

METRIC PROPERTIES OF EIGENFUNCTIONS OF THE LAPLACE OPERATOR ON MANIFOLDS

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In this note, we prove two theorems which express a quasi-symmetry relation between the positive and the negative part of the distribution function of an eigenfunction of the Laplace operator on a Riemannian manifold.

1. An estimate of the volume of a domain on which an eigenfunction of the Laplace operator on a Riemannian surface has constant sign.

Let M be a two-dimensional compact real analytic Riemannian manifold, u_1, u_2, \dots -eigenfunctions of the Laplace operator on M , $\Delta u_i = \lambda_i u_i$.

THEOREM. — *There exists a positive constant C which depends on M such that, for every $i = 1, 2, \dots$,*

$$\text{vol}\{x \in M, u_i(x) > 0\} > C .$$

The proof is based on the two following lemmas.

Let f be a bounded function, continuous on $[0, 1]$. Let us denote by $N(f)$ the number of changes in the sign of the function f on $[0, 1]$.

LEMMA 1. — Let f_n be a sequence of non-zero continuous functions, defined on \mathbf{R} , with values in \mathbf{R} , with support in $[0, 1]$ and assume that $N(f_n)$ is bounded by some fixed number N .

Then there exists a subsequence n_i of \mathbf{N} ($n_i \rightarrow \infty$), real numbers α_{n_i} , such that $\alpha_{n_i} \cdot f_{n_i}$ converges as $i \rightarrow \infty$, for the (weak-)topology of the space of distributions \mathcal{D}' to a non-zero distribution of order less than N .

Remark. — First, we recall that a distribution is said of order less than N if it is a sum of derivatives of order less than N of Radon measures. Moreover, if P is a polynomial of degree N and μ a Radon measure, the set of all $T \in \mathcal{D}'$ satisfying $PT = \mu$ is an N -dimensional affine subspace of the space of all distributions of order less than N .

Proof of Lemma 1 (suggested by Y. Colin de Verdière). — Let $P_n = \prod_{k=1}^N (x - x_k)$ be a sequence of polynomials of degree N such that $P_n \cdot f_n$ is ≥ 0 . By renormalisation and taking a subsequence, we may assume that $\int_0^1 P_n \cdot f_n = 1$, that $P_n \cdot f_n$ converges to a probability measure μ and that P_n converges to a polynomial P of degree exactly N .

Let $T_0 \in \mathcal{D}'$ be such that $PT_0 = \mu$.

Let $T_n = f_n - T_0$, then we get :

$$\lim P_n \cdot T_n = 0 .$$

We introduce now the following decomposition of the space of distributions :

$$\mathcal{D}' = Z_P \oplus W ,$$

where $Z_P = \{T \in \mathcal{D}' | PT = 0\}$, and W is a topological complement of Z_P .

W is a complement to Z_{P_n} if n is big enough. Now we can write uniquely :

$$T_n = z_n + w_n ,$$

where $z_n \in Z_{P_n}$ and $w_n \in W$. Now $P_n \cdot w_n \rightarrow 0$ and we deduce that $w_n \rightarrow 0$, because the multiplication by P_n is uniformly invertible in W .

Now $z_n \in Z_{P_n}$ and Z_{P_n} converges to Z_P .

Two cases are possible :

First case : z_n is bounded and we can extract a convergent subsequence converging to T_1 in Z_P . Then $T_0 + T_1$ is not zero and we get the conclusion.

Second case : z_n is unbounded; then there exists a sequence $\beta_{n_i} \rightarrow 0$ such that $\beta_{n_i} \cdot z_{n_i}$ converges to a non-zero distribution T_1 and then :

$$\beta_{n_i} \cdot f_{n_i}$$

converges to T_1 .

Let us denote by B the unit disk in \mathbf{R}^2 , $S = \partial B$, if f is a continuous function on S then $N(f)$ is the number of changes of sign of the function f on S .

LEMMA 2. — Let u be a harmonic function in B which is continuous in \bar{B} , $u|_S = f$, $u(0) = 0$. Let $N(f) = k < \infty$. Define

$$G_u = \{x \in B, u(x) > 0\}.$$

Then $\text{mes } G_u > C$, where constant $C > 0$ is dependent on k .

