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ON THE BOUNDARY LIMITS OF HARMONIC FUNCTIONS WITH GRADIENT IN L^p

by Yoshihiro MIZUTA

1. Introduction.

Let u be a function harmonic in the half space

$$\mathbf{R}^n_+ = \{x = (x_1, \ldots, x_n) ; x_n > 0\}$$

and satisfying the condition :

$$\int_G |\operatorname{grad} u(x)|^p x_n^\alpha \, dx < \infty \tag{1}$$

for any bounded open set $G \subset \mathbf{R}^n_+$, where $p > 1$ and $\alpha < p - 1$. For $\xi \in \partial \mathbf{R}^n_+$, $\gamma \geqslant 1$ and $a > 1$, set

$$T_\gamma(\xi, a) = \{x \in \mathbf{R}^n_+ ; |x - \xi| < a x_n^{1/\gamma}\}.$$

The existence of nontangential limits of u, that is, the limit of $u(x)$ as $x \longrightarrow \xi$, $x \in T_1(\xi, a)$, was studied by Carleson [1] ($n = p = 2$ and $0 \leqslant \alpha < 1$), Wallin [10] ($p = 2$ and $0 \leqslant \alpha < 1$) and Mizuta [6] in the present situation.

Recently Cruzeiro [2] proved the existence of $\lim u(x)$ as $x \longrightarrow \xi$, $x \in T_\gamma(\xi, a)$, for a harmonic function u satisfying (1) with $p = n$ and $\alpha = 0$. The existence of such limits for Green potentials in $\mathbf{R}^n_+$ was obtained by Wu [11]. Taking these results into account, we give the following theorem :

THEOREM. — *Let u be a function harmonic in $\mathbf{R}^n_+$ and satisfying (1) with $p > 1$ and $\alpha < p - 1$.*

(i) *If $n - p + \alpha > 0$, then for each $\gamma > 1$ there exists a set $E_\gamma \subset \partial \mathbf{R}^n_+$ such that $H_{\gamma(n-p+\alpha)}(E_\gamma) = 0$ and*

$$\lim_{x \to \xi, x \in T_\gamma(\xi, a)} u(x) \tag{2}$$

exists and is finite for any $\xi \in \partial \mathbf{R}^n_+ - E_\gamma$ *and any* $a > 1$.

(ii) *If* $n - p + \alpha = 0$, *then there exists a set* $E \subset \partial \mathbf{R}^n_+$ *such that* $B_{n/p,p}(E) = 0$ *and (2) exists and is finite for any* $\xi \in \partial \mathbf{R}^n_+ - E$, *any* $\gamma \geqslant 1$ *and any* $a > 1$.

(iii) *If* $n - p + \alpha < 0$, *then* $\lim_{x \to \xi, x \in \mathbf{R}^n_+} u(x)$ *exists and is finite for any* $\xi \in \partial \mathbf{R}^n_+$.

Here H_d denotes the d-dimensional Hausdorff measure, and $B_{\beta,p}$ the Bessel capacity of index (β, p) (cf. [3]). In view of [3 ; Theorems 21 and 22], one notes the following results :

a) If $q > 1$ and $n - d \geqslant \beta q$, then $H_d(E) < \infty$ implies $B_{\beta,q}(E) = 0$;

b) If $q > 1$ and $n - d < \beta q$, then $B_{\beta,q}(E) = 0$ implies $H_d(E) = 0$.

Recently Nagel, Rudin and Shapiro [7] studied tangential behaviors of Poisson integrals of potential type functions (see Sec. 4, Remark 2). Their results are not applicable to our case unless $p = 2$.

Remark. – The same result as in the theorem is also valid for a domain Ω for which any function v satisfying

$$\int_\Omega |\operatorname{grad} v(x)|^p \, \delta(x)^\alpha \, dx < \infty, \quad p > 1, \alpha < p - 1, \tag{3}$$

can be extended to a function satisfying (3) with Ω replaced by $\mathbf{R}^n$, where $\delta(x)$ denotes the distance from x to $\partial\Omega$. The special Lipschitz domains in [9; Chap. VI] are typical examples of Ω.

