Annales de l'institut Fourier

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Annales de l'institut Fourier, tome 35, nº 2 (1985), p. 83-106
<http://www.numdam.org/item?id=AIF_1985__35_2_83_0>

Ann. Inst. Fourier, Grenoble
35, 2 (1985), 83-106.

EUCLIDEAN FIELDS HAVING A LARGE LENSTRA CONSTANT

par Armin LEUTBECHER

INTRODUCTION

In this note we present 143 new examples of Euclidean number fields K having a sufficiently large Lenstra constant. This constant, the maximal length of exceptional sequences in K, has been introduced by H. W. Lenstra Jr in [4] which is the basic reference. Lenstra there derives bounds $\alpha_{r,s}$ which guarantee that an algebraic number field K with r real and s complex Archimedean primes is Euclidean provided $M(K) > \alpha_{r,s}\sqrt{|d_K|}$, where d_K is the discriminant of K. In [6] J. Martinet and the author gave several applications of Lenstra's method of exceptional sequences. This paper is a supplement to [6]. The search for fields having a large Lenstra constant again revealed for several signatures n, r fields $K_{n,r}$ whose discriminant is smaller in absolute value than those of all other examples known before ($n = r + 2s$ is the degree).

This article ends with an updated table of the number of all known Euclidean fields counted according to their values of n and $r+s$ respectively (table 5). It is made on the model of table 11 in [4] which had been updated in [5] and [6]. For $n = 4$, $r + s = 2$ the quartic subfield of the 13^{th} cyclotomic field is counted, which was shown to be Euclidean by F. J. van der Linden using different methods [7]. There he also proves that this field and the 5^{th} cyclotomic field are the only complex cyclic fields of degree 4 which are Euclidean with respect to the norm [7], Theorem 10.30.

22 of the new Euclidean fields are imprimitive. Acknowledgement is due to J. Martinet whose advice concerning the subfields I have been following at several places.

Key-words : Units and factorization - Class number, discriminant - Euclidean rings and generalizations.

1. THE GRAPH OF EXCEPTIONAL UNITS

1.1. General observations.

Let R be a commutative ring with $1 \neq 0$ and R^* its group of units. Following [6] we call a sequence $\omega_1, \omega_2, \ldots, \omega_m$ of elements of R « exceptional sequence » if $\omega_i - \omega_j \in R^*$ for each pair of different indices i, j. This gives rise to the following construction. Two elements $a, b \in R$ shall be connected iff $a - b \in R^*$, thus R becomes a graph. The number $M = M(R)$ of vertices of a maximal complete subgraph, i.e. the maximal length of exceptional sequences in case $R = \mathbf{Z}_K$ is the ring of integers of an algebraic number field K, has been called the Lenstra constant of K. We are so free to call it the Lenstra constant of the ring R in this general context. All translations and all multiplications by elements $u \in R^*$ yield automorphisms of R as a graph. Therefore what $M(R)$ is concerned we can restrict the consideration to maximal complete subgraphs which contain 0 and 1 as vertices.

Every ring morphism $\varphi : R \to R'$ having $1_{R'} = \varphi(1_R)$ also gives rise to a morphism of the graph of R into the graph of R'. Therefore

$$M(R_0) \leqslant M(R) \leqslant M(R/I)$$

for each subring R_0 of R with the same unit element and for each ideal $I \neq R$. The elements $u \in R$ which are connected with 0 and 1 simultaneously will be called « exceptional units » following Nagell [9]. The set $E(R)$ of all exceptional units of R is finite in case $R = \mathbf{Z}_K$ is the ring of integers of an algebraic number field, a fact which to my knowledge was first proved by S. Lang [3].

The two transformations $\omega \mapsto 1 - \omega$ and $\omega \mapsto 1/\omega$ are automorphisms of the subgraph $E(R)$ of exceptional units. In general they generate a group of order six. Whenever $E(R)$ is not empty we thus have a group G of order 6 isomorphic to the symmetric group S_3 acting on $E(R)$.

It is convenient to extend the subgraph $\{0\} \cup R^*$ by a further element ∞, which is connected with each of these elements. On this graph R^*

operates by multiplication and also $\omega \mapsto 1/\omega$ is an automorphism. So

$$q_a(\omega) = a/\omega$$

is an automorphism of $\mathrm{R}^* \cup \{0,\infty\}$ for each $a \in \mathrm{R}^*$.

For each $x \in \mathrm{E(R)}$ there is a Kleinian four group operating on the set of neighbours of x, given by its elements of order two:

$$q_x(\omega) = x/\omega, \qquad r_x(\omega) = \frac{\omega - x}{\omega - 1}, \qquad s_x(\omega) = \frac{x(\omega - 1)}{\omega - x}.$$

1.2. Conditioned exceptional sequences.

We are considering number fields $\mathrm{K} = \mathbf{Q}(x)$ where x is a zero of a monic irreducible $f \in \mathbf{Z}[\mathrm{X}]$ in some fixed algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. For each $g \in \mathbf{Z}[\mathrm{X}]$ the element $g(x)$ is in the integral closure $\mathbf{Z}_\mathrm{K}$ of $\mathbf{Z}$ in K, however, $g(x) \in \mathbf{Z}_\mathrm{K}^*$ is possible only for primitive polynomials g. In case $g(x)$ is a unit the canonical ring morphism of $\mathbf{Z}[\mathrm{X}]$ into $\mathbf{Z}_\mathrm{K}$ by substitution $\mathrm{X} \mapsto x$ extends uniquely to a ring morphism φ_f of $\mathrm{Z}[\mathrm{X},1/g]$ into $\mathbf{Z}_\mathrm{K}$.

Starting with the ring of fractions $\mathbf{Z}[\mathrm{X}, 1/g]$ one gets for each complete subgraph S of $\mathbf{Z}[\mathrm{X},1/g]$ the complete subgraph $\varphi_f(\mathrm{S})$ in $\mathbf{Z}_\mathrm{K}$ which has the same number of vertices as S has. This can be done with a fixed g for various f.

The product formula of the resultant $\mathrm{R}(f,g)$ in terms of the zeros x_i of f and y_j of g shows that $g(x)$ is a unit in $\mathbf{Z}_\mathrm{K}$ iff $\mathrm{R}(f,g) = \pm 1$. Given a primitive $g \in \mathbf{Z}[\mathrm{X}]$ and a complete subgraph S of $\mathbf{Z}[\mathrm{X},1/g]$ one has to look for monic polynomials $\mathrm{F} \in \mathbf{Z}[\mathrm{X}]$ for which $\mathrm{R}(\mathrm{F},g) = \pm 1$. Then one has $\mathrm{R}(f,g) = \pm 1$ for each monic irreducible factor f of R. Similarly $\mathrm{R}(f,g) = \pm 1$ is equivalent to $\mathrm{R}(f,g_1) = \pm 1$ for each primitive irreducible factor g_1 of g. In case g_1 is also monic and y is a zero of g_1 the condition $\mathrm{R}(f,g_1) = \pm 1$ is the same as $f(y) \in \mathbf{Z}^*_{\mathrm{Q}(y)}$. This can be read as a system of $m = \deg g_1$ linear equations for the unknown coefficients of f, for which the right hand side depends on a parameter in the unit group $\mathbf{Z}^*_{\mathrm{Q}(y)}$. We shall speak of a condition of the first kind on f. In aid of a short notation we introduce as in [6] the set $\mathrm{U}(f)$ of those integral $y \in \bar{\mathbf{Q}}$, for which $f(y)$ is a unit in $\mathbf{Z}_{\mathrm{Q}(y)}$.

