ANNALES DE L'INSTITUT FOURIER

R. A. ARTINO
J. BARROS-NETO

Matrix triangulation of hypoelliptic boundary value problems

Annales de l'institut Fourier, tome 42, nº 4 (1992), p. 805-824
<http://www.numdam.org/item?id=AIF_1992__42_4_805_0>

Ann. Inst. Fourier, Grenoble
42, 4 (1992), 805-824

MATRIX TRIANGULATION OF HYPOELLIPTIC BOUNDARY VALUE PROBLEMS

by R.A. ARTINO and J. BARROS-NETO

1. Introduction.

Let ω be a domain in $\mathbb{R}^n$ and T a positive real number. On $\omega \times [0, T)$ we consider

$$P(x, t, D_x, D_t) = D_t^m + \sum_{j=1}^{m} P_j(x, t, D_x) D_t^{m-j}, \tag{1.1}$$

where, for each $j = 1, \ldots, m$, $P_j(x, t, D_x)$, is a general pseudodifferential operator depending smoothly on $0 \leq t < T$, and whose symbols are in $S_{\rho,\delta}^{m_j}(\omega \times [0, T))$.

For $1 \leq j \leq \nu$, let

$$B_j(x, D_x, D_t) = \sum_{k=0}^{d_j} B_{jk}(x, D_x) D_t^k,$$

where $B_{jk}(x, D_x)$ are pseudodifferential operators whose symbols $B_{jk}(x, \xi)$ belong to $S_{\rho,\delta}^{b_{jk}}(\omega)$.

We want to consider boundary value problems of the type

$$P(x, t, D_x, D_t) u(x, t) = f(x, t) \tag{1.2}$$

$$B_j(x, D_x, D_t) u(x, 0) = h_j(x), \ j = 1, \ldots, \nu, \tag{1.3}$$

Key words : Hypoelliptic boundary value problems – Parametrix – Regularity up to the boundary.
A.M.S. Classification : 35S15.

with $u, f \in \mathcal{C}^\infty([0,T);\mathcal{D}'(\omega))$, $h_j \in \mathcal{D}'(\omega)$, $j = 1,\dots,\nu$, where $P(x,t,D_x, D_t)$, of the form (1.1) and is hypoelliptic and its inverse lies in some Hörmander class $S^{\ell}_{\rho,\delta}$ with $\rho > \delta$.

Boundary value problems of this type for elliptic operators were studied in Treves [19] using the classical theory of pseudodifferential operators. There it is shown that the ellipticity condition implies that the principal symbol $P_m(x,t,\xi,\tau)$ of P as a polynomial in τ can be factorized. This factorization leads to a factorization of the operator $P(x,t,D_x,D_t)$ itself, modulo a regularizing operator, i.e., $P = P^- P^+ + R$. The properties of the operators $P^\pm$ allow one to reduce (1.2) to two systems of first order equations. The boundary conditions (1.3) can then be adjoined by constructing a matrix valued pseudodifferential operator $\mathcal{B}$ defined on the boundary ω, called the Calderon operator. This type of operator was first introduced by Calderon in [8]. Although the resulting system is not quite uncoupled it is, however, simple enough to show that the regularity up to the boundary of the solutions of (1.2) – (1.3) is equivalent to the hypoellipticiy of $\mathcal{B}$ in ω.

In our paper [3] we showed that if $P(x,t,D_x,D_t)$ is formally hypoelliptic then the total symbol of P, considered as a polynomial in τ can be factorized in the form

$$P(x,t,\xi,\tau) = P^-(x,t,\xi,\tau)P^+(x,t,\xi,\tau),$$

where $P^\pm$ are polynomials in τ all of whose roots lie in the the half-plane $\mathbb{C}^\pm$ when (x,t) belong to a compact subset of $\omega \times [0,T)$ and $|\xi|$ large. This factorization implies that the operator $P(x,t,D_x,D_t)$ can be written in the form

$$P = LP^-P^+ + R,$$

where $P^\pm(x,t,D_x,D_t)$ are pseudodifferential operators whose symbols are given by $P^\pm(x,t,\xi,\tau)$, L is a hypoelliptic pseudodifferential operator, and R is regularizing.

In Section 2 of this paper we show that such a factorization holds for the more general class of hypoelliptic operators whose symbols belong to $S^m_{\rho,\delta}$ as introduced by Hörmander in [13] (see conditions (H1) and (H2) below). We also show, by matrix triangulation, how to reduce the boundary value problem (1.2) – (1.3) to two uncoupled first order systems. In Section 3 we estimate the eigenvalues of the corresponding matrices. In Section 4 we construct a parametrix for first order systems for operators of the type $D_t - \mathcal{A}(t)$ with the symbol of $\mathcal{A}(t)$ satisfying certain hypoellipticity conditions derived from (H1) and (H2). In Section 5 we construct the Calderon operator, $\mathcal{B}(x)$, on the boundary of our domain and show that the boundary value problem (1.2) – (1.3) is hypoelliptic if and only if $\mathcal{B}(x)$ is hypoelliptic in ω. In Section 6 we go back to the constant coefficient case and compare our results with the classical Hörmander's results [10] about hypoellipticity up to the boundary.

The theory of elliptic boundary value problems has a long history. The papers of Agmon–Douglis–Nirenberg [1] are classic and based on the work of Lopatinski [15]. The reduction to pseudodifferential systems on the boundary is due partly to Calderon [8], Agranovich [2]. More extensive work was carried on by Visik-Eskin [20], and Boutet de Monvel [7]. In fact, the pseudodifferential operators that we consider in this paper are clearly related to the ones satisfying the *transmission property* studied by Boutet de Monvel [7]. It seems likely that some of our results can be extended to situations analogous to those considered by Boutet de Monvel, and these are questions that we are now investigating.

2. Factorization of $P(x, t, D_x, D_t)$.

We assume the following hypoellipticity conditions on $P(x, t, \xi, \tau)$ the total symbol of P :

(H1) For each compact subset $K \subset \omega \times [0, T)$ and for all $\alpha, \beta \in \mathbb{Z}_+^{n+1}$, there exist constants $C(K, \alpha, \beta) > 0$ and $M(K) > 0$ such that

$$|D^{\beta}_{(x,t)} D^{\alpha}_{(\xi,\tau)} P(x, t, \xi, \tau)| \leq C(1 + |(\xi, \tau)|)^{-\rho|\alpha|+\delta|\beta|} |P(x, t, \xi, \tau)|$$

for all $|(\xi, \tau)| \geq M$.

(H2) There exists a real number μ such that for each compact set $K \subset \omega \times [0, T)$, there exist positive constants $C = C(K)$, $M = M(K)$, such that

$$|P(x, t, \xi, \tau)| \geq C(1 + |(\xi, \tau)|)^{\mu}$$

for all $|(\xi, \tau)| \geq M$ and all $(x, t) \in K$.

