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A NOTE ON HOWE'S OSCILLATOR SEMIGROUP

by Joachim HILGERT

0. Introduction.

In [KMS] Kramer, Moshinski and Seligman showed that it is possible to extend the projective representation of $\mathrm{Sp}(1,\mathbb{R})$ on the Bargmann-Fock-space defined by the uniqueness of the canonical commutator relations to a subsemigroup with interior in $\mathrm{Sp}(1,\mathbb{C})$ and applied this representation to the nuclear cluster model (see also [K]). The corresponding analytic extension for the symplectic groups of arbitrary dimension was described in [BrK]. Later Brunet [Br] proved that the projective representation can be "integrated" to a contractive representation of a double covering semigroup of the aforementioned complex semigroup. The Shilov boundary of this covering semigroup is the metaplectic group and the representation restricts to the metaplectic representation. Starting from integral operators with Gaussian kernels on $L^2(\mathbb{R}^n)$ Howe constructed in [How2] a semigroup of contractions whose closure contained the metaplectic representation and applied his semigroup to prove certain estimates for pseudo-differential and Fourier integral operators. In this paper we show that the two semigroups are isomorphic via the standard isometry between $L^2(\mathbb{R}^n)$ and the Bargmann-Fock space (*cf.* [Ba2]) and give detailed information on the relation between both constructions. It should be mentioned here that analytic continuations of the type described have been treated in a more general and abstract manner by Ol'shanskii [Ol'1]. For further appearances of related semigroups see [LM], [OlaØ] and [S].

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1. Contraction semigroups and the operator Cayley transform.

Let V be a complex vector space and $B : V \times V \to \mathbb{C}$ a non-degenerate Hermitian form. We consider the semigroup

(1.1) $$S_B = \{g \in \mathrm{Gl}(V) : B(gv, gv) \leq B(v,v), \forall v \in V\}$$

of B–contractions. Its tangent wedge $\mathbf{L}(S_B) = \{x \in \mathrm{gl}(V) : e^{\mathbf{R}^+ x} \subseteq S_B\}$ is then given by (*cf.* [HilHofL])

(1.2) $$\mathbf{L}(S_B) = \{x \in \mathrm{gl}(V) : B(xv, v) + B(v, xv) \leq 0\} .$$

Note that the interior S_B^o of S_B is given by (1.1) with $\leq$ replaced by $<$. Consider the *operator Cayley transform* defined by

$$c_{\mathrm{op}}(x) = (x+1)(x-1)^{-1}$$

whenever the inverse of $x - 1$ exists. We note that $(x+1)(x-1)^{-1} - 1 = (x+1)(x-1)^{-1} - (x-1)(x-1)^{-1} = 2(x-1)^{-1}$ so that we can apply the Cayley transform twice.

1.1. *Remark.* — Set $D_c = \{x \in \mathrm{gl}(V) : \det(x-1) \neq 0\}$.

(i) $c_{\mathrm{op}}^2 : D_c \to D_c$ is the identity.

(ii) $S_B^o \subseteq D_c$.

Proof. — (i) is an elementary calculation and for (ii) we note that, because of $B(gv, gv) < B(v,v)$ for *all* $v \in V$, the transformation g cannot have the eigenvalue 1. □

1.2. PROPOSITION. — *$c_{\mathrm{op}} : \mathbf{L}(S_B) \cap D_c \to S_B \cap D_c$ is a bijection.*

Proof. — Let $v = (x-1)w$ be an arbitrary element of V. Then

$$\begin{aligned} B(c_{\mathrm{op}}(x)v, c_{\mathrm{op}}(x)v) &= B((x+1)(x-1)^{-1}v, (x+1)(x-1)^{-1}v) \\ &= B((x+1)w, (x+1)w) \\ &= B(xw, xw) + B(w,w) + B(xw, w) + B(w, xw) \\ &\leq B(w,w) + B(xw, xw) - B(xw, w) - B(w, xw) \\ &= B((x-1)w, (x-1)w) \\ &= B(v,v) \ . \end{aligned}$$

Conversely, let $g \in S_B \cap D_c$ then $c_{\mathrm{op}}(g) \in D_c$. If $x = c_{\mathrm{op}}(g)$ is not contained in $\mathbf{L}(S_B)$ then there exists a $w \in V$ such that $B(w, xw) + B(xw, w) > 0$. Therefore the calculation above with $v = (x-1)w$ shows that $B(gv, gv) > B(v,v)$ because of $g = (x+1)(x-1)^{-1}$. Thus g cannot be in S_B. □

2. Gauss functions in $L^2(\mathbb{R}^n)$ and $\mathcal{F}_n$.

We call a function on $\mathbb{R}^n$ a function of *Gaussian type* if it is of the form $\xi \mapsto e^{-\frac{1}{2}\xi^t A\xi}$ where A is a symmetric complex matrix. It is integrable if the real part of A is positive definite. We call a function of Gaussian type a *Gaussian function* if it is integrable or, equivalently, if the real part of A is positive definite, *i.e.* if A belongs to the *generalized Siegel upper halfplane* S_n.

Similarly we call a function on $\mathbb{C}^n$ of *Gaussian type* if it is of the form $\zeta \mapsto e^{-\frac{1}{2}\zeta^t A\zeta}$ where A is a symmetric complex matrix. Note that such functions are holomorphic on all of $\mathbb{C}^n$. We will call a function of Gaussian type on $\mathbb{C}^n$ a *Gaussian function* if it belongs to the Bargmann-Fock Hilbert space $\mathcal{F}_n$ of entire functions on $\mathbb{C}^n$ with the L^2-norm given by the measure $d\mu(\zeta) = \pi^{-n}e^{\overline{\zeta}^t\zeta}d\zeta$. We will determine those functions of Gaussian type on $\mathbb{C}^n$ which are Gaussian functions using the isometry $U\ :\ L^2(\mathbb{R}^n) \to \mathcal{F}_n$ given by (*cf.* [Ba1,2])

$$Uf(\zeta) = \int_{\mathbb{R}^n} U(\zeta,\xi)f(\xi)d\xi\ ,$$

where

$$U(\zeta,\xi) = \pi^{-\frac{n}{4}}e^{-\frac{1}{2}(\zeta^2+\xi^2)+\sqrt{2}\zeta^t\xi}\ . \tag{2.1}$$

2.1. LEMMA. — *Let $B \in S_n$ and $f = e^{-\frac{1}{2}\xi^t B\xi} \in L^2(\mathbb{R}^n)$ be the associated Gaussian function. Then $Uf(\zeta) = \pi^{\frac{n}{4}}(\det \frac{B+1}{2})^{-\frac{1}{2}}e^{-\frac{1}{2}\zeta^t c_{\mathrm{op}}(B)^{-1}\zeta}$.*

Proof.

$$\begin{aligned}
Uf(\zeta) &= \int_{\mathbb{R}^n} U(\zeta,\xi)e^{-\frac{1}{2}\xi^t B\xi}d\xi \\
&= \pi^{-\frac{n}{4}}\int_{\mathbb{R}^n} e^{-\frac{1}{2}(\zeta^2+\xi^2)+\sqrt{2}\zeta^t\xi-\frac{1}{2}\zeta^t B\xi}d\xi \\
&= \pi^{-\frac{n}{4}}\int_{\mathbb{R}^n} e^{-\frac{1}{2}\zeta^2}e^{-(\frac{1}{2}(\xi^t B\xi+\xi^t\xi)+\sqrt{2}\zeta^t\xi)}d\xi \\
&= \pi^{-\frac{n}{4}}\int_{\mathbb{R}^n} e^{-\frac{1}{2}\zeta^t\zeta}e^{-(\xi^t\frac{(B+1)}{2}\xi+\sqrt{2}\zeta^t\xi)}d\xi \\
&= \pi^{-\frac{n}{4}}e^{-\frac{1}{2}\zeta^t\zeta}\int_{\mathbb{R}^n} e^{-\pi\xi^t\frac{(B+1)}{2\pi}\xi+2\pi\tau^t\xi}d\xi|\text{ set }\tau = \frac{1}{\pi\sqrt{2}}\zeta \\
&= \frac{\pi^{-\frac{n}{4}}}{(\det\frac{(B+1)}{2\pi})^{\frac{1}{2}}}e^{-\frac{1}{2}\zeta^t\zeta}e^{\pi\tau^t(\frac{(B+1)}{2\pi})^{-1}\tau}|\text{ by [How2], 1.3.4} \\
&= \frac{\pi^{\frac{n}{4}}}{(\det\frac{(B+1)}{2})^{\frac{1}{2}}}e^{-\frac{1}{2}\zeta^t c_{\mathrm{op}}(B)^{-1}\zeta}\ .
\end{aligned}$$

□

Let Ω_m be the Siegel domain consisting of complex symmetric $m \times m-$matrices X with $X^*X < 1$.