In case g_1 is not monic, we call $\mathrm{R}(f,g_1) = \pm 1$ a condition of the second king on f.

2. EXAMPLES IN THE NUMBER FIELD CASE

Starting with a suitable monic $g \in \mathbf{Z}[X]$ and a complete subgraph S in $\mathbf{Z}[X,1/g]$ with M vertices we construct monic polynomials $F \in \mathbf{Z}[X]$ of given degree n and resultant $R(F,g) = \pm 1$. Each irreducible factor f of F with a zero $x \in \bar{\mathbf{Q}}$ gives a field $K = \mathbf{Q}(x)$ and a complete subgraph $\varphi_f(S)$ of $\mathbf{Z}_K$ with likewise M elements. Let d_K denote the discriminant of $\mathbf{Z}_K$, r and s the number of real and complex Archimedean places of K respectively. If

$$M > \alpha_{r,s}\sqrt{|d_K|}$$

$\mathbf{Z}_K$ is already shown to be a Euclidean ring. In case

$$M \leqslant \alpha_{r,s}\sqrt{|d_K|} < L(K)$$

for the least ideal norm $L(K) > 1$ in $\mathbf{Z}_K$, we try to prove $M(K) > \alpha_{r,s}\sqrt{|d_K|}$ by enlarging $\varphi_f(S)$.

In aid of our examples we have used different sets of conditions of the first kind on polynomials $f \in \mathbf{Z}[X]$ which define the fields. For the purpose of formulating these conditions as well as for the description of certain subfields we need some concrete algebraic integers which are put together in table 1 (similar to and compatible with table 10 of [4]).

table 1

symbol	minimal polynomial	discriminant
θ	X^2-X-1	5
α	X^3-X-1	-23
γ	X^3+X-1	-31
η	X^3+X^2-2X-1	49
β	$X^4-X^3-X^2+X+1$	117
ρ	X^4-2X^3+X-1	-275
δ	X^4-X-1	-283
ε	X^4-2X^2+3X-1	-331
τ	X^4+3X^3-2X-1	-643

Besides ζ_m always denotes a primitive $m-th$ root of unity.

To start with

$$\mathrm{S}: 0, 1, x, x+1, x^2$$

is an exceptional sequence on condition $0, 1, -1, \theta \in \mathrm{U}(f)$.

$$\mathrm{T}: 0, 1, x, (x-1)/x, 1/(1-x)$$

is an exceptional sequence on condition $0, 1, \zeta_6 \in \mathrm{U}(f)$. Both have been used in [4]. $\mathrm{S} \cup \{\infty\}$ consists of three orbits of $\langle s_x \rangle$ and $\mathrm{T} \cup \{\infty\}$ consists of two orbits of the subgroup $\langle t \rangle$ of G whose generator t is acting by

$$t(\omega) = \frac{\omega - 1}{\omega}.$$

Several different enlargements of S have been studied in [6]. Thereby frequently use was made of the transformations s_x and q_{x^2+x}. In this paper the larger part of examples is derived from enlargements of T especially by full orbits of $\langle t \rangle$. A smaller part is based on S. In aid of a space saving notation of exceptional sequences we need in both cases some abbreviations. Their relevance is restricted to the last column of the main table 2 and that of table 3. Let $a = 1/t(x)$, $b = 1/t(x+1)$, $c = 1/t(x^2)$, $d = x^2 + x$, $e = x^3 - x$, beta $= x^4 - x^3 - x^2 + x + 1$, eta $= x^3 + x^2 - 2x - 1$. We denote by a dash the s_x image: $a' = s_x(a)$ etc. With this notation the most useful sequences in [6] have been

$$\begin{aligned} &\mathrm{A}: \mathrm{S}, a, a' \\ &\mathrm{B}: \mathrm{S}, b \\ &\mathrm{C}: \mathrm{S}, b', c, d. \end{aligned}$$

In this paper nearly all examples in the S line are based on

$$\mathrm{B}^*: \mathrm{B}, c$$

which is an exceptional sequence on condition

$$0, 1, -1, \theta, \alpha, \sqrt{2} \in \mathrm{U}(f).$$

As for the T line in [6] the following sequence was introduced

$$\mathrm{D}: \mathrm{T}, x - x^2$$

which is exceptional on condition $0, 1, \zeta_4, \zeta_6, \alpha^2 \in U(f)$. The sequence

$$\breve{D}: T, x - x^2, t(x-x^2), \quad t^{-1}(x-x^2)$$

is exceptional on one extra condition, namely $\beta^2 - \beta \in U(f)$. Furthermore the following abbreviations are used

$$u = x/t(x), \quad v = x/t(x-x^2), \quad w = x/t^{-1}(x-x^2),$$
$$y = q_x \circ s_u \circ q_x(x-x^2), \quad z = x/t(v).$$

Here $q_x \circ s_u \circ q_x$ viewed as a homographic map is given by the matrix

$$\begin{pmatrix} -x & x-1 \\ -1 & x \end{pmatrix}.$$

Thus it interchanges ∞ and x, 0 and $t(x)$, 1 and $t^{-1}(x)$. The images of $x - x^2$, $t(x-x^2)$, $t^{-1}(x-x^2)$ are y, $t^{-1}(y)$ and $t(y)$ respectively. Therefore the sequence $\breve{D}'$, consisting of 0,1 and the t orbits of x and y is exceptional on the same conditions as $\breve{D}$ is. Similarly the homographic map $q_{x-x^2} \circ s_\varepsilon \circ q_{x-x^2}$ with $\varepsilon = (x-x^2)/t(x-x^2)$ is given by the matrix

$$\begin{pmatrix} x^2-x & -x^2+x-1 \\ -1 & -x^2+x \end{pmatrix}.$$

It interchanges ∞ and $x - x^2$, 0 and $t(x-x^2)$, 1 and $t^{-1}(x-x^2)$, whereas the images of $x, t(x), t^{-1}(x)$ are $1/t^{-1}(y) = t(1/y)$, $1/y$ and $1/t(y) = t^{-1}(1/y)$ respectively. Therefore the sequence $\breve{D}''$ consisting of 0,1 and the t orbits of $x - x^2$ and $1/y$ again is exceptional on the same conditions as $\breve{D}$ is.

3. COMMENTS ON THE MAIN TABLE

3.1. Explanation.

Table 2 contains all Euclidean fields found since the publication of [6] with the exception of the quartic subfield of $\mathbf{Q}(\zeta_{13})$ mentioned in the introduction. Also two polynomials are listed in degree $n = 10$ whose discriminants already appear in [6].