2. Lemmas.

First we note the following result, which follows readily from the fact in [4 ; p. 165].

LEMMA 1. – *Let* f *be a locally integrable function on* $\mathbf{R}^n$. *For* $\beta > 0$, *we set*

$$A_\beta = \{\xi \in \partial \mathbf{R}^n_+ \,;\, \limsup_{r \downarrow 0} r^{-\beta} \int_{B(\xi,r)} |f(y)|\, dy > 0\},$$

where $B(x, r)$ *denotes the open ball with center at* x *and radius* r. *Then* $H_\beta(A_\beta) = 0$.

By [3 ; Theorem 21] and the result of [4 ; p. 165], we have

LEMMA 2. – *Let* f *be a locally integrable function on* $\mathbf{R}^n$. *For* $p > 1$, *we set*

$$B_p = \{x \,;\, \limsup_{r \downarrow 0} (\log r^{-1})^{p-1} \int_{B(x,r)} |f(y)|\, dy > 0\}.$$

Then $B_{n/p,p}(B_p) = 0$.

Next we prove the following technical result.

LEMMA 3. – *Let* $c_1 > 0$, $c_2 > 0$, $\gamma \geqslant 1$, $p > 1$ *and* $p - n \leqslant \alpha < p - 1$. *Then*

$$\left\{ \int_{\{y \,;\, c_1 x_n < |x-y| < c_2 x_n^{1/\gamma}\}} |x - y|^{p'(1-n)} |y_n|^{-\alpha p'/p} dy \right\}^{1/p'} \leqslant C \begin{cases} x_n^{(p-n-\alpha)/p} & \text{if } n - p + \alpha > 0, \\ (\log x_n^{-1})^{1/p'} & \text{if } n - p + \alpha = 0, \end{cases}$$

where $1/p + 1/p' = 1$ *and* C *is a positive constant independent of* $x = (x_1, \ldots, x_n)$ *with* $0 < x_n < 1/2$.

Proof. – Let $x^* = (0, \ldots, 0, 1)$. By change of variables we see that the left hand side is equal to

$$x_n^{1-n-\alpha/p+n/p'} \left\{ \int_{E_1} |x^* - z|^{p'(1-n)} |z_n|^{-\alpha p'/p} dz \right\}^{1/p'} \leqslant C x_n^{(p-n-\alpha)/p} \left\{ \int_{E_2} (1 + |z|)^{p'(1-n)} |z_n|^{-\alpha p'/p} dz \right\}^{1/p'},$$

where $E_1 = \{z, c_1 < |x^* - z| < c_2\, x_n^{1/\gamma - 1}\}$,
$E_2 = B(0, (c_2 + 1)\, x_n^{1/\gamma - 1})$,

C is a positive constant independent of x with $0 < x_n < 1$. The required inequalities are established by estimating the last integral.

In the same manner we can prove

LEMMA 4. – *Let* $p > 1$ *and* $\alpha < p - n$. *Then*

$$\left\{ \int_{\{y\,;x_n/2 < |x-y| < |x|/2\}} |x-y|^{p'(1-n)} \, |y_n|^{-\alpha p'/p} \, dy \right\}^{1/p'} \leqslant C\,|x|^{(p-n-\alpha)/p}$$

for any $x \in \mathbf{R}^n_+$, *where* C *is a positive constant independent of* x.

Finally we borrow a result from [6 ; Lemma 4].

LEMMA 5. – *Let* $p > 1$, $\alpha < p - 1$ *and* f *be a measurable function on* $\mathbf{R}^n$ *such that* $\int_G |f(y)|^p \, |y_n|^\alpha \, dy < \infty$ *for any bounded open set* $G \subset \mathbf{R}^n$. *If we set*

$$E' = \left\{ \xi \in \partial \mathbf{R}^n_+ \,;\, \int_{B(\xi,1)} |\xi - y|^{1-n} \, |f(y)| \, dy = \infty \right\} ,$$

then $B_{1-\alpha/p,p}(E') = 0$.