It is shown in [13],[19] that with $\delta < \rho$ the above conditions imply that $P(x, t, D_t, D_x)$ is invertible modulo regularizing operators, and is hypoelliptic, i.e., for each $u(x, t)$ in $\mathcal{C}^{\infty}([0, T); \mathcal{D}'(\omega))$,

$$\text{sing supp}\, Pu = \text{sing supp}\, u.$$

Condition (H2) above implies that for each compact set $K \subset \omega \times [0,T)$ the total symbol $P(x,t,\xi,\tau)$, as a polynomial of degree m in τ, has no real zeros for $|\xi|$ large. Therefore, when $n > 1$ (what we will always assume), the number of roots, $\mu^+(\mu^-)$ of $P(x,t,\xi,\tau)$, as a polynomial in τ, with positive (negative) imaginary parts is constant when $|\xi| > M$. A simple connectivity argument shows $\mu^+(\mu^-)$ is the same for each K. We label these roots $\tau_j^\pm(x,t,\xi)$, $j = 1,\dots,\mu^\pm$. Consequently,

$$P(x,t,\xi,\tau) = P^-(x,t,\xi,\tau)P^+(x,t,\xi,\tau),$$

where

$$P^\pm(x,t,\xi,\tau) = \prod_{j=1}^{\mu^\pm}(\tau - \tau_j^\pm(x,t,\xi)) = \tau^{\mu^\pm} + \sum_{j=1}^{\mu^\pm} p_j^\pm(x,t,\xi)\tau^{\mu^\pm - j},$$

for $(x,t) \in K$, $|\xi| > M$.

We have shown in our paper [3] that $P^\pm(x,t,\xi,\tau)$ satisfies (H1) and (H2) with δ replaced by $\delta + \epsilon$, where $\epsilon > 0$ is arbitrary and $C = C(\epsilon,\alpha,\beta,K)$. It follows that $P^\pm(x,t,\xi,\tau)$ belongs to $S^m_{\rho,\delta+\epsilon}$ for suitable m. In that paper, we were unable to prove (perhaps for technical reasons) our result for $\epsilon = 0$, as in the case of elliptic operators. However, since ϵ is arbitrary, we can choose it so that $\delta + \epsilon < \rho$. By doing so the symbols $P^\pm(x,t,\xi,\tau)$ define pseudodifferential operators which are invertible modulo regularizing operators. If we denote by $K(x,t,D_x,D_t)$, $K^\pm(x,t,D_x,D_t)$, the parametrices of P, $P^\pm$, respectively, we have

$$P(x,t,D_x,D_t)K(x,t,D_x) \sim I \text{ and } P^\pm(x,t,D_x,D_t)K^\pm(x,t,D_x) \sim I.$$

Now write $Pu = f$ in the following way :

$$P^+P^-u = \tilde{f} - Ru,$$

where $\tilde{f} = (PK^+K^-)^{-1}f$, with $(PK^+K^-)^{-1}$ the parametrix of PK^+K^- and R regularizing. In fact, it is easily seen that

$$P = LP^-P^+ + R,$$

where $L = PK^+K^-$, and R is regularizing. It is clear that L is pseudodifferential operator of order 0, and has a parametrix.

We can replace the equation $P^- P^+ u = \tilde{f} - Ru$ with the system of equations

$$P^+ u = v \tag{2.1}$$

$$P^- v = \tilde{f} - Ru. \tag{2.2}$$

We in turn reduce each of these equations to a first order system.

Both of these equations are of the type

$$Q(x,t,D_x,D_t)w = D_t^{\mu} w + \sum_{j=1}^{\mu} q_j(x,t,D_x) D_t^{\mu-j} w = g, \tag{2.3}$$

with $\mu = \mu^{\pm}$, $q_j = p_j^{\pm}$, and $g = v$ or $\tilde{f} - Ru$. Letting $w_1 = w,\ w_j = D_t w_{j-1}$ we can write (2.3) as

$$D_t \mathbf{w} - \mathcal{A}(t)\mathbf{w} = \mathbf{g} \tag{2.4}$$

where $\mathbf{w} = (w_1, \ldots, w_\mu)^T,\ \mathbf{g} = (0, \ldots, 0, g)^T$ and $\mathcal{A}(t)$ is the matrix :

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -q_\mu & -q_{\mu-1} & -q_{\mu-2} & \cdots & -q_1 \end{pmatrix} \tag{2.5}$$

$\mathcal{A}$ is a matrix valued pseudodifferential operator whose symbol $\sigma(\mathcal{A}) = a(x,t,\xi)$ belongs to $S^{\bar{m}}_{\rho,\delta+\epsilon}(\omega \times [0,T))$, with $\bar{m} = \max_{1 \le j \le \mu}(\text{order } q_j)$. We also note that since

$$\det\,(\tau I - \sigma(\mathcal{A})) = Q(x,t,\xi,\tau), \tag{2.6}$$

the eigenvalues of $\sigma(\mathcal{A})$ are the roots of $Q(x,t,\xi,\tau)$.

3. Estimates of the eigenvalues of $\sigma(\mathcal{A})$.

Theorem 3.1 below locates the eigenvalues of $\mathcal{A}$ in the complex plane. First we need the following lemma.

Let K be a compact subset of $\omega \times [0,T)$ and

$$N(K) = \{(\zeta,\tau) \in \mathbb{C}^{n+1} : Q(x,t,\zeta,\tau) = 0\},$$

for $(x,t) \in K$. For $(\xi,\tau) \in \mathbb{R}^{n+1}$, let $d((\xi,\tau),N(K))$ be the distance from (ξ,τ) to $N(K)$.

LEMMA 3.1. — *Suppose $Q(x,t,D_x,D_t)$ is a pseudodifferential operator of the type (2.3) whose symbol satisfies (H1) and (H2). For each compact set $K \subset \omega \times [0,T)$, there exists a constant $C = C(K) > 0$ such that*

$$C^{-1} \leq d((\xi,\tau),N(K)) \sum_{j=1}^{\mu} \left(\frac{|D_\tau^j Q(x,t,\xi,\tau)|}{|Q(x,t,\xi,\tau)|} \right)^{1/j} \leq C, \tag{3.1}$$

for $(\xi,\tau) \in \mathbb{R}^{n+1}, (x,t) \in K$ and $Q(x,t,\xi,\tau) \neq 0$.

Proof. — The proof in the case of variable coefficients is similar to that of Lemma 3.1 in [3] by arguing in a neigborhood of a point (x,t) in ω and then using a simple compactness argument for a general K.

