beta_j^\nu \Big(\frac{1}{2}d\rho_k + \frac{i}{2}dt_j\Big) + O(|\zeta - z|)$$

and

$$\overline{\partial}\rho_{k_\nu}|_{\Gamma_K} = \sum_{j=1}^{\ell} \beta_j^\nu \Big(\frac{1}{2}d\rho_k + \frac{i}{2}dt_j\Big) + O(|\zeta - z|)$$

for $(z, \zeta) \in \mathbb{C}^n \times \Gamma_K$ and $|\zeta - z| \to 0$ $(1 \leq \nu \leq \ell)$. Hence, for some constant $C_1 > 0$,

$$\begin{aligned}\Big|\sigma \wedge \bigwedge_{\nu=1}^{m} \partial\rho_{i_\nu}\Big| + \Big|\sigma \wedge \overline{\partial}\rho_{i_1} \wedge \bigwedge_{\nu=2}^{m} \partial\rho_{i_\nu}\Big| &\leq C_1|\sigma^1||\zeta - z|^m \\ &+ C_1|\sigma^2 \wedge d\rho_K||\zeta - z|^{m-1} \\ &+ C_1 \sum_{\substack{J \in P'(\ell) \\ |J| \leq m-1}} \Big|\sigma_J^3 \wedge d\rho_K \wedge \bigwedge_{j \in J} dt_j\Big||\zeta - z|^{m-1-|J|} \\ &+ C_1 \sum_{\substack{J \in P'(\ell) \\ |J| \leq m}} \Big|\sigma_J^4 \wedge \bigwedge_{j \in J} dt_j\Big||\zeta - z|^{m-|J|}\end{aligned} \tag{7.17}$$

for all $(z,\zeta) \in D \times \Gamma_K$, where $\sigma^1, \sigma^2, \sigma_J^3$ and $\sigma_J^4$ are some monomials in $d\zeta_1, \ldots, d\zeta_n$, $d\overline{\zeta}_1, \ldots, d\overline{\zeta}_n$. Moreover, since $\rho_K$ can be used as a local coordinate on $\Gamma_K$, for some $C_2 > 0$, we have the estimates

$$\left|\sigma_J^4 \wedge \bigwedge_{j\in J} dt_j\right| \leq C_2 \sum_{\kappa\in J} \left|\sigma_\kappa^5 \wedge d\rho_K \wedge \bigwedge_{j\in J(\hat{\kappa})} dt_j\right|$$

where $\sigma_\kappa^5$ are again some monomials in $d\zeta_1, \ldots, d\zeta_n, d\overline{\zeta}_1, \ldots, d\overline{\zeta}_n$. Together with (7.17) this implies (7.14). □

**7.6.** *Proof of Theorem 7.2.* — We use the same notations as in Definition 5.2. Let $K = (k_1, \ldots, k_\ell)$. We may restrict ourselves to the case that, for some $1 \leq j_1 < \cdots < j_\kappa \leq n$,

$$\widehat{E}(z,\zeta,\lambda) = \vartheta(z,\zeta,\lambda) d\lambda_0 \wedge d\lambda_{k_2} \wedge \cdots \wedge d\lambda_{k_\ell} \wedge d\overline{\zeta}_{j_1} \wedge \cdots \wedge d\overline{\zeta}_{j_\kappa},$$

where $\vartheta(z,\zeta,\lambda)$ is a complex function of type $O_{\ell-2n+2k+m}$ on $D \times \Gamma_K \times \Delta_{OK}$. Fix a number $0 < \xi < 1$ such that $\vartheta(z,\zeta,\lambda) = 0$ for all $(z,\zeta,\lambda) \in \Delta_{OK}$ with $\xi \leq \lambda_0 \leq 1$.

In addition to the notations introduced at the beginning of this section, in this proof we shall use the following notations :

$\nabla_z$ is one of the operators $\partial/\partial z_1, \ldots, \partial/\partial z_n, \partial/\partial\overline{z}_1, \ldots, \partial/\partial\overline{z}_n$ ;

if $f \in B^\beta_{n,*}(D), 0 \leq \beta < 1$ and $I = (\alpha_1, \ldots, \alpha_\nu) \in P'(n)$, then $f_I$ denotes the coefficient of the form $f$ at the monomial $d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_{\alpha_1} \wedge \cdots \wedge d\overline{\zeta}_{\alpha_\kappa}$;

$$\begin{aligned} \overset{\circ}{\Delta}_{OK} &= \{\lambda \in \Delta_{OK} : 0 \leq \lambda_0 \leq \xi\} \ ; \\ d\lambda_K &= d\lambda_{k_2} \wedge \cdots \wedge d\lambda_{k_\ell} \ ; \\ \sigma &= d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_{j_1} \wedge \cdots \wedge d\overline{\zeta}_{j_\kappa} \ ; \\ f_I &= f_I(\zeta) \text{ and } \Theta = \Theta(\zeta). \end{aligned}$$