2.2. LEMMA. — *$\alpha : X \mapsto c_{\rm op}(X)^{-1}$ is a bijection from S_m to Ω_m with inverse $Y \mapsto -c_{\rm op}(Y)$.*

Proof. — Note first that X and $Y = c_{\rm op}(X)^{-1}$ are symmetric so that $Y^* = \overline{Y} = c_{\rm op}(\overline{X})^{-1}$ and we can calculate

$$\begin{aligned} Y^*Y - 1 &= (\overline{X}-1)(\overline{X}+1)^{-1}(X-1)(X+1)^{-1} - 1 \\ &= (\overline{X}+1)^{-1}(\overline{X}-1)(X-1)(X+1)^{-1} - 1 \\ &= -(\overline{X}+1)^{-1}(4\,\mathrm{Re}\,X)(X+1)^{-1} \\ &= -4((X+1)^{-1})^*\,\mathrm{Re}\,X(X+1)^{-1} \end{aligned}$$

which proves that $Y^*Y - 1$ is negative definite. Similarly we calculate

$$\begin{aligned} 2\,\mathrm{Re}\,X &= (Y+1)(1-Y)^{-1} + (\overline{Y}+1)(1-\overline{Y})^{-1} \\ &= (1-\overline{Y})^{-1}(1+\overline{Y}) + (1+Y)(1-Y)^{-1} \\ &= (1-\overline{Y})^{-1}((1+\overline{Y})(1-Y) + (1-\overline{Y})(1+Y))1-Y)^{-1} \\ &= (1-\overline{Y})^{-1}(1-\overline{Y}Y)(1-Y)^{-1} \end{aligned}$$

which now proves the lemma. □

3. Gauss kernel operators on $L^2(\mathbb{R}^n)$ and $\mathcal{F}_n$.

Let S_{2n} be the Siegel upper halfplane of complex symmetric $2n \times 2n-$matrices with positive definite real part and let

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$$

be an element of S_{2n}. Then we set

$$K_X(\xi,\eta) = e^{-\frac{1}{2}(\xi^t A\xi + 2\xi^t B\eta + \eta^t D\eta)} = e^{-\frac{1}{2}v^t X v}\ , \tag{3.1}$$

where $v^t = (\xi^t, \eta^t)$. The corresponding kernel operator

$$f \longmapsto (\xi \longmapsto \int_{\mathbf{R}^n} K_X(\xi,\eta) f(\eta) d\eta)$$

will be denoted by T_X , *i.e.* we have

$$T_X f(\xi) = \int_{\mathbf{R}^n} e^{-\frac{1}{2}(\xi^t A\xi + 2\xi^t B\eta + \eta^t D\eta)} f(\eta) d\eta\ . \tag{3.2}$$

We want describe how the Gauss kernel operators behave under the transformation $U : L^2(\mathbb{R}^n) \to \mathcal{F}_n$. Thus we consider a fixed T_X and calculate the operator $T = U \circ T_X \circ U^{-1}$. Note that $\mathcal{F}_n$ has a reproducing kernel, so that T has to be a kernel operator. We are going to determine this kernel. In fact we claim that this kernel will be a Gauss kernel whose matrix is the inverse of the Cayley transform of X. The first thing to do is to describe what we mean by a Gauss kernel operator on $\mathcal{F}_n$.

Let

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$$

be an element of Ω_{2n}. Then we set

$$K_X(\zeta, \overline{\omega}) = e^{-\frac{1}{2}(\zeta^t A\zeta + 2\zeta^t B\overline{\omega} + \overline{\omega}^t D\overline{\omega})} = e^{-\frac{1}{2}v^t X v}\ , \tag{3.3}$$

where $v^t = (\zeta^t, \overline{\omega}^t)$. The corresponding kernel operator

$$f \longmapsto (\zeta \longmapsto \int_{\mathbb{C}^n} K_X(\zeta, \omega) f(\omega) d\mu(\omega))$$

will be denoted by T_X, *i.e.* we have

$$T_X f(\zeta) = \int_{\mathbb{C}^n} e^{-\frac{1}{2}(\zeta^t A\zeta + 2\zeta^t B\overline{\omega} + \overline{\omega}^t D\overline{\omega})} f(\omega) d\mu(\omega)\ . \tag{3.4}$$

This motivates the following definition: Let $U_{n,n} : L^2(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n \times \overline{\mathbb{C}^n})$ be given by

(3.5)

$$U_{n,n}F(\zeta, \overline{\omega}) = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-\frac{1}{2}(\zeta^2 + \overline{\omega}^2 + \xi^2 + \eta^2) + \sqrt{2}(\zeta^t \xi + \overline{\omega}^t \eta)} F(\xi, \eta) d\xi d\eta\ .$$

As in the case of U one shows that $U_{n,n}$ is a unitary operator.

3.1. LEMMA. — *$U_{n,n}F = U \circ F \circ U^{-1}$, where we identify the function F with the associated integral operator.*

Proof. — Recall the functions $\chi_a(\xi) = \overline{U(a, \xi)}$ and $e_a = U\chi_a$ which is given by $e_a(\zeta) = e^{\overline{a}^t \zeta}$. It suffices to show that $U \circ F(\chi_a) = U_{n,n}F(e_a)$. For the left hand side we calculate :

$$\begin{aligned} U \circ F(\chi_a)(\zeta) &= \int_{\mathbb{R}^n} U(\zeta, \xi) F(\chi_a)(\xi) d\xi \\ &= \int_{\mathbb{R}^n} U(\zeta, \xi) \int_{\mathbb{R}^n} F(\xi, \eta) \chi_a(\eta) d\eta d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \pi^{-\frac{n}{2}} e^{-\frac{1}{2}(\zeta^2 + \xi^2)} e^{-\frac{1}{2}(\bar{a}^2 + \eta^2)} e^{\sqrt{2}(\zeta^t \xi + \bar{a}^t \eta)} F(\xi, \eta) d\eta d\xi. \end{aligned}$$

For the right hand term we have

$$\begin{aligned} U_{n,n}F(e_a)(\zeta) &= \int_{\mathbb{C}^n} (U_{n,n}F)(\zeta,\overline{\omega})e_a(\omega)d\omega \\ &= \int_{\mathbb{C}^n}\int_{\mathbf{R}^n\times\mathbf{R}^n} \pi^{-\frac{n}{2}} e^{-\frac{1}{2}(\zeta^2+\overline{\omega}^2+\xi^2+\eta^2)+\sqrt{2}(\zeta^t\xi+\overline{\omega}^t\eta)} \\ &\qquad\qquad F(\xi,\eta)e^{\bar{a}^t\omega}d\mu(\omega)d\eta d\xi \\ &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \pi^{-\frac{n}{2}} e^{-\frac{1}{2}(\zeta^2+\xi^2)} e^{-\frac{1}{2}(\bar{a}^2+\eta^2)} e^{\sqrt{2}(\zeta^t\xi+\bar{a}^t\eta)} F(\xi,\eta)d\eta d\xi, \end{aligned}$$

since it follows from [Ba2], (1.6b) that

$$\int_{\mathbb{C}^n} e^{-\frac{1}{2}\overline{\omega}^2+\sqrt{2}\overline{\omega}^t\eta+\bar{a}^t\omega}d\mu(\omega) = \overline{e^{-\frac{1}{2}a^2+\sqrt{2}a^t\eta}}\ .$$