Proof. — Let us assume the contrary. This means that there exists a sequence of harmonic functions u_n in B , $u_n|_S = f_n$, $u_n(0) = 0$, $N(f_n) \leq k$, $\text{mes } G_{u_n} \rightarrow 0$, $n \rightarrow \infty$. According to lemma 1 there exists a real valued sequence α_m and a subsequence f_{n_m} such that, $\alpha_m f_{n_m} \rightarrow \tilde{f} \neq 0$ in the sense of distributions. From the convergence of the distributions $\alpha_m f_{n_m}$ on S it follows that in an arbitrary compact subdomain of B the convergence of functions $\alpha_m u_{n_m}$ is uniform. Let $\alpha_m u_{n_m} \rightarrow U$ in B . From [1] it follows that $U \neq 0$ in B if $\tilde{f} \neq 0$ on S . We have $U(0) = 0$. From the assumption $\text{mes } G_{u_n} \rightarrow 0$, $n \rightarrow \infty$, it follows that $U \leq 0$ in B . Equality $U(0) = 0$ and inequality $U \leq 0$ in B contradicts the maximum principle for harmonic functions.

Proof of the theorem.

1. Let us denote by B_r^x , $x \in M$, r , the geodesic circle on M with centre x and radius r .

There is a constant $C_0 > 0$, such that for every $\varepsilon > 0$ there exists points $x_1 \dots x_N \in M$, $N > C_0/\varepsilon^2$, such that the circles $B_\varepsilon^{x_1} \dots B_\varepsilon^{x_N}$ mutually have no intersections.

2. There exists a constant r_0 , such that for every $x \in M$, $0 < r < r_0$, B_r^x is diffeomorphic to a disk.

3. There is a constant $C_1 > 0$, such that for all $x \in M$, $\lambda > 0$ in the circle $B_{1/C_1\sqrt{\lambda}}^x$ there exists a positive solution of the equation $\Delta u + \lambda u = 0$.

4. Let $x \in M$, $\lambda > 0$, $r = 1/C_1\sqrt{\lambda} < r_0$, u is a solution of the equation $\Delta u + \lambda u = 0$ in B_r^x . Then there exists a diffeomorphism h of the

unit disk B on B_r^x , $h(0) = x$, and a function s in B , $0 < s < \infty$, such that $s.u(h)$ is a harmonic function in B (by a representation theorem in quasiconformal mapping theory, [2]). From the compactness of M it follows that the Jacobian of the mapping h is uniformly bounded.

5. There is a constant $C_2 > 0$, such that for all $x \in M$, $\lambda > 0$ in the circle $B_{1/C_2\sqrt{\lambda}}^x$ every solution $u \neq 0$ of the equation $\Delta u + \lambda u = 0$ changes its sign [3].

6. Let u_i be an eigenfunction, $\Delta u_i = \lambda_i u_i$ on M , γ is a nodal line of the function u_i . For a two-dimensional real analytic manifold the following estimate is true, [4],

$$\text{length } \gamma \leq C_3 \sqrt{\lambda_i}$$

where constant $C_3 > 0$ is dependent on M .

7. Let u_i be an eigenfunction, $\Delta u_i = \lambda_i u_i$ on M . According to 1 we can choose circles $B_\varepsilon^{x_1} \dots B_\varepsilon^{x_n}$ with $\varepsilon = 2/C_2\sqrt{\lambda_i}$. We have $N > C_0 C_2^2 \lambda_i / 4$.

According to 5 there exist points $y_n \in B_{\varepsilon/2}^{x_n}$, $n = 1 \dots N$, such that $u_i(y_n) = 0$.

According to 6 at least $N/2$ points $y_{k_1} \dots y_{k_J}$, $J > N/2$, from the set $\{y_n\}$ have the following property : for all $j = 1 \dots J$ there exist r_j ,

$$\frac{1}{2C_1\sqrt{\lambda_i}} < r_j < \frac{1}{C_1\sqrt{\lambda_i}},$$

such that restriction of the function u_j on $\partial B_{r_j}^{y_{k_j}}$ has no more than

$$\frac{8C_1C_3}{C_2^2C_0}$$

zeros.

According to 4 and lemma 2 for all $j = 1 \dots J$

$$\text{mes}\{x \in B_{r_j}^{y_{k_j}}, u_i(x) > 0\} > C_4 \varepsilon^2.$$

We have $J > C_0/2\varepsilon^2$ and so the theorem is proved.

