

ANNALES DE L'INSTITUT FOURIER

IZU VAISMAN

Remarks on the Lichnerowicz-Poisson cohomology

Annales de l'institut Fourier, tome 40, n° 4 (1990), p. 951-963

http://www.numdam.org/item?id=AIF_1990__40_4_951_0

© Annales de l'institut Fourier, 1990, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

REMARKS ON THE LICHNEROWICZ-POISSON COHOMOLOGY

by Izu VAISMAN

The Lichnerowicz-Poisson (LP) cohomology of a Poisson manifold was defined in [L], and it provides a good framework to express deformation and quantization obstructions [L], [VK], [H], [V2]. The LP cohomology spaces are, generally, very large, and their structure is known only in some particular cases [VK], [X]. The homological algebraic place of these spaces was clarified in [H]. In the present note, we make a number of further remarks on the LP cohomology, most of them related with a certain natural spectral sequence which shows that, in the case of a regular Poisson manifold, the LP cohomology is connected with the cohomology of the sheaves of germs of foliated (i.e., projectable) forms of the symplectic foliation of the manifold (e.g., [V1]).

1. General remarks.

Let M^m be a Poisson manifold with the Poisson bivector Π , and put $\mathcal{V}^0(M) \stackrel{\text{def}}{=} C^\infty(M)$, $\mathcal{V}(M) = \mathcal{V}^1(M) \stackrel{\text{def}}{=} \mathcal{V}^1(M)$ the space of C^∞ vector fields of M , $\mathcal{V}^k(M) \stackrel{\text{def}}{=} \mathcal{V}^k(M)$ the space of k -vector fields (i.e., antisymmetric k -contravariant tensor fields of M), $\mathcal{V}^*(M) \stackrel{\text{def}}{=} \mathcal{V}^*(M)$ the space of Pfaff forms of M , and, finally $\mathcal{L}(M) \stackrel{\text{def}}{=} \bigoplus_{k=0}^m \mathcal{V}^k(M)$ the contravariant Grassmann algebra of M . The bivector Π has an associated morphism $\# : T^*M \rightarrow TM$, defined by $\beta(\alpha^\#) = \Pi(\alpha, \beta)$, $\forall \alpha, \beta \in T^*M$, and it yields the Poisson bracket of functions $\{f, g\} = \Pi(df, dg)$, as well as Hamiltonian vector fields X_f , $\forall f \in \mathcal{V}^0(M)$, given by $X_f g = \{f, g\}$. These fields define a generalized foliation with symplectic leaves called the *symplectic foliation* of (M, Π) (i.e., $\{X_f\}$ generate the tangent spaces of

Key-words : Poisson manifolds - LP cohomology.

A.M.S. Classification : 58A12 - 58F05.

the leaves). It is important to remember that the Poisson bracket induces a bracket of Pfaff forms which is the unique natural extension of the formula $\{df, dg\} = d\{f, g\}$, and is given by

$$(1.1) \quad \{\alpha, \beta\} = L_{\alpha\#} \beta - L_{\beta\#} \alpha - d(\Pi(\alpha, \beta)).$$

The basic Poisson condition $[\Pi, \Pi] = 0$, where $[\ , \]$ denotes the Schouten-Nijenhuis bracket, ensures that $(\mathcal{V}^0(M), \{ , \})$ and $(\mathcal{V}^*(M), \{ , \})$ are Lie algebras. The same condition also shows that the operator $\sigma Q = -[\Pi, Q]$ is a coboundary on $\mathcal{L}(M)$ (i.e., $\sigma^2 = 0$), and the cohomology of the cochain complex (\mathcal{L}, σ) is, by definition, the LP cohomology of (M, Π) . Its spaces will be denoted by $H_{LP}^k(M, \Pi)$. It is also important to remind that, for $Q = \mathcal{V}^k(M)$, one has [BV]

$$(1.2) \quad (\sigma Q)(\alpha_0, \dots, \alpha_k) = \sum_{i=1}^k \alpha_i\#(Q(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k)) + \sum_{i < j=0}^k (-1)^{i+j} Q(\{\alpha_i, \alpha_j\}, \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k),$$

where $\alpha_i \in \mathcal{V}^*(M)$, and $\hat{\ \ }$ denotes the absence of an argument.

