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ON THE DISTRIBUTION OF SCATTERING POLES FOR PERTURBATIONS OF THE LAPLACIAN

by Georgi VODEV (*)

1. Introduction.

In this note we study the distribution of the scattering poles associated to second order differential operators of the form

$$G = c(x)^{-1}\left(- \sum_{i,j=1}^{n} \partial_{x_i}(g_{ij}(x)\partial_{x_j}) + \sum_{j=1}^{n} b_j(x)\partial_{x_j} + a(x)\right)$$

in $\mathbb{R}^n$, $n \geqslant 3$, odd, where the coefficients are such that the following conditions are fulfilled :

(i) The operator G admits a selfadjoint realization, which will be again denoted by G, in the Hilbert space $H = L^2(\mathbb{R}^n; c(x)dx)$ with domain $D(G)$;

(ii) There exists a constant $\rho_0 > 0$ so that for any $u \in D(G)$ such that $u = 0$ for $|x| \leqslant \rho_0$ we have $u \in H^2(\mathbb{R}^n)$ and $Gu = -\Delta u$, Δ being the Laplacian in $\mathbb{R}^n$;

(iii) G is positively definite, i.e. $(Gu,u)_H \geqslant 0$, $\forall u \in D(G)$.

In what follows $\|\ \|$ will denote the norm in $\mathfrak{L}(H,H)$, the space of all linear bounded operators acting from H into H. It is easy to see by (i) and (iii) that the resolvent $R(z) = (G - z^2)^{-1} \in \mathfrak{L}(H,H)$ is well defined and holomorphic in $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and

$$\|R(z)\| \leqslant C(\operatorname{Im} z)^{-2} \quad \text{for} \quad \operatorname{Im} z > 0. \tag{1.1}$$

Choose a function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ for $|x| \leqslant \rho_0 + 1$ and set $R_\chi(z) = \chi R(z) \chi$ for $z \in \mathbb{C}_+$. When

(iv) $R_\chi(z_0)$ is a compact operator in $\mathfrak{L}(H,H)$ for some $z_0 \in \mathbb{C}_+$,

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it is well known that the cutoff resolvent $R_\chi(z)$ admits a meromorphic continuation from $\mathbb{C}_+$ to the entire complex plane $\mathbb{C}$ (see the analysis in the next section). The poles of this continuation are known as scattering poles or resonances and in our case they all are in $\bar{\mathbb{C}}_-$, where $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. Note that if (iv) holds for at least one z_0, it holds for all z_0. Let $\{\lambda_j\}$ be the poles of $R_\chi(z)$, repeated according to multiplicity, and set

$$N(r) = \# \{\lambda_j : |\lambda_j| \leqslant r\}.$$

When the operator G is elliptic, in [8] and [14] (see also [13]) it is proved (without assuming (iii)) that

$$N(r) \leqslant Cr^n + C. \tag{1.2}$$

It also follows from the analysis in [8] and [14] that for hypoelliptic operators, i.e. when we have the estimates

$$\|f\|_{s+2\delta} \leqslant C_s(\|Gf\|_s + \|f\|_s), \quad \forall s \geqslant 0, \forall f \in D(G),\ Gf \in H^s, \tag{1.3}$$

where $0 < \delta < 1$ and $\|\ \|_s$ denotes the norm in the usual Sobolev space H^s, (again without assuming (iii)) the number of the poles satisfies the bound

$$N(r) \leqslant Cr^{n/\delta} + C. \tag{1.4}$$

Note that (1.3) implies (iv) at once. By (1.4) one actually concludes that the less regular the operator G is, the worse bound for $N(r)$ one has. In this work we show that outside a conic neighbourhood of the real axis the number of the scattering poles satisfies a much better estimate than (1.4) no matter how regular the operator G is. It actually has a bound of the type (1.2). To be more precise, given any ε, $0 < \varepsilon \ll 1$, set $\Lambda_\varepsilon = \{z \in \mathbb{C} : \varepsilon \leqslant \arg z \leqslant \pi - \varepsilon\}$ and

$$N(\varepsilon, r) = \# \{\lambda_j : |\lambda_j| \leqslant r,\ -\lambda_j \in \Lambda_\varepsilon\}.$$

Our main result is the following :

THEOREM 1. — *Assume (i)-(iv) fulfilled. Then for any* ε, $0 < \varepsilon \ll 1$, *there exists a constant* $C_\varepsilon > 0$ *so that*

$$N(\varepsilon, r) \leqslant C_\varepsilon r^n + C_\varepsilon. \tag{1.5}$$

The estimate (1.5) shows that to study the counting function $N(r)$ modulo terms $O(r^n)$ for positively definite selfadjoint hypoelliptic operators it suffices to study the number of the scattering poles in a conic ε-neighbourhood of the real axis for any small $\varepsilon > 0$.

