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On the C^∞-singularities of regular holonomic distributions

ON THE $\mathscr{C}^\infty$-SINGULARITIES
OF REGULAR HOLONOMIC DISTRIBUTIONS

by Emmanuel ANDRONIKOF

0. Introduction.

Let M be a real analytic n-dimensional manifold, X a complexification of M, $\mathscr{D}b_M$ the sheaf of Schwartz distributions on M, $\mathscr{D}_X$ the sheaf of differential operators with holomorphic coefficients on X. Say that a distribution u on M is *regular holonomic* if, locally on M, there exists a coherent left ideal $\mathscr{I}$ of $\mathscr{D}_X$ such that

$$\begin{cases} \mathscr{I}u = 0 \\ \mathscr{D}_X/\mathscr{I} \text{ is a regular holonomic } \mathscr{D}_X\text{-module.} \end{cases}$$

The main purpose of this paper is to prove

THEOREM 0.1. — *Let u be a regular holonomic distribution. Then* $WF(u) = WF_A(u)$.

Here $WF(u)$ (resp. $WF_A(u)$) denotes the $\mathscr{C}^\infty$ (resp. analytic) wave front-set of the distribution u.

Example 0.2. — Let $P(x+iy)$ be a polynomial on $\mathbb{C}^n = \mathbb{R}_x^n + i\mathbb{R}_y^n$ which is hyperbolic in the direction $iN \in i\mathbb{R}^n \setminus \{0\}$. Then $u := P(x+i0N)^{-1}$ is a well defined distribution on $\mathbb{R}^n$ which is regular holonomic, as is, more generally, any distribution boundary value of a Nilsson class function.

One may consult the fundamental papers [KK] and [K] on the theory of regular holonomic systems and also [Bj], the latter containing

a wealth of examples of regular holonomic distributions. As an application let us give the

COROLLARY 0.3. − *Let $P(D)$ be a homogeneous differential polynomial on $\mathbb{R}^n$ which is hyperbolic in the direction N and denote by E the forward fundamental solution with support in $\{x; <x, N> \geqslant 0\}$. Then $WF(E) = WF_A(E)$.*

In fact, by the homogeneity assumption, this is equivalent to the similar statement for $\hat{E}$, thus the equality stems from the preceding example (note that E will not be regular holonomic). This extends a result of [H1] for the doubly characteristic case, under the extra assumption of homogeneity.

The idea of proof of theorem 0.1 is to deduce it from a microlocal version of point (i) of the following celebrated theorem of Kashiwara :

THEOREM 0.4 (Kashiwara, *loc. cit.*). − *Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. Then for any $j \geqslant 0$ one has canonical isomorphisms*

(i) $\mathscr{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathscr{A}_M) \xrightarrow{\sim} \mathscr{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathscr{C}^\infty_M),$

(ii) $\mathscr{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathscr{D}b_M) \xrightarrow{\sim} \mathscr{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathscr{B}_M).$

Here $\mathscr{A}_M$ (resp. $\mathscr{C}^\infty_M$, resp. $\mathscr{B}_M$) denotes the sheaf on M of real analytic functions (resp. of indefinitely differentiable functions, resp. of hyperfunctions).

Note that for a distribution u on the underlying real manifold $X_\mathbb{R}$ of a complex manifold X thas is a solution to a regular holonomic $\mathcal{D}_X$-module, we can obtain a more precise result : $WF(u) = WF_A(u) = \mathrm{Car}(\mathcal{D}_X u)$, the characteristic variety of the $\mathcal{D}_X$-module generated by u in $\mathscr{D}b_{X_\mathbb{R}}$ (see [A1]).

Note also that theorem 0.1 has been claimed independently in [Z], but the proof there being erroneous, it compelled us to publish ours.

The author expresses his thanks to J.-E. Björk, for, without his interest in the subject, this would not have appeared.