Now, the definitions given above have some easy consequences such as

a) [X], [VK]. $H_{LP}^0(M, \Pi) = \{f \in C^\infty(M) / \forall g \in C^\infty(M), X_g f = 0\}$. (Since $\sigma f = -X_f$.)

b) [X], [VK]. $H_{LP}^1(M, \Pi) = \mathcal{V}_\pi(M) / \mathcal{V}_{\mathcal{X}}(M)$, where

$$\mathcal{V}_\pi(M) \stackrel{\text{def}}{=} \{X \in \mathcal{V}(M) / L_X \Pi = 0\}, \quad \mathcal{V}_{\mathcal{X}}(M) \stackrel{\text{def}}{=} \{X_f / f \in \mathcal{V}^0(M)\}.$$

(Since $\sigma X = -L_X \Pi$ [L].)

c) [L], $\sigma \Pi = 0$, and Π defines a *fundamental class* $[\Pi] \in H_{LP}^2(M, \Pi)$.

d) The LP cohomology satisfies the Mayer-Vietoris exact sequence property i.e., if U, V are open subsets of M , there is an exact sequence of the form

$$(1.3) \quad \dots \rightarrow H_{LP}^k(U \cup V, \Pi) \rightarrow H_{LP}^k(U, \Pi) \oplus H_{LP}^k(V, \Pi) \rightarrow H_{LP}^k(U \cap V, \Pi) \rightarrow H_{LP}^{k+1}(U \cup V, \Pi) \rightarrow \dots$$

The definition of the arrows and the proof of the exactness are the same as for the de Rham cohomology (e.g., [BT]).

e) [L], [K]. Natural homomorphisms $\rho : H^k(M, \mathbb{R}) \rightarrow H^k_{LP}(M, \Pi)$, which are isomorphisms in the symplectic case, exist. Namely, ρ is defined by the extension of $\#$ to k -forms λ by

$$(1.4) \quad \lambda^\#(\alpha_1, \dots, \alpha_k) = (-1)^k \lambda(\alpha_1^\#, \dots, \alpha_k^\#),$$

since (1.2) shows that $\sigma(\lambda^\#) = (-1)^k(d\lambda)^\#$.

Because of e), it is natural to ask for a covariant interpretation of the whole LP cohomology via a Riemannian metric, and such an interpretation can be obtained by using Koszul's generating operators of the Schouten-Nijenhuis bracket. If we change signs such as to agree with [L], Koszul's formula for $[A, B]$ where $A \in \mathcal{V}^i(M)$, $B \in \mathcal{V}^j(M)$ is [K]

$$(1.5) \quad [A, B] = D_\nabla(A \wedge B) - (D_\nabla A) \wedge B - (-1)^i A \wedge (D_\nabla B),$$

where ∇ is a torsionless linear connection on M , and D_∇ is defined by the coordinatewise formula

$$(1.6) \quad (D_\nabla A)^{h_2, \dots, h_i} = \nabla_k A^{kh_2, \dots, h_i}.$$

If ∇ is the Riemannian connection of a metric g , (1.6) means $D_\nabla = -\#_g \delta_g \#_g^{-1}$, where $\#_g : T^*M \rightarrow TM$ is the well known musical isomorphism, and δ_g is the codifferential of (M, g) . Now, if we denote $\pi = \#_g^{-1} \Pi$, $B = \#_g \lambda$, and take $A = \Pi$ in (1.5), we obtain $\sigma(\#_g \lambda) = \#_g \delta_\pi$, where, if $e(i)$ denotes the exterior (interior) multiplication by a form, one has

$$(1.7) \quad \delta_\pi = \delta_g e(\pi) - e(\pi) \delta_g - e(\delta_g \pi).$$

Hence, $H^k_{LP}(M, \Pi)$ are isomorphic to the cohomology spaces of the Grassmann complex ΛM endowed with the coboundary δ_π .

Of course, π must satisfy the condition $\delta_\pi \pi = 0$, which is equivalent to $[\Pi, \Pi] = 0$ i.e., we must have

$$(1.8) \quad \delta_g(\pi \wedge \pi) = 2\pi \wedge (\delta_g \pi),$$

and this is a new characterization of a Poisson structure which may have some usefulness. For instance, it shows that the parallel 2-forms of a Riemannian manifold (if any) and the harmonic 2-forms of a compact Riemannian symmetric space (where the exterior product of two harmonic forms is again a harmonic form) define Poisson structures. Formulas (1.7), (1.8) may also be used if we are looking for compatible

Poisson structures on a given symplectic manifold M with symplectic form ω i.e., Poisson bivectors Π such that $[\omega^{-1}, \Pi] = 0$ (e.g., [G]). After the choice of a metric g on M , this problem amounts to solving the equations

$$(1.9) \quad \delta_{\#_g^{-1}(\omega^{-1})}\pi = 0, \quad \delta_g(\pi \wedge \pi) = 2\pi \wedge \delta_g\pi,$$

where also, if we ask g to be almost Hermitian ω -compatible, then $\#_g^{-1}(\omega^{-1}) = \omega$. For instance, (1.9) shows that, if M is a compact Hermitian symmetric space, and ω is its Kähler form, then any harmonic form of M defines an ω -compatible Poisson structure. On the other hand, we shall notice that, in case M is compact and oriented, δ_π has the formal adjoint

$$(1.10) \quad d_\pi = i(\pi)d - di(\pi) - i(\delta_g\pi),$$

and we may expect to be able to apply the abstract Hodge decomposition theorem of [LT]. (From the expression of the Schouten-Nijenhuis bracket [L], it follows easily that the complex

$$\dots \rightarrow \mathcal{V}^k(M) \xrightarrow{\sigma} \mathcal{V}^{k+1}(M) \rightarrow \dots$$

is elliptic along the leaves of the symplectic foliation of (M, Π) .)

