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VANISHING THEOREMS FOR COMPACT HESSIAN MANIFOLDS

by Hirohiko SHIMA

Let M be a flat affine manifold with a locally flat affine connection D . Among the Riemannian metrics on M there is an important class of Riemannian metrics which are compatible with the flat affine structure on M . A Riemannian metric g on M is said to be *Hessian* if g has an expression $g = D^2u$ where u is a local C^∞ -function. A flat affine manifold provided with a Hessian metric is called a *Hessian manifold*. A certain geometry of Hessian manifolds has been studied in Shima [10]-[14]. See also Cheng and Yau [2] and Yagi [15].

Hessian manifolds have in a certain sense some analogy with Kählerian manifolds. In this paper, being motivated by the theory of cohomology for Kählerian manifolds we study cohomology groups for Hessian manifolds.

Let F be a locally constant vector bundle over M . We denote by $\Omega^{p,q}(F)$ the space of all sections of $(\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F$, where T^* is the cotangent bundle over M . Since the vector bundle $(\wedge^q T^*) \otimes F$ is locally constant, we can naturally define a complex

$$\dots \xrightarrow{\partial} \Omega^{p-1,q}(F) \xrightarrow{\partial} \Omega^{p,q}(F) \xrightarrow{\partial} \Omega^{p+1,q}(F) \xrightarrow{\partial} \dots$$

We denote by $H^{p,q}(F)$ the p -th cohomology group of the complex. Then we have the following duality theorem analogous to that of Serre [9].

THEOREM. — *Let M be a compact oriented flat affine manifold of dimension n . Then we have*

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$$H^{p,q}(F) \cong H^{n-p, n-q}((K \otimes F)^*),$$

where K is the canonical line bundle over M and $(K \otimes F)^*$ is the dual bundle of $K \otimes F$.

Let F be a locally constant line bundle over M . Choose an open covering $\{U_\lambda\}$ of M such that the local triviality holds on each U_λ . Denote by $\{f_{\lambda\mu}\}$ the constant transition functions with respect to $\{U_\lambda\}$. A fiber metric $a = \{a_\lambda\}$ on F is a collection of positive C^∞ -functions a_λ on U_λ such that

$$a_\mu = f_{\lambda\mu}^2 a_\lambda.$$

Using this we can define a globally defined closed 1-form A and a symmetric bilinear form B by

$$A = -D \log a_\lambda,$$

$$B = -D^2 \log a_\lambda,$$

and we call them the *first Koszul form* and the *second Koszul form* of F with respect to the fiber metric $a = \{a_\lambda\}$ respectively.

A locally constant line bundle F is said to be *positive* (resp. *negative*) if the second Koszul form is positive (resp. negative) definite with respect to a certain fiber metric. It should be remarked that if a compact connected flat affine manifold M admits a locally constant positive (resp. negative) line bundle, then by a theorem of Koszul [6] M is a hyperbolic affine manifold, that is, the universal covering of M is an open convex cone not containing any full straight line.

Kodaira-Nakano's vanishing theorem for compact Kählerian manifolds plays an essential role in the theory of compact Kählerian manifolds. In this paper we prove the following vanishing theorem for a compact Hessian manifold analogous to that of Kodaira-Nakano.

THEOREM. — *Let M be a compact connected oriented Hessian manifold. Denote by K the canonical line bundle over M . Let F be a locally constant line bundle over M .*

(i) *If $2F + K$ is positive, then*

$$H^{p,q}(F) = 0 \quad \text{for } p + q > n.$$

(ii) If $2F + K$ is negative, then

$$H^{p,q}(F) = 0 \quad \text{for } p + q < n.$$

As to vanishing theorem for compact hyperbolic affine manifolds we should mention the following theorem due to Koszul [7].

THEOREM. — *Let M be a compact oriented hyperbolic affine manifold. Then we have*

$$H^{p,q}(1) = 0 \quad \text{for } p, q > 0,$$

where 1 is the trivial line bundle over M .

In § 1 and § 2 a Riemannian metric g is not assumed to be Hessian. We define in § 1 fundamental operators $e(g), i(g), \Pi, *, \partial, \delta$ and \square . In § 2 we define the Laplacian \square_a on $\Omega^{p,q}(F)$, and prove the duality theorem $H^{p,q}(F) \cong H^{n-p,n-q}((K \otimes F)^*)$ and the cohomology isomorphisms $\mathcal{H}^{p,q}(F) \cong H^{p,q}(F) \cong H^p(P^q(F))$. In § 3 we give the local expressions for geometric concepts on Hessian manifolds. In § 4 and § 5 the formulae of Weitzenböck type for \square and \square_a are obtained. In § 6 we prove a vanishing theorem analogous to that of Kodaira-Nakano. In § 7 we mention a vanishing theorem of Koszul type.

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1. The Laplacian \square on $\Omega^{p,q}$.

Let M be a flat affine manifold with a locally flat affine connection D . Then there exist local coordinate systems $\{x^1, \dots, x^n\}$ such that $Ddx^i = 0$, which will be called *affine local coordinate systems*. Throughout this paper the local expressions for geometric concepts on M will be given in terms of affine local coordinate system. From now on we assume further that M is compact, connected and oriented.