1. Statement of the comparison theorem and proof of theorem 0.1.

Let M and X as before, and denote by $\pi : T^*X \to X$ the cotangent bundle of X, and by $\mathring{\pi}$ the restriction of π to $\mathring{T}^*X := T^*X \setminus T^*_X X$, the complement of the zero section of T^*X, with similar notations for

M. As usual we identify $\sqrt{-1}\,T^*M$ to the conormal bundle $T^*_M X$ of M in X and $\sqrt{-1}\,T^*_M M$ to M. Denote by

$$\varpi : \sqrt{-1}\,S^*M \simeq S^*_M X \to M$$

the (imaginary) cotangent sphere bundle of M, and by

$$\alpha : \mathring{T}^*_M X \to S^*_M X$$

the canonical map.

We need to recall now the sheaves of singularities of distributions that were considered by diverse authors.

In order to have statements valid on the whole of the cotangent bundle we may, for a distribution u on M, define here $WF(u)$ (resp. $WF_A(u)$) to be the set of $p \in T^*_M X$ such that, either $p \in \mathring{T}^*_M X$ and p belongs to the $\mathscr{C}^\infty$ (resp. analytic) wave-front set of u, or $p \in \sqrt{-1}\,T^*_M M$ and $p \in \mathrm{supp}\,(u)$. These are closed $\mathbb{R}_{>0}$-conic subsets of $T^*_M X$.

Then we denote by $\mathscr{C}'_M$ (resp. $\mathscr{C}^f_M$) the sheaf on $T^*_M X$ associated to the presheaf

$$\begin{cases} U \text{ open set in } T^*_M X \mapsto \Gamma(M\,;\mathscr{D}b_M)/\{u \in \Gamma(M\,;\mathscr{D}b_M)\,;\,WF(u)\cap U=\emptyset\} \\ (\text{resp. } U \mapsto \Gamma(M\,;\mathscr{D}b_M)/\{u \in \Gamma(M\,;\mathscr{D}b_M)\,;\,WF_A(u)\cap U=\emptyset\}). \end{cases}$$

These sheaves are $\mathscr{D}_X$-modules and are invariant under the action of $\mathbb{R}_{>0}$ and have been introduced by Bony [Bo] under a different name (for a cohomological construction of $\mathscr{C}^f_M$ see [A2]).

These sheaves enjoy the following properties:

$$(1.1) \qquad \begin{cases} \pi_*\mathscr{C}'_M = \mathscr{D}b_M, & \mathring{\pi}_*\mathscr{C}'_M = \mathscr{D}b_M/\mathscr{C}^\infty_M, \\ \pi_*\mathscr{C}^f_M = \mathscr{D}b_M, & \mathring{\pi}_*\mathscr{C}^f_M = \mathscr{D}b_M/\mathscr{A}_M, \end{cases}$$

(1.2) $\begin{cases} \text{there are canonical morphisms} \\ \mathrm{sp}_\infty : \pi^{-1}\mathscr{D}b_M \to \mathscr{C}'_M, \ \mathrm{sp} : \pi^{-1}\mathscr{D}b_M \to \mathscr{C}^f_M, \text{ and } j_\infty : \mathscr{C}^f_M \to \mathscr{C}'_M, \\ \text{such that } j_\infty \circ \mathrm{sp} = \mathrm{sp}_\infty, \text{ and, for any distribution } u \text{ on } M, \\ WF(u) = \mathrm{supp}\,(\mathrm{sp}_\infty(u)) \text{ and } WF_A(u) = \mathrm{supp}\,(\mathrm{sp}(u)). \end{cases}$

(1.3) $\begin{cases} \alpha_*\mathscr{C}'_M \text{ and } \alpha_*\mathscr{C}^f_M \text{ are soft sheaves, in particular} \\ H^j(U;\mathscr{C}'_M) = H^j(U;\mathscr{C}^f_M) = 0 \\ \text{for any open conic subset } U \subset T^*_M X \text{ and } j > 0. \end{cases}$

Properties (1.1) and (1.2) are obvious consequences of the definitions, whereas (1.3) is an immediate consequence of [BS].