Remark. – If $p - \alpha > n$, then one sees that E′ is empty.

3. Proof of the theorem.

Take q such that $q = p$ if $\alpha \leqslant 0$ and $1 < q < p/(\alpha + 1)$ if $\alpha > 0$. Then Hölder's inequality implies that

$$\int_G |\operatorname{grad} u(x)|^q \, dx < \infty$$

for any bounded open set $G \subset \mathbf{R}^n_+$. Hence the function

$$v(x) = \begin{cases} u(x_1, \ldots, x_{n-1}, x_n) & \text{for } x_n > 0, \\ u(x_1, \ldots, x_{n-1}, -x_n) & \text{for } x_n < 0, \end{cases}$$

can be extended to a locally q-precise function w on $\mathbf{R}^n$ in view of [8 ; Theorem 5.6] (for q-precise functions, see Ohtsuka [8 ; Chap. IV] and Ziemer [12]). Define

$$E' = \left\{ \xi \in \partial \mathbf{R}^n_+ \; ; \int_{B(\xi,1)} |\xi - y|^{1-n} \, |\mathrm{grad}\, w(y)| \, dy = \infty \right\}.$$

Then we have $B_{1-\alpha/p,p}(E') = 0$ on account of Lemma 5. We also define A_β and B_p with $f(y) = |\mathrm{grad}\, w(y)|^p \, |y_n|^\alpha$. By Lemmas 1 and 2, we see that $H_\beta(A_\beta) = 0$ and $B_{n/p,p}(B_p) = 0$.

First suppose $n - p + \alpha > 0$. Let $\gamma > 1$ be given. We shall show that (2) exists and is finite for any

$$\xi \in \partial \mathbf{R}^n_+ - (E' \cup A_{\gamma(n-p+\alpha)})$$

and any $a > 1$. Let $\xi \in \partial \mathbf{R}^n_+ - (E' \cup A_{\gamma(n-p+\alpha)})$. Take $N > 1$ such that $\xi \in B(0, N)$, and let ϕ_N be a function in $C_0^\infty(\mathbf{R}^n)$ such that $\phi_N = 1$ on $B(0, 2N)$ and $\phi_N = 0$ on $\mathbf{R}^n - B(0, 3N)$. Set $w_N = \phi_N w$. Then w_N is q-precise on $\mathbf{R}^n$ and satisfies

$$w_N(x) = c \sum_{i=1}^n \int \left(\frac{\partial}{\partial x_i} R_2 \right)(x - y) \frac{\partial w_N}{\partial y_i} \, dy \quad \text{for } x \in \mathbf{R}^n_+,$$

where c is a constant, $R_2(x) = \log(1/|x|)$ if $n = 2$ and $R_2(x) = |x|^{2-n}$ if $n \geqslant 3$. In fact, since w_N is continuously differentiable on $\mathbf{R}^n_+$, the right hand side is continuous on $\mathbf{R}^n_+$ and the required equality holds for any $x \in \mathbf{R}^n_+$ on account of Ohtsuka [8 ; Theorem 9.11].

For $x \in \mathbf{R}^n_+$, we write

$$w_N(x) = w_{N,1}(x) + w_{N,2}(x) + w_{N,3}(x),$$

where

$$w_{N,1}(x) = c \sum_{i=1}^n \int_{B(x, x_n/2)} \left(\frac{\partial}{\partial x_i} R_2 \right)(x - y) \frac{\partial w_N}{\partial y_i} \, dy,$$

$$w_{N,2}(x) = c \sum_{i=1}^n \int_{B(x, |\xi - x|/2) - B(x, x_n/2)} \left(\frac{\partial}{\partial x_i} R_2 \right)(x - y) \frac{\partial w_N}{\partial y_i} \, dy,$$

$$w_{N,3}(x) = c \sum_{i=1}^n \int_{\mathbf{R}^n - B(x, |\xi - x|/2)} \left(\frac{\partial}{\partial x_i} R_2 \right)(x - y) \frac{\partial w_N}{\partial y_i} \, dy.$$

