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THE LEVI PROBLEM FOR COHOMOLOGY CLASSES

by Mihnea COLTOIU

Introduction.

The aim of this paper is to extend some of the results of Andreotti and Norguet from [4] to complex spaces.

The paper is divided into two paragraphs :

1) The local problem

2) The global problem

In the first paragraph we prove the following

THEOREM 1. – *Let* X *be a perfect complex space,* $Y \subset X$ *an open subset,* $x_0 \in \partial Y$ *and* $\mathcal{F}$ *a sheaf which is locally free in a neighbourhood of* x_0. *Suppose* Y *is strongly pseudoconcave in* x_0 *and let* $n_0 = \dim \mathcal{O}_{X,x_0} > 0$. *Then* $H^{n_0 - 1}(Y, x_0, \mathcal{F})$ *contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in* x_0.

When X is a complex manifold this result was proved in [4] using a generalization of an integral formula of E. Martinelli. In the proof of Theorem 1 we use elementary results of local cohomology (one needs only supports consisting of a point) and the local structure theorems of a strongly pseudoconcave domain from [2].

The second paragraph is devoted to the generalization of Theorem 3 from [4]. More precisely we prove

THEOREM 2. – *Let* X *be a complex space and* $Y \subset\subset X$ *an open subset which is strongly* q*-pseudoconvex. Suppose* Y *is strictly* q*-pseudoconvex in every point of* $\partial Y \cap \mathrm{Reg}(X)$ *and let* $\mathcal{F} \in \mathrm{Coh}(X)$ *such that* $\partial Y \subset \mathrm{supp}(\mathcal{F})$. *Then there exists an element in* $H^q(Y, \mathcal{F})$ *which is not extendable in any point of* ∂Y.

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1. The local problem.

Let us briefly recall some definitions from [4] which will be used throughout this paper.

Let $\mathcal{F}$ be a sheaf of vector spaces on a topological space X, $Y \subset X$ an open subset and x_0 a point in ∂Y. Put :

$$H^r(Y, x_0, \mathcal{F}) = \varinjlim_{U \in \mathcal{V}_{x_0}} H^r(Y \cap U, \mathcal{F})$$

$$H^r_+(Y \cup \{x_0\}, \mathcal{F}) = \varinjlim_{U \in \mathcal{V}_{x_0}} H^r(Y \cup U, \mathcal{F})$$

$$H^r(x_0, \mathcal{F}) = \varinjlim_{U \in \mathcal{V}_{x_0}} H^r(U, \mathcal{F})$$

where $\mathcal{V}_{x_0}$ = the set of all open neighbourhoods U of x_0 in X.

We have $H^0(x_0, \mathcal{F}) = \mathcal{F}_{x_0}$ and $H^r(x_0, \mathcal{F}) = \{0\}$ for $r \geqslant 1$ (cf. [6, pp. 192-193]). Consider the natural restriction maps :

$$r_1 : H^r(x_0, \mathcal{F}) \longrightarrow H^r(Y, x_0, \mathcal{F})$$

$$r_2 : H^r_+(Y \cup \{x_0\}, \mathcal{F}) \longrightarrow H^r(Y, \mathcal{F}).$$

An element in $H^r(Y, x_0, \mathcal{F})$ (in $H^r(Y, \mathcal{F})$) will be called extendable in $x_0 \in \partial Y$ if it belongs to the image of the map r_1 (r_2 respectively).

Suppose now that X is a complex space. We say that Y is strongly pseudoconcave in x_0 if there exist an open neighbourhood U of x_0 in X and $\varphi \in C^\infty(U, \mathbf{R})$ a strongly plurisubharmonic function such that $U \cap Y = \{x \in U \mid \varphi(x) > \varphi(x_0)\}$.

If $x_0 \in \mathrm{Reg}(X)$ we say that Y is strictly q-pseudoconvex in x_0 if there exist an open neighbourhood U of x_0 and $\varphi \in C^\infty(U, \mathbf{R})$ such that :

i) $(d\varphi)_{x_0} \neq 0$

ii) $U \cap Y = \{x \in U \mid \varphi(x) < \varphi(x_0)\}$

iii) the restriction of the Levi form $\mathcal{L}(\varphi)$ to the analytic tangent hyperplane to ∂Y at x_0 is nondegenerate and admits precisely q strictly negative eigenvalues.