□

Now we define $\widetilde{U} : \mathcal{B}(L^2(\mathbb{R}^n)) \to \mathcal{B}(\mathcal{F}_n)$ ($\mathcal{B}$ means bounded operator) by

$$\widetilde{U}(T) = U \circ T \circ U^{-1}\ . \tag{3.6}$$

3.2. Proposition. — *Let $X \in S_{2n}$ be of the form*

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}\ .$$

Then we have

$$\widetilde{U}(T_X) = \frac{(4\pi)^{\frac{n}{2}}}{\det(X+1)^{\frac{1}{2}}} T_{c_{\mathrm{op}}(X)^{-1}}\ .$$

Proof. — We can apply Lemma 2.1 with $L^2(\mathbb{R}^{2n})$ and $U_{n,n}$ instead of $L^2(\mathbb{R}^n)$ and U to find

$$U_{n,n}(K_X) = \frac{(4\pi)^{\frac{n}{2}}}{\det(X+1)^{\frac{1}{2}}} K_{c_{\mathrm{op}}(X)^{-1}}\ .$$

From Lemma 3.1 we know that the following diagram is commutative

$$\begin{array}{ccc} L^2(\mathbb{R}^n) & \xrightarrow{T_X} & \mathbf{L}^2(\mathbb{R}^n) \\ \downarrow U & & \downarrow U \\ \mathcal{F}_n & \xrightarrow{c_X T_{c_{\mathrm{op}}(X)^{-1}}} & \mathcal{F}_n\ . \end{array}$$

Here we set $c_X = (4\pi)^{\frac{n}{2}}(\det(X+1))^{-\frac{1}{2}}$. □

Next we recall the multiplication of Gauss kernel operators :

3.3 PROPOSITION.

(i) ([How2], 3.2.2) *Let* $X, Y \in S_{2n}$

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}, \quad Y = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^t & \widetilde{D} \end{pmatrix},$$

then we have

$$T_X \circ T_Y = \frac{(2\pi)^n}{\det(D + \widetilde{A})^{\frac{1}{2}}} T_Z : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

where

$$Z = \begin{pmatrix} A - B(D + \widetilde{A})^{-1}B^t & -B(D + \widetilde{A})^{-1}\widetilde{B} \\ -\widetilde{B}^t(D + \widetilde{A})^{-1}B^t & \widetilde{D} - \widetilde{B}^t(D + \widetilde{A})^{-1}\widetilde{B} \end{pmatrix}.$$

(ii) ([BrK], 3.6) *Let* $X, Y \in S_{2n}$

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}, \quad Y = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^t & \widetilde{D} \end{pmatrix},$$

then we have

$$T_X \circ T_Y = \frac{1}{(\det(1 - \widetilde{A}D))^{\frac{1}{2}}} T_Z : \mathcal{F}_n \longrightarrow \mathcal{F}_n$$

where

$$Z = \begin{pmatrix} A + (B(1 - \widetilde{A}D)^{-1}\widetilde{A}B^t)^s & -B(1 - \widetilde{A}D)^{-1}\widetilde{B} \\ -\widetilde{B}^t(1 - \widetilde{A}D)^{-1}B^t & \widetilde{D} + (\widetilde{B}^t D(1 - \widetilde{A}D)^{-1}\widetilde{B})^s \end{pmatrix}.$$

Here $C^s = \frac{1}{2}(C + C^t)$.

Proof. — This follows from [How2], 3.2.1 (watch out : misprint!) and [BrK], 3.6, respectively. □

Note that our normalization for Gaussian functions is different from Howe's. The reason for this is that the intertwining operator $U_{n,n}$ looks more complicated in Howe's normalization. We fix some notation for further use.

3.4. DEFINITION. — *A function* $F : \mathbb{R}^{2n} \to \mathbb{C}$ *is called* a real Gauss kernel *if it is of the form* $F(v) = ce^{-\frac{1}{2}v^t X v}$ *for some* $X \in S_{2n}$ *and* $c \in \mathbb{C}^\times$. *The set of all real Gauss kernels is denoted by* $GK_{\mathbb{R}}$.

A function $F : \mathbb{C}^n \times \overline{\mathbb{C}}^n \to \mathbb{C}$ *is called* a complex Gauss kernel *if it is of the form* $F(v) = ce^{-\frac{1}{2}v^t X v}$ *for some* $X \in \Omega_{2n}$ *and* $c \in \mathbb{C}^\times$ *with* $v^t = (\zeta^t, \overline{\omega}^t)$. *The set of all complex Gauss kernels is denoted by* $GK_{\mathbb{C}}$.

3.5. PROPOSITION. — *$(GK_{\mathbf{R}}, \circ)$ and $(GK_{\mathbf{C}}, \circ)$ are isomorphic semigroups, where $\circ$ is the composition of integral operators and the isomorphism is given by $U_{n,n}$.*

Proof. — This follows immediately from Lemma 3.1. $\square$

4. The Weyl transform.

In this section we recall yet another version of the semigroup $GK_{\mathbf{R}} \cong GK_{\mathbf{C}}$, this time not as a semigroup of integral operators, but of twisted convolution operators (*cf.* [How2], §7). The isomorphism will be given by the Weyl transform.

The *Weyl transform* maps Schwartz functions on $\mathbf{R}^{2n}$ the kernel operators on $L^2(\mathbf{R}^n)$ via

$$\rho(F) = T_{K_{\rho(F)}}$$

where

$$K_{\rho(F)}(\xi, \eta) = \int_{\mathbf{R}^n} F(\xi - \eta, \tau) e^{\pi i (\xi+\eta)^t \tau} d\tau. \tag{4.1}$$

4.1. PROPOSITION (*cf.* [How2], 13.2). — *Let $v^t = (\xi^t, \eta^t)$ and*

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in S_{2n} .$$

Then

$$K_{\rho(F_X)} = \frac{(2\pi)^{\frac{n}{2}}}{(\det D)^{\frac{1}{2}}} K_{\widetilde{X}}$$

with

$$\widetilde{X} = \begin{pmatrix} A-(B-i\pi)D^{-1}(B^t-i\pi) & -A+(B-i\pi)D^{-1}(B^t+i\pi) \\ -A+(B+i\pi)D^{-1}(B^t-i\pi) & A-(B+i\pi)D^{-1}(B^t+i\pi) \end{pmatrix} \in S_{2n} .$$

Proof. — This follows immediately from [How2], 13.2 if one takes into account the change of normalization. $\square$

We denote the map $X \mapsto \widetilde{X}$ by $\widetilde{\rho} : S_{2n} \to S_{2n}$.

4.2. PROPOSITION (*cf.* [How2], §7 and [How1]). — *Let $S(\mathbf{R}^{2n})$ be the space of Schwartz functions on $\mathbf{R}^{2n}$ then*

$$\rho : (S(\mathbf{R}^{2n}), *_{tw}) \to (S(\mathbf{R}^{2n}), \circ)$$

is an involutive algebra isomorphism, where $*_{tw}$ denotes twisted convolution, i.e.

$$F_1 *_{tw} F_2(v) = \int_{\mathbf{R}^{2n}} F_1(w)F_2(v-w)e^{-\pi i w^t J v}dw$$

with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\circ$ the composition of integral operators on $L^2(\mathbf{R}^n)$.