More precisely, let $\mathscr{C}_M''$ be the sheaf on T_M^*X defined by the exact sequence

$$0 \to \mathscr{C}_M'' \to \mathscr{C}_M^f \to \mathscr{C}_M' \to 0.$$

LEMMA 1.1. — Let $p \in \mathring{T}_M^*X$, $x = \pi(p)$ and $u \in \mathscr{D}b_{M,x}$ such that $p \notin WF(u)$. Then there exists $f \in \mathscr{C}_{M,x}^\infty$ such that $p \notin WF_A(u-f)$.

Proof of lemma 1.1. — Straightforward (e.g. use the Lebeau-Hörmander kernel and a suitable cut-off measure as in [H2], corollary 8.4.13). $\qquad\qquad\square$

Thus the sheaf $\mathscr{C}_M''|_{\mathring{T}_M^*X}$ is identified to the (conical) sheaf associated to the presheaf

$$U \text{ open in } \mathring{T}_M^*X \mapsto \Gamma(M; \mathscr{C}_M^\infty)/\{u \in \Gamma(M;\mathscr{C}_M^\infty) \,;\, WF_A(u) \cap U = \emptyset\}.$$

(Also $\pi_*\mathscr{C}_M'' = 0$, and $\mathring{\pi}_*\mathscr{C}_M'' = \mathscr{C}_M^\infty/\mathscr{A}_M$.) Then it is proven in [BS] that $\alpha_*\mathscr{C}_M''$ and $\alpha_*\mathscr{C}_M^f$ are soft sheaves, hence so is also $\alpha_*\mathscr{C}_M'$, which entails (1.3). (Actually these sheaves enjoy the stronger property of being supple, see [BS].)

To state and prove a microlocal comparison theorem we will make use of the sheaf $\mathscr{E}_X$ on T^*X of microdifferential operators of finite order of [SKK]. Recall that $\mathscr{E}_X$ is a sheaf of rings such that $\mathscr{E}_X|_{T_X^*X} = \mathscr{D}_X$, and $\mathscr{E}_X$ is a flat $\pi^{-1}\mathscr{D}_X$-module. Moreover $\mathscr{C}_M'$ and $\mathscr{C}_M^f$ have a structure of left $\mathscr{E}_X$-modules compatible with the action of $\pi^{-1}\mathscr{D}_X$, and j_∞ is $\mathscr{E}_X$-linear : Bony (*loc. cit.*) has described the explicit action of $\mathscr{E}_X$ and of quantized canonical transformations (i.e. real analytic Fourier integral operators) on $\mathscr{C}_M'$ and $\mathscr{C}_M^f$. The microlocal version of Kashiwara's theorem 0.4 is then

THEOREM 1.2. — Let $\mathscr{M}$ be a regular holonomic $\mathscr{E}_X$-module defined on a neighborhood in T^*X of an open subset $U \subset T_M^*X$. One has canonical isomorphisms

(i) $\mathscr{E}xt_{\mathscr{E}_X}^j(\mathscr{M},\mathscr{C}_M^f)|_U \xrightarrow{\sim} \mathscr{E}xt_{\mathscr{E}_X}^j(\mathscr{M},\mathscr{C}_M')|_U, \quad \forall j,$

(ii) $\mathscr{E}xt_{\mathscr{E}_X}^j(\mathscr{M},\mathscr{C}_M^f)|_U \xrightarrow{\sim} \mathscr{E}xt_{\mathscr{E}_X}^j(\mathscr{M},\mathscr{C}_M)|_U, \quad \forall j,$

In (ii), $\mathscr{C}_M$ denotes the sheaf of Sato microfunctions on M. The morphisms above are induced by the canonical morphisms $\mathscr{C}_M^f \to \mathscr{C}_M'$ and $\mathscr{C}_M^f \to \mathscr{C}_M$.