Finally, we make a remark which will be important for the next sections of this paper. Namely, that there is a Serre-Hochschild spectral sequence associated with the LP cohomology. Let $\mathcal{V}_0^*(M) \stackrel{\text{def}}{=} \ker \# =$ the space of *conormal 1-forms* of the symplectic foliation of (M, Π) . Since the bracket (1.1) satisfies $\{\alpha, \beta\}^\# = [\alpha^\#, \beta^\#]$ [BV], $\mathcal{V}_0^*(M)$ is an abelian ideal of $(\mathcal{V}^*(M), \{, \})$, and we may define the *filtration degree* of $Q \in \mathcal{V}^k(M)$ to be h if $Q(\alpha_1, \dots, \alpha_k) = 0$ as soon as $\geq k - h + 1$ of the arguments are conormal. This yields a differential filtration of the LP complex $\mathcal{L}(M)$, where $S_h^k(M) \stackrel{\text{def}}{=} \text{the space of } k\text{-vector fields of filtration degree } h \text{ is equal to the locally finite span of } \{f_0 X_{f_1} \wedge \dots \wedge X_{f_h} \wedge Y_1 \wedge \dots \wedge Y_{k-h} / f_i \in \mathcal{V}^0(M), Y_j \in \mathcal{V}^1(M)\}$. Now, the spectral sequence which we have in mind, and which we shall denote by $E_r^{p,q}(M, \Pi)$, is the one associated with this filtration i.e., the Serre-Hochschild sequence of the pair of Lie algebras $(\mathcal{V}^*(M), \mathcal{V}_0^*(M), \{, \})$. This sequence converges to $H_{\text{LP}}^*(M, \Pi)$, and one has (e.g., [F])

$$(1.11) \quad E_2^{p,q}(M, \Pi) = H^p(V^*(M) / \mathcal{V}_0^*(M); H^q(\mathcal{V}_0^*(M); C^\infty(M))).$$

2. The regular case.

In the remaining part of this paper we assume that Π is of the constant rank $2n$, and $m = 2n + s$. This is the *regularity condition*, and then the symplectic foliation of (M, Π) , hereafter to be denoted by \mathcal{S} , is regular. Hence, we can and shall define a transversal distribution \mathcal{S}' , and $TM = \mathcal{S}' \oplus T\mathcal{S}$, $T^*M = \mathcal{S}'^* \oplus T^*\mathcal{S}$ induce a bigrading of the covariant and contravariant tensors of M . A tensor whose transversal degree is p and whose leafwise degree is q is said to be of the type (p, q) . We shall denote by $\mathcal{V}^{p,q}(M)$ and $\Lambda^{p,q}(M)$ the spaces of k -vector fields and k -forms ($k = p + q$) of the type (p, q) of M , respectively. For instance, it is easy to understand that $\ker \#$ (i.e., $\mathcal{V}_0^*(M)$) is just $\mathcal{S}'^* =$ the space of the 1-forms of type $(1, 0)$, and that type $\Pi = (0, 2)$. (E.g., see [V1] for details on the bigrading of differential forms.)

Now, if $Q \in \mathcal{V}^k(M)$ is of type $(p, q)(p + q = k)$, and if we use bihomogeneous arguments α_i in (1.2), we see that $\sigma = \sigma' + \sigma''$ where type $\sigma' = (-1, 2)$, type $\sigma'' = (0, 1)$, and, for arguments α of type $(1, 0)$ and β of type $(0, 1)$, one has

$$(2.1) \quad (\sigma' Q)(\alpha_0, \dots, \alpha_{p-2}, \beta_0, \dots, \beta_{q+1}) = \sum_{i < j=0}^{q+1} (-1)^{i+j} Q(\{\beta_i, \beta_j\}, \alpha_0, \dots, \alpha_{p-2}, \beta_0, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_{q+1}),$$

$$(2.2) \quad (\sigma'' Q)(\alpha_0, \dots, \alpha_{p-1}, \beta_0, \dots, \beta_q) = \sum_{i=0}^q (-1)^{p+i} \beta_i^\# (Q(\alpha_0, \dots, \alpha_{p-1}, \beta_0, \dots, \hat{\beta}_i, \dots, \beta_q) + \sum_{i=0}^{p-1} \sum_{j=0}^q (-1)^{p+i+j} Q(\{\alpha_i, \beta_j\}, \alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{p-1}, \beta_0, \dots, \hat{\beta}_j, \dots, \beta_q) + \sum_{i < j=0}^q (-1)^{p+i+j} Q(\alpha_0, \dots, \alpha_{p-1}, \{\beta_i, \beta_j\}'' , \beta_0, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_q)).$$