□

4.3. COROLLARY. — *The Weyl transform yields a canonical isomorphism $\rho : (GK_{\mathbf{R}}, *_{tw}) \to (GK_{\mathbf{R}}, \circ)$.*

□

5. The Bargmann-Brunet-Kramer realization.

In [Ba2] Bargmann gives a realization of the projective representation of the symplectic group coming from the Stone-von Neumann Theorem via kernel operators on $\mathcal{F}_n$. He does not use $\mathrm{Sp}(n, \mathbf{R})$ but the isomorphic group $G = U(n,n) \cap \mathrm{Sp}(n, \mathbf{C})$. Note that G is the set of all complex $2n \times 2n-$matrices of the form

$$g = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} ,$$

where A and B are $n \times n-$blockmatrices, which satisfy

$$AA^* - BB^* = 1$$

$$A^tB = B^tA$$

or, equivalently

$$A^*A - B^t\overline{B} = 1$$

$$A^t\overline{B} = B^*A \ .$$

From this it follows that A is invertible and that the matrices $\overline{B}A^{-1}$ and $-A^{-1}B$ are symmetric. It is shown in [Ba2], §3 that the projective representation of G on $\mathcal{F}_n$ is given by $g \mapsto F_g(\zeta, \overline{\omega})$, where

$$F_g(\zeta,\overline{\omega}) = e^{\frac{1}{2}(\zeta^t\overline{B}A^{-1}\zeta+\zeta^t(A^{-1})^t\overline{\omega}+\overline{\omega}^tA^{-1}\zeta-\overline{\omega}^tA^{-1}B\overline{\omega})} \ . \tag{5.1}$$

This means that F_g is a kernel operator of Gaussian type with matrix

$$X_g = -\begin{pmatrix} \overline{B}A^{-1} & (A^{-1})^t \\ A^{-1} & -A^{-1}B \end{pmatrix} . \tag{5.2}$$

In [BrK] Brunet and Kramer formally extend these kernels by simply replacing $\overline{B}$ by an arbitrary C and then find conditions in which the resulting kernels yield decent operators. Since we know already which kernels of Gaussian type we want (cf. Lemma 3.1), we are lead to the following lemma.

5.1. LEMMA. — *Let $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the Hermitian form given by the matrix*

$$L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and S_B the semigroup of $B-$contractions (cf. (1.1)). Then

$$\varphi\left(\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}\right) = \begin{pmatrix} -(B^t)^{-1} & -(B^t)^{-1}D \\ A(B^t)^{-1} & -B + A(B^t)^{-1}D \end{pmatrix}$$

defines a map $\varphi : \mathcal{D}_\Omega \to S_B^o$ where $\mathcal{D}_\Omega = \{X \in \Omega_{2n} : \det(B) \neq 0\}$. The map φ is invertible with inverse $\psi : S_\beta^o \to \mathcal{D}_\Omega$ given by

$$\psi\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = -\begin{pmatrix} CA^{-1} & (A^t)^{-1} \\ A^{-1} & -A^{-1}B \end{pmatrix} .$$

Proof. — The first thing to note is that for any

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_B^o$$

we have $\det(A) \neq 0$. In fact if $0 \neq \zeta \in \mathbb{C}^n$ with $A\zeta = 0$ and $v^* = (\zeta^*, 0)$ then we calculate

$$\begin{aligned} B(gv, gv) &= v^* g^* L g v \\ &= (\zeta^*, 0)\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ C\zeta \end{pmatrix} \\ &= \zeta^* C^* C \zeta \\ &\geq 0 \end{aligned}$$

and

$$B(v, v) = (\zeta^*, 0)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \zeta \\ 0 \end{pmatrix} = -\zeta^*\zeta < 0$$

which is a contradiction.

Next we show that the image of φ is contained in $\mathrm{Sp}(n,\mathbb{C})$. We write

$$\varphi(X) = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix}$$

and calculate

$$\begin{aligned} \widetilde{B}^t\widetilde{D} &= (-(B^t)^{-1}D)^t(B - A(B^t)^{-1}D) \\ &= D - DB^{-1}A(B^t)^{-1}D \\ &= (D - DB^{-1}A(B^t)^{-1}D)^t \\ &= \widetilde{D}^t\widetilde{B} \ . \end{aligned}$$

Further we have

$$\widetilde{A}^t\widetilde{C} = -B^{-1}A(B^t)^{-1} = -(B^{-1}A(B^t)^{-1})^t = \widetilde{C}^t\widetilde{A}$$

and finally

$$\begin{aligned} \widetilde{A}^t\widetilde{D} - \widetilde{C}^tD &= -B^{-1}(-B + A(B^t)^{-1}D) + B^{-1}A(B^t)^{-1}D \\ &= 1 - B^{-1}A(B^t)^{-1}D + B^{-1}A(B^t)^{-1}D \\ &= 1 \ . \end{aligned}$$

Conversely if

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_B^o$$

an easy calculation shows that CA^{-1} and $-A^{-1}B$ are symmetric matrices. Also it is straightforward to check that $\psi\circ\varphi$ is the identity on $\mathcal{D}_\Omega$. Moreover

$$\begin{aligned} \varphi\circ\psi\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) &= \varphi\left(-\begin{pmatrix} CA^{-1} & (A^t)^{-1} \\ A^{-1} & -A^{-1}B \end{pmatrix}\right) \\ &= \begin{pmatrix} A & B \\ C & (A^t)^{-1} - CA^{-1}A(-A^{-1}B) \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ C & (A^t)^{-1} + (A^t)^{-1}C^tB \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{aligned}$$

so that in order to prove the lemma it now suffices to prove that $\psi(S_B^o) \subseteq \Omega_{2n}$.

First we give a characterization of S_B^o in terms of the blockmatrix decomposition

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

We know that $g \in S_B^o$ if and only if $L - g^*Lg > 0$. This yields

$$
\begin{aligned}
0 &< \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -A^* & C^* \\ -B^* & D^* \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} C^*C - A^*A & C^*D - A^*B \\ -B^*A + D^*C & D^*D - B^*B \end{pmatrix} \\
&= \begin{pmatrix} -1 - C^*C + A^*A & -C^*D + A^*B \\ B^*A - D^*C & 1 - D^*D + B^*B \end{pmatrix} .
\end{aligned}
$$

Now let $X = \psi(g)$ and note that

$$
X = -\begin{pmatrix} CA^{-1} & (A^t)^{-1} \\ A^{-1} & -A^{-1}B \end{pmatrix} = \begin{pmatrix} (A^t)^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} -C^t & -1 \\ -1 & B \end{pmatrix} \tag{5.3}
$$

implies that $XX^* < 1$ if and only if

$$
YY^* = \begin{pmatrix} C^t & 1 \\ 1 & -B \end{pmatrix} \begin{pmatrix} C^t & 1 \\ 1 & -B \end{pmatrix}^* < \begin{pmatrix} A^t\overline{A} & 0 \\ 0 & AA^* \end{pmatrix} .
$$

Taking the complex conjugate we find that $X \in \Omega_{2n}$ if and only if

$$
\widetilde{X} = \begin{pmatrix} C^*C - A^*A + 1 & C^* - B^t \\ C - \overline{B} & 1 + \overline{B}B^t - \overline{A}A^t \end{pmatrix} < 0 .
$$

Next we perform a similarity transformation with the matrix

$$
\begin{pmatrix} 1 & 0 \\ B^*(A^*)^{-1} & (\overline{A})^{-1} \end{pmatrix}
$$

and as result

$$
\begin{pmatrix} 1 & 0 \\ B^*(A^*)^{-1} & (\overline{A})^{-1} \end{pmatrix} \widetilde{X} \begin{pmatrix} 1 & A^{-1}B \\ 0 & (A^t)^{-1} \end{pmatrix}
$$

we find the negative of the matrix $L - g^*Lg$ written above. This proves the lemma. □

5.2. PROPOSITION. — *The set $S_\Omega^\# = \{(cK_X) \in GK_\mathbb{C} : X \in \mathcal{D}_\Omega\}$ is a subsemigroup of $GK_\mathbb{C}$ and the map $\varphi : \mathcal{D}_\Omega \to S_B^o$ induces a semigroup homomorphism $\varphi : S_\Omega^\# \to S_B^o$.*

Proof. — The fact that $S_\Omega^\#$ is a subsemigroup follows from Proposition 3.3, (ii) and the second assertion is proved in [BrK], (3.8). □

Finally we are able to describe the Bargmann-Brunet-Kramer realization fo the *oscillator* or *metaplectic semigroup* [Br]).