We, therefore, assume

$$Q(\xi,\tau) = \tau^\mu + \sum_{j=1}^{\mu} q_j(\xi)\tau^{\mu-j},$$

with $q_j(\xi)$ in some symbol class $S_\rho^{m_j}$. Let $N_\xi = \{\tau \in \mathbb{C} : \ Q(\xi,\tau) = 0\}$, and $d(\xi,\tau)$ be the distance from τ to N_ξ. We prove that if $Q(\xi,\tau) \neq 0$, then there exists a constant $C > 0$ such that

$$C^{-1} \leq d(\xi,\tau) \sum_{j=1}^{\mu} \left| \frac{D_\tau^j Q(\xi,\tau)}{Q(\xi,\tau)} \right|^{1/j} \leq C. \tag{3.2}$$

Set

$$A = A(\xi,\tau) = \sum_{j=1}^{\mu} \left| \frac{D_\tau^j Q(\xi,\tau)}{Q(\xi,\tau)} \right|^{1/j}.$$

Then $|D_\tau^j Q(\xi,\tau)| \leq A^j |Q(\xi,\tau)|$. By Taylor's formula,

$$Q(\xi,\tau+\theta) - Q(\xi,\tau) = \sum_{j=1}^{\mu} \frac{1}{j!} D_\tau^j Q(\xi,\tau)\theta^j.$$

Therefore,

$$|Q(\xi,\tau+\theta) - Q(\xi,\tau)| \leq \sum_{j=1}^{\mu} \frac{1}{j!} (A\theta)^j |Q(\xi,\tau)|.$$

Let $c > 0$ be such that $e^c - 1 < 1$. Choose θ so that $A|\theta| < c$. Thus, $\sum_{j=1}^{\mu} \frac{1}{j!} c^j \leq e^c - 1 < 1$. So, $|Q(\xi, \tau + \theta) - Q(\xi, \tau)| < |Q(\xi, \tau)|$. Hence, $Q(\xi, \tau + \theta) \neq 0$ whenever $|\theta| < c/A$. This implies that $d(\xi, \tau) \geq c/A$, and therefore, the left side of (3.2).

Take $\zeta \in \mathbb{C}$ such that $|\zeta| \leq d(\xi, \tau)/2$. Consider the polynomial in t, $Q(\xi, \tau + t\zeta)$. We are assuming that $Q(\xi, \tau) \neq 0$. If t_i is a root then

$$|t_i \zeta| = |\tau - (\tau + t_i \zeta)| > d(\xi, \tau) \geq 2|\zeta|$$

implies that $|t_i| \geq 2$. If we set $g(t) = \prod_{i=1}^{\mu} (t - t_i)$, we obtain

$$\left| \frac{Q(\xi, \tau + \zeta)}{Q(\xi, \tau)} \right| = \left| \frac{g(1)}{g(0)} \right| = \prod_{i=1}^{\mu} \left| 1 - \frac{1}{t_i} \right| \leq (3/2)^{\mu}.$$

By Cauchy's formula

$$D_\tau^j Q(\xi, \tau) = \frac{j!}{2\pi i} \oint_\gamma \frac{Q(\xi, \tau + \zeta)}{\zeta^{j+1}} \, d\zeta,$$

where $\gamma = \{\zeta : |\zeta| = d(\xi, \tau)/2\}$. The above estimates imply

$$\begin{aligned} |D_\tau^j Q(\xi, \tau)| &\leq \frac{j!}{2\pi} \left(\frac{3}{2} \right)^{\mu} \oint_\gamma \frac{|Q(\xi, \tau)|}{|\zeta|^{j+1}} \, |d\zeta| \\ &\leq \left(\frac{3}{2} \right)^{\mu} j! \left(\frac{2}{d(\xi, \tau)} \right)^j |Q(\xi, \tau)|. \end{aligned}$$

This implies the right hand side of (3.2) and proves the lemma. □

THEOREM 3.1. — *For each compact set $K \subset \omega \times [0, T)$, there exist positive constants $M(K)$, C_1, C_2 such that whenever $|\xi| > M$, the set of zeros, $\tau(x, t, \xi)$, of $Q(x, t, \xi, \tau)$ for $(x, t) \in K$ is contained in the subset of the complex plane defined by :*

$$|\tau| \leq C_1 (1 + |\xi|)^{\bar{m}}, \ |\mathrm{Im}\, \tau| \geq C_2 |\xi|^{\rho}. \tag{3.3}$$

Proof. — Using a well known estimate for the zeros of a polynomial of one variable, we have :

$$|\tau| \leq 1 + \max_{1 \leq j \leq \mu} |q_j(x, t, \xi)|. \tag{3.4}$$

Since $q_j(x,t,\xi)$ belongs to $S^{m_j}_{\rho,\delta}(\omega \times [0,T))$,we obtain the first inequality in (3.3).

It follows from (H1) and (3.1) that for each compact set K there is a constant $C(K) > 0$ such that for $|\xi| > M(K)$,

$$d((\xi,\tau), N(K)) \geq C(1+|\xi|)^{\rho}. \tag{3.5}$$

If $(\zeta,\tau) \in N(K)$, then $d(\mathrm{Re}\,(\zeta,\tau), N(K)) \leq |\mathrm{Im}\,(\zeta,\tau)|$.

It follows from (3.5) that

$$|\mathrm{Re}\,(\zeta)|^{\rho} \leq C|\mathrm{Im}\,(\zeta,\tau)|.$$

Hence, for ξ real, $|\xi| > M(K)$, and $(\xi,\tau) \in N(K)$,

$$|\xi|^{\rho} \leq C|\mathrm{Im}\,\tau|. \tag{3.6}$$

This completes the proof of the theorem. □

4. Evolution operators and their associated parametrices.

Equations (2.1) and (2.2) can be replaced by the first order systems

$$D_t\mathbf{u} - \mathcal{A}^+(t)\mathbf{u} = \mathcal{J}\mathbf{v} \tag{4.1}$$

$$D_t\mathbf{v} - \mathcal{A}^-(t)\mathbf{v} = \mathbf{g} - \mathcal{R}\mathbf{u}, \tag{4.2}$$

where, $\mathbf{u} = (u_1, \ldots, u_{\mu^+})^T$ with $u_1 = u$, $u_j = D_t u_{j-1}$, $j = 2, \ldots, \mu^+$, $\mathcal{J}\mathbf{v}$ is a μ^+-vector with components all zero except the last one equal to v, $\mathbf{v} = (v_1, \ldots, v_{\mu^-})^T$, with $v_1 = v$, $v_j = D_t v_{j-1}$, $j = 2, \ldots, \mu^-$, $\mathbf{g}$ (resp. $\mathcal{R}\mathbf{u}$) a μ^-- vector with components all zero except the last one equal to $\tilde{f}$ (resp. $R\mathbf{u}$).

Definition 4.1. — *For fixed t' such that $0 \leq t' < T$ a pseudodifferential operator*

$$U(t,t') : \mathcal{E}'(\omega;\mathbb{C}^{\mu}) \to \mathcal{D}'(\omega;\mathbb{C}^{\mu})$$

depending smoothly on $t \in [t',T)$ is called a parametrix for the operator $D_t - \mathcal{A}(t)$, if

$$\frac{dU(t,t')}{dt} - \mathcal{A}(t) \circ U(t,t') \sim 0 \text{ in } \omega \times [t',T) \tag{4.3}$$

$$U(t,t')_{|t=t'} \sim I \text{ in } \omega. \tag{4.4}$$

We note that $U(t,t')$ is defined modulo regularizing operators on ω.