Choose an arbitrary Riemannian metric g on M . Let $\Omega^{p,q}$ be the space of all sections of $(\wedge^p T^*) \otimes (\wedge^q T^*)$. We denote the local

expression of $\phi \in \Omega^{p,q}$ by

$$\phi = \frac{1}{p! q!} \sum \phi_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} (dx^{i_1} \wedge \dots \wedge dx^{i_p}) \otimes (dx^{\bar{j}_1} \wedge \dots \wedge dx^{\bar{j}_q}).$$

For simplicity let us fix some notation. We denote as follows :

$$I_p = (i_1, \dots, i_p), \quad i_1 < i_2 < \dots < i_p, \quad i \leq i_\sigma \leq n,$$

$$I_{n-p} = (i_{p+1}, \dots, i_n), \quad i_{p+1} < \dots < i_n, \quad 1 \leq i_\tau \leq n,$$

and $(i_1, \dots, i_p, i_{p+1}, \dots, i_n)$ is a permutation of $(1, \dots, n)$. Then with this notation we write

$$\phi = \sum_{I_p, \bar{J}_q} \phi_{I_p \bar{J}_q} dx^{I_p} \otimes dx^{\bar{J}_q},$$

where $dx^{I_p} = dx^{i_1} \wedge \dots \wedge dx^{i_p}$.

For $\phi, \psi \in \Omega^{p,q}$ we set

$$\begin{aligned} h(\phi, \psi) &= \frac{1}{p! q!} \phi_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} \psi^{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} (*) & (1.1) \\ &= \phi_{I_p \bar{J}_q} \psi^{I_p \bar{J}_q}. (**) \end{aligned}$$

DEFINITION 1.1. — The inner product of $\phi, \psi \in \Omega^{p,q}$ is

$$(\phi, \psi) = \int_M h(\phi, \psi) v,$$

where v is the volume element determined by g .

DEFINITION 1.2. — We define $*$ -operation

$$* : \Omega^{p,q} \longrightarrow \Omega^{n-p, n-q}$$

by $(*\phi)_{i_{n-p} \bar{j}_{n-q}} = (-1)^{pq} \operatorname{sgn}(I_p I_{n-p}) \operatorname{sgn}(\bar{J}_q \bar{J}_{n-q}) G \phi^{I_p \bar{J}_q}$, where $\operatorname{sgn}(I_p I_{n-p})$ is the signature of the permutation $(I_p I_{n-p})$ of $(1, \dots, n)$ and $G = \det(g_{ij})$.

(*) Throughout this paper we use Einstein's convention on indices.

(**) $\phi_{I_p \bar{J}_q} \psi^{I_p \bar{J}_q}$ means $\sum_{I_p, \bar{J}_q} \phi_{I_p \bar{J}_q} \psi^{I_p \bar{J}_q}$.

DEFINITION 1.3. — Let $\phi = \sum \phi_{I_p \bar{J}_q} dx^{I_p} \otimes dx^{\bar{J}_q}$ and

$$\psi = \sum \psi_{K_r \bar{L}_s} dx^{K_r} \otimes dx^{\bar{L}_s}.$$

We set $\phi \wedge \psi = \sum \phi_{I_p \bar{J}_q} \psi_{K_r \bar{L}_s} (dx^{I_p} \wedge dx^{K_r}) \otimes (dx^{\bar{J}_q} \wedge dx^{\bar{L}_s})$.

A straightforward calculation shows

PROPOSITION 1.1. — Let $\phi, \psi \in \Omega^{p,q}$. Then

- (i) $**\phi = (-1)^{n+p+q} \phi$,
- (ii) $\phi \wedge * \psi = (-1)^{pq} h(\phi, \psi) v \otimes v$.

DEFINITION 1.4. — Considering the Riemannian metric g as an element in $\Omega^{1,1}$ we define

$$e(g) : \Omega^{p,q} \longrightarrow \Omega^{p+1,q+1},$$

$$i(g) : \Omega^{p,q} \longrightarrow \Omega^{p-1,q-1},$$

by $e(g)\phi = g \wedge \phi$ for $\phi \in \Omega^{p,q}$ and $i(g) = (-1)^{n+p+q+1} * e(g) *$.

Then $i(g)$ is the adjoint operator of $e(g)$ with respect to the inner product given in Definition 1.1 :

$$(i(g)\phi, \psi) = (\phi, e(g)\psi) \quad \text{for } \phi \in \Omega^{p,q}, \psi \in \Omega^{p-1,q-1}.$$

DEFINITION 1.5. — We set

$$\Pi = \sum_{p,q} (n - p - q) \pi_{p,q},$$

where $\pi_{p,q}$ is the projection from $\sum_{r,s} \Omega^{r,s}$ onto $\Omega^{p,q}$.

PROPOSITION 1.2. — We have

$$[\Pi, e(g)] = -2e(g), \quad [\Pi, i(g)] = 2i(g), \quad [i(g), e(g)] = \Pi.$$

The proof is carried out by a direct calculation and so it is omitted.

DEFINITION 1.6. — *Define*

$$\partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}$$

by $\partial = \sum_k (e(dx^k) \otimes \text{id}) D_k$, where $e(dx^k)$ is a linear map from

$\overset{p}{\wedge} T^*$ to $\overset{p+1}{\wedge} T^*$ given by $e(dx^k) \omega = dx^k \wedge \omega$, id is the identity map on $\overset{p}{\wedge} T^*$ and D_k is the covariant derivation with respect to $\partial/\partial x^k$ for the locally flat affine connection D .

Then we have

$$\partial\partial = 0. \quad (1.2)$$

DEFINITION 1.7. — *Define*

$$\delta : \Omega^{p,q} \longrightarrow \Omega^{p-1,q}$$

by $\delta = (-1)^{n+1} \sqrt{G} * \partial \left(\frac{1}{\sqrt{G}} * \right)$.

PROPOSITION 1.3. — δ is the adjoint operator of ∂ with respect to the inner product given in Definition 1.1;

$$(\partial\phi, \psi) = (\phi, \delta\psi) \quad \text{for } \phi \in \Omega^{p,q}, \psi \in \Omega^{p+1,q}.$$

In Proposition 2.1 we prove the above fact in more general situation and so we omit the proof.