Then

$$Ef(z) = \sum_{\substack{I\in P'(n) \\ |I|=2n-\ell+1-\kappa-m}} \int_{(\zeta,\lambda)\in\Gamma_K\times\overset{\circ}{\Delta}_{OK}} \frac{f_I \vartheta(z,\zeta,\lambda)\Theta \wedge \sigma \wedge d\lambda_0 \wedge d\lambda_K}{\Phi^{k+m}(z,\zeta,\overset{\circ}{\lambda})} \tag{7.18}$$

for all $f \in B^\beta_{n,*}(D), 0 \leq \beta < 1$, and $z \in D$. Now we fix some $I \in P'(n)$ with $|I| = 2n - \ell + 1 - \kappa - m$, and an admissible collection of corners $(\lambda^1, \ldots, \lambda^\ell)$ with $\lambda^1, \ldots, \lambda^\ell \in \Delta_K$, and set

$$\widetilde{\Delta} = \{\lambda \in \overset{\circ}{\Delta}_{OK} : \lambda \in \overset{\circ}{\Delta}(\lambda^1, \ldots, \lambda^\ell)\}$$

and

$$\widetilde{E}f(z) = \int_{(\zeta,\lambda)\in\Gamma_K\times\widetilde{\Delta}} \frac{f_I \vartheta(z,\zeta,\lambda)\Theta \wedge \sigma \wedge d\lambda_0 \wedge d\lambda_K}{\Phi^{k+m}(z,\zeta,\overset{\circ}{\lambda})}$$

for $f \in B^{\beta}_{n,*}(D), 0 \leq \beta < 1$, and $z \in D$. Since, by Lemma 7.4, $\Delta_K$ can be divided into a finite number of admissible simplices, and since (7.18) holds, it is sufficient to find a finite number of $\lambda$-free bounds of first kind $M_1, \dots, M_\omega$ and a finite number of $\lambda$-free bounds of second kind $M'_1, \dots, M'_{\omega'}$ such that

$$\|\widetilde{E}f(z)\| \leq M_1(z) + \cdots + M_\omega(z) \tag{7.19}$$

and

$$\|\nabla_z \widetilde{E}(z)\| \leq M'_1(z) + \cdots + M'_{\omega'}(z) \tag{7.20}$$

for all $f \in B^{\beta}_{n,*}(D), 0 \leq \beta < 1$, and $z \in D$.

Set

$$\varphi(\tau, \mu) = \Big(\tau, (1-\tau)\sum_{\nu=1}^{\ell} \mu_\nu \lambda^\nu, \dots, (1-\tau)\sum_{\nu=1}^{\ell} \mu_\nu \lambda^\nu\Big)$$

for $0 \leq \tau \leq \xi$ and $\mu \in \Delta_{1\dots\ell}$. Then $\varphi$ is a diffeomorphism from $[0,\xi] \times \Delta_{1\dots\ell}$ onto $\widetilde{\Delta}$. Denote by $a(\tau), 0 \leq \tau \leq \delta$, the function with

$$(1-\tau)^{\ell-1} d\tau \wedge \Big(\sum_{\nu=1}^{\ell} \lambda_2^\nu d\mu_\nu\Big) \wedge \cdots \wedge \Big(\sum_{\nu=1}^{\ell} \lambda_\ell^\nu d\mu_\nu\Big) = a(\tau) d\tau \wedge d\mu_{1\dots\ell}$$

on $[0,\xi] \times \Delta_{1\dots\ell}$, where $d\mu_{1\dots\ell} := d\mu_2 \wedge \cdots \wedge d\mu_\ell$. Set

$$\Gamma(z,\zeta,\tau,\mu) = a(\tau)\vartheta\Big(z,\zeta,\varphi(\tau,\mu)\Big),$$

$$\Gamma'(z,\zeta,\tau,\mu) = a(\tau)\nabla_z\vartheta\Big(z,\zeta,\varphi(\tau,\mu)\Big),$$

$$\Gamma''(z,\zeta,\tau,\mu) = -(k+m)a(\tau)\vartheta(z,\zeta,\varphi(\tau,\mu))\nabla_z\Phi\Big(z,\zeta,\sum_{\nu=1}^{\ell}\mu_\nu\lambda^\nu\Big),$$

$$\gamma(z,\zeta,\mu) = \Phi\Big(z,\zeta,\sum_{\nu=1}^{\ell}\mu_\nu\lambda^\nu\Big) - \sum_{\nu=1}^{\ell}\mu_\nu\Phi(z,\zeta,\lambda^\nu)$$

for $(z,\zeta,\tau,\mu) \in D \times \Gamma_K \times [0,\xi] \times \Delta_{1\dots\ell}$, and

$$\Omega(z,\zeta,\tau) = \int_{\mu\in\Delta_{1\dots\ell}} \frac{\Gamma(z,\zeta,\tau,\mu)d\mu_{1\dots\ell}}{\Big(\sum\limits_{\nu=1}^{\ell}\mu_\nu\Phi(z,\zeta,\lambda^\nu) + \gamma(z,\zeta,\mu)\Big)^{k+m}},$$

$$\Omega'(z,\zeta,\tau) = \int_{\mu\in\Delta_{1\dots\ell}} \frac{\Gamma'(z,\zeta,\tau,\mu)d\mu_{1\dots\ell}}{\Big(\sum\limits_{\nu=1}^{\ell}\mu_\nu\Phi(z,\zeta,\lambda^\nu) + \gamma(z,\zeta,\mu)\Big)^{k+m}},$$

$$\Omega''(z,\zeta,\tau) = \int_{\mu\in\Delta_{1\dots\ell}} \frac{\Gamma''(z,\zeta,\tau,\mu)d\mu_{1\dots\ell}}{\Big(\sum\limits_{\nu=1}^{\ell}\mu_\nu\Phi(z,\zeta,\lambda^\nu) + \gamma(z,\zeta,\mu)\Big)^{k+m+1}},$$