Table 2 consists of 5 parts. The first part contains fields K of degree $n = 7$. In column 1 one finds the number r of real places. Column 2 gives the discriminant d_K of K and its factorization. In the third column the coefficients $a_0, a_1, \ldots, a_7$ of an irreducible monic polynomial f are listed, a zero x of which defines $K = \mathbf{Q}(x)$. The next column contains a lower bound on M(K) which guarantees

$$M(K) > \alpha_{r,s}\sqrt{|d_K|}$$

and the last column contains an exceptional sequence in $\mathbf{Z}_K$ in terms of x, which shows the lower bound to be valid. — The other parts are explained similarly. Each of them contains only fields of a fixed signature n, r. An extra column headed by K_0 gives a field generator θ_0 of a proper subfield K_0 of K of maximal degree if K is imprimitive, a blank otherwise. If K is imprimitive, the polynomial f defining K is in $K_0[X]$. The field generator θ_0 is one of the symbols of table 1 or $\theta_0 = \sqrt{-7}$ or in part 5 with quintic subfields K_0, $\theta_0 = \theta_{d_0}$ is defined in the free space of the 3^d column.

We conclude this subsection by some of the bounds $\alpha_{r,s}$ used for the column headed by $M \geqslant$

$n = 7$	$r = 1,$	$s = 3$	and	$r = 3,$	$s = 2:$	$9.848\ 10^{-3}$
				$r = 5,$	$s = 1:$	$7.793\ 10^{-3}$
$n = 8$	$r = 0,$	$s = 4$	and	$r = 2,$	$s = 3:$	$3.955\ 10^{-3}$
				$r = 4,$	$s = 2:$	$3.897\ 10^{-3}$
$n = 9$	$r = 1,$	$s = 4$	and	$r = 3,$	$s = 3:$	$1.563\ 10^{-3}$
$n = 10$	$r = 0,$	$s = 5$	and	$r = 2,$	$s = 4:$	$6.097\ 10^{-4}$

3.2. Exceptional sequences for some extra fields in table 2.

For several fields K of table 2 the exceptional sequence has the shape « S_0 in S_1 *rpb* S_2 » where S_0 is a subset of some conditioned exceptional sequence S_1 and S_2 is a subset of $\mathbf{Z}_K$. This should be read « S_0 replaced by S_2 ». Furthermore, special sequences are used in the following cases :

Part 2 : $n = 8,\ r = 2,\ d = -5\,371\,171$

$$0, 1, x, x^2, x^2 - 1, (x+1)/x, x^2/(x+1), (x^3+x^2-x-1)/x^2, (x^4-x^2)/(x^3-x-1), (x^4-2x^2)/(x^3-x-1).$$

Part 3 : $n = 8$, $r = 4$, $d = 15\,297\,613$.

In [6] an exceptional sequence of length 14 was given, namely

$$A, d, d', 1 - b', s_x(1-b'), e + 1, (e+1)/x, s_x((e+1)/x).$$

This sequence can be enlarged by the following three elements

$$x^4 - 2x^2 + 1, \quad x^3/(x^3-x^2-x+1), \quad (x^4-2x^2-x+1)/(x^2-x),$$

a fact found by the student G. Niklasch on occasion of an advanced course at the Technische Universität München in 1983.

Part 4 : $n = 9$, $r = 1$, $d = 35\,686\,793$

$$0, 1, x, x + 1, (x+1)/x^2, 1/(x^2-1), -1/(x^3-x^2-x), \\ (x^3+x^2-x-1)/x^2, (x^4-2x^2-x+1)/(x^4-2x^2), \\ (x^4-2x^2-x+1)/(x^3-x^2-x).$$

Part 5 : $n = 10$, $r = 0$, $d = -292\,693\,979$.

This is an example to show how more complicated exceptional sequences were found. At first one has the exceptional sequence $\bar{D}, v$. In the second step it was observed that the pair $t^{-1}(x-x^2), v$ can be replaced by the triplet $w, t \circ q_x \circ t^{-1}(v), s_w(t(x-x^2))$ and testing this new sequence again for possible enlargements it was found that the element x can be substituted by the two elements $s_w(x)$ $t(s_u(t(u)))$. Application of t^{-1} and rearrangement gives

$$0, 1, x, (x-1)/x, x - x^2, 1/(x^2-x+1), x/(x^2+1), -1/(x^3-x^2+x-1), \\ (x^3-x^2-1)/(x^3-x^2+x-1), (-x^4+2x^3-x^2+x)/(x^2-x+1), \\ (x^3-x^2)/(x^4-x^3+2x^2-2x+1).$$

By a similar procedure in the case of discriminant $d = -298\,482\,287$ an exceptional sequence of length 11 was found: in $D, v, t(y), z, t^{-1}(s_w(t(x-x^2)))$ the triplet $t(x), v, t(y)$ can be replaced by

$$u, x/s_w(t(x-x^2)), t(s_w(x)), t(x/t^{-1}(s_w(x-x^2))).$$

3.3. Subfields.

Our construction of the fields K with prescribed exceptional sequences yields irreducible monic polynomials $f \in \mathbf{Z}[X]$ defining $K = \mathbf{Q}(x)$ by a zero x of f. The discriminant of f may signal a possible proper subfield $K_0 \neq \mathbf{Q}$ of K.

TABLE 2 - PART 1

$n = 7$

r	d_K	$a_0, a_1, \ldots$ a_7	$M \geq$	exceptional sequence
1	$-397\,703 = -499 \cdot 797$	-1, 1, -2, 1, 0, 1, 0, 1	7	D, y
1	$-397\,991 = -11 \cdot 97 \cdot 373$	-1, 3, -7, 11, -12, 9, -5, 1	7	$D, t^{-1}(y)$
1	$-456\,107$ prime	-1, 4, -7, 10, -9, 7, -4, 1	7	$D, t^{-1}(y)$
3	$909\,929 = 19 \cdot 83 \cdot 577$	1, 0, -2, 1, 4, -2, -2, 1	10	a' in A rpb $e, e/x, q_d(c)$, beta/x
5	$-2\,616\,839 = -61 \cdot 42\,899$	-1, -1, 3, 6, -3, -5, 1, 1	13	q_d closure of $C, s_x(c/x)$

TABLE 2 - PART 2

$n = 8 \quad , \quad r = 2$

d_K	K_0	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
$-4\,542\,739$ prime		-1, 1, -1, -1, 4, -6, 5, -3, 1	9	$\bar{D}, u$
$-4\,570\,091 = -1\,249 \cdot 3\,659$		-1, 3, -4, 3, 0, -4, 4, -3, 1	9	$D, u, t^{-1}(y), x/s_u(t(x))$
$-4\,711\,123 = -43 \cdot 331^2$	ε	1, $-(\varepsilon-1)^2/\varepsilon^2$, 1	9	$\bar{D}, v$
$-4\,761\,667 = -23 \cdot 207\,029$		-1, 1, -1, -1, 3, -4, 4, -3, 1	9	D, v, w, y
$-4\,931\,267 = -11 \cdot 67 \cdot 6\,691$		-1, 2, -4, 4, -3, 1, 1, -2, 1	9	$\bar{D}, v$
$-5\,107\,019$ prime		1, -6, 13, -21, 24, -20, 12, -5, 1	9	$\bar{D}', v$