5.3. PROPOSITION. — *The set* $S_\Omega = \{(cK_X) \in S_\Omega^\# : c^2 = \det(-B)\}$, *where* $X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$, *is a subsemigroup of* $S_\Omega^\#$ *and the semigroup homomorphism* $\varphi : S_\Omega \to S_B^o$ *is a double covering.*

Proof. — Again the semigroup property follows from Proposition 3.3, (ii) while the rest is obvious. □

6. Howes' realization.

Consider the Hermitian form $B_{\mathbf{R}}$ on $\mathbf{C}^n$ given by the matrix

$$iJ = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{6.1}$$

The subsemigroup of $\mathrm{Sp}(n, \mathbf{C})$ consisting of all elements which are contractions w.r.t. $B_{\mathbf{R}}$ will be denoted by $S_{B_{\mathbf{R}}}$. Note that it follows from (1.2) that the edge of $\mathbf{L}(S_{B_{\mathbf{R}}})$ is $\mathrm{Sp}(n, \mathbf{R})$. In fact we have

$$B_{\mathbf{R}}(Xv, v) + B_{\mathbf{R}}(v, Xv) = 2\,\mathrm{Re}(B_{\mathbf{R}}(v, Xv)) = 2\,\mathrm{Re}(iv^* JXv) .$$

6.1. LEMMA. — *The map* $\beta : \mathrm{Mat}(2n, \mathbf{C}) \to \mathrm{Mat}(2n, \mathbf{C})$ *defined by* $\beta(X) = -\frac{i}{\pi}JX$ *induces a linear isomorphism* $\beta : S_{2n} \to \mathrm{int}\,\mathbf{L}(S_{B_{\mathbf{R}}})$ *which maps the set* $\mathcal{D}_{tw} = \{X \in S_{2n} : \det(X + i\pi J) \neq 0\}$ *onto* D_c *(cf. Remark 1.1).*

Proof. — Note first that β maps symmetric matrices into $\mathrm{sp}(n, \mathbf{C})$ since $\beta(X)^t J + J\beta(X) = -\frac{i}{\pi}X^t J^t J - \frac{i}{\pi}J^2 X = \frac{i}{\pi}XJ^2 + \frac{i}{\pi}X = 0$. Conversely, if $\beta(X) \in \mathrm{sp}(n, \mathbf{C})$ then from the above we see that $-\frac{i}{\pi}X^t + \frac{i}{\pi}X = 0$, so that X is symmetric. Moreover, if X is purely imaginary then $\beta(X)$ is real, whence, in order to prove $\beta(S_{2n}) \subseteq \mathbf{L}(S_{B_{\mathbf{R}}})$, it only remains to show that $\beta(X) \in \mathbf{L}(S_{B_{\mathbf{R}}})$ for *real* $X \in S_{2n}$. We calculate

$$\begin{aligned} B_{\mathbf{R}}(\beta(X)v, v) + B_{\mathbf{R}}(v, \beta(X)v) &= 2\,\mathrm{Re}(B_{\mathbf{R}}(v, \beta(X)v)) \\ &= -2\,\mathrm{Re}(iv^* J\beta(X)v) \\ &= \frac{2}{\pi}\,\mathrm{Re}(v^* JJXv) \\ &= -\frac{2}{\pi}\,\mathrm{Re}(v^* Xv) \\ &= -\frac{2}{\pi}(v^* Xv) < 0 \end{aligned}$$

since X is positive definite. Since we have $\beta(X) \in i\,\mathrm{sp}(n,\mathbb{R})$ if and only if X is real, the above calculation also shows that X is positive definite if $\beta(X) \in \operatorname{int} \mathbf{L}(S_{B_\mathbb{R}}) \cap i\,\mathrm{sp}(n,\mathbb{R})$. The last assertion follows from $\det(-\frac{i}{\pi}JX - 1) = \det(-\frac{i}{\pi}J)\det(X + i\pi J)$. $\square$

Next we describe Howe's realization of the *oscillator* or *metaplectic semigroup*.

6.2. Proposition ([How2], §12). — *The set $S_{tw}^{\#} = \{(cK_X) \in GK_\mathbf{R} : X \in \mathcal{D}_{tw}\}$ is a subsemigroup of $(GK_\mathbf{R}, *_{tw})$ and the map $(c, X) \mapsto c_{\mathrm{op}}(-\frac{i}{\pi}JX)$ induces a semigroup homomorphism $S_{tw}^{\#} \to S_{B_\mathbb{R}}^{o}$. Moreover the set*

$$S_{tw} = \left\{(cK_X) \in S_{tw}^{\#} : c^2 = \frac{\det(X + i\pi J)}{(2\pi)^{2n}}\right\}$$

is a subsemigroup of $S_{tw}^{\#}$ and the semigroup homomorphism $c_{\mathrm{op}} \circ \beta : S_{tw} \to S_{B_\mathbb{R}}^{o}$ is a double covering.

Proof. — This follows immediately from [How2], §§8, 11 and 12 taking into account the change of normalization. $\square$

7. Intertwining operators.

The obvious question at this point is how the two realizations of Bargmann-Brunet-Kramer and Howe are related. We start by showing that the domains $\mathcal{D}_\Omega$ and $\mathcal{D}_{tw}$ are well behaved under the transformation $\alpha \circ \widetilde{\rho} : S_{2n} \to \Omega_{2n}$ (*cf.* Proposition 4.2 and Lemma 2.1).

7.1. Lemma. — *Let*

$$\mathcal{D}_0 = \left\{X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in S_{2n} : \det B \neq 0\right\}.$$

Then the maps $\alpha : S_{2n} \to \Omega_{2n}$ and $\widetilde{\rho} : S_{2n} \to S_{2n}$ induce bijections $\alpha : \mathcal{D}_0 \to \mathcal{D}_\Omega$ and $\widetilde{\rho} : \mathcal{D}_{tw} \to \mathcal{D}_0$, respectively.

Proof. — Let

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in S_{2n} \backslash \mathcal{D}_{tw}$$

and

$$\widetilde{\rho}(X) = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^t & \widetilde{D} \end{pmatrix} .$$

Then there exists a $0 \neq v \in \mathbb{C}^{2n}$ such that

$$\begin{pmatrix} A & B + i\pi \\ B^t - i\pi & D \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \tag{7.1}$$

where $v^t = (\xi^t, \eta^t)$. This can be rewritten as

$$A\xi + (B + i\pi)\eta = 0$$
$$(B^t - i\pi)\xi + D\eta = 0 .$$

But since $X \in S_{2n}$ the matrix D is invertible we have

$$\eta = -D^{-1}(B^t - i\pi)\xi$$

which implies that $\xi \neq 0$ and

$$A\xi = (B + i\pi)D^{-1}(B^t - i\pi)\xi . \tag{7.2}$$

But this simply means that $\widetilde{B}^t\xi = 0$ and hence $\widetilde{\rho}(X) \in S_{2n} \backslash \mathcal{D}_0$.

Conversely, if $\det \widetilde{B} = 0$ we find a $\xi \neq 0$ with (7.2) and setting $\eta = -D^{-1}(B^t - i\pi)\xi$ yields (7.1) with $v^t = (\xi^t, \eta^t)$ which in turn shows that $X \in S_{2n} \backslash \mathcal{D}_{tw}$.

In order to show that α has the required properties we suppose that

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in S_{2n} \backslash \mathcal{D}_0$$

and note that this implies the existence of a $0 \neq \xi \in \mathbb{C}^n$ such that $B\xi = 0$. Then we have

$$(X + 1)\begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} A + 1 & B \\ B^t & D + 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ D\xi + \xi \end{pmatrix}$$

and hence

$$\alpha(X)\begin{pmatrix} 0 \\ D\xi + \xi \end{pmatrix} = \begin{pmatrix} 0 \\ D\xi - \xi \end{pmatrix} .$$

Note that $\operatorname{Re} D > 0$ since $X \in S_{2n}$ so that $D\xi + \xi \neq 0$ which now shows that $\alpha(X) \in \Omega_{2n} \backslash \mathcal{D}_\Omega$.