for $(z,\zeta,\tau) \in D \times \Gamma_K \times [0,\xi]$. Then

$$\|\widetilde{E}f(z)\| \le \int_{\zeta\in\Gamma_K} \max_{0\le\tau\le\xi} |\Omega(z,\zeta,\tau)| \|f\| |\Theta \wedge \sigma| \tag{7.21}$$

and

$$\|\nabla_z \widetilde{E}f(z)\| \le \int_{\zeta\in\Gamma_K} \max_{0\le\tau\le\xi} |\Omega'(z,\zeta,\tau)+\Omega''(z,\zeta,\tau)| \|f\| |\Theta \wedge \sigma| \tag{7.22}$$

for all $f \in B^{\beta}_{n,*}(D), 0 \le \beta < 1$, and $z \in D$. Since $\vartheta(z,\zeta,\lambda)$ is of type $0_{\ell-2n+2k+m}$, and $\nabla_z\Phi(z,\zeta,\lambda)$ is of type $0_0$ on $\Delta_{OK}$, there is a constant $K_1 > 0$ such that

$$|\Gamma(z,\zeta,\tau,\mu)|, |\nabla^{i_1\cdots i_s}_{j_1\cdots j_s}\Gamma(z,\zeta,\tau,\mu)| \le \frac{K_1}{|\zeta - z|^{2n-\ell-2k-m}}, \tag{7.23}$$

$$|\Gamma'(z,\zeta,\tau,\mu)|, |\nabla^{i_1\cdots i_s}_{j_1\cdots j_s}\Gamma'(z,\zeta,\tau,\mu)| \le \frac{K_1}{|\zeta - z|^{2n-\ell-2k-m+1}}, \tag{7.24}$$

$$|\Gamma''(z,\zeta,\tau,\mu)|, |\nabla^{i_1\cdots i_s}_{j_1\cdots j_s}\Gamma''(z,\zeta,\tau,\mu)| \le \frac{K_1}{|\zeta - z|^{2n-\ell-2k-m}}, \tag{7.25}$$

for all $(z,\zeta,\tau,\mu) \in D \times \Gamma_K \times [0,\xi] \times \Delta_{1\cdots\ell}$ and for all $1 \le s \le \ell+2$ and $1 \le i_\nu, j_\nu \le \ell$ with $i_\nu \ne j_\nu (1 \le \nu \le s)$. (The operators $\nabla^{i_1\cdots i_s}_{j_1\cdots j_s}$ here are considered with respect to the variable $\mu$.)

Now we are going to estimate $\Omega, \Omega', \Omega''$ by means of Theorem 6.1. First note that $\rho_{\lambda^\nu}(\zeta) = \rho_K$ for all $\zeta \in \Gamma_K$ and $1 \le \nu \le \ell$, and that there is a constant $c_0 > 0$ such that $-\rho_{\lambda^\nu}(z) \ge c_0 d$ for all $z \in D$ and $1 \le \nu \le \ell$. Therefore it follows from estimate (4.10) that

$$\operatorname{Re}\Phi(z,\zeta,\lambda^\nu) \ge c_0(|\rho_K| + d) + \frac{\alpha}{2}|\zeta - z|^2 \tag{7.26}$$

for all $(z,\zeta) \in D \times \Gamma_K$ and $1 \le \nu \le \ell$.

$$C_* = K_1|\zeta - z|^{-2n+\ell+2k+m}$$

$$\delta = c_0(|\rho_K| + d) \text{ and } \varepsilon = \frac{\alpha}{2}|\zeta - z|^2,$$

$$\Phi_j = \Phi(z,\zeta,\lambda^j) \text{ for } j = 1,\dots,\ell,$$

$$\gamma(\mu) = \gamma(z,\zeta,\tau,\mu) \text{ and } \Gamma(\mu) = \Gamma(z,\zeta,\tau,\mu) \text{ for } \mu \in \Delta_{1\cdots\ell},$$

then, by (7.7), (7.8), (7.23) and (7.26), conditions (6.1)-(6.5) are fulfilled (with $\mu$ instead of $\lambda$) and it follows from Theorem 6.1 that with $C_p := (3p)!2^{7p}$

$$|\Omega(z,\zeta,\tau)| \le \begin{cases} \dfrac{C_{k+1}C_*}{(\delta+\varepsilon)^{k+1}} & \text{if } m = 1 \\ \min\limits_{\substack{J\in P'(\ell) \\ |J|\le k+m-1}} \dfrac{C_{k+m}C_*}{\prod\limits_{j\in J}|\Phi_j|(\delta+\varepsilon)^{k+m-|J|}} & \text{if } m \ge 2. \end{cases}$$

In the same way (using (7.24) and (7.25)) then one obtains

$$|\Omega'(z,\zeta,\tau)| \le \begin{cases} \dfrac{C_{k+1}C_*|\zeta-z|^{-1}}{(\delta+\varepsilon)^{k+1}} & \text{if } m=1 \\ \min\limits_{\substack{J\in P'(\ell)\\ |J|\le k+m-1}} \dfrac{C_{k+m}C_*|\zeta-z|^{-1}}{\prod\limits_{j\in J}|\Phi_j|(\delta+\varepsilon)^{k+m-|J|}} & \text{if } m\ge 2 \end{cases}$$

and

$$|\Omega''(z,\zeta,\tau)| \le \begin{cases} \dfrac{C_{k+2}C_*}{(\delta+\varepsilon)^{k+2}} & \text{if } m=1 \\ \min\limits_{\substack{J\in P'(\ell)\\ |J|\le k+m}} \dfrac{C_{k+m}C_*}{\prod\limits_{j\in J}|\Phi_j|(\delta+\varepsilon)^{k+m-|J|+1}} & \text{if } m\ge 2. \end{cases}$$