Let us also recall that a complex space X is called perfect if $\mathcal{O}_{X,x}$ is Cohen-Macauley for any $x \in X$. We denote by $H^{\cdot}_{x_0}(X, \cdot)$ the cohomology groups with support in $\{x_0\}$. In order to prove Theorem 1 we shall need the following statement

PROPOSITION 1. – *Let* X *be a perfect complex space (not necessarily reduced),* $x_0 \in X$ *and* $n_0 = \dim \mathcal{O}_{X,x_0} > 0$. *put* $L_{x_0} = \varinjlim_{U \in \mathcal{V}_{x_0}} H^{n_0}_{x_0}(U, \mathcal{O}_X)$. *Then* $\dim_{\mathbf{C}} L_{x_0} = \infty$.

The above proposition is an immediate consequence of [5, pp. 86, Corollaire 4.5.].

Remark 1. – If $U \in \mathcal{V}_{x_0}$ we have the exact sequence

$$H^{n_0-1}(U, \mathcal{O}_X) \longrightarrow H^{n_0-1}(U\backslash\{x_0\}, \mathcal{O}_X) \longrightarrow H^{n_0}_{x_0}(U, \mathcal{O}_X) \longrightarrow H^{n_0}(U, \mathcal{O}_X).$$

Taking inductive limit we get

$$L_{x_0} \cong H^{n_0-1}(X\backslash\{x_0\}, x_0, \mathcal{O}_X) \quad \text{for } n_0 \geqslant 2.$$

THEOREM 1. – *Let* X *be a perfect complex space,* $Y \subset X$ *an open subset,* $x_0 \in \partial Y$ *and* $\mathcal{F}$ *a sheaf which is locally free in a neighbourhood of* x_0. *Suppose* Y *is strongly pseudoconcave in* x_0 *and let* $n_0 = \dim \mathcal{O}_{X,x} > 0$. *Then* $H^{n_0-1}(Y, x_0, \mathcal{F})$ *contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in* x_0.

Proof. – Obviously, we may suppose $\mathcal{F} = \mathcal{O}_X$. Since the problem is local we also may suppose that X is a closed analytic subset of some open set $G \subset \mathbf{C}^N$ and that

$$Y = \{x \in X \mid \varphi(x) > \varphi(x_0)\},$$

where $\varphi \in C^\infty(G, \mathbf{R})$ is a strongly plurisubharmonic function.

Writing the Taylor expansion of φ at x_0 we get:

$$\varphi(x) = \varphi(x_0) + 2\,\mathrm{Re} f(x) + \mathcal{L}(\varphi)(x) + O(\|x - x_0\|^3)$$

where f is a polynomial of degree two in x and $\mathcal{L}(\varphi)$ is the Levi form. Let $g = f|_X$ and $Z_0 = \{x \in X \mid g(x) = 0\}$.

Replacing G by a smaller subset we may suppose that $Z_0 \backslash \{x_0\} \subset Y$. Moreover, using the perturbation argument in [7, pp. 357-358], we may suppose that the image of g in $\mathcal{O}_{X,x}$ is not a zero-divisor for any $x \in X$. Consider the space $(Z_0, \mathcal{O}_{Z_0})$ where $\mathcal{O}_{Z_0} = \mathcal{O}_X / g\, \mathcal{O}_X$. Since X is a perfect space and the image of g in $\mathcal{O}_{X,x}$ is not a zero-divisor for any $x \in X$ it follows that $(Z_0, \mathcal{O}_{Z_0})$ is also perfect.

Put $n_0 = \dim \mathcal{O}_{X,x_0}$, hence $n_0 - 1 = \dim \mathcal{O}_{Z_0,x_0}$, and let $L_{x_0} = \varinjlim_{U' \in \mathcal{V}'_{x_0}} H^{n_0-1}_{x_0}(U', \mathcal{O}_{Z_0})$ where $\mathcal{V}'_{x_0}$ = the set of all open neighbourhoods U' of x_0 in Z_0.