Remember that type $\alpha = (1, 0)$ means $\alpha \in \mathcal{V}_0^*(M)$, and that the latter is an ideal of $\mathcal{V}^*(M)$. On the other hand, we denoted by $\{ , \}'$, $\{ , \}''$ the type $(1, 0)$ and $(0, 1)$ components of $\{ , \}$. Particularly, if type $X = (1, 0)$, we get easily

$$(2.3) \quad \{\beta_1, \beta_2\}'(X) = (L_X \pi)(\beta_1, \beta_2).$$

In this section we use the type decomposition of σ in order to indicate a recurrent computational process of the LP cohomology which, in fact, is similar to the one used in [VK] for the case where \mathcal{L} is a fibration. Take $Q \in \mathcal{V}^k(M)$, and decompose it as

$$(2.4) \quad Q = Q^{k,0} + Q^{k-1,1} + \dots + Q^{0,k},$$

where the indices denote the type of the components. Then, $\sigma Q = 0$ means

$$(2.5) \quad \sigma'' Q^{i,k-i} + \sigma' Q^{i+1,k-i-1} = 0 \quad (i=0, \dots, k).$$

For $i = k$, (2.5) gives $\sigma'' Q^{k,0} = 0$, and, on the other hand, $(Q + \tilde{Q})^{k,0} = Q^{k,0}$, $\forall \tilde{Q} \in \mathcal{V}^{k-1}(M)$. Therefore, there exist homomorphisms

$$(2.6) \quad p_{k,0}: H_{LP}^k(M, \Pi) \rightarrow \mathcal{V}_0^{k,0}(M),$$

where $\mathcal{V}_0^{k,0}(M)$ is the space of σ'' -closed k -vectors of type $(k,0)$, and, furthermore, (2.5) shows that $\text{im } p_{k,0}$ consists of k -vectors $Q^{k,0} \in \mathcal{V}_0^{k,0}(M)$ which satisfy the following sequence of existence conditions of k -vectors $Q^{k-1,1}, \dots, Q^{0,k}$ such that

$$\begin{aligned} (c_1) \quad \sigma' Q^{k,0} &= \sigma''\text{-exact} \stackrel{\text{def}}{=} -\sigma'' Q^{k-1,1}, \\ (c_2) \quad \sigma' Q^{k-1,1} &= \sigma''\text{-exact} \stackrel{\text{def}}{=} -\sigma'' Q^{k-2,2}, \\ &\dots\dots\dots \\ (c_k) \quad \sigma' Q^{1,k-1} &= \sigma''\text{-exact} \stackrel{\text{def}}{=} -\sigma'' Q^{0,k}. \end{aligned}$$

In this case we shall say that $\sigma' Q^{k,0}$ satisfies k times the σ'' -exactness condition, and we shall denote by $\mathcal{V}_{0(k)}^{k,0}(M)$ the space of such $Q^{k,0}$. If we also denote $\ker p_{k,0} = {}^0 H_{LP}^k(M, \Pi)$ = the space of k -dimensional LP cohomology classes whose cocycles are (2.4) with $Q^{k,0} = 0$, we obtain the result of the first recurrence step

$$(2.7) \quad H_{LP}^k(M, \Pi) \approx {}^0 H_{LP}^k(M, \Pi) \oplus \mathcal{V}_{0(k)}^{k,0}(M).$$

Now, in the next step we have to compute ${}^0 H_{LP}^k(M, \Pi)$, and for this purpose we take the subcomplex ${}^0 \mathcal{L}(M)$ of $\mathcal{L}(M)$ consisting of multivectors Q with a vanishing $(\cdot, 0)$ component, and denote by $H^k({}^0 \mathcal{L}(M))$ its cohomology spaces. Then ${}^0 H_{LP}^k(M, \Pi)$ is the image of $H^k({}^0 \mathcal{L}(M))$ with respect to the inclusion: ${}^0 \mathcal{L}(M) \subseteq \mathcal{L}(M)$. It is clear that the complex $\mathcal{L}(M)/{}^0 \mathcal{L}(M)$ has coboundary zero, therefore, $H^k(\mathcal{L}/{}^0 \mathcal{L}) = (\mathcal{L}/{}^0 \mathcal{L})^k = \mathcal{V}^{k,0}(M)$. This gives us the exact sequence

$\mathcal{V}^{k-1,0}(M) \xrightarrow{\sigma} H^k({}^0\mathcal{L}(M)) \xrightarrow{\iota^*} H^k(\mathcal{L}(M))$, and we get

$$(2.8) \quad {}^0H_{LP}^k(M, \Pi) \approx H^k({}^0\mathcal{L}(M)) / \sigma(\mathcal{V}^{k-1,0}(M)).$$

Hence, the second step will have to consist of an analysis of $H^k({}^0\mathcal{L}(M))$, which can be made in the same way as in step 1, and resulting in a formula similar to (2.7), and so on.