If $\mathcal{A}(t) = \mathcal{A}^+(t)$ then we can prove existence of the parametrix as follows. The operator $U(t,t')$ is defined by

$$U(t,t')u = (2\pi)^{-n}\int e^{ix\xi}\mathcal{U}(x,t,t',\xi)\hat{u}(\xi)\,d\xi,$$

for all $u \in C_c^\infty(\omega)$, where $\mathcal{U}(x,t,t',\xi)$ is the symbol of $U(t,t')$. We actually construct a formal symbol

$$\mathcal{U}(x,t,t',\xi) = \sum_{j=0}^{\infty}\mathcal{U}_j(x,t,t',\xi)$$

from which a true symbol can later be constructed by use of cut-off functions in the standard way. Proceeding formally, we write

$$(D_t - \mathcal{A}(t))U(t,t')u = (2\pi)^{-n}\int e^{ix\xi}(D_t - a(x,t,D_x+\xi))\mathcal{U}(x,t,\xi)\hat{u}(\xi)\,d\xi,$$

and we require for each $0 \le t < T$ that,

$$\begin{aligned}&(D_t - a(x,t,D_x+\xi))\mathcal{U}(x,t,t',\xi)\\ &= \Big(D_t - \sum_{\alpha\in\mathbb{Z}^n}\frac{1}{\alpha!}\partial_\xi^\alpha a(x,t,\xi)D_x^\alpha\Big)\mathcal{U}(x,t,t',\xi) = 0,\end{aligned} \tag{4.5}$$

and $\mathcal{U}(x,t,t',\xi) = I$ (identity matrix). The term involving the summation symbol is denoted by $a(x,t,\xi)\odot\mathcal{U}(x,t,t',\xi)$.

Let $\lambda(\xi) = (1+|\xi|)^\rho$, and consider the expression

$$zI - \lambda^{-1}a(x,t,\xi) = \lambda^{-1}(z\lambda I - a(x,t,\xi)).$$

It follows from Theorem 3.1 that for each compact set $K \subset \omega\times[0,T)$ there exist positive constants M, C_1, C_2 such that if $(x,t)\in K$ and $|\xi|\ge M$, the eigenvalues of the matrix $\lambda^{-1}a(x,t,\xi)$ lie in $\mathbb{C}^+$ inside the circle

$$|z| \le C_1(1+|\xi|)^{\bar{m}-\rho}$$

and in the half-plane $\mathrm{Im}z \ge C_2$. For any $R \ge M$ and $R \le |\xi| \le R+1$, let Γ_R be a contour in the upper half-plane that encircles the eigenvalues of the matrix $\lambda^{-1}a(x,t,\xi)$ for $(x,t)\in K$. In view of the previous remarks we could take the length of Γ_R to be less than $2\pi(R+2)^{\bar{m}-\rho}$.

We are going to represent $\mathcal{U}$ as

$$\mathcal{U}(x,t,t',\xi) = (2\pi i)^{-1}\oint_{\Gamma_R} e^{i\lambda(t-t')z}k(x,t,\xi;z)\,dz,$$

where k is a suitable formal symbol $\sum_{j=1}^{\infty} k_j$.

Since $k(x,t,\xi,z)$ is going to be a holomorphic function of z, it follows that $\mathcal{U}$ remains the same if the contour Γ_R is changed but still encircles the eigenvalues.

We can take for our compact set K the closure of $\mathcal{O} = D \times [0, T_0)$, with $T_0 < T$ and D a bounded open set in ω. We can write equation (4.5) as

$$\oint_{\Gamma_R} e^{i\lambda(t-t')z}(D_t k + \lambda z k - a(x,t,\xi) \odot k)\, dz = 0. \tag{4.6}$$

We want to solve, in the sense of formal symbols, the equation

$$D_t k + \lambda z k - a(x,t,\xi) \odot k = \lambda I, \tag{4.7}$$

which automatically implies (4.6). Also from the results below it will follow that this k will also satisfy

$$(2\pi i)^{-1} \oint_{\Gamma_R} k(x,t,\xi; z)\, dz = I, \tag{4.8}$$

for all $(x,t) \in \mathcal{O}$, $|\xi| > M$. First rewite (4.7) as

$$k = E[I - \lambda^{-1}(D_t k - a \odot k + ak)], \tag{4.9}$$

with

$$E = (zI - \lambda^{-1} a(x,t,\xi))^{-1},$$

and

$$\lambda^{-1}((a \odot k) - ak)) = \sum_{\alpha \neq 0} \frac{1}{\alpha!} \lambda^{-1} \partial_\xi^\alpha a D_x^\alpha k.$$

We define k by successive approximations as follows :

$$k = \sum_{j=0}^{\infty} k_j \tag{4.10}$$

with

$$k_0 = E = (zI - \lambda^{-1} a(x,t,\xi))^{-1} \tag{4.11}$$

and

$$k_j = -E\lambda^{-1}\left[D_t k_{j-1} - \sum_{1\le|\alpha|\le j}\frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha k_{j-|\alpha|}\right]. \tag{4.12}$$

If we set $\bar{M} = \bar{m} - \rho$, then it is clear that $k_0 = E \in S_{\rho,\delta}^{-\bar{M}}$. Next, one can show by induction that $k_j \in S_{\rho,\delta}^{m_j}$, where $m_j = -\bar{M} - j(\inf(\rho,\bar{m}) - \delta)$. This follows easily if one notes that if $j \ge 1$, each k_j is a finite sum of terms of the form

$$E(b_1E)(b_2E)\cdots(b_rE),$$

with r varying from term to term but always ≥ 2, and each b_i is a bounded linear operator on $\mathbb{C}^\mu$, independent of z, depending smoothly on $t \in [0,T)$. Moreover, for $|\xi| > M,\ k_j(x,t,\xi;z)$ is a C^∞ function of (t,z) for $0 \le t < T_0$ holomorphic for all $z \in \mathbb{C}^+$ such that $\mathrm{Im} z > C_2,\ |z| \le C_1(1+|\xi|)^{\bar{m}-\rho}$.

Since for fixed, but arbitrary, $(x,t) \in \mathcal{O}$ and $|\xi| > M,\ \lambda^{-1}a(x,t,\xi)$ and b_i are bounded linear operators on $\mathbb{C}^\mu$, we obtain :

$$(2\pi i)^{-1}\oint_{\Gamma_R} E(z)\,dz = I$$

$$\oint_{\Gamma_R} E(z)b_1E(z)\cdots b_rE(z)\,dz = 0,$$

with $E(z) = (zI - \lambda^{-1}a(x,t,\xi))^{-1}$. These imply (4.8).