Setting $t_j = \operatorname{Im}\Phi(z,\zeta,\lambda^j)$ and taking into account that then, by (7.26),

$$|\Phi(z,\zeta,\lambda^j)| \ge \min\Big(1, c_0, \frac{\alpha}{2}\Big)(|t_j| + |\rho_K| + d + |\zeta-z|^2),$$

this implies that there is a constant $K_2>0$ (depending only on $K_1, c_0, \alpha, k$ and $m$) such that

$$\begin{aligned} &\max_{0\le\tau\le\xi}|\Omega(z,\zeta,\tau)| \\ &\le \begin{cases} \dfrac{K_2|\zeta-z|^{-2n+\ell+1}}{|\rho_K|+d+|\zeta-z|^2} & \text{if } m{=}1 \\ \min\limits_{\substack{J\in P'(\ell)\\ |J|\le k+m-1}} \dfrac{K_2|\zeta-z|^{-2n+\ell-m+2|J|+2}}{(|\rho_K|+d+|\zeta-z|^2)\prod\limits_{j\in J}(|t_j|+d+|\zeta-z|^2)} & \text{if } m{\ge}2, \end{cases} \end{aligned} \tag{7.27}$$

$$\begin{aligned} &\max_{0\le\tau\le\xi}|\Omega'(z,\zeta,\tau)| \\ &\le \begin{cases} \dfrac{K_2|\zeta-z|^{-2n+\ell}}{|\rho_K|+d+|\zeta-z|^2} & \text{if } m{=}1 \\ \dfrac{K_2|\zeta-z|^{-2n+\ell+1}}{(|\rho_K|+d+|\zeta-z|^2)^2} & \text{if } m{=}2 \\ \min\limits_{\substack{J\in P'(\ell)\\ |J|\le k+m-2}} \dfrac{K_2|\zeta-z|^{-2n+\ell-m+2|J|+3}}{(|\rho_K|+d+|\zeta-z|^2)^2\prod\limits_{j\in J}(|t_j|+d+|\zeta-z|^2)} & \text{if } m{\ge}3 \end{cases} \end{aligned} \tag{7.28}$$

and

$$\begin{aligned} &\max_{0\le\tau\le\xi}|\Omega''(z,\zeta,\tau)| \\ &\le \begin{cases} \dfrac{K_2|\zeta-z|^{-2n+\ell+1}}{(|\rho_K|+d+|\zeta-z|^2)^2} & \text{if } m=1 \\ \min\limits_{\substack{J\in P'(\ell)\\ |J|\le k+m-1}} \dfrac{K_2|\zeta-z|^{-2n+\ell-m+2|J|+2}}{(|\rho_K|+d+|\zeta-z|^2)^2\prod\limits_{j\in J}(|t_j|+d+|\zeta-z|^2)} & \text{if } m\ge 2, \end{cases} \end{aligned} \tag{7.29}$$

for all $(z, \zeta) \in D \times \Gamma_K$ with $z \neq \zeta$. By Lemma 7.5 there are monomials $\sigma_0$ and $\sigma_J$ in $d\zeta_1, \dots, d\zeta_n, d\overline{\zeta}_1, \dots, d\overline{\zeta}_n$ and a constant $K_3 > 0$ such that on $\Gamma_K$ one has the estimate

$$|\Theta \wedge \sigma| \tag{7.30}$$
$$\leq \begin{cases} K_3|\sigma_0 \wedge d\rho_K| & \text{if } m=1 \\ K_3|\sigma_0 \wedge d\rho_K||\zeta - z|^{m-1} + K_3 \sum\limits_{\substack{J \in P'(\ell) \\ |J| \leq m-1}} \left|\sigma_J \wedge d\rho_K \wedge \bigwedge\limits_{j \in J} dt_j\right| |\zeta - z|^{m-1-|J|} & \text{if } m \geq 2. \end{cases}$$

Together with (7.27) and (7.29) this implies that, if $K_4 = K_2 K_3$ then

$$\begin{aligned} \max_{0 \leq \tau \leq \xi} |\Omega(z, \zeta, \tau)||\Theta \wedge \sigma| &\leq K_4 \frac{|\sigma_0 \wedge d\rho_K|}{(|\rho_K| + d + |\zeta - z|^2)|\zeta - z|^{2n-\ell-1}} \\ &+ K_4 \sum_{\substack{J \in P'(\ell) \\ |J| \leq m-1}} \frac{\left|\sigma_J \wedge d\rho_K \wedge \bigwedge\limits_{j \in J} dt_j\right|}{(|\rho_K| + d + |\zeta - z|^2) \prod\limits_{j \in J} (|t_j| + d + |\zeta - z|^2)|\zeta - z|^{2n-\ell-|J|-1}} \end{aligned} \tag{7.31}$$

and

$$\begin{aligned} \max_{0 \leq \tau \leq \xi} |\Omega''(z, \zeta, \tau)||\Theta \wedge \sigma| &\leq K_4 \frac{|\sigma \wedge d\rho_K|}{(|\rho_K| + d + |\zeta - z|^2)^2|\zeta - z|^{2n-\ell-1}} \\ &+ K_4 \sum_{\substack{J \in P'(\ell) \\ |J| \leq m-1}} \frac{\left|\sigma_J \wedge d\rho_K \wedge \bigwedge\limits_{j \in J} dt_j\right|}{(|\rho_K| + d + |\zeta - z|^2)^2 \prod\limits_{j \in J} (|t_j| + d + |\zeta - z|^2)|\zeta - z|^{2n-\ell-|J|-1}} \end{aligned} \tag{7.32}$$