DEFINITION 1.8. — *We define*

$$\square : \Omega^{p,q} \longrightarrow \Omega^{p,q}$$

by $\square = \partial\delta + \delta\partial$, and call it the Laplacian. $\phi \in \Omega^{p,q}$ is said to be \square -harmonic if $\square\phi = 0$.

2. The Laplacian \square_q on $\Omega^{p,q}(F)$.

Let F be a locally constant vector bundle over M . Choose an open covering $\{U_\lambda\}$ of M such that the local triviality holds

on each U_λ . Let $\{\xi_\lambda^1, \dots, \xi_\lambda^m\}$ be fiber coordinate systems such that the transition functions $\{f_{\lambda\mu}\}$ defined by

$$\xi_\lambda^i = \sum_j f_{\lambda\mu}{}^i{}_j \xi_\mu^j$$

are constants. A fiber metric $a = \{a_\lambda\}$ on F is a collection of $m \times m$ positive definite symmetric matrices $a = (a_{\lambda ij})$ such that each $a_{\lambda ij}$ is a C^∞ -function on U_λ and

$$a_\lambda = {}^t f_{\mu\lambda} a_\mu f_{\mu\lambda}$$

holds.

Let $\Omega^{p,q}(F)$ denote the space of all sections of $(\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F$.

Using fiber coordinate systems $\{\xi_\lambda^i\}$ we express an element $\phi \in \Omega^{p,q}(F)$ as $\phi = \{\phi_\lambda^i\}$.

DEFINITION 2.1. — Define

$$\partial : \Omega^{p,q}(F) \longrightarrow \Omega^{p+1,q}(F)$$

by $\partial \{\phi^i\} = \{\partial \phi^i\}$. (*)

We have then

$$\partial \partial = 0. \tag{2.1}$$

DEFINITION 2.2. — The inner product of $\phi, \psi \in \Omega^{p,q}(F)$ is

$$(\phi, \psi) = \int_M \sum a_{ij} h(\phi^i, \psi^j) v.$$

DEFINITION 2.3 — Define

$$\delta_a : \Omega^{p,q}(F) \longrightarrow \Omega^{p-1,q}(F)$$

by $\delta_a \{\phi^i\} = \left\{ (-1)^{n+1} \sum_{i,k} \sqrt{G} a^{ij} * \partial \left(\frac{a_{jk}}{\sqrt{G}} * \phi^k \right) \right\}$, where a^{ij} is the (i, j) -component of $(a_{ij})^{-1}$.

(*) For brevity the subscripts λ, μ, \dots are dropped where no confusion will arise.

PROPOSITION 2.1. — δ_a is the adjoint operator of ∂ with respect to the inner product given in Definition 2.2;

$$(\partial\phi, \psi) = (\phi, \delta_a \psi) \quad \text{for } \phi \in \Omega^{p-1, q}(\mathbb{F}), \psi \in \Omega^{p, q}(\mathbb{F}).$$

Proof. — Since $\sum_{i,j} a_{ij} \phi^i \wedge * \psi^j$ is globally defined on M , there exists $(n-1)$ -form ω on M such that $\omega \otimes v = \sum a_{ij} \phi^i \wedge * \psi^j$. Then

$$\partial(\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,$$

where $\alpha = d \log \sqrt{G}$, and

$$\begin{aligned} \partial(\sum a_{ij} \phi^i \wedge * \psi^j) &= (-1)^{pq} \sum a_{ij} h(\partial\phi^i, \psi^j) v \otimes v + (-1)^{n-q} \sum \phi^i \wedge ** \partial(a_{ij} * \psi^j). \end{aligned}$$

Since

$$\delta_a \psi^i = -(-1)^{n+1} * (\alpha \wedge * \psi^i) + (-1)^{n+1} \sum a^{jk} * \partial(a_{jk} * \psi^k),$$

we have

$$\begin{aligned} (\alpha \wedge \omega + d\omega) \otimes v &= (-1)^{pq} \sum a_{ij} h(\partial\phi^i, \psi^j) v \otimes v + (-1)^{n-q} \sum a_{ij} \phi^i \wedge ** (\alpha \wedge * \psi^j) \\ &\quad + (-1)^{q+1} \sum a_{ij} \phi^i \wedge * \delta_a \psi^j \\ &= (-1)^{pq} \sum a_{ij} h(\partial\phi^i, \psi^j) v \otimes v + (\alpha \wedge \omega) \otimes v \\ &\quad + (-1)^{pq-1} \sum a_{ij} h(\phi^i, \delta_a \psi^j) v \otimes v, \end{aligned}$$

and so

$$d\omega = (-1)^{pq} (\sum a_{ij} h(\partial\phi^i, \psi^j) - \sum a_{ij} h(\phi^i, \delta_a \psi^j)) v.$$

Therefore

$$0 = \int_M d\omega = (-1)^{pq} ((\partial\phi, \psi) - (\phi, \delta_a \psi)).$$

Q.E.D.

DEFINITION 2.4. — We define

$$\square_a : \Omega^{p, q}(\mathbb{F}) \longrightarrow \Omega^{p, q}(\mathbb{F})$$

by $\square_a = \partial\delta_a + \delta_a\partial$, and call it the Laplacian. $\phi \in \Omega^{p,q}(F)$ is said to be \square_a -harmonic if $\square_a\phi = 0$.

DEFINITION 2.5. — We set

$$\mathfrak{H}^{p,q}(F) = \{\phi \in \Omega^{p,q}(F) \mid \square_a\phi = 0\}.$$

THEOREM 2.2. — We have the following duality:

$$\mathfrak{H}^{p,q}(F) \cong \mathfrak{H}^{n-p,n-q}((K \otimes F)^*),$$

where K is the canonical line bundle over M and $(K \otimes F)^*$ is the dual bundle of $K \otimes F$.