TABLE 2 - PART 2

$n = 8$, $r = 2$

d_K	K_0	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
$-5\,118\,587 = -29 \cdot 176\,503$		1, -2, 7, -11, 14, -13, 8, -4, 1	9	q_x closure of $D, t^{-1}(y)$
$-5\,155\,867 = -449 \cdot 11\,483$		-1, 1, -3, 2, -1, -1, 3, -2, 1	9	$\bar{D}, x/y$
$-5\,204\,491$ prime		1, 3, -1, -5, 2, 4, -3, -1, 1	10	$B, d/c, d/(x \cdot c), d/s_x(d/c)$, beta
$-5\,233\,147$ prime		-1, 0, 5, 2, -4, -1, -1, 0, 1	10	1 in C rpb $x^3, s_x(c/x)$
$-5\,272\,027 = -317 \cdot 16\,631$		-1, -1, -3, -2, 8, 3, -5, -1, 1	10	$B^*, a, b', c/x$
$-5\,344\,939 = -521 \cdot 10\,259$		-1, -4, -1, 8, 7, -5, -5, 1, 1	10	$C, s_x(d/c)$
$-5\,346\,947 = -839 \cdot 6\,373$		-1, -4, -3, 4, 8, -1, -5, 0, 1	10	$B^*, c/x, d/b', d/s_x(d/c)$
$-5\,359\,051$ prime		-1, -4, 0, 11, 5, -9, -4, 2, 1	10	$S, c/x, d/c, e, s_e(x^2)$, e/eta
$-5\,365\,936 = -67 \cdot 283^2$	δ	$1, -1-\delta^2.$ 1	10	$\bar{D}, t$ orbit of u
$-5\,369\,375 = -5^4 \cdot 11^2 \cdot 71$	ρ	$\rho^2-\rho-1, -\rho,$ 1	10	$B^*, c'/x, (d \cdot x)/c', e$
$-5\,371\,171 = -13 \cdot 413\,167$		-1, -3, -3, 2, 6, 2, -4, -1, 1	10	cf § 3.2
$-5\,420\,747$ prime		1, 3, 0, -7, 2, 6, -4, -1, 1	10	$B^*, c', c/x, s_x(e/\mathrm{eta})$
$-5\,525\,731 = -17 \cdot 325\,043$		-1, 3, -6, 9, -9, 7, -3, 0, 1	10	$\bar{D}$, t orbit of x/y
$-5\,671\,691 = -193 \cdot 29\,387$		-1, 0, 0, -3, 1, 5, -2, -2, 1	10	$0, 1, x, x^2, b', c', e/x, e, x \cdot e$, beta
$-5\,697\,179$ prime		-1, 0, 2, 1, 2, -1, -3, 0, 1	10	$B^*, d', e, r_x(e^2/x^2)$

TABLE 2 - PART 2

$n = 8$, $r = 2$

d_K	K_0	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
$-5\,756\,875 = -5^4 \cdot 61 \cdot 151$	θ	$1+\theta, -1-\theta, 2+\theta, -2-\theta, 1$	10	$x-x^2$ in $\bar{D}$ rpb $v, x\cdot y, 1/(x^3-x^2)$
$-5\,761\,792 = -2^8 \cdot 71 \cdot 317$		-1, -4, -4, 6, 8, -2, -5, 0, 1	10	B,b',c',d/b',d/c
$-5\,832\,251 = -131 \cdot 211^2$		1, 0, -5, -3, 7, 3, -4, -1, 1	10	B,c',d/c',e',d/e'
-5 902 219 prime		-1, -2, 0, 2, 1, 1, -2, -1, 1	10	$B^*, b/x, a/x^2, s_x((x+1)/c)$
$-5\,979\,187 = -31 \cdot 192\,877$		1, -1, 0, 0, 2, 2, -3, -1, 1	10	1 in B rpb $c', d/c', e, d/e, s_x(c\cdot d')$
$-6\,067\,456 = -2^8 \cdot 137 \cdot 173$		1, 2, -1, -2, 3, 0, -3, 0, 1	10	$B^*, b', c', (x\cdot d)/c$
$-6\,347\,227 = -53 \cdot 119\,759$		-1, -5, -3, 8, 8, -5, -5, 1, 1	10	$B^*, c/x, d/c', s_x(d/c)$
-6 350 923 prime		-1, -3, 0, 6, 2, -4, -3, 1, 1	10	$x+1, b'$ in C rpb $e/x, d/b', s_x((x+1)/c)$

TABLE 2 - PART 3

$n = 8$, $r = 4$

d_K	K_0	a_0, a_1, a_2	$M \geq$	exceptional sequence
$15\,297\,613 = 37 \cdot 643^2$	τ	$\tau, -1-\tau^{-1}, 1$	16	cf. § 3.2

TABLE 2 - PART 4

$n = 9 \quad , \quad r = 1$

d_K	K_0	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
29 510 281 = 101·292 181		-1, 3, -6, 8, -8, 5, -3, 1, -1, 1	9	D,t crbit of x/y
30 073 129 = 353·85 193		1, -1, 1, 1, -4, 7, -8, 7, -4, 1	9	$\bar{D}$,t orbit of v
30 453 593 = 137·222 289		-1, 3, -7, 11,-15, 16,-14, 10, -5, 1	9	$\bar{D}$,t orbit of v
30 544 313 = 47·649 879		1, -2, 4, -3, 1, 2, -4, 3, -2, 1	9	D,y,t(x)/y,$-1/x^2$
30 861 161 prime		-1, 3, -9, 15,-21, 23,-19, 12, -5, 1	9	$\bar{D}$,t orbit of v
31 042 889 = 37·47·17 851		-1, 3, -8, 12,-15, 15,-12, 8, -4, 1	9	$\bar{D}'$,t orbit of v
31 508 353 = 359·87 767		-1, 3, -6, 10,-12, 13,-10, 7, -4, 1	9	$\bar{D}$,t orbit of v
31 759 433 = 179·177 427		-1, 4, -8, 14,-18, 19,-16, 11, -5, 1	9	$\bar{D}$,t orbit of v
32 029 433 prime		1, -4, 9,-12, 12, -6, 0, 3, -3, 1	9	D,t orbit of $x/t^{-1}(y)$
32 031 161 = 61·525 101		-1, 2, -5, 9,-12, 13,-12, 8, -4, 1	9	$\bar{D}$,t orbit of v
32 058 553 prime		-1, 3, -7, 11,-14, 14,-11, 6, -3, 1	9	$\bar{D}$,t orbit of v
32 206 049 = 23^3·2647	α	-1, α^2, $-\alpha-1$,1	9	$\bar{D}$,t orbit of v
32 344 469 = 53·89·6857		-1, 2, -5, 7,-10, 12,-11, 8, -4, 1	9	$\bar{D}'$,t orbit of v
32 768 213 = 3 413,9 601		-1, 3, -5, 9,-11, 13,-12, 8, -4, 1	9	D,u,$t^{-1}(y)$,$x/s_u(t(x))$