for all $(z, \zeta) \in D \times \Gamma_K$ with $z \neq \zeta$. Finally, we observe that from (7.30) one can obtain also the following (weaker) assertion : there exist monomials $\sigma'_0$ and $\sigma'_J$ in $d\zeta_1, \dots, d\zeta_n, d\overline{\zeta}_1, \dots, d\overline{\zeta}_n$ and a constant $K'_3 > 0$ such that

$$|\Theta \wedge \sigma| \leq \begin{cases} K'_3|\sigma'_0 \wedge d\rho_K| & \text{if } m=1,2 \\ K'_3|\sigma'_0 \wedge d\rho_K||\zeta - z|^{m-2} \\ \quad + K'_3 \sum\limits_{\substack{J \in P'(\ell) \\ |J| \leq m-2}} \left|\sigma'_J \wedge d\rho_K \wedge \bigwedge\limits_{j \in J} dt_j\right| |\zeta - z|^{m-2-|J|} & \text{if } m \geq 3. \end{cases}$$

Together with (7.28) this implies, if $K'_4 = K_2 K'_3$ then

$$\begin{aligned} \max_{0 \leq \tau \leq \xi} |\Omega'(z, \zeta, \tau)||\Theta \wedge \sigma| &\leq K'_4 \frac{|\sigma'_0 \wedge d\rho_K|}{(|\rho_K| + d + |\zeta - z|^2)|\zeta - z|^{2n-\ell}} \\ &+ K'_4 \frac{|\sigma'_0 \wedge d\rho_K|}{(|\rho_K| + d + |\zeta - z|^2)^2|\zeta - z|^{2n-\ell-1}} \\ &+ K'_4 \sum_{\substack{J \in P'(\ell) \\ |J| \leq m-2}} \frac{\left|\sigma'_J \wedge d\rho_K \wedge \bigwedge\limits_{j \in J} dt_j\right|}{(|\rho_K| + d + |\zeta - z|^2)^2 \prod\limits_{j \in J} (|t_j| + d + |\zeta - z|^2)|\zeta - z|^{2n-\ell-|J|-1}} \end{aligned} \tag{7.33}$$

for all $(z,\zeta) \in D \times \Gamma_K$ with $z \neq \zeta$. Now (7.19) follows from (7.21) and (7.31), and (7.20) follows from (7.22) and (7.32), (7.33). □

## 8. The Range-Siu trick and end of proof of estimates (*i.e.* of Theorem 4.12).

**8.1.** LEMMA. — *Let $M$ be a $\lambda$-free bound of first kind (see Definition 7.1), let $0 \le \beta < 1$ and $\varepsilon > 0$. Then there exists a constant $C > 0$ such that*

$$\|Mf(z)\| \le C\|f\|_{-\beta}\big(1 + [\mathrm{dist}(z,\partial D)]^{1/2-\beta-\varepsilon}\big) \tag{8.1}$$

*for all $f \in B^\beta_*(D)$ and $z \in D$.*

*Proof.* — We use the same notations as in Definition 7.1, we set $\ell := |K|$, and we denote all positive constants by the same letter $C$. Since $\mathrm{dist}(\zeta,\partial D) \le C|\rho_K|$ for $\zeta \in \Gamma_K$ and since $M$ is of the form (7.1) or (7.4),

$$\|Mf(z)\| \le C\|f\|_{-\beta}\int_{\zeta\in\Gamma_K} \frac{|\rho_K|^{-\beta}\left|\sigma\wedge d\rho_K \wedge \bigwedge\limits_{\nu=1}^{s} dt_\nu\right|}{(|\rho_K| + d + |\zeta-z|^2)\prod\limits_{\nu=1}^{s}(|t_\nu| + d + |\zeta-z|^2)|\zeta-z|^{2n-\ell-s-1}}$$

for all $f \in B^\beta_*(D)$ and $z \in D$, where $s = 0$ if $M$ is defined by (7.1).

Since $\rho_K$ can be used as local coordinate on $\Gamma_K$, now we can apply the Range-Siu trick (see the proof of Proposition (3.7) in [RS], where this is described in detail) which consists in replacing the functions $t_\nu$ by appropriate quadratic polynomials in local coordinates containing $\rho_K$. In this way one obtains that

$$\|Mf(z)\| \le C\|f\|_{-\beta}\sum_{\kappa=1}^{s+1}\int_{\substack{y\in\mathbf{R}^{2n-\ell+1}\\ |y|<C}} \frac{|y_1|^{-\beta}dy_1\wedge\cdots\wedge dy_{2n-\ell+1}}{\prod\limits_{\nu=1}^{\kappa}(|y_\nu| + d + |y|^2)|y|^{2n-\ell-\kappa}}$$

for all $f \in B^\beta_*(D)$ and $z \in D$. Since we may assume that $\beta > 0$, this implies that