Note that, on the zero section, point (i) is void (since $\mathscr{C}_M^f|_M = \mathscr{C}_M'|_M = \mathscr{D}b_M$), whereas (ii) is a restatement of Kashiwara's theorem 0.4 (ii).

Proof of theorem 0.1 *from theorem* 1.2. — If $\mathcal{M}$ is a regular holonomic $\mathcal{D}$-module of the form $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ and u a distribution on M such that $\mathcal{I}u = 0$, then u is canonically identified to a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)$. Since $\mathcal{E}_X \mathcal{I} \operatorname{sp}(u) = \mathcal{E}_X \operatorname{sp}(\mathcal{I}u) = 0$ and $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X}(\mathcal{D}_X/\mathcal{I}) = \mathcal{E}_X/\mathcal{E}_X\mathcal{I}$, $\operatorname{sp}(u)$ defines a global section of $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{M}, \mathcal{C}_M^f)$. Sameley $\operatorname{sp}_\infty(u)$ defines a global section of $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{M}, \mathcal{C}_M')$. Then point (i) of theorem 1.2 entails $\operatorname{supp}(\operatorname{sp}(u)) = \operatorname{supp}(\operatorname{sp}_\infty u)$ whence the equality in theorem 0.1.   □

As another application of theorem 1.2 one gets

COROLLARY 1.3. — *Let $\mathcal{M}$ be a regular holonomic $\mathcal{E}_X$-module defined in a neighborhood of $p \in T_M^*X$, and assume that the pair $(T_M^*X, \operatorname{Car}\mathcal{M})$ is positive at p in the sense of* [MS] *(see also* [S]). *Then*

$$\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{C}_M^f)_p = \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{C}_M')_p = 0, \quad \forall j > 0.$$

In fact we know that the hypothesis implies that $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{C}_M)_p = 0$ for all $j > 0$ by [HS] ($\mathcal{M}$ need not be regular), then it remains to apply theorem 1.2. Note also that if $p \in M$ (i.e. $\mathcal{M}$ is a regular $\mathcal{D}_X$-module such that the positivity condition holds at p), we get in particular

$$\mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{D}b_M)_p = 0, \quad \forall j > 0.$$

2. Recalling the Generic Position theorem.

To derive theorem 1.2 from theorem 0.4 we are going to make essential use of the following deep result of Kashiwara-Kawaï in [KK].

Let X be a complex manifold of dimension n and set $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$. Recall that if $\mathcal{M}$ is any coherent $\mathcal{D}_M$-module, its characteristic variety $\operatorname{Car}\mathcal{M} \subset T^*X$ is a closed analytic $\mathbb{C}^\times$-conic subset set which coïncides with $\operatorname{supp}(\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{M})$, and which is, by the definition, Lagrangian when $\mathcal{M}$ is holonomic ([SKK]). If $\mathcal{M}$ is regular holonomic, then $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{M}$ is a regular holonomic $\mathcal{E}_X$-module.

Let Λ be a conic (i.e. locally $\mathbb{C}^\times$-conic) analytic Lagrangian subset of T^*X. Following [KK] we say that Λ is in a generic position at $p \in \Lambda \cap \mathring{T}^*X$ if $\pi^{-1}\pi(p) \cap \Lambda = \mathbb{C}^\times p$ in a neighborhood of p. (It implies that there exists a (singular) complex hypersurface S in X defined in a

neighborhood of $\pi(p)$ such that $\Lambda = T_S^* X$ in the neighborhood of p, where $T_S^* X$ is the closure in $T^* X$ of the conormal bundle to the regular part of S.)

THEOREM 2.1 (Kashiwara-Kawaï, *loc. cit.*). – *Let $\mathcal{M}$ be a regular holonomic $\mathcal{E}_X$-module such that* Car $\mathcal{M} = \Lambda$, *and $p \in \Lambda \cap \overset{\circ}{T}^* X$. Suppose that Λ is in a generic position at p. Then there is a regular holonomic $\mathcal{D}_X$-module $\mathcal{L}$ defined in a neighborhood of $\pi(p)$ such that*

(i) $\mathcal{L}_{\pi(p)} = \mathcal{M}_p$,

(ii) $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{L} \xrightarrow{\sim} \mathcal{M}$ *in a neighborhood of p*,

(iii) Car $\mathcal{L} \subset \Lambda \cup T_X^* X$ *in a neighborhood of $\pi^{-1}\pi(p)$*.