Proof. — For $\psi = \{\psi^j\} \in \Omega^{p,q}(F)$ we set

$$\psi_i^* = \sum_j \frac{a_{ij}}{\sqrt{G}} * \psi^j. \tag{2.2}$$

Then we have $\psi^* = \{\psi_i^*\} \in \Omega^{n-p,n-q}((K \otimes F)^*)$. It follows from Proposition 1.1 (i)

$$\psi^j = (-1)^{n+p+q} \sum_i \sqrt{G} d^i * \psi_i^*. \tag{2.3}$$

Thus the map $\psi \rightarrow \psi^*$ is a linear isomorphism from $\Omega^{p,q}(F)$ onto $\Omega^{n-p,n-q}((K \otimes F)^*)$.

Let $\phi \in \Omega^{p,q}(F)$ and $\psi^* \in \Omega^{n-p,n-q}((K \otimes F)^*)$. Then

$\sum_i \sqrt{G} \phi^i \wedge \psi_i^*$ is globally defined on M . Hence there exists a C^∞ -function $k(\phi, \psi^*)$ on M such that

$$\sum_i \sqrt{G} \phi^i \wedge \psi_i^* = k(\phi, \psi^*) v \otimes v.$$

We set

$$\langle \phi, \psi^* \rangle = (-1)^{pq} \int_M k(\phi, \psi^*) v.$$

Since

$$k(\phi, \psi^*) v \otimes v = \sum_{i,j} a_{ij} \phi^i \wedge * \psi^j = (-1)^{pq} \sum_{i,j} a_{ij} h(\phi^i, \psi^j) v \otimes v,$$

we have

$$\langle \phi, \psi^* \rangle = (\phi, \psi) \quad \text{for } \phi, \psi \in \Omega^{p,q}(F).$$

Define the inner product of $\psi^*, \phi^* \in \Omega^{n-p, n-q}((K \otimes F)^*)$ by

$$(\psi^*, \phi^*) = \int_M \sum G^{ij} h(\psi_i^*, \phi_j^*) v.$$

Since

$$\begin{aligned} \sum_{i,j} G^{ij} h(\psi_i^*, \phi_j^*) v \otimes v &= \sum_{i,j} a_{ij} h(*\psi^i, *\phi^j) v \otimes v \\ &= (-1)^{pq} \sum_{i,j} a_{ij} \phi^j \wedge *\psi^i = \sum_{i,j} a_{ij} h(\phi^j, \psi^i) v \otimes v, \end{aligned}$$

we obtain

$$(\psi^*, \phi^*) = (\phi, \psi) \quad \text{for } \phi, \psi \in \Omega^{p,q}(F).$$

Let $\phi \in \Omega^{p-1,q}(F)$ and $\psi^* \in \Omega^{n-p, n-q}((K \otimes F)^*)$. Then

$\sum_i \sqrt{G} \phi^i \wedge \psi_i^*$ is globally defined on M and hence there exists $(n-1)$ -form ω on M such that

$$\sum_i \sqrt{G} \phi^i \wedge \psi_i^* = \omega \otimes v.$$

Since

$$\begin{aligned} \partial \left(\sum_i \sqrt{G} \phi^i \wedge \psi_i^* \right) &= \sum_i \{ \alpha \wedge \sqrt{G} \phi^i \wedge \psi_i^* + \sqrt{G} \partial \phi^i \wedge \psi_i^* + (-1)^{p-1} \sqrt{G} \phi^i \wedge \partial \psi_i^* \} \\ &= (\alpha \wedge \omega) \otimes v + \sum_i \{ k(\partial \phi^i, \psi_i^*) + (-1)^{p-1} k(\phi^i, \partial \psi_i^*) \} v \otimes v, \end{aligned}$$

and

$$\partial(\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,$$

we obtain

$$d\omega = \sum_i \{ k(\partial \phi^i, \psi_i^*) + (-1)^{p-1} k(\phi^i, \partial \psi_i^*) \} v.$$

Therefore

$$\begin{aligned} 0 &= \int_M d\omega \\ &= (-1)^{pq} \langle \partial \phi, \psi^* \rangle + (-1)^{p-1+(p-1)q} \langle \phi, \partial \psi^* \rangle. \end{aligned}$$

This implies

$$\langle \partial \phi, \psi^* \rangle = (-1)^{p+q} \langle \phi, \partial \psi^* \rangle.$$

Using these facts we obtain

$$\begin{aligned} \langle \phi^*, \partial \psi^* \rangle &= \langle \phi, \partial \psi^* \rangle = (-1)^{p+q} \langle \partial \phi, \psi^* \rangle = (-1)^{p+q} \langle \partial \phi, \psi \rangle \\ &= (-1)^{p+q} \langle \phi, \delta_a \psi \rangle = (-1)^{p+q} \langle \phi^*, (\delta_a \psi)^* \rangle, \end{aligned}$$

hence

$$\partial \psi^* = (-1)^{p+q} (\delta_a \psi)^* \quad \text{for } \psi \in \Omega^{p,q}(F). \quad (2.4)$$

By the same way we have

$$\begin{aligned} \langle \psi^*, \delta_a \phi^* \rangle &= \langle \partial \psi^*, \phi^* \rangle = \langle \phi, \partial \psi^* \rangle = (-1)^{p+q} \langle \partial \phi, \psi^* \rangle \\ &= (-1)^{p+q} \langle \partial \phi, \psi \rangle = (-1)^{p+q} \langle (\partial \phi)^*, \psi^* \rangle, \end{aligned}$$

hence

$$\delta_a \phi^* = (-1)^{p+q} (\partial \phi)^*.$$

Thus

$$\delta_a \psi^* = (-1)^{p+q+1} (\partial \psi)^* \quad \text{for } \psi \in \Omega^{p,q}(F). \quad (2.5)$$

(2.4) and (2.5) imply that ψ^* is harmonic if and only if ψ is harmonic.

Q.E.D.