The idea for the proofs of polynomial bounds of the scattering poles originates from Melrose [4] (see also [2], [5], [11], [12], [13], [14], [17]). One first needs to find an entire family of compact operators, $K(z)$, so that $(1-K(z))R_\chi(z)$ is an entire operator-valued function and $1 - K(z)$ is invertible for at least one $z \in \mathbb{C}$. Thus one concludes that the poles of $R_\chi(z)$, with multiplicity, are among the poles of $(1-K(z))^{-1}$ and hence among the zeros of an entire function $h(z) = \det(1-K(z)^p)$, where $p \geqslant 1$ is an integer taken so that $K(z)^p$ is trace class. Thus the problem is reduced to obtaining suitable estimates for $|h(z)|$.

To prove (1.5) we need to find a family $K(z)$ as above so that $(1-K(z))^{-1}$ can be expressed in terms of $R(z)$ for $z \in \mathbb{C}_+$ (see (2.5)), and $K(z) - K(-z)$ is trace class for any $z \in \mathbb{C}$. This enables us to characterize the poles of $R_\chi(z)$ in $\mathbb{C}_-$, with multiplicity, as zeros of a function $h(z)$, defined and holomorphic in $\mathbb{C}_-$, such that for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ so that

$$(1.6) \qquad |h(-z)| \leqslant C_\gamma \exp(C_\gamma |z|^n) \qquad \text{for} \qquad \operatorname{Im} z \geqslant \gamma .$$

Then, we derive (1.5) from (1.6) and a classical result due to Carleman (see Lemma 2).

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2. Representation of the cutoff resolvent.

Denote by G_0 the selfadjoint realization of $-\Delta$ in the Hilbert space $H_0 = L^2(\mathbb{R}^n)$ and let $R_0(z)$ denote the outgoing resolvent of $-\Delta - z^2$, $z \in \mathbb{C}$. Then $R_0(z) = (G_0 - z^2)^{-1} \in \mathfrak{L}(H_0, H_0)$ for $z \in \mathbb{C}_+$ and as is well-known the kernel of $R_0(z)$ is given in terms of Hankel's functions by

$$(2.1) \quad R_0(z)(x,y) = (i/4)(z/2\pi|x-y|)^{(n-2)/2} H^{(1)}_{(n-2)/2}(z|x-y|) .$$

It is easy to see that $\chi R_0(z)\chi \in \mathfrak{L}(H_0, H_0)$ for all $z \in \mathbb{C}$ and it forms an entire family of compact pseudodifferential operators of order -2. Using this together with the assumption (iv) we shall build the meromorphic continuation of the cutoff resolvent of G. Set $Q = G - G_0$ and fix a $z_0 \in \mathbb{C}_+$. Clearly, for all $z \in \mathbb{C}_+$ we have

$$(2.2) \qquad R(z) = R_0(z) + R(z)QR_0(z)$$

and

$$R(z) = R(z_0) + (z^2 - z_0^2)R(z)R(z_0). \tag{2.3}$$

Combining (2.2) and (2.3) yields

$$R(z)(1-(z^2-z_0^2)QR_0(z)R(z_0)) = R(z_0) + (z^2-z_0^2)R_0(z)R(z_0)$$

for $z \in \mathbb{C}_+$. Multiplying the both sides of this identity by χ, since $Q = \chi Q$, we get

$$R_\chi(z)(1-K(z)) = R_\chi(z_0) + K_1(z) \quad \text{for} \quad z \in \mathbb{C}_+, \tag{2.4}$$

where

$$K(z) = (z^2-z_0^2)QR_0(z)R(z_0)\chi$$
$$K_1(z) = (z^2-z_0^2)\chi R_0(z)R(z_0)\chi.$$