TABLE 2 - PART 4

$n = 9 \quad , \quad r = 1$

d_K	K_0	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
32 855 993 = 113·290 761		-1, 3, -8, 14, -21, 23, -19, 12, -5, 1	9	$\bar{D}'$,t orbit of v
32 894 473 = 17·1 934 969		1, -3, 6, -9, 9, -6, 2, 2, -2, 1	9	$\bar{D}$,x/y
32 987 233 prime		1, -3, 7, -11, 13, -11, 7, -2, -1, 1	9	$\bar{D}'$,t orbit of w
33 121 433 = 1 321·25 073		-1, 3, -6, 10, -14, 14, -12, 8, -4, 1	9	$\bar{D}$,v
33 445 561 = 449·74 489		-1, 1, 3, -4, -7, 5, 6, -3, -2, 1	10	B,c',e',s_b(e'),beta
33 571 261 = 43·857·911		-1, 3, -7, 11, -15, 15, -13, 9, -4, 1	10	$\bar{D}$,t orbit of v
33 860 761 = $11^2 \cdot 23^4$	α	$-\alpha-1$, $\alpha^2+\alpha-1$, $-\alpha^2$, 1	10	1 in B rpb d/c,c',e,x·e,$s_x(a/x^2)$
33 984 793 prime		-1, 2, -6, 10, -13, 14, -12, 8, -4, 1	10	T,t orbits of v,w
34 090 153 = 71·480 143		1, -2, 3, -2, -1, 5, -6, 5, -3, 1	10	$\bar{D}'$,w,x/s_w(t(x))
34 349 041 prime		-1, 2, -5, 8, -10, 10, -8, 5, -3, 1	10	$\bar{D}'$,t orbit of w
34 573 709 prime		1, -1, 0, 4, -9, 12, -11, 8, -4, 1	10	$\bar{D}'$,t orbit of v
34 590 113 = 509·67 957		-1, 2, -6, 10, -14, 15, -12, 8, -4, 1	10	$\bar{D}'$,t orbit of v
34 628 113 = 13·2 663 701		-1, 4, -10, 19, -27, 29, -25, 15, -6, 1	10	$\bar{D}$,t orbit of x/$s_w(x-x^2)$
35 028 793 = $23^3 \cdot 2$ 879	α	$-\alpha$, α^{-2}, $-\alpha^{-2}-1$, 1	10	$\bar{D}$,t orbit of w

TABLE 4 - PART 4

$n = 9 \quad , \quad r = 1$

d_K	K_0	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
35 050 633 = 43·311·2 621		-1, 2, -5, 8, -11, 12, -10, 6, -3, 1	10	$\bar{D}$,t orbit of w
35 234 033 = 59·347·1 721		-1, 2, -4, 5, -5, 4, -3, 2, -2, 1	10	D,$t^{-1}(x-x^2)$,v,$t^{-1}(w)$,z
35 357 129 = 41·862 369		-1, 3, -7, 11, -15, 16, -13, 8, -4, 1	10	$\bar{D}$,t orbit of v
35 607 973 = 5 501·6 473		-1, 2, -5, 8, -12, 14, -13, 10, -5, 1	10	$\bar{D}$,t orbit of v
35 666 053 = 787·45 319		-1, 3, -6, 11, -14, 16, -14, 9, -4, 1	10	$\bar{D}'$,t orbit of v
35 678 113 = 199·179 287		-1, 4, -10, 19, -25, 26, -20, 11, -4, 1	10	$t^{-1}(x)$ in $\bar{D}$ rpb x/y,t(x)/y,x/t(y)
35 686 793 prime		-1, 2, -3, -8, 5, 10, -2, -5, 0, 1	10	cf. § 3.2
35 935 321 = 29·337·3 677		-1, 4, -10, 19, -25, 27, -22, 13, -5, 1	10	$\bar{D}'$,t orbit of v
36 055 441 = 41·879 401		-1, 3, -7, 10, -13, 13, -11, 8, -4, 1	10	T,w,$t^{-1}(z)$,t orbit of v
36 722 413 = 7^2·13·57 649		-1, 3, -6, 12, -16, 18, -16, 11, -5, 1	10	T,t$(x-x^2)$,u,v,x·y,x/t(w)
36 743 849 = 37·71^2·197		-1, 2, -5, 9, -13, 15, -14, 9, -4, 1	10	$\bar{D}$,t orbit of w
37 009 129 = 839·44 111		1, -1, 0, 5, -12, 17, -17, 12, -5, 1	10	$\bar{D}$,t orbit of $x/s_w(t^{-1}(x))$
37 065 113 = 5 849·6 337		-1, 2, -5, 9, -14, 16, -15, 11, -5, 1	10	$\bar{D}$,t orbit of v
37 086 373 = 23·1 612 451		-1, 3, -8, 14, -19, 20, -17, 10, -4, 1	10	$\bar{D}$,v,$s_w(t(x))$

TABLE 2 - PART 4

$n = 9$, $r = t$

d_K	K_0	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
37 232 393 = 11·157·21 559		1, -3, 7, -10, 10, -6, 1, 3, -3, 1	10	$\bar{D}'$, t orbit of v
37 354 501 prime		-1, 2, -5, 7, -9, 10, -9, 6, -3, 1	10	T, t(v), $x \cdot t^{-1}(w)$, t orbit of w
38 114 257 = 457·83 401		1, 5, 3, -10, -8, 10, 5, -5, -1, 1	10	B, c', d/c, (x+1)/c, $d/s_x(d/c)$
38 118 173 = 967·39 419		-1, 2, -4, 6, -7, 7, -7, 5, -3, 1	10	D, $t^{-1}(x-x^2)$, v, $t^{-1}(w)$, z
38 525 297 prime		-1, 4, -12, 21, -29, 30, -23, 13, -5, 1	10	$\bar{D}'$, t orbit of $s_w(x-x^2)$
38 577 961 = 19·2 030 419		-1, 3, -7, 12, -15, 15, -11, 6, -2, 1	10	$\bar{D}''$, t orbit of x/y
38 709 673 prime		-1, 3, -7, 11, -16, 18, -16, 11, -5, 1	10	$\bar{D}$, t orbit of v
39 067 993 prime		-1, 2, -6, 10, -14, 16, -14, 10, -5, 1	10	$\bar{D}$, t orbit of v
39 319 073 = 127·309 599		1, -3, 6, -8, 8, -5, 2, 1, -2, 1	10	$\bar{D}$, t orbit of $s_w(t(x))$
39 655 225 = 25·1 586 209		-1, 3, -6, 10, -13, 13, -12, 8, -4, 1	10	$\bar{D}$, v, w
39 724 513 prime		1, -2, 3, -2, -1, 5, -7, 7, -4, 1	10	$\bar{D}$, $x/s_w(t(x))$, $t(s_w(t(x)))$
40 167 241 prime		-1, 1, -2, 2, -2, 3, -4, 4, -3, 1	10	$\bar{D}$, t orbit of $-x^3+x^2+1$
40 172 057 prime		-1, 4, -9, 14, -18, 17, -13, 8, -4, 1	10	$\bar{D}$, t orbit of $s_w(t(x))$
40 237 633 = 4 253·9 461		1, -2, 4, -4, 1, 4, -7, 7, -4, 1	10	$\bar{D}'$, t orbit of v