The arrow in (ii) is the $\mathcal{E}_X$-linear morphism induced by the scalar extension of (i).

To achieve the generic position in our situation, we also need the following.

LEMMA 2.2. – *Let M be a real analytic manifold, X a complex neighborhood of M, Λ a complex analytic locally $\mathbb{C}^\times$-conic Lagrangian subset of $T^* X$ and a point $p \in \Lambda \cap \overset{\circ}{T}_M^* X$.*

Then there is a real analytic canonical transformation φ defined on a neighborhood of $p \in T_M^* X$ whose complexification $\varphi^{\mathbb{C}}$ puts Λ into a generic position at $\varphi^{\mathbb{C}}(p)$.

Proof. – We proceed as in [KK] (proof of Corollary 1.6.4 of *loc. cit.*). Let $\rho = T_p(\mathbb{R}_{>0}p) \subset T_p T^* M$ the Euler line through p, and E the real symplectic vector space $E = \rho^\perp/\rho$. By [KS] the set $C = (C_p(\Lambda \cap T_M^* X) \cap \rho^\perp + \rho)/\rho$ is an analytic isotropic cone in E and we may find a Lagrangian plane λ in E such that $\lambda \cap C = \{0\}$. Thus there is a Lagrangian plane $\mu \subset T_p T^* M$ such that $\mu \supset \rho$ and $\mu \cap C_p(\Lambda \cap T_M^* X) \subset \rho$. If we choose a real analytic canonical transformation φ defined in a neighborhood of p such that $T_p\varphi(\mu)$ is the tangent space to the fiber of $T^* M$ at p, $\varphi^{\mathbb{C}}(\Lambda)$ will be in a generic position at $\varphi^{\mathbb{C}}(p) = \varphi(p)$, $\varphi^{\mathbb{C}}$ denoting a complexification of φ in a neighborhood of p. $\square$

J.-E. Björk has pointed out to us that lemma 2.2 can also be proven by a suitable real analytic change of coordinates followed by a Legendre-type canonical transformation (see [Bj]) – a classical technique in the non-singular case (see [H2]).

3. Proof of theorem 1.1.

Proof of (i). — We have to prove that for any $j \geqslant 0$ and any $p \in U$

(3.1) $\quad \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}^f_M)_p \to \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}'_M)_p$ is an isomorphism.

Put $\Lambda = \operatorname{Car}\mathscr{M}(:= \operatorname{supp}\mathscr{M})$ and $x = \pi(p)$. Note that there is nothing to prove if $p \in M$. Also, if $p \notin \Lambda$, both terms in (3.1) are zero since $\mathscr{M}_p = 0$.

Hence we may assume $p \in \Lambda \cap \mathring{T}^*X$.

1) First assume

(3.2) $\qquad\qquad \Lambda$ is in a generic position at p.

Making use of theorem 2.1 we may suppose $\mathscr{M} = \mathscr{E}_X \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{L}$ where $\mathscr{L}$ is a regular holonomic $\mathscr{D}_X$-module defined in a neighborhood of x; Let $p^a \in T^*_M X$ be the image of p by the antipodal map of $T^*_M X$. Let us first prove

(3.3) $\quad \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}^f_M)_p \oplus \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}^f_M)_{p^a} \simeq \mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{L}, \mathscr{D}b_M/\mathscr{A}_M)_x,$

and

(3.4) $\quad \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}'_M)_p \oplus \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}'_M)_{p^a} \simeq \mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{L}, \mathscr{D}b_M/\mathscr{C}^\infty_M)_x.$

In the following, the notation RT means, as is usual, the right derived functor of a left exact functor T.