Moreover, since $R(z)$ is well defined in $\mathbb{C}_+$, it is easy to see by (2.4) that $1 - K(z)$ is invertible in $\mathfrak{L}(H,H)$ for all $z \in \mathbb{C}_+$ and

$$(1-K(z))^{-1} = 1 + (z^2-z_0^2)QR_0(z)(R_0(z_0)+R_0(z_0)QR(z))\chi \tag{2.5}$$

for $z \in \mathbb{C}_+$. Now, since $R_0(z)$ and $R(z)$ are holomorphic in $\mathbb{C}_+$ with values in $\mathfrak{L}(H,H)$ and since $QR_0(z) = QR_0(z_0)(1+(z^2-z_0^2)R_0(z))$ for $z \in \mathbb{C}_+$, we deduce from (2.5) that $(1-K(z))^{-1}$ is holomorphic in $\mathbb{C}_+$ with values in $\mathfrak{L}(H,H)$. Moreover, by (1.1), which clearly holds with $R(z)$ replaced by $R_0(z)$ as well, for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ so that

$$\|(1-K(z))^{-1}\| \leqslant C_\gamma(1+|z|)^4 \quad \text{for} \quad \operatorname{Im} z \geqslant \gamma. \tag{2.6}$$

Now let us see that the operator-valued functions $K(z)$ and $K_1(z)$, defined in $\mathbb{C}_+$, extend analytically to the entire $\mathbb{C}$ with values in the compact operators in $\mathfrak{L}(H,H)$. We shall consider $K(z)$ only, since $K_1(z)$ is treated similarly. Using that $R(z_0) = R_0(z_0) + R_0(z_0)QR(z_0)$ it is easy to see that

$$K(z) = (z^2-z_0^2)QR_0(z)R_0(z_0)\chi(1+QR_\chi(z_0)) \tag{2.7}$$

for $z \in \mathbb{C}_+$. Choose functions $\chi_1,\ \chi_2 \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_1 = 1$ on $\operatorname{supp} Q$, $\chi_2 = 1$ on $\operatorname{supp}\chi_1$ and $\chi = 1$ on $\operatorname{supp}\chi_2$. After a standard computation (2.7) takes the form

$$K(z) = (z^2-z_0^2)QR_0(z)R_0(z_0)\chi K_2 + (z^2-z_0^2)QR_0(z)\chi K_3 R_\chi(z_0) \tag{2.8}$$

for $z \in \mathbb{C}_+$, where

$$K_2 = 1 + [\chi_2, G_0] R_0(z_0)[\chi_1, G_0] R_0(z_0) Q R_\chi(z_0),$$
$$K_3 = \chi_1 R_0(z_0)\, Q + \chi_2 R_0(z_0)[\chi_1, G_0] R_0(z_0) Q.$$

Here [,] denotes the comutator. Clearly, we have K_2, $K_3 \in \mathfrak{L}(H,H)$. Further on, by a similar computation, for $z \in \mathbb{C}_+$, one obtains

$$\begin{aligned}(z^2 - z_0^2) Q R_0(z) R_0(z_0)\chi \\ = (K_4 + (z^2 - z_0^2) K_5)\chi R_0(z)\chi - K_4 \chi R_0(z_0)\chi\end{aligned} \tag{2.9}$$

and

$$Q R_0(z)\chi = (K_4 + (z^2 - z_0^2) K_5)\chi R_0(z)\chi + K_5, \tag{2.10}$$

where

$$K_4 = Q R_0(z_0)[G_0, \chi_1] R_0(z_0)[G_0, \chi_2],$$
$$K_5 = Q R_0(z_0)\chi_1 + Q R_0(z_0)[G_0, \chi_1] R_0(z_0)\chi_2.$$

Clearly, K_4, $K_5 \in \mathfrak{L}(H,H)$. Thus, by (2.8)-(2.10) we deduce

$$K(z) = K_6(z)\chi R_0(z)\chi K_2 + K_7(z)\chi R_0(z)\chi K_8 + K_9(z) \tag{2.11}$$

for $z \in \mathbb{C}_+$, where

$$K_6(z) = K_4 + (z^2 - z_0^2) K_5,$$
$$K_7(z) = (z^2 - z_0^2) K_6(z),$$
$$K_8 = K_3 R_\chi(z_0),$$
$$K_9(z) = -K_4 \chi R_0(z_0)\chi K_2 + (z^2 - z_0^2) K_5 K_3 R_\chi(z_0).$$