We now want to estimate the symbols

$$\mathcal{U}_j(x,t,t',\xi) = \frac{1}{2\pi i}\oint_{\Gamma_R} e^{i\lambda(t-t')z}k_j(x,t,\xi;z)dz. \tag{4.13}$$

For $z \in \Gamma_R$, we have

$$|D_\xi^\alpha D_t^r(e^{i\lambda(t-t')z})| \le C(t-t')^{-N}(1+|\xi|)^{-|\alpha|}\lambda^{(r-N)}, \tag{4.14}$$

for arbitrary r, N in $\mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^n$. Since $k_j \in S_{\rho,\delta}^{m_j}$, we have

$$|D_x^\beta D_\xi^\alpha D_t^r k_j(x,t,\xi;z)| \le C(1+|\xi|)^{m_j-\rho|\alpha|+(|\beta|+r)\delta}. \tag{4.15}$$

Using Leibniz's formula we can write $D_x^\beta D_\xi^\alpha D_t^r(e^{i\lambda(t-t')z}k_j(x,t,\xi;z))$ as a linear combination of products of the type $D_x^{\beta'}D_\xi^{\alpha'}D_t^r(k_j)D_\xi^{\alpha''}D_t^{r''}(e^{i\lambda(t-t')z})$ each of which can be estimated by

$$C(t-t')^{-N}(1+|\xi|)^{m_j-\rho|\alpha'|-|\alpha''|+(|\beta|+r')\delta}\lambda^{r''-N}, \tag{4.16}$$

by virtue of the estimates (4.14) and (4.15). Since $0 \le \delta < \rho \le 1$, and $\lambda = (1+|\xi|)^\rho$, it follows that

$$(1+|\xi|)^{r'\delta}\lambda^{r''-N} \le (1+|\xi|)^{r\delta+\rho(r-N)}. \tag{4.17}$$

By combining (4.16) and (4.17) we get

$$D_x^\beta D_\xi^\alpha D_t^r(e^{i\lambda(t-t')z}k_j)| \le C(t-t')^{-N}(1+|\xi|)^{m_j-\rho|\alpha|+(|\beta|+r)\delta+(r-N)\rho},$$

and hence the following estimate for the symbol (4.13)

$$\begin{aligned}|D_x^\beta D_\xi^\alpha D_t^r \mathcal{U}_j(x,t,t',\xi)| \\ \le C(t-t')^{-N}(1+|\xi|)^{m_j-\rho|\alpha|+(|\beta|+r)\delta+(r-N)\rho}\oint_{\Gamma_R}|dz|.\end{aligned}$$

Since $\oint_{\Gamma_R} \le CR^{\bar{m}-\rho}$ and $|\xi| \sim R$, we finally obtain

$$\begin{aligned}|D_x^\beta D_\xi^\alpha D_t^r \mathcal{U}_j(x,t,t',\xi)| \\ \le C(t-t')^{-N}(1+|\xi|)^{m_j-|\alpha|\rho+(|\beta|+r)\delta+(r-N)\rho}R^{\bar{m}-\rho} \\ \le C(t-t')^{-N}(1+|\xi|)^{\bar{m}+m_j-|\alpha|\rho+(|\beta|+r)\delta+(r-N-1)\rho}.\end{aligned} \tag{4.18}$$

The lemma that follows summarizes our results.

LEMMA 4.1. — *To every $K \in \Omega$, there is a constant $c > 0$ such that to every pair of n-tuples $\alpha, \beta \in \mathbb{Z}^n$ and to every pair of integers r and N, there exists a constant $C = C(\alpha, \beta, r, K)$ such that*

$$\begin{aligned}|D_x^\beta D_\xi^\alpha D_t^r \mathcal{U}_j(x,t,t',\xi)| \\ \le C(t-t')^{-N}(1+|\xi|)^{\bar{m}+m_j-|\alpha|\rho+(|\beta|+r)\delta+(r-N)\rho}\end{aligned} \tag{4.19}$$

for all $(x,t) \in K$ and $|\xi| > c$.

It follows from this lemma that using a standard proceedure (see [19]) one can construct a true symbol $\mathcal{U} \sim \sum \mathcal{U}_j$, for $U(t,t')$.

Remark 1. — Since the eigenvalues of $\lambda^{-1}a^-(x,t,\xi)$, where $a^-(x,t,\xi) = \sigma(\mathcal{A}^-)$, lie in the negative half-plane $\mathbb{C}^-$ we could solve the backward Cauchy problem

$$D_tU^- - \mathcal{A}^-(t)U^- \sim 0, \text{ in } \omega \times (0,t'], \tag{4.20}$$

$$U^-(t,t')_{|t=t'} \sim I, \text{ in } \omega, \tag{4.21}$$

where

$$U^-(t,t')u = (2\pi)^{-n} \int e^{ix\xi} \mathcal{U}^-(x,t,t',\xi)\hat{u}(\xi)\,d\xi, \tag{4.22}$$

$$\mathcal{U}^-(x,t,t',\xi) = (2\pi)^{-1} \oint_{\Gamma_R} e^{i(t-t')\lambda z} k^-(x,t,\xi;z)\,dz. \tag{4.23}$$

Remark 2. — Going back to our operator (1.1) let us assume that the coefficients are defined on $\omega \times (-T,T)$, and conditions (H1) and (H2) are satisfied. Let $V^+(t,t')$ (resp. $V^-(t,t')$) be a parametrix, defined on $\omega \times [t',T)$ (resp. $\omega \times (-T,t']$), of the forward (resp. backward) Cauchy problem for $D_t - \mathcal{A}^+(t)$ (resp. $D_t - \mathcal{A}^-(t)$). Set $V^+(t,0) = V^+(t)$ and $V^-(t,0) = V^-(t)$. By Lemma 4.1 $V^+(t)$ (resp. $V^-(t)$) is a regularizing operator whenever $t \neq 0$. As $t \to 0\pm$ the symbols of $V^+(t)$ and $V^-(t)$ converge in $S^{\bar{m}+m_j}_{\rho,\delta}(\omega)$. Furthermore, $V^+(0) \oplus V^-(0) = I$.

It follows that we can solve modulo regularizing operators the inhomogeneous equations :

$$D_t\mathbf{u} - \mathcal{A}^+(t)\mathbf{u} = \mathbf{f} \text{ in } \omega \times [0,T) \tag{4.24}$$

$$\mathbf{u}_{|t=0} = \mathbf{g} \text{ in } \omega, \tag{4.25}$$

using

$$\mathbf{u} \sim U^+(t,0)\mathbf{g} + \int_0^t U^+(t,t')\mathbf{f}(t')\,dt'. \tag{4.26}$$

Likewise, for $T_0 < T$, we can solve

$$D_t\mathbf{u} - \mathcal{A}^-(t)\mathbf{u} = \mathbf{f} \text{ in } \omega \times [0,T_0), \tag{4.27}$$

$$\mathbf{u}_{|t=T_0} = \mathbf{g} \in \omega, \tag{4.28}$$

using

$$\mathbf{u} \sim U^-(t,T_0)\mathbf{g} - \int_t^{T_0} U^-(t,t')\mathbf{f}(t')\,dt'. \tag{4.29}$$

5. The Calderon operator.

We now adjoin the boundary conditions (1.3) to (4.1) and (4.2). This is done in the following standard way. Since P is a monic polynomial in