Conversely, if

$$Y = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^t & \widetilde{D} \end{pmatrix} \in \Omega_{2n} \backslash \mathcal{D}_\Omega$$

then there exists a $0 \neq \xi \in \mathbb{C}^n$ such that $\widetilde{B}\xi = 0$ so that

$$(X - 1)\begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} \widetilde{A} - 1 & \widetilde{B} \\ \widetilde{B}^t & \widetilde{D} - 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{D}\xi - \xi \end{pmatrix} .$$

This time we know that $\widetilde{D}\xi - \xi \neq 0$ since $\widetilde{D} < 1$ because of $Y \in \Omega_{2n}$. Therefore

$$c_{\mathrm{op}}(Y)\begin{pmatrix} 0 \\ D\xi - \xi \end{pmatrix} = \begin{pmatrix} 0 \\ D\xi + \xi \end{pmatrix}$$

proves the claim. □

We want to show that $U_{n,n} \circ \rho$ maps S_{tw} bijectively onto S_Ω . To do that we introduce "intermediate" semigroups in $(GK_{\mathbf{R}}, \circ)$.

7.2. PROPOSITION. — *The sets* $S_0^{\#} = \{(cK_X) \in GK_{\mathbf{R}} : X \in \mathcal{D}_0\}$ *and*

$$S_0 = \left\{(cK_X) \in S_0^{\#} : c^2 = \det(-\frac{B}{2\pi})\right\} ,$$

where

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} ,$$

are subsemigroups of $(GK_{\mathbf{R}}, \circ)$.

Proof. — This follows from Proposition 3.3, (i). □

7.3. LEMMA. — *Let*

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with D invertible then

$$\det X = \det D \det(A - BD^{-1}C) .$$

Proof. — (*cf.* [BrK], 2.3.4)

$$\begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

□

7.4. LEMMA. — *Let*

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in S_{2n}$$

and

$$(c_{\mathrm{op}}X)^{-1} = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^t & \widetilde{D} \end{pmatrix} .$$

If now

$$(X+1)^{-1} = \begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{B}^t & \widehat{D} \end{pmatrix} ,$$

then we have

(i) $\widetilde{B} = -2\widehat{B}$;

(ii) $\widehat{A} = ((A+1) - B(1+D)^{-1}B^t)^{-1}$;

(iii) $\widehat{A}B = \widehat{B}(D+1) = 0$.

In particular we have

$$\det(X+1) = (\det \widehat{A})^{-1} \det(D+1) \ .$$

Proof.

$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^t & \widetilde{D} \end{pmatrix} = \begin{pmatrix} A-1 & B \\ B^t & D-1 \end{pmatrix} \begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{B}^t & \widehat{D} \end{pmatrix}$$

which shows that $\widetilde{B} = (A-1)\widehat{B} + B\widehat{D}$. Moreover

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A+1 & B \\ B^t & D+1 \end{pmatrix} \begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{B}^t & \widehat{D} \end{pmatrix}$$

shows $0 = (A+1)\widehat{B} + B\widehat{D}$ so that we find $\widetilde{B} = (A-1)\widehat{B} - (A+1)\widehat{B} = -2\widehat{B}$. Similarly we find

$$\widehat{A}(A+1) + \widehat{B}B^t = 1$$

$$\widehat{A}B + \widehat{B}(D+1) = 0$$

which implies (iii) and $\widehat{A}B(D+1)^{-1} + \widehat{B} = 0$ and hence also (ii). The last claim now follows from Lemma 7.3 applied to $X+1$. □

7.5. PROPOSITION. — *$(cK_X) \in S_0$ if and only if $U_{n,n}(cK_X) \in S_\Omega$.*

Proof. — Recall that $cK_X \in S_0$ if and only if $c^2 = \det\left(-\frac{B}{2\pi}\right)$, where

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in S_{2n} \ .$$

From Proposition 3.2 we know that

$$U_{n,n}(cK_X) = \frac{c(4\pi)^{\frac{n}{2}}}{\det(X+1)^{\frac{1}{2}}} K_{c_{\mathrm{op}}(X)^{-1}} \ .$$

Then, using Lemma 7.4, we calculate

$$\begin{aligned} \left(\frac{c(4\pi)^{\frac{n}{2}}}{\det(X+1)^{\frac{1}{2}}}\right)^2 &= \frac{\det(-\frac{B}{2\pi})(4\pi)^n}{\det(X+1)} = \frac{2^n \det(-B)\det(\widehat{A})}{\det(D+1)} \\ &= \frac{2^n \det(-\widehat{A}B)}{\det(D+1)} = 2^n \det(\widehat{B}) = \det(-\widetilde{B}) \end{aligned}$$

which proves one half of the proposition. The converse direction is proved by following the above calculation backwards. □

7.6. PROPOSITION. — *$(cK_X) \in S_{tw}$ if and only if $\rho(cK_\xi) \in S_0$.*

Proof. — Recall from Proposition 4.1 that

$$\rho(cK_\xi) = \frac{c(2\pi)^{\frac{n}{2}}}{\det(D)^{\frac{1}{2}}} K_{\widetilde{\rho}(X)}$$

where

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in S_{2n} .$$

Using Lemma 7.3 we calculate

$$\begin{aligned}\left(\frac{c(2\pi)^{\frac{n}{2}}}{\det(D)^{\frac{1}{2}}}\right)^2 &= \frac{(2\pi)^{-2n}\det(X+i\pi J)(2\pi)^n}{\det(D)} \\ &= (2\pi)^{-n}\det(A-(B+i\pi)D^{-1}(B^t-i\pi)) \\ &= \det\left(-\frac{-A+(B+i\pi)D^{-1}(B^t-i\pi)}{2\pi}\right)\end{aligned}$$

which proves one half of the proposition in view of Proposition 4.1. The converse is again obtained by following this computation backwards. □

It will turn out to be useful to consider a certain automorphism of the semigroup S_0.

7.7. LEMMA. — *The map $\theta : S_0 \to S_0$, given by*

$$\theta(cK_X) = (2\pi)^{\frac{n}{2}} cK_{2\pi X} ,$$

is an automorphism.

Proof. — In order to prove that θ is a bijection let

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and calculate

$$((2\pi)^{\frac{n}{2}}c)^2 = (2\pi)^n c^2 = (2\pi)^n \det\left(-\frac{B}{2\pi}\right) = \det\left(-\frac{2\pi B}{2\pi}\right) .$$

The multiplicativity of the map follows immediately from Proposition 3.3, (i). □

We denote the map induced on $\mathcal{D}_0$ by θ, *i.e.* the multiplication with 2π, by $\widetilde{\theta}$.

If we now collect the diverse mappings we obtain the following diagram

$$\begin{array}{ccccccccc}
(GK_{\mathbf{R}}, *_{tw}) & \supseteq & S_{tw} & \longrightarrow & \mathcal{D}_{tw} & \xrightarrow{\beta} & \operatorname{int}\mathbf{L}(S_{B_{\mathbf{R}}}) & \xrightarrow{c_{op}} & S^o_{B_{\mathbf{R}}} \\
\downarrow\rho & & \downarrow\rho & & \downarrow\widetilde{\rho} & & & & \\
(GK_{\mathbf{R}}, \circ) & \supseteq & S_0 & \longrightarrow & \mathcal{D}_0 & & & & \\
\downarrow\theta & & \downarrow\theta & & \downarrow\widetilde{\theta} & & & & \\
(GK_{\mathbf{R}}, \circ) & \supseteq & S_0 & \longrightarrow & \mathcal{D}_0 & & & & \\
\downarrow U_{n,n} & & \downarrow U_{n,n} & & \downarrow\alpha & & & & \\
(GK_{\mathbf{C}}, \circ) & \supseteq & S_\Omega & \longrightarrow & \mathcal{D}_\Omega & & \xrightarrow{\varphi} & & S^o_B\,.
\end{array}$$

8. Completing the diagram.

We want to complete the above diagram by filling in an isomorphism between $S^o_{B_{\mathbf{R}}}$ and S^o_B such that the diagram commutes. We start by calculating $\varphi \circ \alpha$.