Since $R\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}^f_M)$ is represented by a (bounded) complex whose cohomology is constant on the fibers of α, we have

$$\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}^f_M)_p = H^j(R\mathscr{H}om_{\alpha_*\mathscr{E}_X}(\alpha_*\mathscr{M}, \alpha_*\mathscr{C}^f_M))_{\alpha(p)}.$$

On the other hand, (3.2) implies $\varpi^{-1}(x) \cap \alpha(\Lambda) = \{\alpha(p), \alpha(p^a)\}$, and, since ϖ is proper, we may write

$\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}^f_M)_p \oplus \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}^f_M)_{p^a}$

$$\begin{aligned}
&= H^j R\Gamma(\varpi^{-1}(x); R\mathscr{H}om_{\alpha_*\mathscr{E}_X}(\alpha_*\mathscr{M}, \alpha_*\mathscr{C}^f_M)) \\
&= H^j(R\varpi_* R\mathscr{H}om_{\alpha_*\mathscr{E}_X}(\alpha_*\mathscr{M}, \alpha_*\mathscr{C}^f_M))_x \\
&= H^j(R\varpi_* R\mathscr{H}om_{\varpi^{-1}\mathscr{D}_X}(\varpi^{-1}\mathscr{L}, \alpha_*\mathscr{C}^f_M))_x \\
&= H^j(R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}, R\varpi_*(\alpha_*\mathscr{C}^f_M)))_x \\
&= H^j(R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}, \mathscr{D}b_M/\mathscr{A}_M))_x,
\end{aligned}$$

the last equality because $R\varpi_*(\alpha_*\mathscr{C}_M^f) = R\varpi_* R\alpha_* \mathscr{C}_M^f = R\mathring{\pi}_* \mathscr{C}_M^f = \mathscr{D}b_M/\mathscr{A}_M$ by (1.3).

The same argument, mutadis mutandis, proves (3.4).

Apply then the functor $R\mathscr{H}om(\mathscr{L},\cdot)$ to the exact sequence

$$(3.5) \qquad 0 \to \mathscr{C}_M^\infty/\mathscr{A}_M \to \mathscr{D}b_M/\mathscr{A}_M \to \mathscr{D}b_M/\mathscr{C}_M^\infty \to 0.$$

Since $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L},\mathscr{C}_M^\infty/\mathscr{A}_M) = 0$ by Kashiwara's theorem 0.4 (i), we get

$$R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L},\mathscr{D}b_M/\mathscr{A}_M)_x \;\overset{\sim}{\to}\; R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L},\mathscr{D}b_M/\mathscr{C}_M^\infty)_x.$$

In view of (3.3) and (3.4), it proves (3.1) in this case.

2) To prove (3.1) in the general case, one notes that the statement is invariant by real analytic quantized canonical transformations. Then lemma 2.2 ensures the existence of a real analytic canonical transformation φ that brings Λ into a generic position at p, hence by choosing any analytic Fourier integral operator defined over φ, we may assume that condition (3.2) holds, and we are done.

Proof of (ii). — The proof is similar. Under assumption (3.2) one gets also

$$(3.6) \quad \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M},\mathscr{C}_M)_p \oplus \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M},\mathscr{C}_M)_{p^a} \simeq \mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{L},\mathscr{B}_M/\mathscr{A}_M)_x.$$

Then, since $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L},\mathscr{B}_M/\mathscr{D}b_M) = 0$ (Kashiwara's theorem (0.4) (ii)), by the exact sequence

$$0 \to \mathscr{D}b_M/\mathscr{A}_M \to \mathscr{B}_M/\mathscr{A}_M \to \mathscr{B}_M/\mathscr{D}b_M \to 0,$$

one gets

$$R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L},\mathscr{D}b_M/\mathscr{A}_M)_x \;\overset{\sim}{\to}\; R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L},\mathscr{B}_M/\mathscr{A}_M)_x,$$

which, in view of (3.3) and (3.6), entails (ii) of theorem 1.2 (under assumption (3.2)); then we conclude as in 2) above.   □