Since w_N is harmonic on $B(0, 2N) \cap \mathbf{R}^n_+$, $w_{N,1}(x) = 0$ for $x \in B(0, N) \cap \mathbf{R}^n_+$. It follows from our assumption $\xi \notin E'$ that $\int |\xi - y|^{1-n} |\operatorname{grad} w_N(y)|\, dy < \infty$. Hence Lebesgue's dominated convergence theorem gives

$$\lim_{x \to \xi, x \in \mathbf{R}^n_+} w_{N,3}(x) = c \sum_{i=1}^{n} \int \left(\frac{\partial}{\partial x_i} R_2\right)(\xi - y) \frac{\partial w_N}{\partial y_i}\, dy .$$

For $w_{N,2}$ we apply Hölder's inequality to obtain by Lemma 3,

$$|w_{N,2}(x)|$$

$$\leqslant \text{const.} \left\{ \int_{\{y; x_n/2 < |x-y| < |\xi - x|/2\}} |x-y|^{p'(1-n)}\, |y_n|^{-\alpha p'/p}\, dy \right\}^{1/p'}$$

$$\times \left\{ \int_{B(x, |\xi - x|/2)} |\operatorname{grad} w_N(y)|^p\, |y_n|^\alpha\, dy \right\}^{1/p}$$

$$\leqslant \text{const.} \left\{ x_n^{p-\alpha-n} \int_{B(\xi, 2ax_n^{1/\gamma})} |\operatorname{grad} w(y)|^p\, |y_n|^\alpha\, dy \right\}^{1/p}$$

for $x \in B(0, N) \cap T_\gamma(\xi, a)$. Since $\xi \notin A_{\gamma(n-p+\alpha)}$,

$$\lim_{x \to \xi, x \in T_\gamma(\xi, a)} w_{N,2}(x) = 0 .$$

Thus

$$\lim_{x \to \xi, x \in T_\gamma(\xi, a)} u(x) = \lim_{x \to \xi, x \in T_\gamma(\xi, a)} w_N(x) = \lim_{x \to \xi, x \in T_\gamma(\xi, a)} w_{N,3}(x),$$

which is finite.

If $n - p + \alpha = 0$, then we can prove that (2) exists and is finite for any $\xi \in \partial \mathbf{R}^n_+ - (E' \cup B_p)$, any $\gamma \geqslant 1$ and any $a > 1$.

If $n - p + \alpha < 0$, then E' is empty, so that similar arguments yield the required assertion with the aid of Lemma 4.

4. Remarks.

Remark 1. – If $p > n$, $p - \alpha - n > 0$ and u is a locally p-precise function on $\mathbf{R}^n_+$ satisfying (1), then $u(x)$ has a finite limit as $x \longrightarrow \xi$, $x \in \mathbf{R}^n_+$, for any $\xi \in \partial \mathbf{R}^n_+$.

Remark 2. – Nagel, Rudin and Shapiro [7] studied tangential behaviors of functions of the form

$$(P_{x_n} * (K * g))(x'),\ x = (x', x_n) \in \mathbf{R}^n_+ ,$$

where P is the Poisson kernel in $\mathbf{R}^n_+$, K is a nonnegative kernel, which is radial and decreasing, and g is a function in $L^p(\mathbf{R}^{n-1})$. The function in our theorem has a boundary value in the Lipschitz space $\Lambda^{p,p}_\beta(\mathbf{R}^{n-1})$ with $\beta = 1 - (\alpha + 1)/p$ locally, provided $-1 < \alpha < p + 1$ (cf. [9; Chap. VI, §§ 4.3, 4.5]). We do not know whether functions f in $\Lambda^{p,p}_\beta(\mathbf{R}^{n-1})$ can be written as $f = K * g$, where K is an appropriate kernel function which is determined independently of f and $g \in L^p(\mathbf{R}^{n-1})$.