TABLE 2 - PART 4

$n = 9$, $r = 1$

d_K	K_o	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
40 466 809 prime		-1, 2, -5, 8, -10, 11, -10, 7, -4, 1	10	$\bar{D}'$,t orbit of z
40 617 649 = 19·2 137 771		-1, 2, -4, 6, -7, 7, -6, 4, -3, 1	10	t(x) in $\bar{D}$ rpb u,v,$t^{-1}(s_w(t(x)))$
41 135 713 = 83·495 611		-1, 3, -8, 14, -20, 21, -18, 12, -5, 1	11	$\bar{D}$,t orbit of v
41 678 873 prime		-1, 2, -6, 9, -13, 15, -13, 9, -4, 1	11	$\bar{D}''$,t orbit of w
41 832 233 prime		-1, 3, -6, 11, -13, 14, -12, 8, -4, 1	11	$\bar{D}'$,t orbit of z
41 872 193 = 11·41·227·409		-1, 3, -7, 10, -13, 14, -12, 8, -4, 1	11	T,t orbits of v,w
41 940 761 = 479·87 559		-1, 4, -9, 16, -21, 22, -18, 12, -5, 1	11	$\bar{D}$,t orbit of $s_w(x-x^2)$
41 995 553 prime		1, -1, 0, 4, -10, 15, -15, 11, -5, 1	11	$\bar{D}''$,t orbit of $x/s_w(t^{-1}(x))$
42 156 761 = 127·331 943		-1, 3, -8, 13, -18, 20, -17, 11, -5, 1	11	$\bar{D}$,t orbit of v
42 173 713 = 487·86 599		1, -2, 3, -2, -2, 7, -9, 8, -4, 1	11	$\bar{D}$,t orbit of $s_w(t^{-1}(x))$
45 175 393 = 73·618 841		1, -1, 2, -1, -1, 3, -4, 4, -3, 1	11	T,t orbits of v,z
46 002 241 = 59·779 699		-1, 3, -8, 15, -23, 26, -23, 15, -6, 1	11	$\bar{D}'$,t orbit of v
47 100 349 = 41·59·19 471		-1, 3, -7, 13, -17, 19, -15, 9, -4, 1	11	$\bar{D}'$,t orbit of v
48 504 241 = 2 719·17 839		1, -3, 7, -11, 12, -9, 4, 1, -2, 1	11	$\bar{D}$,t orbit of $s_w(t^{-1}(x))$

TABLE 2 - PART 5

$n = 10 \quad , \quad r = 0$

d_K	K_0	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
$-209\ 352\ 647 = -23 \cdot 7^2 \cdot 431^2$	θ_{3017}	$-\theta, \theta^4+\theta^2-1, 1;$ $\quad \epsilon^5-\theta^2+1=0$	10	$\bar{D}$,t orbit of v
$-212\ 309\ 359 = -31 \cdot 2\ 617^2$	θ_{2617}	$-\theta^4+2\theta, \theta^4-\theta-1, 1;$ $\quad \theta^5+\theta^2-2\theta^2-1=0$	10	$\bar{D}$,t orbit of v
$-214\ 816\ 879 = -79 \cdot 17^2 \cdot 97^2$	θ_{1649}	$\theta, \theta^2-\theta, 1;$ $\theta^5-2\theta^4+3\theta^3-4\epsilon^2+2\theta-1=0$	10	$\bar{D}$,t orbit of v
$-215\ 067\ 767$ prime		1, -4, 10, -17, 25, -28, 26, -20, 12, -5, 1	10	$\bar{D}$,t orbit of v
$-216\ 670\ 707 = -3^5 \cdot 11^2 \cdot 7\ 369$	ζ_3	$-\zeta_3, \zeta_3, -1-2\zeta_3, 2+2\zeta_3, -2-\zeta_3, 1$	10	$\bar{D}'$,t orbit of w
$-220\ 506\ 347 = -37 \cdot 677 \cdot 8\ 803$		1, -2, 6, -9, 13, -14, 12, -9, 4, -2, 1	10	$\bar{D}'$,t orbit of z
$-224\ 198\ 759 = -71 \cdot 1\ 777^2$	θ_{1777}	$1, \theta+\theta^{-1}-1, 1;$ $\theta^5-\theta^4+2\theta^3-\theta^2+\theta-1=0$	10	$\bar{D}$,t orbit of u
$-226\ 173\ 952 = -2^{10} \cdot 220\ 873$	ζ_4	$-1, 1-\zeta_4, -2+\zeta_4, 2-\zeta_4, -2+\zeta_4, 1$	10	$\bar{D}$,t orbit of z
$-226\ 876\ 987 = -43 \cdot 2\ 297^2$	θ_{2297}	$-\theta^4+\theta^3+\theta, \theta^4-\theta-1, 1;$ $\quad \theta^5-\epsilon^4+\theta^3-\theta^2-1=0$	10	0,1,t orbits of $x-x^2$,u,x/y
$-227\ 991\ 979 = -599 \cdot 380\ 621$		1, -3, 8, -13, 18, -20, 18, -13, 8, -4, 1	10	$\bar{D}$,t orbit of v
$-229\ 203\ 619$ prime		1, -2, 1, 12, -2, -20, 3, 13, -3, -3, 1	10	B,c',d/a',s_x(d/a'),s_x(x·d/a')
$-233\ 043\ 179 = -97 \cdot 463 \cdot 5\ 189$		1, -3, 8, -14, 19, -20, 17, -11, 6, -3, 1	10	$\bar{D}''$,t orbit of x/y
$-236\ 438\ 047$ prime		1, -3, 8, -15, 22, -26, 26, -20, 12, -5, 1	10	$\bar{D}$,t orbit of v
$-243\ 226\ 747$ prime		1, -3, 9, -16, 24, -28, 25, -18, 10, -4, 1	10	$\bar{D}''$,t orbit of t(x)/y