8.1. Lemma. — *Let* $X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in \mathcal{D}_0$ *then we have*

$$(X+1)^{-1} = \begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{B}^t & \widehat{D} \end{pmatrix}$$

where

$$\begin{aligned}
\widehat{B} &= (B^t - (D+1)B^{-1}(A+1))^{-1}\,, \\
\widehat{A} &= -(B^t)^{-1}(D+1)\widehat{B}^t = -\widehat{B}(D+1)B^{-1}\,, \\
\widehat{D} &= -B^{-1}(A+1)\widehat{B} = -\widehat{B}^t(A+1)(B^t)^{-1}\,.
\end{aligned}$$

Proof.

$$\begin{pmatrix} A+1 & B \\ B^t & D+1 \end{pmatrix}\begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{B}^t & \widehat{D} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

implies

$$B^t\widehat{A} + (D+1)\widehat{B}^t = 0$$

$$(A+1)\widehat{B} + B\widehat{D} = 0$$

so that

$$\widehat{A} = -(B^t)^{-1}(D+1)\widehat{B}^t$$

$$\widehat{D} = -B^{-1}(A+1)\widehat{B}$$

whence

$$(A+1)\widehat{A}+B\widehat{B}^t=1$$

and

$$-(A+1)(B^t)^{-1}(D+1)\widehat{B}^t+B\widehat{B}^t=1$$

which proves the claim. □

8.2. LEMMA. — *Let*

$$X=\begin{pmatrix} A & B \\ B^t & D\end{pmatrix}\in\mathcal{D}_0$$

and

$$(X+1)^{-1}=\begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{B}^t & \widehat{D}\end{pmatrix}$$

then we have

$c_{\mathrm{op}}(X)^{-1}=$

$$\begin{pmatrix} 2\widehat{B} & 0 \\ 0 & 2\widehat{B}^t\end{pmatrix}\begin{pmatrix} \frac{1}{2}(B^t+(D+1)B^{-1}(1-A)) & -1 \\ -1 & \frac{1}{2}(B+(A+1)(B^t)^{-1}(1-D))\end{pmatrix}.$$

Proof.

$$\begin{aligned}(X+1)^{-1}&=\begin{pmatrix} -\widehat{B}(D+1)B^{-1} & \widehat{B} \\ \widehat{B}^t & -\widehat{B}^t(A+1)(B^t)^{-1}\end{pmatrix}\\ &=\begin{pmatrix} -\widehat{B} & 0 \\ 0 & -\widehat{B}^t\end{pmatrix}\begin{pmatrix} (D+1)B^{-1} & -1 \\ -1 & (A+1)(B^t)^{-1}\end{pmatrix}.\end{aligned}$$

Now, using Lemma 8.1 we calculate

$$\begin{aligned}c_{\mathrm{op}}(X)^{-1}&=1-2(X+1)^{-1}\\ &=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-\begin{pmatrix} -2\widehat{B} & 0 \\ 0 & -2\widehat{B}^t\end{pmatrix}\begin{pmatrix} (D+1)B^{-1} & -1 \\ -1 & (A+1)(B^t)^{-1}\end{pmatrix}\\ &=\begin{pmatrix} -2\widehat{B} & 0 \\ 0 & -2\widehat{B}^t\end{pmatrix}\left(\begin{pmatrix} -\frac{1}{2}\widehat{B}^{-1} & -1 \\ -1 & -\frac{1}{2}(\widehat{B}^t)^{-1}\end{pmatrix}-\begin{pmatrix} (D+1)B^{-1} & -1 \\ -1 & (A+1)(B^t)^{-1}\end{pmatrix}\right)\\ &=\begin{pmatrix} 2\widehat{B} & 0 \\ 0 & 2\widehat{B}^t\end{pmatrix}\begin{pmatrix} \frac{1}{2}(\widehat{B}^{-1}+2(D+1)B^{-1} & -1 \\ -1 & \frac{1}{2}((\widehat{B}^t)^{-1}-2(A+1)(B^t)^{-1})\end{pmatrix}\\ &=\begin{pmatrix} 2\widehat{B} & 0 \\ 0 & 2\widehat{B}^t\end{pmatrix}\begin{pmatrix} \frac{1}{2}(B^t-(D+1)B^{-1}(A-1) & -1 \\ -1 & \frac{1}{2}(B-(A+1)(B^t)^{-1}(D-1))\end{pmatrix}.\end{aligned}$$

□

Now we can calculate $\varphi \circ \alpha$.

8.3. LEMMA. — *Let* $X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in \mathcal{D}_0$ *then we have*

$$\delta \circ \alpha(X) = \frac{1}{2} \begin{pmatrix} B-(A+1)(B^t)^{-1}(D+1) & B-(A+1)(B^t)^{-1}(D-1) \\ -B+(A-1)(B^t)^{-1}(D+1) & -B+(A-1)(B^t)^{-1}(D-1) \end{pmatrix}.$$

Proof. — Let $\varphi \circ \alpha(X) = \begin{pmatrix} R & S \\ T & V \end{pmatrix}$ then

$$\begin{aligned} \alpha(X) &= \psi \circ \varphi \circ \alpha(X) \\ &= \begin{pmatrix} (R^t)^{-1} & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} -T^t & -1 \\ -1 & S \end{pmatrix}. \end{aligned}$$

Therefore we have $R^{-1} = 2\widehat{B}^t$ (notation as above) which implies

$$\begin{aligned} R &= \frac{1}{2}(\widehat{B}^t)^{-1} = \frac{1}{2}(B - (A+1)(B^t)^{-1}(D+1)) \\ S &= \frac{1}{2}(B - (A+1)(B^t)^{-1}(D-1)) \\ T &= \frac{1}{2}(-B + (A-1)(B^t)^{-1}(D+1). \end{aligned}$$

In order to find V we have to verify the equation $R^tV = 1 + T^tS$.

$$\begin{aligned} &\frac{1}{4}(B^t - (D+1)B^{-1}(A+1))(-B + (A-1)(B^t)^{-1}(D-1)) \\ &= \frac{1}{4}(-B^tB + B^t(A-1)(B^t)^{-1}(D-1) \\ &\qquad +(D+1)B^{-1}(A+1)B-(D+1)B^{-1}(A^2-1)(B^t)^{-1}(D-1)) \\ &= \frac{1}{4}(-B^tB-(D+1)B^{-1}(A^2-1)(B^t)^{-1}(D-1)+(D+1)B^{-1}(A-1)B \\ &\qquad +B^t(A+1)(B^t)^{-1}(D-1))-2(D-1)+2(D+1) \\ &= \frac{1}{4}(-B^t+(D+1)B^{-1}(A-1))(B-(A+1)(B^t)^{-1}(D-1))+1. \end{aligned}$$

□

Next we consider the *geometric* Cayley transform $c_{\text{geo}} : \mathrm{Sp}(n,\mathbb{C}) \to \mathrm{Sp}(n,\mathbb{C})$ given by $c_{\text{geo}}(g) = h_o g h_o^{-1}$ where

$$h_o = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

8.4. LEMMA. — *Let* $g = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{C})$ *then*

$$c_{\text{geo}}(g) = \begin{pmatrix} \widetilde{A} + i\widetilde{B} - i\widetilde{C} + \widetilde{D} & \widetilde{A} - i\widetilde{B} - i\widetilde{C} - \widetilde{D} \\ \widetilde{A} + i\widetilde{B} + i\widetilde{C} - \widetilde{D} & \widetilde{A} - i\widetilde{B} + i\widetilde{C} + \widetilde{D} \end{pmatrix}.$$

Proof. — This is a straight forward calculation.

8.5. LEMMA. — *Let* $X \in \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in \mathcal{D}_0$ *then*

$$c_{\text{geo}}^{-1} \circ \varphi \circ \alpha(X) = \begin{pmatrix} -(B^t)^{-1}D & i(B^t)^{-1} \\ i(B - A(B^t)^{-1}D) & -A(B^t)^{-1} \end{pmatrix}.$$

Proof. — This follows easily from Lemmas 8.3 and 8.4. □

The result of Lemma 8.5 allows us to calculate also the image under the operator Cayley transform.

8.6. LEMMA. — *Let* $X \in \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in \mathcal{D}_0$ *then*

$$c_{\text{op}} \circ c_{\text{geo}}^{-1} \circ \varphi \circ \alpha(X) = \begin{pmatrix} -i\widehat{B}(A-B+B^t-D) & 2\widehat{B} \\ 2iD+2(D+B^t)\widehat{B}(B+D) & i(A-B^t+B-D)\widehat{B} \end{pmatrix},$$

where $\widehat{B} = -i(A+B+B^t+D)^{-1}$.