Consider the exact sequence of sheaves on Y

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot g} \mathcal{O}_X \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0. \tag{1}$$

If $U \subset X$ is an open neighbourhood of x_0, then (1) together with the long exact sequence of cohomology provide the exact sequence

$$H^q(Y \cap U, \mathcal{O}_X) \longrightarrow H^q(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \longrightarrow H^{q+1}(Y \cap U, \mathcal{O}_X) \tag{2}$$

where $U' = U \cap Z_0$ (recall that by choice of Z_0 we have $Y \cap U' = U' \setminus \{x_0\}$).

Consider first the case $n_0 \geqslant 3$. Making $q = n_0 - 2$ in (2) and taking inductive limit we get the exact sequence

$$H^{n_0-2}(Y, x_0, \mathcal{O}_X) \longrightarrow H^{n_0-2}(Z_0 \setminus \{x_0\}, x_0, \mathcal{O}_{Z_0}) \longrightarrow H^{n_0-1}(Y, x_0, \mathcal{O}_X). \tag{3}$$

By [2, Théorème 9] we get $H^{n_0-2}(Y, x_0, \mathcal{O}_X) = 0$. Since $H^{n_0-2}(Z_0 \setminus \{x_0\}, x_0, \mathcal{O}_{Z_0}) \cong L_{x_0}$, Proposition 1 implies that $\dim_{\mathbf{C}} H^{n_0-1}(Y, x_0, \mathcal{O}_X) = \infty$ hence the theorem is proved for $n_0 \geqslant 3$.

For $n_0 = 1$ the theorem is obvious, hence to conclude the proof we only have to deal with the case $n_0 = 2$. If $U \subset X$ is an open neighbourhood of x_0, then by (1) and the long exact sequence of cohomology we get the exact sequence

$$H^0(Y \cap U, \mathcal{O}_X) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \longrightarrow H^1(Y \cap U, \mathcal{O}_X) \tag{4}$$

where $U' = U \cap Z_0$.

By [2, Théorème 10] there exists a fundamental system of Stein neighbourhoods U of x_0 in X such that the restriction map $H^0(U, \mathcal{O}_X) \longrightarrow H^0(Y \cap U, \mathcal{O}_X)$ is bijective. The commutative diagram

$$\begin{array}{ccc} H^0(U, \mathcal{O}_X) & \xrightarrow{\sim} & H^0(Y \cap U, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ H^0(U', \mathcal{O}_{Z_0}) & \longrightarrow & H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \end{array}$$

and the surjectivity of the map $H^0(U, \mathcal{O}_X) \longrightarrow H^0(U', \mathcal{O}_{Z_0})$ imply that

$$\operatorname{Im}(H^0(Y \cap U, \mathcal{O}_X) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}))$$
$$= \operatorname{Im}(H^0(U', \mathcal{O}_{Z_0}) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0})),$$

hence there is a natural injection $H^1_{x_0}(U', \mathcal{O}_{Z_0}) \longrightarrow H^1(Y \cap U, \mathcal{O}_X)$. Taking inductive limit it follows that the map $L_{x_0} \longrightarrow H^1(Y, x_0, \mathcal{O}_X)$ is injective, hence by Proposition 1 we get $\dim_{\mathbf{C}} H^1(Y, x_0, \mathcal{O}_X) = \infty$, and we are done.

COROLLARY 1 [4, Proposition 6]. – *Let* Y *be an open subset of a complex manifold* X, $x_0 \in \partial Y$ *and suppose* Y *is strictly* q*-pseudoconvex in* x_0. *Let* $\mathcal{F}$ *be a sheaf which is locally free in a neighbourhood of* x_0. *Then* $H^q(Y, x_0, \mathcal{F})$ *contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in* x_0.

Proof. – We may suppose $\mathcal{F} = \mathcal{O}_X$ and $q > 0$ (the case $q = 0$ is obvious).

By definition of strictly q-pseudoconvexity it immediately follows that :

i) Y is strongly q-pseudoconvex in a neighbourhood of x_0.

ii) In some neighbourhood of x_0 there exists an analytic submanifold B containing x_0 such that $\dim B = q + 1$ and $B \cap Y$ is strongly pseudoconcave in x_0. By [2, Théorème 5] we deduce that the map

$$H^q(Y, x_0, \mathcal{O}_X) \longrightarrow H^q(B \cap Y, x_0, \mathcal{O}_B)$$

is surjective and using Theorem 1 we get $\dim_{\mathbf{C}} H^q(Y, x_0, \mathcal{O}_X) = \infty$.