TABLE 2 - PART 5

$n = 10$, $r = 0$

d_K	K_0	$a_0, a_1, \ldots$	$M \geq$	exceptional sequence
$-243\,415\,027 = -17 \cdot 14\,318\,531$		1, -2, 5, -9, 14, -18, 19, -16, 11, -5, 1	10	$\bar{D}, t$ orbit of v
$-244\,175\,707 = -19^2 \cdot 676\,387$		1, -3, 7, -12, 17, -19, 19, -15, 9, -4, 1	10	$\bar{D}, t$ orbit of v
$-246\,071\,287 = -7^5 \cdot 11^4$	$\sqrt{-7}$	$1, -3/2+\sqrt{-7}/2, 3/2-\sqrt{-7}/2, -2+\sqrt{-7}, -1-\sqrt{-7}, 1$	10	$\bar{D}'', t$ orbit of x/y
$-250\,679\,267$ prime		1, -2, 5, -8, 11, -13, 12, -9, 6, -3, 1	10	$\bar{D}, t$ orbit of w
$-252\,486\,127 = -787 \cdot 320\,821$		1, -3, 8, -14, 19, -21, 19, -14, 9, -4, 1	10	$\bar{D}', v, w$
$-254\,121\,139 = -149 \cdot 991 \cdot 1\,721$		1, -3, 6, -9, 10, -9, 8, -6, 5, -3, 1	10	$\bar{D}, t$ orbit of $s_w(t^{-1}(x))$
$-260\,270\,739 = -3^5 \cdot 163 \cdot 6\,571$	ζ_3	$-1, 1-\zeta_3, -2+\zeta_3, 2-2\zeta_3, -2+\zeta_3, 1$	10	$t(x), x-x^2$ in $\bar{D}$ rpb $v, w, x \cdot y, s_x(t^2(y))$
$-262\,864\,627 = -353 \cdot 744\,659$		1, -3, 7, -12, 17, -20, 20, -15, 9, -4, 1	10	$\bar{D}', t$ orbit of v
$-262\,909\,719 = -3^5 \cdot 73 \cdot 14\,821$	ζ_3	$-\zeta_3, 1+2\zeta_3, -2-3\zeta_3, 2+2\zeta_3, -2-\zeta_3, 1$	10	$\bar{D}', v, w$
$-268\,043\,264 = -2^{10} \cdot 261\,761$	ζ_4	$\zeta_4, 1-\zeta_4, -2+2\zeta_4, 2-\zeta_4, -2, 1$	10	$D, t^{-1}(x-x^2), v, t^{-1}(w), z$
$-274\,127\,275 = -5^2 \cdot 59 \cdot 185\,849$		1, -3, 8, -15, 22, -25, 23, -16, 9, -4, 1	11	$\bar{D}', t$ orbit of v
$-274\,550\,779 = -19 \cdot 14\,450\,041$		1, -3, 8, -14, 20, -23, 22, -16, 9, -4, 1	11	$\bar{D}, t$ orbit of v
$-277\,010\,267 = -107 \cdot 1\,609^2$	Θ_{1609}	$\theta, \theta^2-\theta, 1;\ \theta^5-2\theta^4+3\theta^3-3\theta^2+3\theta-1 = 0$	11	$\bar{D}', t$ orbit of $s_w(t(s_w(x)))$
$-281\,268\,947$ prime		1, -3, 7, -13, 20, -24, 25, -20, 12, -5, 1	11	$\bar{D}'', t$ orbit of w

TABLE 2 - PART 5

$n = 10$, $r = 0$

d_K	K_0	$a_0, a_1, \dots$	$M \geq$	exceptional sequence
$-283\,885\,691 = -8\,941 \cdot 31\,751$		1, -2, 5, -8, 11, -13, 14, -12, 8, -4, 1	11	$\bar{D}'$, t orbit of z
$-284\,152\,779 = -3^5 \cdot 1\,169\,353$	ζ_3	ζ_3, $-\zeta_3$, $-1+\zeta_3$, $1-\zeta_3$, -2, 1	11	D, t(x-x²), v, w, s_w(t(x)), t(z)
$-285\,267\,739$ prime		1, -3, 8, -13, 17, -18, 16, -12, 8, -4, 1	11	$\bar{D}'$, t orbit of v
$-290\,491\,759$ prime		1, -3, 9, -16, 24, -29, 28, -22, 13, -5, 1	11	$\bar{D}$, t orbit of z
$-292\,693\,979 = -127 \cdot 239 \cdot 9\,643$		1, -3, 7, -12, 17, -18, 17, -13, 8, -4, 1	11	cf. § 3.2
$-293\,025\,059$ prime		1, -2, 4, -6, 8, -10, 12, -11, 8, -4, 1	11	$\bar{D}'$, t orbit of v
$-298\,482\,287 = -461 \cdot 691 \cdot 937$		1, -2, 6, -10, 15, -19, 19, -15, 9, -4, 1	11	cf. § 3.2
$-300\,400\,367 = -97 \cdot 3\,096\,911$		1, -3, 7, -13, 19, -22, 22, -17, 11, -5, 1	11	$\bar{D}$, t orbit of v
$-303\,827\,627 = -499 \cdot 608\,873$		1, -3, 9, -17, 25, -29, 27, -20, 12, -5, 1	11	$\bar{D}'$, t orbit of v
$-306\,979\,327 = -67 \cdot 4\,581\,781$		1, -2, 4, -5, 5, -4, 4, -4, 4, -3, 1	11	$\bar{D}$, t orbit of x/s_w(t(x))
$-318\,337\,619 = -33 \cdot 197 \cdot 19\,469$		1, -3, 8, -14, 21, -24, 23, -18, 10, -4, 1	11	$\bar{D}$, t orbit of v
$-325\,246\,087 = -103 \cdot 1\,777^2$	θ_{1777}	$\theta-\theta^2+\theta^3, \theta^2, 1$; $\theta^5-\theta^4+2\theta^3-\theta^2+\theta-1=0$	11	0, 1, t orbits of x, w, s_w(t(x-x²))

The existence of K_0 is proven via a factorization of f over a possible candidate for a subfield which in turn can be guessed from the discriminant of K.

There are four imprimitive fields in Part 2. In part 4 one finds the first three examples of imprimitive Euclidean fields of degree 9. They all have the same cubic subfield $\mathbf{Q}(\alpha)$. Part 5 of the main table contains 14 totally imaginary imprimitive fields of degree $n = 10$. Seven of them are quadratic extensions of quintic fields (each of which with one real place) and seven of them are extensions of imaginary quadratic fields. The extension K of $K_0 = \mathbf{Q}(\sqrt{-7})$ of discriminant $-7^5.11^4$ is of special interest. $\pi = 2 + \sqrt{-7}$ is a prime of K_0 lying above 11, and the polynomial f defining K is congruent to $(x-2)^5 \pmod{\pi}$. Therefore K belongs to the classe of extensions L of K_0 unramified outside of (π) and tamely ramified above (π). Their root discriminant $|d_L|^{1/N}$ is less than $\sqrt{7}\sqrt{11} < 8.8$ (N being the degree of L over $\mathbf{Q}$). On the other hand the unconditioned lower bounds Odl_N of totally imaginary extensions L of $\mathbf{Q}$ of degree N are increasing with N and $\mathrm{Odl}_{20} > 9.8$. From this one concludes that K is the ray class field of K_0 with conductor (π).

4. LARGE LENSTRA CONSTANTS AND SMALL DISCRIMINANTS

As already noticed in [6] the experiments upon number fields of degree $n \leqslant 10$ revealed for various signatures n, r coincidence of small discriminant compared with the lower bounds of Odlyzko on the one hand and large Lenstra constant on the other hand. This observation has been confirmed by new examples in the meantime. For fields of degree 7 Diaz y Diaz determined the first 4 minima of discriminants in case of one real place [1] and the first 6 minima in case of three real places [2]. All these fields had been shown to be Euclidean fields because of their large Lenstra constant before [4], [6].