Proof. — Let

$$(c_{\text{geo}}^{-1} \circ \varphi \circ \alpha(X) - 1)^{-1} = \begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{pmatrix}.$$

Then

$$(-(B^t)^{-1}D - 1)\widehat{B} + i(B^t)^{-1}\widehat{D} = 0$$

and hence

$$\widehat{D} = -i(D + B^t)\widehat{B}.$$

Moreover

$$i(B - A(B^t)D)\widehat{B} + (-A(B^t)^{-1} - 1)\widehat{D} = 1$$

so that

$$(iB - iA(B^t)^{-1}D + iA(B^t)^{-1}D + iD + iB^t + iA)\widehat{B} = 1$$

which then gives

$$\widehat{B} = -i(A + B + B^t + D)^{-1}.$$

Further

$$(-(B^t)^{-1}D - 1)\widehat{A} + i(B^t)^{-1}\widehat{C} = 1$$

which shows

$$\widehat{C} = -i(B^t + (D + B^t)\widehat{A}).$$

Finally we use

$$i(B - A(B^t)^{-1}D)\widehat{A} + (-A(B^t)^{-1} - 1)\widehat{C} = 0$$

to obtain

$$0 = \widehat{B}^{-1}\widehat{A} + i(A + B^t).$$

From this we find

$$\widehat{A} = -i\widehat{B}(A + B^t)$$

and

$$\begin{aligned}\widehat{C} &= -i(B^t - i(D + B^t)\widehat{B}(A + B^t)) \\ &= -iB^t - (D + B^t)\widehat{B}(A + B^t) \\ &= -iB^t - (D + B^t)\widehat{B}(A + B + B^t + D) + (D + B^t)\widehat{B}(B + D) \\ &= iD + (D + B^t)\widehat{B}(D + B).\end{aligned}$$

Now in order to prove the lemma we only have to note that

$$c_{\mathrm{op}} \circ c_{\mathrm{geo}}^{-1} \circ \varphi \circ \alpha(X) = 1 + 2\begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{pmatrix}$$

and insert the above formulas. □

8.7. LEMMA. — *Let*

$$Y = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & -\widetilde{A}^t \end{pmatrix} \in \operatorname{int} \mathbf{L}(S_{B_{\mathbf{R}}})$$

then

$$\widetilde{\theta}^{-1} \circ \widetilde{\rho} \circ \beta^{-1}(Y) = \frac{i}{2}\begin{pmatrix} -\widetilde{C} - (\widetilde{A}^t - 1)\widetilde{B}^{-1}(\widetilde{A} - 1) & \widetilde{C} + (\widetilde{A}^t - 1)\widetilde{B}^{-1}(\widetilde{A} + 1) \\ \widetilde{C} + (\widetilde{A}^t + 1)\widetilde{B}^{-1}(\widetilde{A} - 1) & -\widetilde{C} - (\widetilde{A}^t + 1)\widetilde{B}^{-1}(\widetilde{A} + 1) \end{pmatrix}.$$

Proof. — $\beta^{-1}(Y) = -\pi i J Y$ so the claim follows from Proposition 4.1. □

8.8. LEMMA. — *Let* $X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in \mathcal{D}_0$ *then*

$$\beta \circ \widetilde{\rho}^{-1} \circ \widetilde{\theta}(X) = \begin{pmatrix} -i\widehat{B}(A - B + B^t - D) & 2\widehat{B} \\ 2iD + 2(B^t + D)\widehat{B}(B + D) & i(A + B - B^t - D)\widehat{B} \end{pmatrix},$$

where $\widehat{B} = -i(A + B + B^t + D)^{-1}$.

Proof. — We use the notation of Lemma 8.7. Then

$$X = \frac{i}{2}\begin{pmatrix} -\widetilde{C} - (\widetilde{A}^t - 1)\widetilde{B}^{-1}(\widetilde{A} - 1) & \widetilde{C} + (\widetilde{A}^t - 1)\widetilde{B}^{-1}(\widetilde{A} + 1) \\ \widetilde{C} + (\widetilde{A}^t + 1)\widetilde{B}^{-1}(\widetilde{A} - 1) & -\widetilde{C} - (\widetilde{A}^t + 1)\widetilde{B}^{-1}(\widetilde{A} + 1) \end{pmatrix}.$$

This shows

$$\frac{2}{i}(A+B) = 2(\widetilde{A}^t - 1)\widetilde{B}^{-1}$$

$$\frac{2}{i}(B^t+D) = -2(\widetilde{A}^t + 1)\widetilde{B}^{-1}$$

and hence $-2\widehat{B}^{-1} = -4\widetilde{B}^{-1}$, i.e.

$$\widetilde{B} = 2\widehat{B}\ .$$

Moreover we find $2\widetilde{A}^t\widetilde{B}^{-1} = \frac{1}{i}(A+B-B^t-D)$ which means

$$\widetilde{A}^t = -i(A+B-B^t-D)\widehat{B}\ .$$

Adding all four entries of X with appropriate signs yields

$$\frac{2}{i}(-A+B+B^t-D) = 4\widetilde{C} + 4\widetilde{A}^t\widetilde{B}^{-1}\widetilde{A}\ .$$

Thus we calculate

$$\begin{aligned}
2\widetilde{C} &= \frac{1}{i}(-A+B+B^t-D) - \frac{1}{i}(A+B-B^t-D)\widetilde{A}\\
&= i(A-B-B^t+D) - i(A+B-B^t-D)i\widehat{B}(A+B^t-B-D)\\
&= 2i(D-B^t) - 2(B^t+D)\widehat{B}(A-B+B^t-D)\\
&= 4iD + 4(B^t+D)\widehat{B}(B+D)\ .
\end{aligned}$$

□

Comparing the results of Lemma 8.6 and Lemma 8.8 we find the following theorem.

8.9. THEOREM. — *The following diagram is commutative*

$$\begin{tikzcd}
(GK_{\mathbf{R}}, *_{tw}) \arrow[d, "\rho"] & \supseteq & \mathcal{S}_{tw} \arrow[r] \arrow[d, "\rho"] & \mathcal{D}_{tw} \arrow[r, "\beta"] \arrow[d, "\widetilde{\rho}"] & \operatorname{int}\mathbf{L}(S_{B_{\mathbf{R}}}) \arrow[r, "c_{\mathrm{op}}"] & S^0_{B_{\mathbf{R}}} \arrow[ddd, "c_{\mathrm{geo}}"] \\
(GK_{\mathbf{R}}, \circ) \arrow[d, "\theta"] & \supseteq & \mathcal{S}_0 \arrow[r] \arrow[d, "\theta"] & \mathcal{D}_0 \arrow[d, "\widetilde{\theta}"] & & \\
(GK_{\mathbf{R}}, \circ) \arrow[d, "U_{n,n}"] & \supseteq & \mathcal{S}_0 \arrow[r] \arrow[d, "U_{n,n}"] & \mathcal{D}_0 \arrow[d, "\alpha"] & & \\
(GK_{\mathbf{C}}, \circ) & \supseteq & \mathcal{S}_\Omega \arrow[r] & \mathcal{D}_\Omega \arrow[rr, "\varphi"] & & S^o_B\ .
\end{tikzcd}$$

Proof. — Note that $c_{\mathrm{geo}}(g)$ is a B–contraction (*cf.* Lemma 5.1) if and only if g is a $h_o^*Bh_o$–contraction, since h_o is unitary. But one easily verifies that $h_o^*Lh_o = iJ$ so that $h_o^*Bh_o = B_{\mathbf{R}}$ (*cf.* (6.1)). The claim then follows from Lemma 8.6 and Lemma 8.8. □

Added in proof: The Fock realization of the oscillator semigroup has been studied independently by G.B. Folland (*cf.* G.B. Folland, Harmonic analysis in phase space, Ann. Math. Studies, 122, Princeton Univ. Press, 1989).

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