DEFINITION 2.6. — We set

$$H^{p,q}(F) = \{ \phi \in \Omega^{p,q}(F) \mid \partial \phi = 0 \} / \{ \partial \psi \mid \psi \in \Omega^{p-1,q}(F) \}.$$

A q -form ω on M is said to be D -parallel if $D\omega = 0$. Let us denote by $P^q(F)$ the sheaf over M of germs of F -valued D -parallel q -forms.

DEFINITION 2.7. — We denote by $H^p(P^q(F))$ the p -th cohomology group of M with coefficients on $P^q(F)$.

THEOREM 2.3. — We have the following isomorphisms:

$$\mathcal{H}^{p,q}(F) \cong H^{p,q}(F) \cong H^p(P^q(F)).$$

Proof. – By the theory of harmonic integral we have

$$\mathcal{H}^{p,q}(F) \cong H^{p,q}(F).$$

Let $A^{p,q}(F)$ denote the sheaf over M of germs of sections of $(\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F$. Then

$$0 \longrightarrow P^q(F) \longrightarrow A^{0,q}(F) \xrightarrow{\delta} A^{1,q}(F) \xrightarrow{\delta} A^{2,q}(F) \xrightarrow{\delta} \dots$$

is a fine resolution of $P^q(F)$. Thus we have $H^{p,q}(F) \cong H^p(P^q(F))$.

Q.E.D.

3. Hessian metrics on affine local coordinate systems.

Let M be a Hessian manifold with a locally flat affine connection D and a Hessian metric g . We denote by ∇ the Riemannian connection for g . In this section we shall express various geometric concepts on the Hessian manifold M in terms of affine local coordinate systems. Let us denote by D_k and ∇_k the covariant derivations with respect to $\partial/\partial x^k$ for D and ∇ respectively. Since the Christoffel symbol Γ_{jk}^i for g is the difference between the components of affine connections ∇ and D , we may consider that Γ_{jk}^i is a tensor field. We have then

$$\Gamma_{jk}^i = \frac{1}{2} g^{is} D_k g_{sj}, \quad (3.1)$$

$$D_k g_{ij} = 2\Gamma_{ijk}, \quad D_k g^{ij} = -2\Gamma_k^{ij},$$

$$\Gamma_{ijk} = \Gamma_{jik} = \Gamma_{ikj}.$$

DEFINITION 3.1. – We define a 1-form α and a symmetric bilinear form β by

$$\alpha = D \log \sqrt{G},$$

$$\beta = D^2 \log \sqrt{G},$$

where $G = \det(g_{ij})$, and call them the first Koszul form and the second Koszul form of M respectively.

Then we have

$$\begin{aligned} \alpha_i &= \Gamma^r_{ir}, \\ \beta_{ij} &= D_j \Gamma^r_{ir}. \end{aligned} \tag{3.2}$$

DEFINITION 3.2. — Let γ_k be the derivation of the algebra of tensor fields defined by

$$\gamma_k = \nabla_k - D_k.$$

Let T^p_q be the space of tensor fields of type (p, q) defined on M .

DEFINITION 3.3. — We define certain covariant derivations $\nabla'_k, \bar{\nabla}'_{\bar{k}}$ on $T^p_q \otimes T^r_s$ by

$$\begin{aligned} \nabla'_k &= (2\gamma_k) \otimes \text{id} + D_k, \\ \bar{\nabla}'_{\bar{k}} &= \text{id} \otimes (2\gamma_{\bar{k}}) + D_{\bar{k}}, \end{aligned}$$

where id are the identity transformations.

Notice that

$$\nabla_k = \frac{1}{2} (\nabla'_k + \bar{\nabla}'_{\bar{k}}), \quad \text{where } k = \bar{k}.$$

LEMMA 3.1. — For the Hessian metric g we have

$$\begin{aligned} \nabla'_k g_{\bar{i}\bar{j}} &= 0, & \bar{\nabla}'_{\bar{k}} g_{\bar{i}\bar{j}} &= 0, \\ \nabla'_k g^{\bar{i}\bar{j}} &= 0, & \bar{\nabla}'_{\bar{k}} g^{\bar{i}\bar{j}} &= 0. \end{aligned}$$

Proof. — By (3.1) we obtain

$$\nabla'_k g_{\bar{i}\bar{j}} = D_k g_{\bar{i}\bar{j}} - 2\Gamma^m_{ki} g_{m\bar{j}} = 2\Gamma_{\bar{i}\bar{k}} - 2\Gamma_{\bar{j}ki} = 0.$$

Similarly we can prove the other equalities.

Q.E.D.

DEFINITION 3.4. — Considering γ_i as tensor fields of type (1.1) we define tensor fields γ and S by

$$\begin{aligned} \gamma &= \sum_i \gamma_i \otimes dx^i, \\ S &= D\gamma. \end{aligned}$$

The component of S is given by

$$S^i_{jkl} = D_k \Gamma^i_{jl}.$$

LEMMA 3.2. — $S_{ijkl} = S_{kjil} = S_{klij} = S_{iljk}$.

Proof. — Let $g_{ij} = D_i D_j u$. By (3.1) we have

$$\begin{aligned} S_{ijkl} &= g_{ip} D_k \Gamma^p_{jl} = g_{ip} D_k (g^{pq} \Gamma_{qjl}) = g_{ip} (D_k g^{pq}) \Gamma_{qjl} + g_{ip} g^{pq} D_k \Gamma_{qjl} \\ &= -2\Gamma^q_{ik} \Gamma_{qjl} + D_k \Gamma_{ijl} = -2g^{qr} \Gamma_{irk} \Gamma_{qjl} + D_k \Gamma_{ijl} \\ &= \frac{1}{2} D_i D_j D_k D_l u - \frac{1}{2} g^{qr} (D_r D_i D_k u) (D_q D_j D_l u). \end{aligned}$$

This proves the Lemma.