In table 3 of this section we gather some improvements on the lower bound of Lenstra's constant for the fields with least known discriminant in signatures $(n,r) = (6,2)$, $(6,4)$, ((7,1), (7,3), (7,5), (8,0), (8,2) and for two extra fields which are abelian. This table is explained in the same way as table 2 is. The only difference is that one finds in the column headed by $M \geqslant$ the best known lower bound on $M(K)$.

TABLE 3 LARGE LENSTRA CONSTANTS

n,r	d	$a_0, a_1, \ldots$	M ≥	exceptional sequence
6,2	28 037	1, -2, -3, 4, 1, -3, 1	12	$A, 1-b', d/(1-b'), e/a, s_x(e/x^2), s_x(x\cdot(1-b'))$
6,4	-92 779	-1, 0, 4, 2, -4, -1, 1	13	$B^*, a, b', c/x, s_b(d), -x^4+2x^3+x^2-x, (2x^2+x-1)/x^2$
6,6	13^5	-1, -3, 6, 4, -5, -1, 1	13	$A, d, d/a, d/a', s_x(d/a), s_x(d/a'), d/s_x(d/a')$
7,1	-184 607	-1, 2, -4, 6, -7, 5, -3, 1	12	$\bar{D}, v, t$ orbit of u
7,3	612 233	-1, -3, 1, 5, 0, -4, 0, 1	15	$B^*, 1/t(e), e'/x, d/c, c'/x, c'/b, s_b(b'), s_x(b'/b)$, beta
7,5	-2 306 599	-1, 1, 3, 1, -1, -3, 0, 1	13	$C, s_x(c/x), s_x(c'/x), d\cdot x/c', x^3$
8,0	1 257 728	1, -2, 4, -6, 8, -8, 7, -4, 1	12	$\bar{D}, v, t(v), s_w(t(x)), x/s_w(t^{-1}(x))$
8,0	$3^4\cdot 5^6$	1, -1, 0, 1, -1, 1, 0, -1, 1	12	$T, -1/x^2, t$ orbits of x^2, x^3+1
8,2	-4 296 211	1, -4, 9, -16, 20, -19, 13, -6, 1	12	$\bar{D}, u, t$ orbit of $x/s_w(t^{-1}(x))$

TABLE 4 SMALL DISCRIMINANTS

n,r	d	$a_0, a_1, \ldots$	
8,6	-65 106 259	1, 0, -5, -1, 7, 4, -4, -2, 1	2.28 %
9,1	29 510 281	-1, 3, -6, 8, -8, 5, -3, 1, -1, 1	0.92 %
9,3	-109 880 167	1, -3, 4, -3, -3, 9, -12, 10, -5, 1	1.02 %
9,5	453 771 377	-1, 0, 3, 0, 1, 2, -3, -3, 1, 1	1.64 %
9,7	-1 904 081 383	-1, -6, -6, 5, 11, 5, -6, -5, 1, 1	1.85 %
10,0	209 352 647	1, -3, 7, -11, 16, -19, 19, -16, 11, -5, 1	1.00 %
11,1	-5 869 649 839	-1, 3, -9, 17, -26, 33, -35, 30, -21, 12, -5, 1	1.22 %

TABLE 5

r+s \ n	1	2	3	4	5	6	7	8	9	10	Total
1	1	5									6
2		16	52	35							103
3			57	11	12	28					108
4				9	10	33	39	43			134
5					1	11	25	59	80	42	218
6						3	3	1	0	0	7
	1	21	109	55	23	75	67	103	80	42	576

Number of known Euclidean fields (april 1984)

After that we give in table 4 fields $K_{n,r}$ for seven different signatures (n,r) defined by an algebraic integer x whose minimal polynomial $\sum_{m=0}^{n} a_m X^m$ has the same discriminant as $K_{n,r}$ has and which is smaller in absolute value than the examples for that signature known before. These polynomials showed up in tests of constructing fields with prescribed exceptional sequences. The last column contains the relative excess of the root discriminant $|d_{n,r}|^{1/n}$ over the lower bound of $|d|^{1/n}$ given by Odlyzko under assumption of the generalized Riemann hypothesis. In degree $n \leqslant 10$ this bound is taken from [6], table 4, and for $n = 11$ table III of [8] has been used.

We finish by one example of a field K of signature 7,5 which might have a sufficiently large Lenstra constant. The polynomial $f = X^7 - 2X^6 - 2X^5 + 4X^4 + X - 1$ of discriminant $d = -2\,932\,823 = -17 \,.\, 172\,519$ is irreducible mod 2, and the field $K = \mathbf{Q}(x)$ generated by a zero x of f has 5 real places. The bound $\alpha_{7,5}$ given by Lenstra shows that $M(K) \geqslant 14$ would suffice to prove K to be Euclidean, but we only know $M(K) \geqslant 13$ from the exceptional sequence

$$0,\ 1,\ (x-1)/x,\ x-1,\ x^2-x,\ (x^2-1)/x,\ x^2-1,\ 1/(x^2-1),\\ x/(x^2-1),\ x^2/(x^2-1),\ x^2-x-1,\ 1/(x^2-x-1),\\ (-x^2+x)/(x^3-2x^2-x+1).$$

BIBLIOGRAPHY

[1] F. Diaz y Diaz, Valeurs minima du discriminant des corps de degré 7 ayant une seule place réelle, *C.R.A.S.*, Paris, 296 (1983), 137-139.

[2] F. Diaz y Diaz, Valeurs minima du discriminant pour certains types de corps de degré 7, *Ann. de l'Inst. Fourier*, 34-3 (1984), 29-38.

[3] S. Lang, Integral points on curves, *Publ. IHES*, (1960), N° 6.

[4] H. W. Lenstra jr., Euclidean number fields of large degree, *Invent. Math.*, 38 (1977), 237-254.

[5] H. W. Lenstra jr., Euclidean number fields, *Math. Intelligencer* 2, no. 1 (1979), 6-15; no. 2 (1980), 73-77, 99-103.

[6] A. Leutbecher and J. Martinet, Lenstra's constant and Euclidean number fields, *Astérisque*, 94 (1982), 87-131.

[7] F. J. van der Linden, *Euclidean rings with two infinite primes*, Thesis, Amsterdam, (1984).

[8] J. Martinet, Petits discriminants des corps de nombre, J. V. Armitage (éd.), Journées Arithmétiques 1980, Cambridge University Press, *LMS Lecture Notes séries*, 56 (1982), 151-193.

[9] T. Nagell, Sur un type particulier d'unités algébriques, *Ark. Mat.*, 8 (1969), 163-184.

[10] M. Pohst, On the computation of number fields of small discriminants including the minimum discriminants of sixth degree fields, *J. of Number Th.*, 14 (1982), 99-117.

Manuscrit reçu le 21 mai 1984.

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