If $g \in L^p(\mathbf{R}^n)$, $\beta = 1 - (\alpha + 1)/p$ and $-1 < \alpha < p - 1$, then

$$F(x') = \int g_{1-\alpha/p}((x', 0) - y)\, g(y)\, dy$$

belongs to $\Lambda^{p,p}_\beta(\mathbf{R}^{n-1})$, where g_ℓ denotes the Bessel kernel of order ℓ (cf. [9; Chap. VI, § 4.3]). Hence $u(x) = P_{x_n} * F(x')$ satisfies

$$\int_0^\infty \left[x_n^{k-\beta} \left\{ \int_{\mathbf{R}^{n-1}} \left| \left(\frac{\partial}{\partial x_n}\right)^k u(x', x_n) \right|^p dx' \right\}^{1/p} \right]^p x_n^{-1}\, dx_n < \infty ,$$

where k is an integer greater than β, in view of [9; p. 152]. This implies, by the observation given after Lemma 4′ in [9; Chap. V], that u satisfies

$$\int_{\mathbf{R}^n_+} |\operatorname{grad} u(x)|^p\, x_n^\alpha\, dx < \infty. \tag{1)'$$

Thus our theorem is applicable to this function u.

Remark 3. – In case $p - \alpha - n = 0$, our theorem gives the best possible result as to the size of the exceptional set as the next proposition shows.

PROPOSITION. – *Let* E *be a compact set in* $\partial \mathbf{R}^n_+$ *with* $B_{1-\alpha/p,p}(E) = 0$, *where* $p > 1$ *and* $-1 < \alpha < p - 1$. *Then there exists a function* u *which is harmonic in* $\mathbf{R}^n_+$ *and satisfies (1)′ such that* $\lim_{x_n \downarrow 0} u(x', x_n)$ *does not exist for any* $(x', 0) \in E$.

Proof. – Since $B_{1-\alpha/p,p}(E) = 0$, we can find a nonnegative function $f \in L^p$ such that

$$F(x') \equiv \int g_{1-\alpha/p}((x',0)-y)f(y)\,dy = \infty$$

for any $(x',0) \in E$. As seen above, $P_{x_n} * F(x')$ satisfies $(1)'$.

Take a_1, b_1 and c_1 such that

$$0 < a_1 < 1,\ 0 < b_1 < c_1 < 1$$

and $P_{a_1} * F_1(x') > 1$ for $(x',0) \in E$, where

$$F_1(x') = \int_{\{b_1 < |y_n| < c_1\}} g_{1-\alpha/p}((x',0)-y)f(y)\,dy.$$

We proceed inductively and obtain $\{a_j\}$, $\{b_j\}$ and $\{c_j\}$ such that $0 < a_{j+1} < a_j$, $0 < c_{j+1} < b_j < c_j$, $P_{a_k} * F_j(x') < 2^{-j}$ if $k < j$ and $(x',0) \in E$ and $P_{a_j} * F_j(x') > j + \sum_{k=1}^{j-1} M_k$ if $(x',0) \in E$, where

$$F_j(x') = \int_{\{b_j < |y_n| < c_j\}} g_{1-\alpha/p}((x',0)-y)f(y)\,dy$$

and
$$M_k = \max\{F_k(x');\ (x',0) \in \partial \mathbf{R}^n_+\}.$$

Define

$$u(x',x_n) = \sum_{j=1}^{\infty} (-1)^j P_{x_n} * F_j(x').$$

Then one sees easily that $\{u(x',a_j)\}$ does not converge as $j \longrightarrow \infty$ for any $(x',0) \in E$. Since u satisfies $(1)'$ and is harmonic in $\mathbf{R}^n_+$, u is the required function.

Remark 4. – Let $p > 1$, $p - n < \alpha < p - 1$ and $\alpha > -1$. Then we can find a function u harmonic in $\mathbf{R}^n_+$ and satisfying the following conditions :

(i) u has a finite nontangential limit at 0;

(ii) $\limsup_{x \to 0,\, x \in T_\gamma(0,a)} u(x) = \infty$ for any $\gamma > 1$ and any $a > 1$;

(iii) $\int_{\mathbf{R}^n_+} |\operatorname{grad} u(x)|^p x_n^\alpha\,dx < \infty$.