For $k = 1$, we get easily

$$(2.9) \quad {}^0H_{LP}^1(M, \Pi) = \{X \in \mathcal{V}^{0,1}(M) / \sigma'' X = 0\} / \sigma''(\mathcal{V}^0(M)).$$

For $k = 2$, we have first

$$(2.10) \quad H^2({}^0\mathcal{L}(M)) = \frac{\{Q^{1,1} + Q^{0,2} / \sigma'' Q^{1,1} = 0, \sigma'' Q^{0,2} + \sigma' Q^{1,1} = 0\}}{\{\sigma'' X^{0,1}\}},$$

and the analysis which gave (2.7) now yields

$$(2.11) \quad H^2({}^0\mathcal{L}(M)) \approx {}''H^2(\mathcal{L}^{0,*}(M)) \oplus \mathcal{V}_{0(1)}^{1,1}(M),$$

where $\mathcal{L}^{0,*}(M) = \bigoplus_k \mathcal{V}^{0,k}(M)$, and ${}''H$ is its cohomology with respect to σ'' , and

$$(2.12) \quad \mathcal{V}_{0(1)}^{1,1}(M) = \{Q^{1,1} / \sigma'' Q^{1,1} = 0 \text{ and } \sigma' Q^{1,1} = \sigma''\text{-exact}\}.$$

(We shall see in Section 3 that, if the foliation \mathcal{S} is either transversally Riemannian or transversally symplectic, then

$${}''H^i(\mathcal{L}^{0,*}(M)) \approx H^i(M, \Phi^0(\mathcal{S})),$$

where $\Phi^0(\mathcal{S})$ is the sheaf of germs of functions which are constant along the leaves of \mathcal{S} .) Summing up the results we get

$$(2.13) \quad H_{LP}^2(M, \Pi) \approx ({}''H^2(\mathcal{L}^{0,*}(M)) \oplus ((\mathcal{V}_{0(1)}^{1,1}(M)) / \sigma(\mathcal{V}^{1,0}(M)))) \oplus \mathcal{V}_{0(2)}^{2,0}(M),$$

Etc.

3. The spectral sequence.

In this section we continue to refer to a regular Poisson manifold (M, Π) , and use the notation introduced in Section 2, while we are focussing on the spectral sequence $E_r^{p,q}(M, \Pi)$ defined at the end of Section 1. We have :

PROPOSITION 3.1. — *The first terms of the LP Serre-Hochschild spectral sequence of a regular Poisson manifold (M, Π) are given by*

$$(3.1) \quad \begin{aligned} E_0^{pq}(M, \Pi) &= E_1^{pq}(M, \Pi) = \mathcal{V}^{q,p}(M), \\ E_2^{pq}(M, \Pi) &= H^p(\oplus \mathcal{V}^{q,*}, \sigma''). \end{aligned}$$

The reader can prove this by noticing that the h -filtering subcomplex of $\mathcal{L}(M)$ as defined in Section 1 is equal to $S_h(M) = \bigoplus_{i \geq h} \bigoplus_p \mathcal{V}^{p,i}(M)$, and then following the usual definition of E_r^{pq} . Here, we just prefer to observe that $\{\mathcal{L}(M) = \bigoplus \mathcal{W}^{i,j}(M), \sigma = \Sigma d_{hk}\}$, where $\mathcal{W}^{i,j}(M) = \mathcal{V}^{j,i}(M)$, and the terms of σ are $d_{01} = 0, d_{10} = \sigma'', d_{2,-1} = \sigma'$, is a double semipositive cochain complex in the sense of [V1], p. 76-77, and then (3.1) follows from this reference.

Now, let G be a metric of the vector bundle \mathcal{S}'^* of Section 2, and let $\tilde{\#} \stackrel{\text{def}}{=} \#_G \oplus \# : \mathcal{S}'^* \oplus T^*\mathcal{S} \rightarrow \mathcal{S}' \oplus T\mathcal{S}$ be the corresponding musical isomorphism also extended to $\Lambda^k(M) \rightarrow \mathcal{V}^k(M)$. Then, if λ is a differential form of type (p, q) , $\lambda^{\tilde{\#}}$ is a multivector of the same type, and we have

$$(3.2) \quad \begin{aligned} (\tilde{\#}^{-1}\sigma''\lambda^{\tilde{\#}})(X_0, \dots, X_{p-1}, Y_0, \dots, Y_q) \\ = (-1)^{q+1}(\sigma''\lambda^{\tilde{\#}})(\#_G^{-1}X_0, \dots, \#_G^{-1}X_{p-1}, \#^{-1}Y_0, \dots, \#^{-1}Y_q). \end{aligned}$$