Clearly, these four operators are analytic $\mathfrak{L}(H,H)$-valued functions. Now, since $\chi R_0(z)\chi$ forms an entire family of compact operators and by (iv) so does $K_9(z)$, by (2.11) we can extend $K(z)$ analytically to the entire $\mathbb{C}$. Then, since $K(z_0) = 0$, by Fredholm theorem, $(1 - K(z))^{-1}$ is a meromorphic $\mathfrak{L}(H,H)$-valued function on $\mathbb{C}$. Thus, by (2.4) we obtain the desired meromorphic continuation of $R_\chi(z)$. Moreover, clearly the poles of this continuation coincide, with multiplicity, with the poles of $(1 - K(z))^{-1}$. Thus, since $1 - K(z)$ is invertible for $z \in \mathbb{C}_+$, we have that all the poles are in $\bar{\mathbb{C}}_-$. Now, for $z \in \mathbb{C}_+$, we have

$$1 - K(-z) = (1 - K(z))(1 - T(z)), \tag{2.12}$$

where

$$T(z) = (1 - K(z))^{-1}(K(-z) - K(z)).$$

By (2.11) we have

$$T(z) = T_1(z)\chi S(z)\chi K_2 + T_2(z)\chi S(z)\chi K_8 \tag{2.13}$$

where

$$S(z) = R_0(-z) - R_0(z)$$
$$T_1(z) = (1-K(z))^{-1}K_6(z)$$
$$T_2(z) = (1-K(z))^{-1}K_7(z).$$

By (2.6), for any $\gamma > 0$, we get

$$\|T_j(z)\| \leqslant C_\gamma(1+|z|)^8 \qquad \text{for} \qquad \operatorname{Im} z \geqslant \gamma, \ j = 1, 2. \tag{2.14}$$

On the other hand, by (2.1) and the well known properties of the Hankel functions, we have the following formula for the kernel of $S(z)$:

$$\begin{aligned} S(z)(x,y) &= (i/2)(z/2\pi|x-y|)^{(n-2)/2}J_{(n-2)/2}(z|x-y|) \\ &= (i/2)(2\pi)^{-n+1}z^{n-2}\int_{\mathbb{S}^{n-1}} \exp\left(iz\langle x-y, w\rangle\right) dw, \quad x, y \in \mathbb{R}^n, \end{aligned} \tag{2.15}$$

where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. Denote by $\tilde{S}(z)$ the operator with kernel given by the integral above. Now it is easy to see by (2.15) that $\chi S(z)\chi$ forms an entire family of trace class operators in $\mathfrak{L}(H,H)$. Hence, by (2.13), $T(z)$ is holomorphic in $\mathbb{C}_+$ with values in the trace class operators in $\mathfrak{L}(H,H)$. Now, by (2.12) it is easy to see that $1 - T(z)$ is invertible in $\mathfrak{L}(H,H)$ for those $z \in \mathbb{C}_+$ for which so is $1 - K(-z)$, and then we have

$$(1-K(-z))^{-1} = (1-T(z))^{-1}(1-K(z))^{-1}. \tag{2.16}$$

Since $(1-K(z))^{-1}$ is holomorphic in $\mathbb{C}_+$, by (2.16) we conclude that the poles of $(1-K(-z))^{-1}$ lying in $\mathbb{C}_+$, with multiplicity, coincide with the poles of $(1-T(z))^{-1}$. Introduce the function

$$h(z) = \det(1-T(z)),$$

which is well defined and holomorphic in $\mathbb{C}_+$. Now, by the above analysis we conclude that if λ_j, $\lambda_j \in \mathbb{C}_-$, is a scattering pole, then $-\lambda_j$ is a zero of $h(z)$ with the corresponding multiplicity. Thus we can characterize the scattering poles as zeros of $h(-z)$, $z \in \mathbb{C}_-$. Notice that the fact that $T(z)$ is trace class does not depend on whether (iv) is fulfilled or not. Hence the function $h(z)$ is always defined, under the conditions (i)-(iii), and holomorphic in $\mathbb{C}_+$. Now we are going to study the distribution of the zeros of $h(z)$ without assuming (iv). Note that in general the zeros of $h(z)$ may accumulate at points on the real axis. Let $\{z_j\} \subset \mathbb{C}_+$ be the zeros of $h(z)$, repeated according to multiplicity, and given $0 < \varepsilon,\ \delta \ll 1,\ r \gg 1$, set

$$N(\varepsilon,\delta,r) = \#\{z_j : \delta \leqslant |z_j| \leqslant r, z_j \in \Lambda_\varepsilon\}.$$