Q.E.D.

LEMMA 3.3. — $\beta_{ij} = S^r_{rij} = S^r_{ijr}$.

Proof. — $\beta_{ij} = D_j \alpha_i = D_i \alpha_j = D_i \Gamma^r_{rj} = S^r_{rij}$. By Lemma 3.2 we have $S^r_{rij} = g^{rp} S_{prij} = g^{rp} S_{ijpr} = S^r_{ijr}$.

Q.E.D.

4. The local expression for \square .

From now on we always assume that M is a compact connected oriented Hessian manifold.

PROPOSITION 4.1. — Let $\phi \in \Omega^{p,q}$. Then we have

$$(\partial\phi)_{i_1 \dots i_{p+1} \bar{j}_q} = \sum_{\sigma} (-1)^{\sigma-1} \nabla'_{i_{\sigma}} \phi_{i_1 \dots \hat{i}_{\sigma} \dots i_{p+1} \bar{j}_q},$$

where \hat{i}_{σ} means "omit i_{σ} ".

Proof. — By Definition 1.6 we have

$$(\partial\phi)_{i_{p+1} \bar{j}_q} = \sum_{\sigma=1}^{p+1} (-1)^{\sigma-1} D_{i_{\sigma}} \phi_{i_1 \dots \hat{i}_{\sigma} \dots i_{p+1} \bar{j}_q}. \tag{4.1}$$

Using this and (3.1) we obtain the proposition.

Q.E.D.

PROPOSITION 4.2. — Let $\phi \in \Omega^{p,q}$. Then we have

$$(\delta\phi)_{1_{p-1}\bar{1}_q} = -g^{s\bar{r}} \bar{\nabla}'_r \phi_{1_{p-1}\bar{1}_q} + \alpha^s \phi_{s1_{p-1}\bar{1}_q}.$$

Proof. — Let $\psi \in \Omega^{p-1,q}$. By (4.1) and Green's theorem we have

$$(\phi, \partial\psi) = - \int_M D_r(\phi^{r1_{p-1}\bar{1}_q} \sqrt{G}) \frac{1}{\sqrt{G}} \psi_{1_{p-1}\bar{1}_q} v.$$

Thus we obtain

$$\begin{aligned} (\delta\phi)^{1_{p-1}\bar{1}_q} &= -D_r \phi^{r1_{p-1}\bar{1}_q} - \alpha_r \phi^{r1_{p-1}\bar{1}_q} \\ &= -\nabla_r \phi^{r1_{p-1}\bar{1}_q} + \alpha_r \phi^{r1_{p-1}\bar{1}_q}. \end{aligned}$$

This completes the proof.

Q.E.D.

THEOREM 4.1. — Let $\phi \in \Omega^{p,q}$. Then we have

$$\begin{aligned} (\square\phi)_{1_p\bar{1}_q} &= -g^{s\bar{r}} \bar{\nabla}'_r \nabla'_s \phi_{1_p\bar{1}_q} + \alpha^s \nabla'_s \phi_{1_p\bar{1}_q} - \sum_{\sigma} \beta^s_{i_\sigma} \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{1}_q} \\ &\quad + 2 \sum_{\sigma, \tau} S^{i_\sigma i_\tau} \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{1}_1 \dots (i)_\tau \dots \bar{1}_q}, \end{aligned}$$

where $(s)_\sigma$ means "substitute s for σ -th place".

Proof. — Using Proposition 4.1, Proposition 4.2 and $\nabla'_i \alpha^j = \beta^j_i$ we obtain

$$\begin{aligned} (\partial\delta\phi)_{1_p\bar{1}_q} &= -g^{s\bar{r}} \sum_{\sigma} \nabla'_{i_\sigma} \bar{\nabla}'_r \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{1}_q} + \sum_{\sigma} \beta^s_{i_\sigma} \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{1}_q} \\ &\quad + \sum_{\sigma} \alpha^s \nabla'_{i_\sigma} \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{1}_q}, \end{aligned}$$

$$\begin{aligned} (\delta\partial\phi)_{1_p\bar{1}_q} &= -g^{s\bar{r}} \bar{\nabla}'_r (\nabla'_s \phi_{1_p\bar{1}_q} - \sum_{\sigma} \nabla'_{i_\sigma} \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{1}_q}) \\ &\quad + \alpha^s (\nabla'_s \phi_{1_p\bar{1}_q} - \sum_{\sigma} \nabla'_{i_\sigma} \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{1}_q}), \end{aligned}$$

and so

$$\begin{aligned}
 (\square\phi)_{i_p \bar{j}_q} &= -g^{s\bar{r}} \bar{\nabla}'_{\bar{r}} \nabla'_s \phi_{i_p \bar{j}_q} + \alpha^s \nabla'_s \phi_{i_p \bar{j}_q} \\
 &\quad - g^{s\bar{r}} \sum_{\sigma} [\nabla'_{i_\sigma}, \bar{\nabla}'_{\bar{r}}] \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{j}_q} \\
 &\quad + \sum_{\sigma} \beta^s_{i_\sigma} \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{j}_q}.
 \end{aligned}$$