$$\begin{aligned}\|Mf(z)\| &\le C\|f\|_{-\beta}\sum_{\kappa=1}^{s+1}\int_{\substack{y\in\mathbf{R}^{2n-\ell}\\ |y|<C}} \frac{dy_1\wedge\cdots\wedge dy_{2n-\ell}}{(d+|y|^2)^\beta\prod\limits_{\nu=1}^{\kappa-1}(|y_\nu| + d + |y|^2)|y|^{2n-\ell-\kappa}}\\
&\le C\|f\|_{-\beta}\sum_{\kappa=1}^{s+1}\int_{\substack{y\in\mathbf{R}^{2n-\ell-\kappa+1}\\ |y|<C}} \frac{[1+|\ell n(d+|y|^2)|]^{\kappa-1}dy_1\wedge\cdots\wedge dy_{2n-\ell-\kappa}}{(d+|y|^2)^\beta|y|^{2n-\ell-\kappa}}\\
&\le C\|f\|_{-\beta}\int_0^C \frac{dr}{(d+r^2)^{\beta+\varepsilon}}\end{aligned}$$

for all $f \in B_*^{\beta}(D)$ and $z \in D$. Since we may assume also that $\beta + \varepsilon \neq 1/2$, this implies (8.1). □

**8.2. Lemma.** — *Let $M$ be a $\lambda$-free bound of second kind (see Definition 7.1), let $0 \leq \beta < 1$ and $\varepsilon > 0$. Then there exists a constant $C > 0$ such that*

$$\|Mf(z)\| \leq C\|f\|_{-\beta}[\operatorname{dist}(z, \partial D)]^{-1/2-\beta-\varepsilon} \tag{8.2}$$

*for all $f \in B_*^{\beta}(D)$ and $z \in D$.*

*Proof.* — If $M$ is defined by (7.2), then as in the proof of Lemma 8.1 we obtain that

$$\|Mf(z)\| \leq C\|f\|_{-\beta} \int_{\substack{y \in \mathbf{R}^{2n-\ell+1} \\ |y|<C}} \frac{|y_1|^{-\beta} dy_1 \wedge \cdots \wedge dy_{2n-\ell+1}}{(|y_1| + d + |y|^2)|y|^{2n-\ell}}.$$

Since we may assume that $\beta + \varepsilon < 1$, this implies that

$$\begin{aligned}\|Mf(z)\| &\leq C\|f\|_{-\beta} \int_{\substack{y \in \mathbf{R}^{2n-\ell+1} \\ |y|<C}} \frac{|y_1|^{-\beta-\varepsilon} dy_1 \wedge \cdots \wedge dy_{2n-\ell+1}}{(|y_1| + d + |y|^2)|y|^{2n-\ell-\varepsilon}} \\ &\leq C\|f\|_{-\beta} \int_{\substack{y \in \mathbf{R}^{2n-\ell} \\ |y|<C}} \frac{dy_1 \wedge \cdots \wedge dy_{2n-\ell}}{d^{\beta+\varepsilon}|y|^{2n-\ell-\varepsilon}} \leq C\|f\|_{-\beta} d^{-\beta-\varepsilon}\end{aligned}$$

for all $f \in B_*^{\beta}(D)$ and $z \in D$. If $M$ is defined by (7.3) or (7.5), then as in the proof of Lemma 8.1 we obtain that

$$\|Mf(z)\| \leq C\|f\|_{-\beta} \sum_{\kappa=1}^{s+1} \int_{y \in \mathbf{R}^{2n-\ell+1}} \frac{|y_1|^{-\beta} dy_1 \wedge \cdots \wedge dy_{2n-\ell+1}}{(|y_1| + d + |y|^2)^2 \prod_{\nu=2}^{\kappa} (|y_\nu| + d + |y|^2)|y|^{2n-\ell-\kappa}}$$

for all $f \in B_*^{\beta}(D)$ and $z \in D$, where $s = 0$ in the case (7.3). This implies that

$$\begin{aligned}\|Mf(z)\| &\leq C\|f\|_{-\beta} \sum_{\kappa=1}^{s+1} \int_{y \in \mathbf{R}^{2n-\ell}} \frac{dy_1 \wedge \cdots \wedge dy_{2n-\ell}}{(d + |y|^2)^{1+\beta} \prod_{\nu=2}^{\kappa} (|y_\nu| + d + |y|^2)|y|^{2n-\ell-\kappa}} \\ &\leq C\|f\|_{-\beta} \sum_{\kappa=1}^{s+1} \int_{y \in \mathbf{R}^{2n-\ell-\kappa+1}} \frac{[1 + |\ell n(d + |y|^2)|]^{\kappa-1} dy_1 \wedge \cdots \wedge dy_{2n-\ell-\kappa+1}}{(d + |y|^2)^{1+\beta}|y|^{2n-\ell-\kappa}} \\ &\leq C\|f\|_{-\beta} \int_0^{\infty} \frac{dr}{(d + r^2)^{1+\beta+\varepsilon}} \leq C\|f\|_{-\beta} d^{-1/2-\beta-\varepsilon}\end{aligned}$$

for all $f \in B_*^{\beta}(D)$ and $z \in D$. □

*Proof of Theorem 4.12.* — We denote all positive constants by the same letter $C$. By Theorem 5.4, $H$ is a finite sum of operators of type $m$. By Lemma 5.3 and Ascoli's theorem operators of type 0 admit estimates which are even stronger than those stated in Theorem 4.12. Therefore it is sufficient to prove that Theorem 4.12 holds for all operators of type $m$ with $m \geq 1$ (at the place of $H$).