In this relation, and in the sequel, we agree that type $X = (1, 0)$ and type $Y = (0, 1)$. Furthermore, in order to compute $\sigma''\lambda^{\tilde{\#}}$ by (2.2) we establish first

$$\{\#^{-1}Y_i, \#^{-1}\tilde{Y}_j\}^{\#} = \{\#^{-1}Y_i, \#^{-1}Y_j\}^{\#} = [Y_i, Y_j]$$

(remember that $\{\alpha, \beta\}^{\#} = [\alpha^{\#}, \beta^{\#}]$ [BV]), and using (1.1))

$$\{\#_G^{-1}X_i, \#^{-1}Y_j\}(X) = - (L_{Y_j}G^*)(X_i, X) - G^*([Y_j, X_i], X),$$

where G^* is the dual metric of G on \mathcal{S}' . If these formulas are used, and the result is compared with the formula of the \mathcal{S} -leafwise exterior differential d_f [V1], p. 184, one gets

$$(3.3) \quad \begin{aligned} (\tilde{\#}^{-1}\sigma''\lambda^{\tilde{\#}})(X_0, \dots, X_{p-1}, Y_0, \dots, Y_q) \\ = - (d_f\lambda)(X_0, \dots, X_{p-1}, Y_0, \dots, Y_q) \\ + \sum_{i=0}^{p-1} \sum_{j=0}^q (-1)^{p+i+j} \lambda([(L_{Y_j}G^*)(X_i, \cdot)]^{\#G}, \\ X_0, \dots, \hat{X}_i, \dots, X_{p-1}, Y_0, \dots, \hat{Y}_j, \dots, Y_q). \end{aligned}$$

Remark. – The same result holds if G is a symplectic structure on \mathcal{S}'^* .

This computation leads to

PROPOSITION 3.2. – *If the symplectic foliation \mathcal{S} of the regular Poisson manifold (M, Π) is either transversally Riemannian or transversally symplectic, one has*

$$(3.4) \quad E_2^{pq}(M, \Pi) = E_1^{pq}(\mathcal{S}) = H^p(M, \Phi^q(\mathcal{S}))$$

where $E_r^{pq}(\mathcal{S})$ is the spectral sequence of the foliation \mathcal{S} (e.g., [KT]), and $\Phi^q(\mathcal{S})$ is the sheaf of germs of \mathcal{S} -foliated q -forms of M (e.g., [V1]). Particularly, (3.4) holds if \mathcal{S} is a fibration.

Indeed, under the hypotheses, $L_Y G = 0$ in (3.3), and in view of (3.1) we get an isomorphism $E_2^{pq}(M, \Pi) = H^p(\oplus \Lambda^{q,*}(M), d_f)$. But then (3.4) is known [V1], p. 216, 222, 77. (Remember that an \mathcal{S} -foliated q -form is a q -form which, locally, is the pull-back of a form of a local transversal manifold of the foliation \mathcal{S} .)

Now, let us define an interesting special class of Poisson manifolds. A vector field V of M is \mathcal{S} -foliated if it sends leaves to leaves or, equivalently, $\forall Y \in T\mathcal{S}, [V, Y] \in T\mathcal{S}$. For instance, this happens if V is an infinitesimal automorphism of Π i.e., $L_V \Pi = 0$, a condition which is easily seen to be equivalent to each of the following two conditions, where $f, g \in C^\infty(M)$,

$$(3.5) \quad V\{f, g\} = [V, X_f](g) - [V, X_g](f),$$

$$(3.6) \quad [V, X_f] = X_{V(f)}.$$

A regular Poisson structure Π of M will be called *transversally constant* if \mathcal{S} has a transversal distribution \mathcal{S}' such that every local foliate vector field $V \in \mathcal{S}'$ is a local infinitesimal automorphism of Π . For instance, if $M = S \times N$, and Π is defined by a symplectic structure of S , the distribution $\mathcal{S}' = TN$ has this property. Particularly, the existence of the local canonical coordinates of Π in the sense of [L] p. 256-257, shows that every regular Poisson manifold is locally transversally constant. Another example is the *Dirac bracket* defined as follows. Let (M, ω) be a symplectic manifold endowed with a foliation \mathcal{F} such that ω induces symplectic structures of its leaves. These induced structures yield a Poisson bivector Π such that $\mathcal{S}(\Pi) = \mathcal{F}$, and $\{ , \}_\Pi$ is the Dirac bracket of (M, ω, \mathcal{F}) . It follows that every \mathcal{F} -foliate vector

field V which is ω -orthogonal to \mathcal{F} is an infinitesimal automorphism of Π . Indeed, for such V , (3.5) is equivalent to $(L_V\omega)(X_f, X_g) = 0$, and this is an easy consequence of $d\omega = 0$. Using this definition, we have