Let us calculate the third term on the right-hand of the above formula. Since $[\nabla'_i, \bar{\nabla}'_{\bar{j}}]$ is a derivation of the algebra of tensor fields which maps every function to 0 and since

$$\begin{aligned}
 [\nabla'_i, \bar{\nabla}'_{\bar{j}}] \xi_k &= 2S^p_{i\bar{j}k} \xi_p, \\
 [\nabla'_i, \bar{\nabla}'_{\bar{j}}] \xi_{\bar{k}} &= -2S^p_{i\bar{j}\bar{k}} \xi_{\bar{p}},
 \end{aligned}$$

we have

$$\begin{aligned}
 [\nabla'_{i_\sigma}, \bar{\nabla}'_{\bar{r}}] \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{j}_q} &= \sum_{\tau} 2S^m_{i_\sigma \bar{i}_\tau} \phi_{i_1 \dots (s)_\sigma \dots (m)_\tau \dots i_p \bar{j}_q} \\
 &\quad + 2S^m_{i_\sigma \bar{r}s} \phi_{i_1 \dots (m)_\sigma \dots i_p \bar{j}_q} \\
 &\quad - \sum_{\tau} 2S^{\bar{m}}_{\bar{r} i_\sigma \bar{\tau}} \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{j}_1 \dots (\bar{m})_\tau \dots \bar{j}_q}.
 \end{aligned}$$

Thus, by Lemma 3.2 and 3.3 we obtain

$$\begin{aligned}
 g^{s\bar{r}} \sum_{\sigma} [\nabla'_{i_\sigma}, \bar{\nabla}'_{\bar{r}}] \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{j}_q} &= 2 \sum_{\sigma} \beta^m_{i_\sigma} \phi_{i_1 \dots (m)_\sigma \dots i_p \bar{j}_q} \\
 &\quad - 2 \sum_{\sigma, \tau} S^{\bar{m}s}_{i_\sigma \bar{\tau}} \phi_{i_1 \dots (s)_\sigma \dots i_p \bar{j}_1 \dots (\bar{m})_\tau \dots \bar{j}_q}.
 \end{aligned}$$

This completes the proof.

Q.E.D.

Example. – For the Hessian metric g we have

$$(\square g)_{i\bar{j}} = -\beta_{i\bar{j}}.$$

Thus the Hessian metric g is \square -harmonic if and only if the second Koszul form $\beta = 0$. Therefore, by [12] the following conditions are equivalent :

- (i) g is \square -harmonic.
- (ii) The first Koszul form $\alpha = 0$.
- (iii) The second Koszul form $\beta = 0$.
- (iv) g is locally flat.

5. The local expression for \square_a .

Let F be a locally constant line bundle over a compact connected oriented Hessian manifold M , and let a be a fiber metric on F .

PROPOSITION 5.1. — We have

$$\delta_a = \delta + i(A),$$

where $A = -D \log a$ and $(i(A) \phi)_{i_1 \dots i_{p-1} \bar{j}_q} = A^r \phi_{r i_1 \dots i_{p-1} \bar{j}_q}$ for $\phi \in \Omega^{p,q}(F)$.

Proof. — By Definition 1.2, 1.7 and 2.3 we have

$$\begin{aligned} \delta_a &= (-1)^{n+1} \frac{\sqrt{G}}{a} * \partial \left(\frac{a}{\sqrt{G}} * \right) \\ &= (-1)^n * e(A) * + (-1)^{n+1} \sqrt{G} * \partial \left(\frac{1}{\sqrt{G}} * \right) \\ &= i(A) + \delta, \end{aligned}$$

where

$$(e(A) \phi)_{i_1 \dots i_{p+1} \bar{j}_q} = \sum_{\sigma} (-1)^{\sigma-1} A_{i_{\sigma}} \phi_{i_1 \dots \hat{i}_{\sigma} \dots i_{p+1} \bar{j}_q}$$

for $\phi \in \Omega^{p,q}(F)$.

Q.E.D.

DEFINITION 5.1. — For $\phi \in \Omega^{p,q}(F)$ we set

$$\bar{\nabla}'_{\bar{r}}{}^{i(a)} \phi = \frac{1}{a} \bar{\nabla}'_{\bar{r}}(a\phi).$$

THEOREM 5.1. — Let $\phi \in \Omega^{p,q}(F)$. Then we have

$$\begin{aligned} (\square_a \phi)_{i_p \bar{j}_q} &= -g^{\bar{s}r} \bar{\nabla}'_{\bar{r}}{}^{i(a)} \nabla'_s \phi_{i_p \bar{j}_q} + \alpha^s \nabla'_s \phi_{i_p \bar{j}_q} \\ &\quad + \sum_{\sigma} (-\beta^s_{i_{\sigma}} + B^s_{i_{\sigma}} \phi_{i_1 \dots (s)_{\sigma} \dots i_p \bar{j}_q} \\ &\quad + 2 \sum_{\sigma, \tau} S^{\bar{s}r} i_{\sigma} \bar{j}_{\tau} \phi_{i_1 \dots (s)_{\sigma} \dots i_p \bar{j}_1 \dots (\bar{r})_{\tau} \dots \bar{j}_q}. \end{aligned}$$

Proof. — By Proposition 5.1 we have

$$\square_a = \square + i(A) \partial + \partial i(A).$$

A straightforward calculation shows

$$\begin{aligned} (i(A) \partial \phi)_{i_p \bar{j}_q} + (\partial i(A) \phi)_{i_p \bar{j}_q} \\ = g^{\bar{s}r} A_{\bar{r}} \nabla'_s \phi_{i_p \bar{j}_q} + \sum_{\sigma=1}^p B^r_{i_{\sigma}} \phi_{i_1 \dots (r)_{\sigma} \dots i_p \bar{j}_q}. \end{aligned}$$

Thus our assertion follows from the above facts and Theorem 4.1.

Q.E.D.

6. A vanishing theorem of Kodaira-Nakano type.

Let θ be a symmetric covariant tensor field of degree 2. Considering θ as an element in $\Omega^{1,1}$ we define

$$\begin{aligned} e(\theta) : \Omega^{p,q} &\longrightarrow \Omega^{p+1,q+1}, \\ i(\theta) : \Omega^{p,q} &\longrightarrow \Omega^{p-1,q-1}, \end{aligned}$$

by $e(\theta) \phi = \theta \wedge \phi$ for $\phi \in \Omega^{p,q}$ and $i(\theta) = (-1)^{n+p+q+1} * e(\theta) *$.