Remark 2. – Let φ be a strongly plurisubharmonic function in some neighbourhood U of the origin in $\mathbf{C}^n$ $(n \geqslant 2)$, $(d\varphi)_0 \neq 0$ and put $Y = \{z \in U \mid \varphi(z) > \varphi(0)\}$. In suitable coordinates the Taylor expansion of φ at 0 has the form

$$\varphi(z) = \varphi(0) + 2\operatorname{Re} z_1 + \sum_{1 \leqslant j, k \leqslant n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(0)\, z_j \bar{z}_k + O(\|z\|^3).$$

Put exactly as in [4]

$$\psi_\alpha = \Big(\sum_{1\leq j\leq n} z_j^{\alpha_j}\bar z_j^{\alpha_j}\Big)^{-n} \sum_{1\leq j\leq n} (-1)^{j-1}\bar z_j^{\alpha_j} \bigwedge_{\substack{1\leq k\leq n\\ k\neq j}} d(\bar z_k^{\alpha_k}).$$

By [4, Proposition 5] it follows that the images of the differential forms $\psi_{\alpha+1}$ $(\alpha\in\mathbf{N}^n)$ in $H^{n-1}(Y\cap U, \mathcal{O})$ are linearly independent. Let M be the linear span of the above images.

We shall now investigate the relation between M and the vector space considered in the proof of Theorem 1 (which we denote now by L_1). Recall that L_1 is the kernel of the map α_1 = multiplication by z_1,

$$\alpha_1 : H^{n-1}(Y\cap U, \mathcal{O}) \longrightarrow H^{n-1}(Y\cap U, \mathcal{O}).$$

In the same way we define α_k = multiplication by z_1^k,

$$\alpha_k : H^{n-1}(Y\cap U, \mathcal{O}) \longrightarrow H^{n-1}(Y\cap U, \mathcal{O})$$

and put $L_k = \ker\alpha_k$, $L = \bigcup_{k=1}^{\infty} L_k$. We claim that $M\subset L$. To prove this inclusion we use the relation $z_1^{\alpha_1}\psi_\alpha = \bar\partial\mu_\alpha$ where

$$\mu_\alpha = \frac{1}{n-1}\Big(\sum_{1\leq j\leq n} z_j^{\alpha_j}\bar z_j^{\alpha_j}\Big)^{1-n} \wedge \sum_{2\leq j\leq n} (-1)^j \bar z_j^{\alpha_j} d(\bar z_k^{\alpha_k}).$$

This equality shows that the image of $\psi_{\alpha+1}$ in $H^{n-1}(Y\cap U, \mathcal{O})$ is contained in L_{α_1+1}, hence $M\subset L$.

2. The global problem.

α) Let U be an open subset of $\mathbf{C}^n$ and $\varphi\in C^\infty(U, \mathbf{R})$. Recall that φ is called strongly q-pseudoconvex $(0\leq q\leq n-1)$ if the Levi form $\mathcal{L}(\varphi)$ has at least $(n-q)$ strictly positive eigenvalues at any point in U. Using local embeddings in the Zarisky tangent space one easily extends the notion of strongly q-pseudoconvex function in the case of complex spaces (for details see [1, pp. 12-13]).

Remark 3. – Let X be a complex space and $\varphi : X \longrightarrow \mathbf{R}$ a strongly q-pseudoconvex function. For any $x\in X$ put $\mu(x) = \min\dim X_x^i$ where X_x^i are the irreducible components of X_x (X_x denotes the germ of X in x). From the above definitions it immediately follows that $q < \min\limits_{x \in X} \mu(x)$.

To state our theorem recall the following definition : an open subset $Y \subset\subset X$ is called strongly q-pseudoconvex if there exist an open neighbourhood V of ∂Y and $\varphi \in C^\infty(V, \mathbf{R})$ a strongly q-pseudoconvex function such that $V \cap Y = \{x \in V \mid \varphi(x) < 0\}$.