D_t, we can divide

$$B_j = \sum_{k=0}^{d_j} B_{jk}(x, D_x)D_t^k, \ 1 \leq j \leq \nu,$$

by P and obtain

$$B_j = Q'_j P + B'_j, \ \deg B'_j \leq m - 1.$$

Thus (1.3) can be replaced by

$$B'_j u = h_j - Q'_j f_{|t=0}, 1 \leq j \leq \nu. \tag{5.1}$$

Now divide each B'_j by P^+ and obtain

$$B'_j = Q_j P^+ + B^\#_j, \ \deg B^\#_j \leq \mu^+ - 1.$$

Since $P^+ u = v$ we can replace (5.1) by

$$B^\#_j u = h_j - Q'_j f_{|t=0} - Q_j v|_{t=0}, \tag{5.2}$$

and note that the degree of $Q_j \leq (\mu^- - 1)$. With

$$B^\#_j u = \sum_{k=0}^{\mu^+-1} B^\#_{jk}(x, t, D_x)D_t^k u = \sum_{k=1}^{\mu^+} B^\#_{j,k-1}(x, t, D_x)u_k, \tag{5.3}$$

we can write (5.2) as

$$\mathcal{B}\mathbf{u}(0) = \mathbf{h} - \mathcal{Q}\mathbf{v}(0), \tag{5.4}$$

where, $\mathcal{B}$ is a $\nu \times \mu^+$ matrix with entries $B^\#_{j,k-1}(x, 0, D_x)$, $\mathbf{h}$ is the ν–vector whose components are $h_j - Q'_j f_{|t=0}$, and $\mathcal{Q}\mathbf{v}$ is a ν–vector whose components are $Q_j v_{|t=0}$.

Definition 5.1. — *The matrix valued pseudodifferential operator $\mathcal{B}$ defined on the boundary ω is called the Calderon operator of the boundary value problem (1.2) – (1.3).*

We have thus transformed the boundary value problem (1.2) – (1.3) into the equivalent system :

$$D_t\mathbf{v} - \mathcal{A}^-(t)\mathbf{v} = \mathbf{g} - \mathcal{R}\mathbf{u}, \tag{5.5}$$

$$D_t\mathbf{u} - \mathcal{A}^+(t)\mathbf{u} = \mathcal{J}\mathbf{v} \tag{5.6}$$

$$\mathcal{B}\mathbf{u}(0) = \mathbf{h} - \mathcal{Q}\mathbf{v}(0). \tag{5.7}$$

DEFINITION. 5.2. — *The boundary value problem defined by (5.5) – (5.7) is said to be hypoelliptic if given any open set $\mathcal{O} \subset \omega$ and data*

$$\mathbf{g} \in \mathcal{C}^\infty([0,T];\mathcal{D}'(\omega,\mathbb{C}^{\mu^-})),\ \ \mathbf{h} \in \mathcal{D}'(\omega,\mathbb{C}^{\mu^+})$$

whose restriction to $\mathcal{O}$ are smooth, then every solution $(\mathbf{u},\mathbf{v})$ of (5.5)–(5.7) with

$$\mathbf{u} \in \mathcal{C}^\infty([0,T];\mathcal{D}'(\omega;\mathbb{C}^{\mu^+})) \tag{5.8}$$

$$\mathbf{v} \in \mathcal{C}^\infty([0,T];\mathcal{D}'(\omega;\mathbb{C}^{\mu^-})) \tag{5.9}$$

is indeed smooth in $\mathcal{O}$ for $t < T$, i.e.,

$$\mathbf{u} \in \mathcal{C}^\infty(\mathcal{O} \times [0,T);\mathbb{C}^{\mu^+}) \tag{5.10}$$

$$\mathbf{v} \in \mathcal{C}^\infty(\mathcal{O} \times [0,T);\mathbb{C}^{\mu^-}). \tag{5.11}$$

Although the system (5.5) – (5.7) is not quite uncoupled due to the occurrence of $\mathbf{u}$ on the right side of (5.5), much can be said about the solutions of this system and equivalently about the original system (1.2) – (1.3). In particular, one can study regularity.

If one neglects the $\mathcal{R}\mathbf{u}$ term in (5.5), we can apply the results of the previous section to solutions $(\mathbf{u}^\#,\mathbf{v}^\#)$ of the system :

$$D_t\mathbf{v} - \mathcal{A}^-(t)\mathbf{v} = \mathbf{g}, \tag{5.12}$$

$$D_t\mathbf{u} - \mathcal{A}^+(t)\mathbf{u} = \mathcal{J}\mathbf{v} \tag{5.13}$$

$$\mathcal{B}\mathbf{u}(0) = \mathbf{h} - \mathcal{Q}\mathbf{v}(0) \tag{5.14}$$

on $\omega \times [0,T)$. We first assume that $\mathcal{A}(t)$ and $f(x,t)$ are smooth with respect to t on the closed interval $[0,T]$. This is no great restriction since for most applications this amounts to taking T to be slightly smaller. Using (4.27) we can solve the backward Cauchy problem starting at $t = T$ of (5.12) and obtain :

$$\mathbf{v}^\#(t) \sim U^-(t,T)\mathbf{v}^\#(T) - \int_t^T U^-(t,t')\mathbf{g}(t')\,dt', \tag{5.15}$$

where $U^-(t,t')$ is the relevant parametrix and $\mathbf{v}^\#(T)$ is arbitrary. We then put (5.15) into (5.13), and using (4.24) represent the solution of the forward Cauchy problem starting at $t = 0$, with initial data $\mathbf{u}^\#(0)$, for $\mathbf{u}$.

One obtains similar to (5.15)

$$\mathbf{u}^{\#}(t) \sim U^{+}(t,0)\mathbf{u}^{\#}(0) + \int_0^t U^{+}(t,t')\mathcal{J}\mathbf{v}^{\#}(t')\,dt'. \tag{5.16}$$

We can then show that if $(\mathbf{u}, \mathbf{v})$ are solutions of (5.5) – (5.7), with

$$\mathbf{g} \in \mathcal{C}^{\infty}([0,T); \mathcal{D}'(\omega; \mathbb{C}^{\mu^-})),$$

and if $(\mathbf{u}^{\#}, \mathbf{v}^{\#})$ are solutions of (5.12)– (5.14) defined by (5.15) and (5.16), then

$$\mathbf{v} - \mathbf{v}^{\#} \in \mathcal{C}^{\infty}(\omega \times [0,T); \mathbb{C}^{\mu^-}). \tag{5.17}$$

Moreover, if

$$\mathbf{u}(0) - \mathbf{u}^{\#}(0) \in \mathcal{C}^{\infty}(\omega \times [0,T); \mathbb{C}^{\mu^+}), \tag{5.18}$$

then

$$\mathbf{u} - \mathbf{u}^{\#} \in \mathcal{C}^{\infty}(\omega \times [0,T); \mathbb{C}^{\mu^+}). \tag{5.19}$$

Using these facts we can proceed as in Treves [19] and show the following result.