We have the following :

THEOREM 2. – *Assume (i)-(iii) fulfilled. Then, for any* $\varepsilon,\ \delta,\ r$ *as above there exists a constant* $C_{\varepsilon,\delta} > 0$, *independent of* r, *so that*

$$N(\varepsilon,\delta,r) \leqslant C_{\varepsilon,\delta} r^n \quad \text{for} \quad r \geqslant 1. \tag{2.17}$$

When (iv) is fulfilled the number of the scattering poles in $\{z \in \mathbb{C} : |z| \leqslant \delta\}$ is finite for any $\delta > 0$, and hence (1.5) is obtained as an immediate consequence of (2.17).

3. Proof of Theorem 2.

We start with the following :

LEMMA 1. – *Under the assumptions (i)-(iii), for any* $\gamma > 0$ *there exists a constant* $C_\gamma > 0$ *so that*

$$|h(z)| \leqslant C_\gamma \exp(C_\gamma |z|^n) \quad \text{for} \quad \operatorname{Im} z \geqslant \gamma. \tag{3.1}$$

Proof. – The estimate (3.1) is established in the same way as in [13] (see also [17]). Here we shall sketch the proof. Given a compact operator A, $\mu_j(A)$ will denote the characteristic values of A, i.e. the eigenvalues of $(A^*A)^{1/2}$, repeated according to multiplicity and ordered to form a nonincreasing sequence. First, recall some well known properties of $\mu_j(A)$:

$$\mu_j(A) \leqslant \|A\|, \quad \forall j, \tag{3.2}$$

$$\{\mu_j(AB), \mu_j(BA)\} \leqslant \mu_j(A)\|B\|, \quad \forall j, \tag{3.3}$$

$$\mu_j\left(\sum_{i=1}^k A_i\right) \leqslant \sum_{i=1}^k \mu_{j_k}(A_i), \quad \forall j, \tag{3.4}$$

where $j_k \sim [j/k]$, $[a]$ denotes the integer part of a. By (2.13)-(2.15) and (3.2)-(3.4) it is easy to see that

$$\mu_j(T(z)) \leqslant C_\gamma(1+|z|)^{n+6}\mu_{j_2}(\chi\tilde{S}(z)\chi) \qquad \text{for} \qquad \operatorname{Im} z \geqslant \gamma. \tag{3.5}$$

On the other hand, clearly we have

$$\|\chi\tilde{S}(z)\chi\| \leqslant C\exp(C|z|), \quad \forall z \in \mathbb{C}. \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$\mu_j(T(z)) \leqslant C_\gamma \exp(C|z|), \quad \forall j, \qquad \text{for} \qquad \operatorname{Im} z \geqslant \gamma. \tag{3.7}$$

Further on, we shall show that there exists a constant $C > 0$ so that

$$\mu_j(\chi\tilde{S}(z)\chi) \leqslant Ce^{-|z|}j^{-n/(n-1)} \quad \text{if} \quad j \geqslant C|z|^{n-1}, \quad \forall z \in \mathbb{C}. \tag{3.8}$$

This is actually proved in [13], but for the sake of completeness we shall repeat the key points. The key observation is the representation

$$\tilde{S}(z) = S_1(z)S_2(z), \tag{3.9}$$

where $S_1(z)$ is the operator with kernel $S_1(z)(x,w) = \exp(iz\langle x,w\rangle)$, $S_2(z)$ is the operator with kernel $S_2(z)(w,x) = \exp(-iz\langle x,w\rangle)$, $x \in \mathbb{R}^n$, $w \in \mathbb{S}^{n-1}$. Then, using (3.3) and (3.9) we have

$$\mu_j(\chi\tilde{S}(z)\chi) \leqslant \|\chi S_1(z)\|_1\|(1-\Delta_w)^m S_2(z)\chi\|_2\mu_j((1-\Delta_w)^{-m}), \ \forall j, \tag{3.10}$$

for any integer $m \geqslant 1$, where Δ_w denotes the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$, $\| \|_1$ and $\| \|_2$ denote the norms in $\mathscr{L}(L^2(\mathbb{S}^{n-1}), L^2(\mathbb{R}^n))$ and $\mathscr{L}(L^2(\mathbb{R}^n), L^2(\mathbb{S}^{n-1}))$, respectively. On the other hand, we have with a constant $C > 0$,