If $\mathcal{F} \in \mathrm{Coh}(X)$ and $Y \subset\subset X$ is strongly q-pseudoconvex we have [2, Théorème 11] $\dim_{\mathbf{C}} H^r(Y, \mathcal{F}) < \infty$ if $r \geqslant q + 1$.

As we already annouced in the introduction the aim of this paragraph is to prove the following

THEOREM 2. – *Let X be a complex space and $Y \subset\subset X$ an open subset which is strongly q-pseudoconvex. Suppose Y is strictly q-pseudoconvex in every point of $\partial Y \cap \mathrm{Reg}(X)$ and let $\mathcal{F} \in \mathrm{Coh}(X)$ such that $\partial Y \subset \mathrm{supp}(\mathcal{F})$. Then there exists an element in $H^q(Y, \mathcal{F})$ which is not extendable in any point of ∂Y.*

β) LEMMA 1. – *Let $Y \subset\subset X$ be an open subset such that Y is strongly q-pseudoconvex and let $A \subset X$ be an analytic closed subset such that $\dim_x A < \dim_x X$ for any $x \in A$. Then $\partial Y \setminus A$ is dense in ∂Y.*

Proof. – Let V be an open neighbourhood of ∂Y and $\varphi \in C^\infty(V, \mathbf{R})$ a strongly q-pseudoconvex function such that $V \cap Y = \{x \in V \mid \varphi(x) < 0\}$. Let's make a couple of remarks :

1) For any point $x \in A$ with X_x irreducible there exists a fundamental system of open neighbourhoods $(U_i)_{i \in \mathbf{N}}$ of x such that $U_i \setminus A$ is connected.

2) For any point $x \in \partial Y$ there exists a germ of analytic set Q_x passing through x, $\dim_x Q_x \geqslant 1$ and $\varphi | Q_x$ is strongly plurisubharmonic.

Assertion 1) is well known and 2) may be deduced from [8, pp. 46, Corollary 4] using the condition $q < \min\limits_{x \in \partial Y} \dim \mathcal{O}_{X,x}$ (which is a consequence of Remark 3). Let's show now that $\partial Y \setminus A$ is dense in ∂Y.

a) Take first $x_0 \in \partial Y \cap A$ such that X_{x_0} is irreducible and let $(U_i)_{i\in\mathbf{N}}$ be a fundamental system of open neighbourhoods of x_0 such that $U_i\setminus A$ is connected and $U_i \subset V$. We must prove that for any i $\partial Y \cap U_i \not\subset A \cap U_i$. If there existed an i_0 such that $\partial Y \cap U_{i_0} \subset A \cap U_{i_0}$ we would get

$$U_{i_0}\setminus A = [(U_{i_0} \cap Y)\setminus A] \cup [(U_{i_0} \cap \complement\, \overline{Y})\setminus A]$$

and since $U_{i_0}\setminus A$ is connected we would get $(U_{i_0} \cap \complement\, \overline{Y})\setminus A = \emptyset$, hence $U_{i_0} \subset \overline{Y}$. In particular we would have $\varphi \leqslant 0$ on U_{i_0}.

Since $\varphi(x_0) = 0$ and $\varphi|_{Q_{x_0}}$ is strongly plurisubharmonic the maximum principle yields a contradiction and we are done.

b) Take now $x_0 \in \partial Y \cap A$ and suppose that X_{x_0} is not irreducible. Let $X_{x_0} = \bigcup_{i=1}^{k_0} X^i_{x_0}$ be the decomposition of X_{x_0} into irreducible components. One may easily deduce that there exist $i_0 \in \{1, \dots, k_0\}$ and an open neighbourhood $U = U(x_0)$ of x_0 such that $X^{i_0}_{x_0}$ is induced in U by an irreducible subspace $Z = Z(x_0)$ with $x_0 \in \partial(Y \cap Z)$. On the other hand by Remark 3 we get that $q < \dim Z$. If we put $A' = A \cap Z$ and $\varphi' = \varphi|_Z$ it follows that $\dim A' < \dim Z$ and φ' is strongly q-pseudoconvex. Hence there exists a germ of analytic set Q'_{x_0} passing through x_0 with $\dim_{x_0} Q'_{x_0} \geqslant 1$, $Q'_{x_0} \subset Z$ and $\varphi'|_{Q'_{x_0}}$ is strongly plurisubharmonic. Since Z_{x_0} is irreducible the same reasoning as in a) shows that we may find a sequence $(x_n)_{n\in\mathbf{N}}$, $x_n \longrightarrow x_0$ and $x_n \in \partial(Y \cap Z)\setminus A'$. Lemma 1 is completely proved.