PROPOSITION 3.3. — *If Π is transversally constant, $\sigma' = 0$, and*

$$(3.7) \quad H_{LP}^k(M, \Pi) = \bigoplus_{k=0}^q E_2^{k-q, q}(M, \Pi).$$

Proof. — Of course, the proposition refers to σ' of (2.1) taken with respect to the distribution \mathcal{S}' involved in the definition of a transversally constant Poisson structure. Let us use the notation of (2.1), and evaluate there $\{\beta_i, \beta_j\}'_p(X_p)$ ($p \in M, X_p \in \mathcal{S}'_p$). This may be done by extending X_p to a local foliate (1,0)-vector field X , and using (2.3). Since Π is transversally constant, $L_X\Pi = 0$ and we get $\sigma' = 0$. Then, (3.7) follows from (3.1). Q.e.d.

We shall finish by giving various corollaries of Propositions 3.1, 3.2, 3.3.

COROLLARY 3.1. — *If (M, Π) is a transversally constant Poisson manifold whose symplectic foliation is either transversally Riemannian or transversally symplectic, one has*

$$(3.8) \quad H_{LP}^k(M, \Pi) = \bigoplus_{q=0}^k E_1^{q, k-q}(\mathcal{S}) = \bigoplus_{q=0}^k H^q(M, \Phi^{k-q}(\mathcal{S})).$$

COROLLARY 3.2. — *Let Π be a Dirac bracket of a symplectic manifold (M, ω) endowed with a leafwise symplectic foliation \mathcal{S} , and its ω -orthogonal distribution \mathcal{S}' . Assume that the bihomogeneous components of ω with respect to the decomposition $TM = \mathcal{S}' \oplus T\mathcal{S}$ are closed. Then, again, formula (3.8) holds good.*

Proof. — Being a Dirac bracket, Π is transversally constant. On the other hand, if $\omega = \omega_{(2,0)} + \omega_{(0,2)}$; the hypothesis $d\omega_{(2,0)} = 0$ implies $(L_Y\omega_{(2,0)})(X_1, X_2) = 0$ for $(Y \in T\mathcal{S}, X_{1,2} \in \mathcal{S}')$, and we see that $\omega_{(2,0)}$ defines a transversal symplectic structure of \mathcal{S} . Q.e.d.

COROLLARY 3.3 [X]. — *Let Π be the Poisson structure defined on $M = S \times N$ by a fixed symplectic structure of S , and assume that S has finite Betti numbers. Then one has*

$$(3.9) \quad H_{LP}^k(M, \Pi) = \bigoplus_{q=0}^k [H^q(S, \mathbb{R}) \otimes \Lambda^{k-q}(N)].$$

This result follows from (3.8) and from

PROPOSITION 3.4. — *Let \mathcal{F} be the foliation of $M = F \times N$ by the leaves $F \times \{x\}$ ($x \in N$), and assume that F has finite Betti numbers. Then*

$$(3.10) \quad H^q(M, \Phi^p(\mathcal{F})) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N).$$

Proof. — For $q = 0$ the result was proven in [E] by a spectral sequence argument. Generally, we have the following straightforward argument. By the foliated de Rham theorem [V1], p. 216, we have

$$(3.11) \quad H^q(M, \Phi^p(\mathcal{F})) = \frac{\ker [d_f : \Lambda^{p,q}(M) \rightarrow \Lambda^{p,q+1}(M)]}{\text{im} [d_f : \Lambda^{p,q-1}(M) \rightarrow \Lambda^{p,q}(M)]}.$$

In our case, $\Lambda^{p,q}(M)$ is isomorphic to the space $\Lambda^q(F, \Lambda^p(N))$ of $\Lambda^p(N)$ -valued q -forms on F by the mapping which sends $\lambda \in \Lambda^{p,q}(M)$ to $\tilde{\lambda} \in \Lambda^q(F, \Lambda^p(N))$ defined by

$$(\tilde{\lambda}_y(Y_1, \dots, Y_q))_x(X_1, \dots, X_p) = (-1)^p \lambda_{(x,y)}(X_1, \dots, X_p, Y_1, \dots, Y_q),$$

$y \in F$, $x \in N$, $Y_i \in T_y F$, $X_j \in T_x N$. Moreover, this isomorphism sends d_f to the exterior differential of $\Lambda^p N$ -valued forms. Hence (3.11) becomes

$$H^q(M, \Phi^p(\mathcal{F})) = H^q(F, \Lambda^p(N)) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N),$$

where the last equality follows from the hypothesis on F . Q.e.d.