THEOREM 5.1. — *The system (5.5) – (5.7) (or equivalently the system (1.2) – (1.3)) is hypoelliptic if and only if the Calderon operator $\mathcal{B}$ defined on the boundary ω is hypoelliptic.*

Proof. — Suppose $\mathcal{B}$ is hypoelliptic. Since $U^{-}(t,t')$, is pseudolocal we derive from (5.15) that $\mathbf{v}^{\#}$ is smooth in $\mathcal{O} \times [0,T)$. By (5.17) we have that $\mathbf{v}$ is also smooth in $\mathcal{O} \times [0,T)$. In particular, $\mathbf{v}(0) \in \mathcal{C}^{\infty}(\mathcal{O}; \mathbb{C}^{\mu^-})$. Since $\mathcal{B}$ is hypoelliptic the relation $\mathcal{B}\mathbf{u}(0) = \mathbf{h} - \mathcal{Q}\mathbf{v}(0)$, implies $\mathbf{u}(0) \in \mathcal{C}^{\infty}(\mathcal{O}, \mathbb{C}^{\mu^+})$. Similarly, $\mathbf{u}^{\#}(0) \in \mathcal{C}^{\infty}(\mathcal{O}, \mathbb{C}^{\mu^+})$. It follows from (5.19) that $\mathbf{u}$ is smooth.

Now suppose that $\mathcal{B}$ is not hypoelliptic. Then there exists a distribution $\mathbf{u}_0$ in ω valued in $\mathbb{C}^{\mu^+}$ whose restriction to $\mathcal{O}$ is not smooth but $\mathcal{B}\mathbf{u}_0$ is $\mathcal{C}^{\infty}$. We construct $\mathbf{u}, \mathbf{v}, \mathbf{g}, \mathbf{h}$ which satisfy (5.5) – (5.7) for which $\mathbf{g}(x,0)$ and $\mathbf{h}(x)$ are smooth in $\mathcal{O}$ but $\mathbf{u}(0)$ is not smooth in $\mathcal{O}$. Let $\mathbf{w}(t) = U^{+}(t,0)\mathbf{u}_0$. Then $\mathbf{w}(t)$ is $\mathcal{C}^{\infty}$ for $t > 0$ since $U^{+}(t,0)$ is regularizing for $t > 0$. Since

$$(D_t - \mathcal{A}^{+}(t))\mathbf{w} \sim \mathcal{A}^{+}(t)U^{+}(t,0)\mathbf{u}_0 - \mathcal{A}^{+}(t)U^{+}(t,0)\mathbf{u}_0 = 0,$$

and $U^{+}(0,0) = I$, we have that

$$\mathbf{v}_{\#} = (D_t - \mathcal{A}^{+}(t))\mathbf{w} \in \mathcal{C}^{\infty}(\omega \times [0,T]; \mathbb{C}^{\mu^+}).$$

Note that $v_{\#}^j = D_t w^j - w^{j+1}$ for $j < \mu^+$. Let

$$u^{\mu^+} = w^{\mu^+} \tag{5.20}$$

$$u^j = w^j + \int_0^t [(u^{j+1} - w^{j+1})(t') - v_{\#}^j(t')]\, dt' \tag{5.21}$$

for $j < \mu^+$. By (descending) induction we can show that $\mathbf{u} - \mathbf{w}$ belongs to $\mathcal{C}^\infty(\omega \times [0,T]; \mathbb{C}^{\mu^+})$, and $D_t u^j = u^{j+1}$, for $j < \mu^+$. Now let χ be the last (i.e. the μ^+ one) component of $D_t\mathbf{u} - \mathcal{A}^+(t)\mathbf{u}$, and define

$$\mathbf{v} = (\chi, v^2, \dots, v^{\mu^-})$$

with $v^j = D_t v^{j-1}$. Then

$$D_t\mathbf{u} - \mathcal{A}^+(t)\mathbf{u} = \mathcal{J}\mathbf{v}$$

is automatically satisfied. Since $\mathbf{u} = \mathbf{w} + (\mathbf{u} - \mathbf{w})$

$$D_t\mathbf{u} - \mathcal{A}^+(t)\mathbf{u} = \mathbf{v}_{\#} + (D_t - \mathcal{A}^+(t))(\mathbf{u} - \mathbf{w}),$$

therefore, $D_t\mathbf{u} - \mathcal{A}^+(t)\mathbf{u} \in \mathcal{C}^\infty(\omega \times [0,T]; \mathbb{C}^{\mu^+})$, which by definition of $\mathbf{v}$ implies that $\mathbf{v} \in \mathcal{C}^\infty(\omega \times [0,T]; \mathbb{C}^{\mu^-})$. Finally, letting

$$\mathbf{g} = (D_t - \mathcal{A}^-(t))\mathbf{v} + \mathcal{R}\mathbf{u},$$

we have that both $\mathbf{g}$ and $\mathbf{h} = \mathcal{B}\mathbf{u}_0 + \mathcal{Q}\mathbf{v}(0)$ are $\mathcal{C}^\infty \in \mathcal{O}$. However, $\mathbf{u}(0) = \mathbf{w}(0) = \mathbf{u}_0$ is not in $\mathcal{C}^\infty$. Thus (5.5) – (5.7) is not hypoelliptic.

$\square$

6. An example.

Let

$$P = P(D_x, D_t) = D_t^m + \sum_{j=1}^m P_j(D_x) D_t^{m-j}$$

be a *hypoelliptic* partial differential operator in $\mathbb{R}^{n+1}$ $(n > 1)$ with constant coefficients and let

$$P(\xi, \tau) = \tau^m + \sum_{j=1}^m P_j(\xi)\tau^{m-j}$$

be the total symbol of P.

If P is hypoelliptic, then it is well known that there is a constant $M > 0$ such that the number of τ zeros with *positive imaginary* part of $P(\xi,\tau)$ remains constant for all $|\xi| > M$. As before, denote by μ^+ the number of such zeros and by $\tau_j^+(\xi)$, $1 \leq j \leq \mu^+$, the zeros (counting multiplicites) with positive imaginary part.

Let $\Omega = \omega \times [0,T)$, with ω an open subset in $\mathbb{R}^n$ and $T > 0$, and consider the boundary value problem

$$P(D_x, D_t)u(x,t) = f(x,t), \text{ in } \Omega, \tag{6.1}$$

$$B_j(D_x, D_t)u(x,0) = h_j(x), \ j = 1,\ldots,\mu^+, \text{ in } \omega \tag{6.2}$$

where for simplicity we assume that $\deg B_j \leq m-1$, for all $j = 1,\ldots,\mu^+$.

The boundary value problem (6.1) – (6.2) is said to be *hypoelliptic* in Ω if every solution $u(x,t)$ belongs to $\mathcal{C}^\infty(\Omega)$, whenever $f \in \mathcal{C}^\infty(\Omega)$ and $h_j \in \mathcal{C}^\infty(\omega)$, $1 \leq j \leq \mu^+$.