Let $E$ be such an operator and $0 \leq \beta < 1$. Then, by Theorem 7.2, $E$ can be estimated by a finite sum of $\lambda$-free bounds of first kind. Hence, if $\varepsilon > 0$, then by Lemma 8.1

$$\|Ef(z)\| \leq C\|f\|_{-\beta}\big(1 + [\mathrm{dist}(z, \partial D)]^{1/2-\beta-\varepsilon}\big) \tag{8.3}$$

for all $f \in B^{\beta}_{n,*}(D)$ and $z \in D$. Further let $\nabla_z$ be one of the operators $\partial/\partial z_1, \dots, \partial/\partial z_n, \partial/\partial \overline{z}_1, \dots, \partial/\partial \overline{z}_n$. Then, by Theorem 7.2, $\nabla_z E$ can be estimated by a finite sum of $\lambda$-free bounds of second kind. Therefore Lemma 8.2 implies that if $\varepsilon > 0$, then

$$\|\nabla_z Ef(z)\| \leq C\|f\|_{-\beta}[\mathrm{dist}(z, \partial D)]^{-1/2-\beta-\varepsilon} \tag{8.4}$$

for all $f \in B^{\beta}_{n,*}(D)$ and $z \in D$.

Now let $0 \leq \beta < 1/2$. Then (8.3) in particular implies that

$$\|Ef\|_0 \leq C\|f\|_{-\beta} \tag{8.5}$$

for all $f \in B^{\beta}_{n,*}(D)$. It is well-known (see, e.g., Proposition 2 in Appendix 1 of [HeLe1]) that (8.4) and (8.5) together imply that

$$E\big(B^{\beta}_{n,*}(D)\big) \subseteq \bigcap_{0<\varepsilon\leq 1/2-\beta} C^{1/2-\beta-\varepsilon}_{n,*}(\overline{D})$$

and that $E$ is bounded as operator from $B^{\beta}_{n,*}(D)$ to each $C^{1/2-\beta-\varepsilon}_{n,*}(\overline{D}), 0 < \varepsilon \leq 1/2 - \beta$. By Ascoli's theorem it follows that $E$ is even compact as operator from $B^{\beta}_{n,*}(D)$ to each $C^{1/2-\beta-\varepsilon}_{n,*}(\overline{D}), 0 < \varepsilon \leq 1/2 - \beta$. Hence part (i) of Theorem 4.12 is proved.

To prove part (ii), we assume that $1/2 \leq \beta < 1$. Then by (8.3)

$$E\big(B^{\beta}_{n,*}(D)\big) \subseteq \bigcap_{\varepsilon>0} B^{\beta+\varepsilon-1/2}_{n,*}(D)$$

and $E$ is bounded as operator from $B^{\beta}_{n,*}(D)$ into each $B^{\beta+\varepsilon-1/2}(D)$, $\varepsilon > 0$. Moreover, it follows from (8.4) and Ascoli's theorem that, for each domain $\Omega \subset\subset D, E$ is bounded as operator from $B^{\beta}_{n,*}(D)$ to $C^{0}_{n,*}(\overline{\Omega})$. Together this implies that $E$ is compact as operator from $B^{\beta}_{n,*}(D)$ to each $B^{\beta+\varepsilon-1/2}_{n,*}(D), \varepsilon > 0$. □

## 9. Globalization.

In this section $E$ is a holomorphic vector bundle over an $n$-dimensional complex manifold $X$, and $D \subset\subset X$ is a strictly $q$-convex $C^2$ intersection, $0 \le q \le n-1$ (see Definition 0.1). Further, we denote by $C^{\alpha}_{n,r}(\overline{D}, E), B^{\alpha}_{n,r}(D, E)$ etc. the Banach spaces of $E$-valued differential forms on $D$ which one obtains canonically extending the definitions from sect. 1.16.

**9.1.** THEOREM. — *There exist linear operators*

$$\widetilde{T}_r : \bigcup_{0\le\beta<1} B^{\beta}_{n,r}(D, E) \longrightarrow C^{0}_{n,r-1}(D, E)$$

*and*

$$K_r : \bigcup_{0\le\beta<1} B^{\beta}_{n,r}(D, E) \longrightarrow C^{0}_{n,r}(D, E)$$

*for $n - q \le r \le n$ such that the following holds :*

(i) *If $n - q \le r \le n$, then*

$$f = d\widetilde{T}_r f + \widetilde{T}_{r+1} df + K_r f \tag{9.1}$$

*for all $f \in B^{\beta}_{n,r}(D, E), 0 \le \beta < 1$, such that $df$ also belongs to $B^{\beta}_{*}(D, E)$. (For $r = n$, the term $T_{r+1}df$ must be omitted.)*

(ii) *If $0 \le \beta < 1/2$ and $0 < \varepsilon \le 1/2 - \beta$, then, for all $n - q \le r \le n, \widetilde{T}_r$ and $K_r$ are compact operators from $B^{\beta}_{n,r}(D, E)$ into $C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D}, E)$ resp. $C^{1/2-\beta-\varepsilon}_{n,r}(\overline{D}, E)$.*

(iii) *If $1/2 \le \beta < 1$ and $\varepsilon > 0$, then, for all $n - q \le r \le n, \widetilde{T}_r$ and $K_r$ are compact operators from $B^{\beta}_{n,r}(D, E)$ into $B^{\beta+\varepsilon-1/2}_{n,r-1}(D, E)$ resp. $B^{\beta+\varepsilon-1/2}_{n,r}(D, E)$.*