Note that to establish theorem 0.1 we would have only needed to prove that the morphism $\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M},\mathscr{C}_M^f) \to \mathscr{H}om_{\mathscr{E}_X}(\mathscr{M},\mathscr{C}_M')$ is injective, which would have been slightly simpler (i.e. it would not involve the cohomological properties (1.3) of $\mathscr{C}_M^f$) and $\mathscr{C}_m'$); this is a remark of Björk. On the other and, the statements in theorem 1.2 carry much more information.

Let us also mention here that part (ii) of theorem 1.2 can be partially generalized in the following manner.

PROPOSITION 3.1. − Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module, and let $F \in D^b_{\mathbb{R}-c}(X)$. Then the canonical morphism

$$R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, T-\mu hom(F, \mathcal{O}_X)) \to R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \mu hom(F, \mathcal{O}_X))$$

is an isomorphism.

Here $D^b_{\mathbb{R}-c}(X)$ denotes the subcategory of the derived category of bounded complexes of sheaves on X with $\mathbb{R}$-constructible cohomology, and $\mu hom\,(\cdot, \cdot)$ is the functor defined by Kashiwara and Schapira that microlocalizes $R\mathcal{H}om\,(\cdot, \cdot)$ (see [KS]). The functor $T-\mu hom\,(\cdot, \mathcal{O}_X)$ is the tempered version of $\mu hom\,(\cdot, \mathcal{O}_X)$ of [A2].

Proof. − Recall first a Kashiwara comparison theorem (which generalizes (ii) of theorem 0.4).

THEOREM 3.2 (Kashiwara, loc. cit.). − Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module, and let $F \in D^b_{\mathbb{R}-c}(X)$. Then the canonical morphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, T-Hom\,(F, \mathcal{O}_X)) \to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, RHom\,(F, \mathcal{O}_X))$$

is an isomorphism.

We have denoted by $T-hom\,(\cdot, \mathcal{O}_X)$ Kashiwara's functor of tempered cohomology with values in $\mathcal{O}_X$ of [K] (which is also denoted sometimes $RH(\cdot)$ or $\Psi(\cdot)$). Let $p \in T^*X$, $x := \pi(p), j \in \mathbb{Z}$. We have to prove the isomorphism

$$(3.7) \qquad \begin{cases} H^j(R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, T-\mu hom\,(F, \mathcal{O}_X)))_p \\ \to H^j(R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \mu hom\,(F\mathcal{O}_X)))_p. \end{cases}$$

Taking a local coordinate system we may suppose $X = \mathbb{C}^n$, $p = (x; \xi)$. We have the following germ formulas for any $j \in \mathbb{Z}$ (see [A2] and [KS] respectively):

$$(3.8) \begin{cases} H^j(T-\mu hom\,(F, \mathcal{O}_X))_p \\ \qquad\qquad = \lim_{\overrightarrow{U,\gamma}} H^j(R\Gamma(U; T-Hom\,(\varphi_\gamma^{-1}R\varphi_{\gamma*}(F_U), \mathcal{O}_X))) \\ \quad \text{and} \\ H^j(\mu hom\,(F, \mathcal{O}_X))_p = \lim_{\overrightarrow{U,\gamma}} H^j(R\Gamma(U; RHom\,(\varphi_\gamma^{-1}R\varphi_{\gamma*}(F_U), \mathcal{O}_X))) \end{cases}$$

where U ranges through the family of open neighborhoods of x and γ ranges through the family of closed convex proper cones of $\mathbb{R}^{2n} \simeq X_\mathbb{R}$ such that $\gamma \subset \{v \in X; \langle v, \xi \rangle < 0\} \cup \{0\}$. The morphism φ_γ is associated to the γ-topology on X.

Then (3.7) follows from theorem 3.2 and (3.8).   □

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