To see this, let $x^{(j)} = (2^{-j}, 0, \ldots, 0$ and define

$$f_j(y) = \begin{cases} a_j |x^{(j)} - y|^{-1} & \text{if } y \in B(x^{(j)}, 2^{-j-2}) - \mathbf{R}^n_+, \\ 0 & \text{otherwise,} \end{cases}$$

where a_j is a positive number determined later. Setting

$$u(x) = \int (x_n - y_n)|x - y|^{-n} f(y)\, dy,$$

where $f = \sum_{j=1}^{\infty} f_j$, we note the following facts :

a) $u(0) \leqslant \int |y|^{1-n} f(y)\, dy \leqslant C_1 \sum_{j=1}^{\infty} a_j$;

b) If $x = (2^{-j}, 0, \ldots, 0, x_n)$ and $0 < x_n < 2^{-j-1}$, then $u(x) \geqslant C_2\, a_j \log(2^{-j}/x_n)$;

c) $\int f(y)^p\, |y_n|^\alpha\, dy \leqslant C_3 \sum_{j=1}^{\infty} a_j^p\, 2^{-j(n-p+\alpha)}$,

where C_1, C_2 and C_3 are positive constants independent of x and j.

Now we choose $\{a_j\}$ so that (a) and (c) are finite but $\limsup_{j \to \infty} j a_j = \infty$. Then (a) implies that u has a nontangential limit at 0, and (iii) follows from (c) and [5 ; Lemma 6](*). By (b) and the construction of $\{a_j\}$, (ii) is fulfilled. Thus u satisfies (i), (ii) and (iii).

Remark 5. – If $p > 1$ and $\alpha = p - n > -1$, then for each $\gamma > 1$ there exists a function u harmonic in $\mathbf{R}^n_+$ such that :

(i) u has a finite nontangential limit at 0 ;

(ii) $\limsup_{x \to 0,\, x \in T_{\gamma'}(\xi, a)} u(x) = \infty$ for any $\gamma' \geqslant \gamma$ and any $a > 1$;

(iii) $\int_{\mathbf{R}^n_+} |\text{grad}\, u(x)|^p\, x_n^\alpha\, dx < \infty$.

In fact we modify above f_j by setting

$$f_j(y) = \begin{cases} a_j |x^{(j)} - y|^{-1} & \text{if } y \in B(x^{(j)}, 2^{-j-2}) - B(x^{(j)}, 2^{-\gamma j}) - \mathbf{R}^n_+, \\ 0 & \text{otherwise,} \end{cases}$$

(*) One notes that the conclusion of [5 ; Lemma 6] is true in case $p > 1$ and $-1 < \alpha < p - 1$ if g in the lemma has compact support.

and consider $u(x) = \sum_{j=1}^{\infty} \int (x_n - y_n)\,|x-y|^{-n} f_j(y)\,dy$. Then as in Remark 3 we can choose $\{a_j\}$ such that (i), (ii) and (iii) hold.

Remark 6. – Let $p > 1$ and h be a positive function on the interval $(0,\infty)$ such that $\lim_{r\downarrow 0} h(r)(\log r^{-1})^{p-1} = 0$. Define for $\xi \in \partial \mathbf{R}_+^n$,

$$T_h(\xi) = \left\{ x \in \mathbf{R}_+^n \,;\, \log \frac{|x-\xi|}{x_n} \leqslant h(|x-\xi|)^{-1/(p-1)} \right\}.$$

If u is a function harmonic in $\mathbf{R}_+^n$ and satisfying (1) with $\alpha = p - n > -1$, then $\lim_{x \to \xi, x \in T_h(\xi)} u(x)$ exists and is finite for any $\xi \in \partial \mathbf{R}_+^n - (E' \cup B_h)$, where

$$B_h = \left\{ \xi \in \partial \mathbf{R}_+^n \,;\, \limsup_{r\downarrow 0} h(r)^{-1} \int_{B(\xi,r)\cap \mathbf{R}_+^n} |\mathrm{grad}\, u(x)|^p x_n^\alpha\, dx > 0 \right\}.$$

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