$$\mu_j((1-\Delta_w)^{-m}) \leqslant C^m j^{-2m/l}, \tag{3.11}$$

where $l = \dim \mathbb{S}^{n-1} = n - 1$, and

$$\|\chi S_1(z)\|_1 \leqslant C\exp(C|z|), \tag{3.12}$$

$$\begin{aligned}\|(1-\Delta_w)^m S_2(z)\chi\|_2 &\leqslant C\sup_{x,w}|\chi(x)(1-\Delta_w)^m(e^{-z\langle x,w\rangle})| \\ &\leqslant C^{2m+1}(|z|^{2m}+(2m)^{2m})e^{C|z|}.\end{aligned} \tag{3.13}$$

Thus, by (3.10)-(3.13),

$$\mu_j(e^{|z|}\chi\tilde{S}(z)\chi) \leqslant C^{2m+1}(|z|^{2m}+(2m)^{2m})e^{C|z|}j^{-2m/l}, \tag{3.14}$$

with a new constant $C > 0$. Now, (3.8) is an easy consequence of (3.14) (see [13], [17]).

Thus, by (3.5) and (3.8), we have

$$\mu_j(T(z)) \leqslant C_\gamma j^{-n/(n-1)} \quad \text{if} \quad j \geqslant C|z|^{n-1}, \quad \text{for} \quad \operatorname{Im} z \geqslant \gamma, \tag{3.15}$$

with new constants C_γ, $C > 0$. Now, it is a straightforward calculation that (3.7) and (3.15) together with Weyl's convexity estimate imply (3.1) (see [13], [17]. The proof of Lemma 1 is completed.

To derive (2.17) from (3.1), instead of Jensen's inequality, we shall use the following classical result (see [9], Section 3, Carleman's theorem).

LEMMA 2. — *Given* $r > r_0 > 0$, *set* $\Omega = \{z \in \mathbb{C} : r_0 \leqslant |z| \leqslant r, \operatorname{Im} z \geqslant 0\}$. *Let* $f(z)$ *be a function holomorphic in* Ω *and let* $r_1 \exp(i\varphi_1), r_2 \exp(i\varphi_2), \ldots, r_k \exp(i\varphi_k)$ *be the zeros of* $f(z)$ *in* Ω *repeated according to multiplicity. Then,*

$$\sum_{j=1}^{k} (r_j^{-1} - r_j r^{-2}) \sin \varphi_j = (\pi r)^{-1} \int_0^\pi \log |f(re^{i\varphi})| \sin \varphi \, d\varphi$$
$$+ (2\pi)^{-1} \int_{r_0}^{r} (t^{-2} - r^{-2}) \log |f(t) f(-t)| \, dt$$
$$- (\pi r_0)^{-1} \int_0^\pi \log |f(r_0 e^{i\varphi})| \sin \varphi \, d\varphi .$$

Note that each term in the sum above is $\geqslant 0$. Fix ε, δ, $0 < \varepsilon$, $\delta \ll 1$, and let $r > 6$. Let

$$z_1 = r_1 \exp(i\varphi_1), \qquad z_2 = r_2 \exp(i\varphi_2), \ldots, z_k = r_k \exp(i\varphi_k)$$

be the zeros of $h(z)$, repeated according to multiplicity, satisfying the conditions: $3 \leqslant r_j \leqslant r/2$; $\varepsilon \leqslant \varphi_j \leqslant \pi - \varepsilon$. Clearly,

$$N(\varepsilon;\delta,r/2) \leqslant k + N(\varepsilon,\delta,3). \tag{3.16}$$

Set $f(z) = h(z + i\gamma)$ where $\gamma = \sin \varepsilon$. Clearly, $f(z)$ is holomorphic in $\bar{\mathbb{C}}_+$ and by (3.1) we have

$$|f(z)| \leqslant C_\varepsilon \exp(C_\varepsilon |z|^n), \quad \forall z \in \bar{\mathbb{C}}_+ . \tag{3.17}$$

Moreover, $z'_j = z_j - i\gamma$, $j = 1, \ldots, k$, are zeros of $f(z)$. Set $r'_j = |z'_j|$ and $\varphi'_j = \arg z'_j$. It is easy to check that $2 \leqslant r'_j \leqslant 2r/3$ and $\sin \varphi'_j \geqslant$