Hypoelliptic boundary value problems were completely characterized by Hörmander in [10] (see also [4]) in the following way. The function

$$\mathcal{C}(\xi) = \det(B_j(\xi, \tau_i^+(\xi)))/\prod_{j<k}(\tau_j^+(\xi) - \tau_k^+(\xi)), \tag{6.3}$$

called the *characteristic function* of the problem (6.1) – (6.2), is well defined for $|\xi| > M$, and extends analytically to a suitable open set $\mathcal{A}$ in $\mathbb{C}^n$ that contains the set $\{\xi \in \mathbb{R}^n : |\xi| > M\}$. Problem (6.1) – (6.2) is hypoelliptic if and only if the following algebraic condition, called *Hörmander's condition*, holds :

$$\zeta \in \mathbb{C}^n, \ \mathcal{C}(\zeta) = 0, \ |\zeta| \to +\infty \text{ imply } |\text{Im}\zeta| \to +\infty. \tag{6.4}$$

On the other hand, according to Theorem 5.1, problem (6.1) – (6.2) is hypoelliptic if and only if the Calderon operator $\mathcal{B}$ associated with it is hypoelliptic. Since we are in the constant coefficient case, the hypoellipticity of the operator $\mathcal{B}$ can be expressed by an algebraic condition which we are going to determine. Following the notations of Section 5, we first divide each $B_j(\xi,\tau)$ by $P^+(\xi,\tau)$

$$B_j = Q_j P^+ + B_j^\#, \tag{6.5}$$

where $\deg B_j^\# \leq \mu^+ - 1$, and then replace the boundary conditions (6.2) by

$$B_j^\# u_{|t=0} = h_j - (Q_j v)_{|t=0}, \ 1 \leq j \leq \mu^+, \tag{6.6}$$

where $v = P^+u$. Next we can write

$$B_j^{\#}u = \sum_{k=1}^{\mu^+} B_{j,k-1}^{\#}(D_x)u^k,$$

so that the Calderon operator (5.4) is given by the square matrix

$$\mathcal{B} = (B_{j,k-1}^{\#})_{\substack{1\le j\le \mu^+ \\ 1\le k\le \mu^+}}.$$

Now going back to the characteristic function $\mathcal{C}(\xi)$ and taking into account (6.5), we rewrite (6.3) as

$$\mathcal{C}(\xi) = \det(B_j^{\#}(\xi, \tau_i^+(\xi)))/\prod_{j<k}(\tau_j^+(\xi) - \tau_k^+(\xi)).$$

But the matrix

$$(B_j^{\#}(\xi, \tau_i^+(\xi)))_{\substack{1\le j\le \mu^+ \\ 1\le i\le \mu^+}}$$

is clearly the product of the matrix $(B_{j,k-1}^{\#}(\xi))_{\substack{1\le j\le \mu^+ \\ 1\le k\le \mu^+}}$ and the Vandermonde determinant $V(\tau_1^+(\xi), \cdots, \tau_{\mu^+}^+(\xi))$ in the roots $\tau_j^+(\xi)$, $1 \le j \le \mu^+$. Therefore,

$$\mathcal{C}(\xi) = \det \mathcal{B}(\xi).$$

As a consequence we conclude that a necessary and sufficient condition for the Calderon operator $\mathcal{B}$ to be hypoelliptic is that its determinant satisfies Hörmander's algebraic condition (6.4). Of course, this result could have been obtained by direct methods without having to rely on the characteristic function of the boundary value problem.

BIBLIOGRAPHY

[1] S. AGMON, A. DOUGLIS, L. NIRENBERG, Estimates Near the Boundary for Solutions of Elliptic Partial Differential Operators Satisfying General Boundary Conditions, I Comm. Pure Appl. Math., 12(1959), 623-727; II Comm. Pure Appl. Math., 17 (1964), 35-92.

[2] M.S. AGRANOVICH, Boundary Value Problems for Systems of First Order Pseudodifferential Operators, Russian Math. Surveys, 24 (1969), 59-126.

[3] R. A. ARTINO & J. BARROS-NETO, The Factorization of Hypoelliptic Pseudodifferential Operators, to appear.

[4] R. A. ARTINO & J. BARROS-NETO, Hypoelliptic Boundary Value Problems, Lecture Notes in Pure and Applied Mathematics, vol. 53, Marcel Dekker, Inc., New York, (1980).

[5] R. A. ARTINO & J. BARROS-NETO, A Construction of a Parametrix for an Elliptic Boundary Value Problem, to appear in Portugaliae Mathematica.

[6] J. BARROS-NETO, The Parametrix of Regular Hypoelliptic Boundary Value Problems, Ann. Sc. Norm. Sup. di Pisa, vol. XXVI, Fasc. I (1972), 247-268.

[7] L. BOUTET DE MONVEL, Boundary Value Problems for Elliptic Pseudodifferential Equations, Acta Math., 126 (1971), 11-51.

[8] A. P. CALDERON, Boundary Value Problems for Elliptic Equations, Proceedings of the Joint Soviet – American Symposium on Partial Differential Equations, Novosibirsk Acad. Sci. USSR, 1-4 (1963).

[9] I.G. ESKIN, Boundary Value Problems for Elliptic Pseudodifferential Equations, Nauka, Moscow, 1973.

[10] L. HÖRMANDER, On the Regularity of the Solutions of Boundary Value Problems, Acta Math., 99 (1958), 225-264.

[11] L. HÖRMANDER, The Analysis of Linear Partial Differential Operators II, Springer-Verlag, Berlin (1983).

[12] L. HÖRMANDER, On the Interior Regularity of Solutions of Partial Differential Operators, Comm. Pure and Appl. Math., vol XI (1958).

[13] L. HÖRMANDER, Pseudodifferential Operators and Hypoelliptic Equations, Proc. of Symposia in Pure Mathematics, vol. X (1967), 138-183.

[14] H. KUMANO-GO, Pseudodifferential Operators, The MIT Press, Cambridge, Mass. and London, England, 1981.

[15] Y. LOPATINSKI, On a Method of Reducing Boundary Problems for a System of Differential Equations of Elliptic Type to Regular Equations, Ukrain. Math. Z., 5 (1953), 123-151.

[16] B. MALGRANGE, Sur une Classe d'Opérateurs Différentiels Hypoelliptiques, Bull. Soc. Math. France, 85 (1957), 282-306.

[17] X. J. PING, The Factorization of Hypoelliptic Operators with Constant Strength (unpublished).

[18] M. E. TAYLOR, Pseudodifferential Operators, Princeton University Press, Princeton, N.J. (1981).

[19] F. TREVES, Introduction to Pseudodifferential Operators and Fourier Integral Operators (2 vols.), Plenum, New York, 1980.

[20] M.I. VISIK, G.I. ESKIN, Convolution Equations in a Bounded Region, Russian Math. Surveys, 20-3 (1965), 86-152.

Manuscrit reçu le 30 mai 1991,
révisé le 6 novembre 1991.

R.A. ARTINO,
The City College
of the City University of New York
New York, NY 10031 (USA)
&
J. BARROS-NETO,
Rutgers University
Dept. of Mathematics
Hill Center
New Brunswick, NJ 08503 (USA).