2. An estimate of the relation between the positive and the negative extrema of an eigenfunction.

Let M be a n -dimensional compact smooth Riemannian manifold, u_1, u_2, \dots -eigenfunctions of the Laplace operator on M , $\Delta u_i = \lambda_i u_i$.

THEOREM 2. — *There exists a positive constant C which depends only on n and a positive constant N which depends on M such that, for every $i > N$,*

$$\frac{1}{C} < \frac{\sup_M u_i(x)}{|\inf_M u_i(x)|} < C .$$

We denote by $B_r \subset \mathbb{R}^n$ the ball centered at 0 of radius r .

In B_r we consider a uniformly elliptic second order operator L defined by

$$L = \sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) , \tag{2.1}$$

where a_{ij} is a symmetric positive definite matrix in B_r . If the eigenvalues of the matrix $\|a_{ij}(x)\|$ lie on the segment $[e^{-1}, e]$, $e \geq 1$ we say that the operator L has an ellipticity constant e .

LEMMA 3. — *Let u be a solution of the equation*

$$a(x)Ly + \lambda u = 0$$

in the ball B_1 , $1/A < a(x) < A$, $A > 0$, L is an elliptic operator with the ellipticity constant e , λ is a constant such that $|\lambda| < C$. Let us assume that $u(x_0) > 0$ and that there exists $x_0 \in B_{1/2}$ with $u(x_0) = 0$. Then

$$|\inf_{B_1} u| > \delta u(0) ,$$

where the constant $\delta > 0$, $\delta = \delta(n, A, e, C)$.

Proof.

1. We shall prove Lemma 2 under the assumption that $\lambda = 0$. Denote

$$\varphi_1 = \sup\{0, u \mid_{\partial B_1}\}$$

$$\varphi_2 = \inf\{0, u \mid_{\partial B_1}\} .$$

Let u_1, u_2 be the solutions of the following Dirichlet problems :

$$Lu_1 = 0 \quad \text{in } B_1, \quad u_1 \mid_{\partial B_1} = \varphi_1,$$

$$Lu_2 = 0 \quad \text{in } B_2, \quad u_2 \mid_{\partial B_1} = \varphi_2.$$

Then, $u = u_1 + u_2$, $u_1 > 0$ in B_1 , $u_1(0) \geq u(0)$. From the Harnack inequality [5] it follows that there exists a constant $\delta > 0$, $\delta = \delta(n, e)$ such that

$$u_1 \mid_{B_{1/2}} > \delta u_1(0) .$$

Since $u(x_0) = 0$, $x_0 \in B_{1/2}$, then

$$\inf \varphi_2 < -\delta u_1(0) < -\delta u(0) .$$

2. Let $\lambda \notin 0$. Let us make a cylindric extension of the functions $u(x), a(x)$ and the operator L in the new coordinate x_{n+1} . After this extension we shall keep the notations u, a, L . Denote

$$v = ue^{\sqrt{\lambda}x_{n+1}}$$

clearly the function v is a solution of the elliptic equation

$$aLv + \frac{\partial^2 v}{\partial x_{n+1}^2} = 0 .$$

Now the statement of Lemma 3 follows from the assertion 1 to the function v in the unit ball in \mathbb{R}^{n+1} .

Proof of Theorem 2.

1. There are constant $C_1 = C_1(M) > 0$, $C_2 = C_2(M) > 0$ such that for all $x \in M$, $\lambda > C_2$ any solution of the equation $\Delta u + \lambda u = 0$ in the ball $B_{C_1/\sqrt{\lambda}}^x$ change its sign.

2. There exists a constant $N > C_2$, $N = N(M)$, such that for all $x \in M$ there exists a diffeomorphism

$$d : B_{2C_1/\sqrt{\lambda}}^x \subset M \rightarrow B_1 \subset \mathbb{R}^n$$

such that the equation $\Delta u + \lambda u = 0$ in $B_{2C_1/\sqrt{\lambda}}^x$ viewed in the ball B_1 has the form

$$a(x)Lu + \lambda'u = 0 \tag{2.2}$$

where L is an elliptic operator of the type (2.1), $e = 2$, $A = 2$, $|\lambda| < C = C(n) > 0$. We can obtain such a diffeomorphism d if we introduce in the ball $B_{2C_1/\sqrt{\lambda}}^x$ a normal coordinate system. Applying Lemma 3 to the solution u of the equation (2.2) we obtain the statement of Theorem 2.

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