*Proof.* — By Lemma 2.4 there exists a finite number of open sets $U_1, \dots, U_m \subseteq X$ such that $\overline{D} \subseteq U_1 \cup \dots \cup U_m$ and each $U_j \cap D, 1 \le j \le m$, is a local $q$-convex domain. Moreover, we may assume that $E$ is trivial over some neighborhood of each $\overline{U_j \cap D}, 1 \le j \le m$. Let $H_j$ be the operators which are induced in

$$\bigcup_{0\le\beta<1} B^{\beta}_{n,*}(D, E)$$

by the operators which exist by sect. 4 for each $U_j \cap D$. Choose non-negative $C^{\infty}$ functions $\chi_j$ with compact support in $U_j$ such that $\chi_1 + \dots + \chi_m = 1$

in a neighborhood of $\overline{D}$. Set

$$\tilde{T}_r f = \sum_{j=1}^{m} \chi_j H_j(f|_{U_j \cap D})$$

and

$$K_r f = \sum_{j=1}^{m} d_{\chi_j} \wedge H_j(f|_{U_j \cap D})$$

for $n-q \leq r \leq n, f \in B^{\beta}_{n,r}(D), 0 \leq \beta < 1$. □

**9.2. LEMMA.** — *For each neighborhood $\Theta \subseteq X$ of $\partial D$ there exist a neighborhood $U \subseteq \Theta$ of $\partial D$ and a real $C^2$ function $\rho$ on $U$ whose Levi form has at least $q+1$ positive eigenvalues at each point in $U$ such that*

$$\{z \in U : \rho(z) < -1\} \cup (D \setminus U) \subset\subset D \subset\subset \{z \in U : \rho(z) < 1\} \cup D. \tag{9.2}$$

*Proof.* — Let $\rho_1, \ldots, \rho_N$ be the functions from Definition 0.1. Choose $\beta > 0$ and set

$$\varphi_1 = \rho_1, \ \varphi_2 = \max_{\beta}(\rho_1, \rho_2), \ldots, \varphi_N = \max_{\beta}(\varphi_{N-1}, \rho_N),$$

where $\max_{\beta}(\cdot,\cdot)$ is defined as in Definition 4.12 in [HeLe2]. Then it is easy to compute that the Levi form of $\varphi_{N-1}$ has at least $q+1$ positive eigenvalues at each point in $U$. Further, it is easy to see that (9.2) holds if $\beta$ is sufficiently small and, for some positive number $C, \rho := C\varphi_{N-1}$. □

**9.3. LEMMA.** — *Let $n-q \leq r \leq n, 0 \leq \beta < 1$, and let $f \in B^{\beta}_{n,r}(D,E)$ be a form which is exact on $D$. Then :*

(i) *If $0 \leq \beta < 1/2$, then there exists $u \in \bigcap_{0<\varepsilon\leq 1/2-\beta} C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D}, E)$ with $f = du$.*

(ii) *If $1/2 \leq \beta < 1$, then there exists $u \in \bigcap_{\varepsilon>0} B^{\beta+\varepsilon-1/2}_{n,r-1}(D,E)$ with $f = du$.*

*Proof.* — By means of Theorem 4.12 and Grauert's "Beulenmethode" (see, e.g., the proof of Theorem 2.3.5 in [HeLe1]), we find a closed continuous $E$-valued form $\tilde{f}$ in a neighborhood of $\overline{D}$ as well as a form

$$\tilde{u} \in \bigcap_{0<\varepsilon\leq 1/2-\beta} C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D}, E) \text{ resp. } \tilde{u} \in \bigcap_{\varepsilon>0} B^{\beta+\varepsilon-1/2}_{n,r-1}(D,E)$$

such that $f - d\tilde{u} = \tilde{f}$ on $D$. Since $f$, and therefore $\tilde{f}$ is exact on $D$, then it follows from Lemma 9.2 and classical Andreotti-Grauert theory (see, e.g., Theorem 12.14 in [HeLe2]) that the equation $dv = \tilde{f}$ has a Hölder continuous (with each exponent $< 1$) solution $v$ in some neighborhood of $\overline{D}$. Set $u = v - \tilde{u}$. □

**9.4.** *Proof of Theorem 0.2.* — The proofs of parts (i) and (ii) are analogous; we restrict ourselves to part (i).

Let $\widetilde{T}_r, K_r, n-q \leq r \leq n$, be the operators from Theorem 9.1. Then, by (9.1), $d\widetilde{T}_r = I - K_r$ on $B^{\beta}_{n,r}(D,E) \cap \ker d$. Since the operators $K_r$ are compact, this implies that the operators $d\widetilde{T}_r$ restricted to $B^{\beta}_{n,r}(D,E) \cap \ker d$ are Fredholm operators with zero index. Therefore we can find finite dimensional linear operators $L_r : B^{\beta}_{n,r}(D,E) \cap \ker d \to B^{\beta}_{n,r}(D,E) \cap \ker d$ such that the operators $d\widetilde{T}_r + L_r$ are invertible.

Moreover, by the Andreotti-Grauert theorem (see [AnG] or, e.g., Theorem 12.16 in [HeLe2]), all forms in $B^{\beta}_{n,r}(D,E) \cap \ker d$ are exact. Therefore, by Lemma 9.3, we can find finite dimensional linear maps

$$F_r : B^{\beta}_{n,r}(D,E) \cap \ker d \longrightarrow \bigcap_{0<\varepsilon\leq 1/2-\beta} C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D},E)$$

such that $L_r = dF_r$. It remains to set

$$T_r = (\widetilde{T}_r + F_r)(d\widetilde{T}_r + L_r)^{-1}. \qquad \square$$

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