Then $i(\theta)$ is the adjoint operator of $e(\theta)$ with respect to the inner product in Definition 1.1 and 2.2.

In this section we always assume that F is a locally constant line bundle over M .

PROPOSITION 6.1. — *We have*

- (i) $[\square_a, e(g)] = e(B + \beta)$,
- (ii) $[\square_a, i(g)] = -i(B + \beta)$.

The proof follows from a straightforward calculation and so it is omitted.

PROPOSITION 6.2. — *Suppose $\square_a \phi = 0$. Then we have*

- (i) $(e(B + \beta) i(g) \phi, \phi) \leq 0$.
- (ii) $(i(g) e(B + \beta) \phi, \phi) \geq 0$.
- (iii) $([i(g), e(B + \beta)] \phi, \phi) \geq 0$.

Proof. — By Proposition 6.1 (i) we have $\square_a e(g) \phi = e(B + \beta) \phi$. Thus we have

$$0 \leq (\square_a e(g) \phi, e(g) \phi) = (e(B + \beta) \phi, e(g) \phi) = (i(g) e(B + \beta) \phi, \phi),$$

which implies (ii). By the same way, since $\square_a i(g) \phi = -i(B + \beta) \phi$ we obtain

$$\begin{aligned} 0 \leq (\square_a i(g) \phi, i(g) \phi) &= (-i(B + \beta) \phi, i(g) \phi) \\ &= (\phi, -e(B + \beta) i(g) \phi), \end{aligned}$$

which shows (i). (iii) follows from (i) and (ii).

Q.E.D.

THEOREM 6.1. — *Let M be a compact connected oriented Hessian manifold. Denote by K the canonical line bundle over M . Let F be a locally constant line bundle over M .*

- (i) *If $2F + K$ is positive, then*

$$H^{p,q}(F) = 0 \quad \text{for } p + q > n.$$

- (ii) *If $2F + K$ is negative, then*

$$H^{p,q}(F) = 0 \quad \text{for } p + q < n.$$

Proof. — Suppose $2F + K$ is negative. Then $B + \beta$ is negative definite. Therefore $g' = -(B + \beta)$ gives a Hessian metric on M . If we denote by β' the Koszul form on M with respect to g' , then there exists a positive C^∞ -function f on M such that

$$\beta' = \beta + D^2 \log f.$$

If B is a Koszul form of F with respect to a fiber metric $a = \{a_\lambda\}$, then the Koszul form B' of F with respect to the fiber metric $a' = \{fa_\lambda\}$ satisfies

$$B' + \beta' = B + \beta = -g'.$$

Therefore if we use $-(B + \beta)$ as a Hessian metric, the formula in Proposition 6.2 (iii) is reduced to

$$([i(g), -e(g)] \phi, \phi) \geq 0 \quad \text{for } \phi \in \mathcal{H}^{p,q}(F).$$

Thus by Proposition 1.2 we have

$$(n - p - q) (\phi, \phi) \leq 0 \quad \text{for } \phi \in \mathcal{H}^{p,q}(F).$$

Therefore, if $n - p - q > 0$ then $\phi = 0$. Hence (ii) is proved. (i) follows from (ii) and Theorem 2.2

Q.E.D.

7. A vanishing theorem of Koszul type.

In this section we mention a vanishing theorem of Koszul type. Let M be a compact oriented hyperbolic affine manifold. Then there exists a canonical Hessian metric g and a unique Killing vector field H on M such that

$$D_X H = X, \tag{7.1}$$

for all vector field X on M [7]. The following theorem is essentially due to Koszul.

THEOREM 7.1. — *Let F be a locally constant vector bundle over a compact hyperbolic affine manifold. If there exist a fiber metric $a = \{a_{ij}\}$ and a constant $c (\neq -2q)$ such that*

$$Ha_{ij} = ca_{ij},$$

then we have

$$H^{p,q}(F) = 0, \quad \text{for } p > 0 \quad \text{and} \quad q \geq 0.$$

The proof of this theorem is nearly the same as Koszul [7], and so we omit the proof.

COROLLARY 7.1. — *Let M be a compact oriented hyperbolic affine manifold. Then we have*

$$H^{p,q}(1) = 0, \quad \text{for } p, q > 0,$$

where 1 is the trivial vector bundle over M .

The tensor bundle $\otimes^r T \otimes^s T^*$ satisfies the condition of Theorem 7.1 if $q - r + s \neq 0$.

We give another example of locally constant vector bundle over M which satisfies the conditions of Theorem 7.1. Let Ω be an open convex cone in \mathbf{R}^n with vertex 0 not containing any full straight line. Suppose that a discrete subgroup Γ of $GL(n, \mathbf{R})$ acts properly discontinuously and freely on Ω such that $M = \Gamma \backslash \Omega$ is compact. Assume further that there exist a linear mapping from Ω to the space of all $m \times m$ positive definite real symmetric matrices and a homomorphism from Γ to $GL(m, \mathbf{R})$, which are denoted by the same letter ρ , such that

$$\rho(\gamma x) = \rho(\gamma) \rho(x) {}^t \rho(\gamma) \quad \text{for } \gamma \in \Gamma, x \in \Omega.$$

We denote by F_ρ the vector bundle over M associated with the universal covering $\Omega \rightarrow M$ and ρ . Let U be an evenly covered open set in M . Choosing a section σ on U we set

$$a = (\rho \circ \sigma)^{-1}.$$

Then a is a fiber metric on F_ρ and we have

$$Ha = -a.$$

Therefore

COROLLARY 7.2. — *We have*

$$H^{p,q}(F_\rho) = 0 \quad \text{for } p > 0 \quad \text{and} \quad q \geq 0.$$

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