$2^{-1}\sin\varepsilon$, $j = 1, \ldots, k$. Hence

$$(3.18) \qquad (r_j'^{-1} - r_j' r^{-2}) \sin \varphi_j' \geqslant (5\gamma/12) r^{-1}, \qquad j = 1, \ldots, k.$$

Now, applying Lemma 2 to $f(z)$ with $r_0 = 2$ and using (3.17) and (3.18), we get

$$\begin{aligned}(5\gamma/12) r^{-1} k &\leqslant \sum_{j=1}^{k} (r_j'^{-1} - r_j' r^{-2}) \sin \varphi_j' \\ &\leqslant (\pi r)^{-1} \int_0^{\pi} \log |f(re^{i\varphi})| \sin \varphi \, d\varphi \\ &+ (2\pi)^{-1} \int_2^r (t^{-2} - r^{-2}) \log |f(t) f(-t)| \, dt + C_\varepsilon \\ &\leqslant C_\varepsilon' r^{-1} (r^n + 1) + C_\varepsilon' \int_2^r t^{-2} (t^n + 1) \, dt + C_\varepsilon \\ &\leqslant C_\varepsilon'' r^{-1} (r^n + 1).\end{aligned}$$

Hence

$$(3.19) \qquad k \leqslant (12/5\gamma) C_\varepsilon'' (r^n + 1).$$

Now (2.17) follows from (3.16) and (3.19) at once.

BIBLIOGRAPHY

[1] C. Bardos, J. Guilot, J. Ralston, La relation de Poisson pour l'équation des ondes dans un ouvert non borné. Application à la théorie de la diffusion, Commun. Partial Differ. Equations, 7 (1982), 905-958.

[2] A. Intissar, A polynomial bound on the number of scattering poles for a potential in even dimensional space $\mathbb{R}^n$, Commun. Partial Differ. Equations, 11 (1986), 367-396.

[3] P. D. Lax, R. S. Phillips, Scattering Theory, Academic Press, 1967.

[4] R. B. Melrose, Polynomial bounds on the number of scattering poles, J. Func. Anal., 53 (1983), 287-303.

[5] R. B. Melrose, Polynomial bounds on the distribution of poles in scattering by an obstacle, Journées « Équations aux dérivées partielles », Saint-Jean-de-Monts (1984).

[6] R. B. Melrose, Weyl asymptotics for the phase in obstacle scattering, Commun. Partial Differ. Equations, 13 (1988), 1431-1439.

[7] J. Sjöstrand, Geometric bounds on the number of resonances for semiclassical problems, Duke Math. J., 60 (1990), 1-57.

[8] J. SJÖSTRAND, M. ZWORSKI, Complex scaling and the distribution of scattering poles, J. Amer. Math. Soc., (1991), 729-769.

[9] E. C. TITCHMARSH, The Theory of Functions, Oxford University Press, 1968.

[10] B. VAINBERG, Asymptotic Methods in Equations of Mathematical Physics, Gordon and Breach Sci. Publ., 1988.

[11] G. VODEV, Polynomial bounds on the number of scattering poles for symmetric systems, Ann. Inst. Henri Poincaré (Physique théorique), 54 (1991), 199-208.

[12] G. VODEV, Polynomial bounds on the number of scattering poles for metric perturbations of the Laplacian in $\mathbb{R}^n$, $n \geqslant 3$, odd, Osaka J. Math., 28 (1991), 441-449.

[13] G. VODEV, Sharp polynomial bounds on the number of scattering poles for metric perturbation of the Laplacian in $\mathbb{R}^n$, Math. Ann., 291 (1991), 39-49.

[14] G. VODEV, Sharp bounds on the number of scattering poles for perturbations of the Lapacian, Commun. Math. Phys., 145 (1992), to appear.

[15] M. ZWORSKI, Distribution of poles for scattering in the real line, J. Func. Anal., 73 (1987), 227-296.

[16] M. ZWORSKI, Sharp polynomial bounds on the number of scattering poles of radial potentials, J. Func. Anal., 82 (1989), 370-403.

[17] M. ZWORSKI, Sharp polynomial bounds on the number of scattering poles, Duke Math. J., 59 (1989), 311-323.

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