Remark. — If $M = S \times N$ of Corollary 3.3 is given a Poisson structure Π which has the symplectic foliation $S \times \{x\}$ ($x \in N$), but where each leaf has a different symplectic structure (e.g., the structure studied in [X]), Π is no more transversally constant, but we may use Propositions 3.2. and 3.4, and get

$$(3.12) \quad E_2^{pq}(M, \Pi) = H^p(S, \mathbb{R}) \otimes \Lambda^q(N).$$

COROLLARY 3.4. — *Let (M, Π) be an arbitrary regular Poisson manifold. Then every $x \in M$ has a connected open neighbourhood Y such that*

$$(3.13) \quad H_{LF}^k(U, \Pi|_U) = \Gamma(\Phi^k(\mathcal{S}|_U)),$$

i.e., the space of the \mathcal{S} -foliated k -forms over U .

Indeed, we may take $U = S \times N$ where S is contractible, and such that the product coordinates are canonical for Π in the sense of [L], p. 257. Then Corollary 3.3 holds on U , and we get (3.13). We shall say that such a neighbourhood U is LP-simple.

COROLLARY 3.5 (*The LP Poincaré Lemma* [L]). — *Let (M, Π) be a regular Poisson manifold, and $x \in M$. Then, there exists an open neighbourhood U of x in M such that, if $Q \in \mathcal{V}^k(U)$ and $\sigma Q = 0$, one has $Q = A + \sigma B$ for some $B \in \mathcal{V}^{k-1}(U)$ and a k -vector field A over U which is projectable to a k -vector field of a local transversal submanifold of \mathcal{S} in U .*

Proof. — Take U LP-simple, and with Π -canonical coordinates. The latter define a bigrading, and we may write $Q = \sum_{p=0}^k (\lambda^{p,k-p})^{\tilde{\#}}$, where λ are differential forms, and $\tilde{\#}$ is like in (3.2). The use of the canonical coordinates makes $\Pi|_U$ transversally constant and transversally Riemannian hence, by Proposition 3.3 and formula (3.3), $\sigma = \sigma''$, and $\sigma Q = 0$ is equivalent to $d_f \lambda^{p,k-p} = 0$ ($k=0, \dots, p$). But d_f satisfies a local Poincaré lemma [V1], p. 215, hence, there are local forms μ such that $\lambda^{p,k-p} = d_f \mu^{p,k-p-1}$ for $k-p > 0$, while $\lambda^{k,0}$ is a foliate form. The conclusion follows by using again (3.3). Q.e.d

BIBLIOGRAPHY

- [BV] K. H. BHASKARA and K. VISWANATH, Poisson algebras and Poisson manifolds, Pitman Research Notes in Math., 174, Longman Sci., Harlow and New York, 1988.
- [BT] R. BOTT and L. W. TU, Differential forms in algebraic topology, Graduate Texts in Math., 82, Springer-Verlag, New York, Heidelberg, Berlin, 1982.
- [E] A. EL KACIMI ALAOU, Sur la cohomologie feuilletée, *Composition Math.*, 49 (1983), 195-215.
- [F] D. B. FUKS, Cohomology of infinite dimensional Lie algebras, Consultants Bureau, New York and London, 1986.
- [G] D. GUTKIN, Variétés bistructurées et opérateurs de récursion, *Ann. Inst. H. Poincaré*, 43 (1985), 349-357.
- [H] J. HUEBSCHMANN, Poisson cohomology and quantization, *J. für Reine und Angew. Math.*, 408 (1990), 57-113.
- [K] J. L. KOSZUL, Crochet de Schouten — Nijenhuis et cohomologie, In : *E. Cartan et les mathématiques d'aujourd'hui*, Soc. Math. de France, Astérisque, hors série, (1985), 257-271.

- [KT] F. KAMBER and Ph. TONDEUR, *Foliations and metrics*, Progress in Math., 32, Birkhäuser, Boston, 1983, 103-152.
- [L] A. LICHNEROWICZ, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Diff. Geometry, 12 (1977), 253-300.
- [LT] J. A. A. LÓPEZ and Ph. TONDEUR, *Hodge decomposition along the leaves of a Riemannian foliation*, Preprint, Urbana-Illinois, 1989.
- [V1] I. VAISMAN, *Cohomology and differential forms*, M. Dekker Inc., New York, 1973.
- [V2] I. VAISMAN, *On the geometric quantization of Poisson manifolds*, Preprint, Haifa, 1990.
- [VK] Yu. M. VOROB'EV and M. V. KARASEV, *Poisson manifolds and their Schouten bracket*, Funct. Analysis and its Applications, 22(1) (1988), 1-9.
- [X] P. XU, *Poisson cohomology of regular Poisson manifolds*, Preprint, Berkeley, 1990.

Manuscrit reçu le 6 novembre 1990.

Izu VAISMAN,
Department of Mathematics
University of Haifa
Mount Carmel
Haifa 31 905 (Israël).