COROLLARY 2. – *Let* $Y \subset\subset X$ *be an open subset such that* Y *is strongly* q*-pseudoconvex and let* $\mathcal{F} \in \mathrm{Coh}(X)$ *such that* $\partial Y \subset \mathrm{supp}(\mathcal{F})$. *Then there exists an open subset* $D \subset X$ *such that :*

a) $D \subset \mathrm{Reg}(X)$

b) $\mathcal{F}|_D$ *is locally free of* rank $\geqslant 1$ (*the rank not being necessarily constant*)

c) $\partial Y \cap D$ *is dense in* ∂Y.

Proof. – Put $A_1 = \{x \in X \mid \mathcal{F}_x$ is not a free $\mathcal{O}_{X,x}$-module$\}$. It is well known that A_1 is an analytic closed subset of X and $\dim_x A_1 < \dim_x X$ for any $x \in A_1$. Put $D_1 = X\setminus(A_1 \cup \mathrm{Sing}(X))$

and $D = D_1 \cap \operatorname{supp}(\mathcal{F})$. By Lemma 1 we immediately deduce that D satisfies conditions a), b), c) and we are done.

γ) Let X be a complex space, $\mathcal{F} \in \operatorname{Coh}(X)$, $\mathcal{U} = (U_i)_{i\in\mathbf{N}}$ a locally finite open covering of X. Put :

$Z^p(\mathcal{U}, \mathcal{F}) =$ the group of p-cocycles with values in $\mathcal{F}$, with its natural topology of Fréchet space

$H^p(\mathcal{U}, \mathcal{F}) =$ the p-th group of Čech cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$

$H^p(X, \mathcal{F}) =$ the p-th cohomology group of $\mathcal{F}$ computed using the canonical resolution of Godement

$\Theta_{\mathcal{U}} : H^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$ the natural maps between the above groups.

If U_i is Stein for any i then $\Theta_{\mathcal{U}}$ are isomorphisms. Let now $X' \subset X$ be an open subset and $\mathcal{U}' = (U'_i)_{i\in\mathbf{N}}$ the covering defined by $U'_i = U_i \cap X'$. We have a commutative diagram :

$$\begin{array}{ccc} H^p(\mathcal{U}, \mathcal{F}) & \xrightarrow{\Theta_{\mathcal{U}}} & H^p(X, \mathcal{F}) \\ \downarrow & & \downarrow \\ H^p(\mathcal{U}', \mathcal{F}) & \xrightarrow{\Theta_{\mathcal{U}'}} & H^p(X', \mathcal{F}) \end{array}$$

Suppose now X is a complex manifold and E is a holomorphic vector bundle over X. Put $\mathcal{F} = \mathcal{O}(E)$ which is a locally free sheaf on X. Let $\mathcal{E}^{p,q}(E)$ be the sheaf of germs of C^∞ E-valued forms of type (p,q). Consider the Dolbeault resolution

$$0 \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{E}^{0,0}(E) \xrightarrow{\bar\partial} \mathcal{E}^{0,1}(E) \xrightarrow{\bar\partial} \cdots$$

Put :

$$Z^p(X, E) = \ker\{\Gamma(X, \mathcal{E}^{0,p}(E)) \xrightarrow{\bar\partial} \Gamma(X, \mathcal{E}^{0,p+1}(E))\}$$

with its natural topology of Fréchet space

$$B^p(X, E) = \operatorname{Im}\{\Gamma(X, \mathcal{E}^{0,p-1}(E)) \xrightarrow{\bar\partial} \Gamma(X, \mathcal{E}^{0,p}(E))\}$$

$$H^p_{\bar\partial}(X, E) = Z^p(X, E)/B^p(X, E).$$

Let $\psi = (\psi_i)_{i\in\mathbf{N}}$ be a partition of unity with respect to $\mathcal{U} = (U_i)_{i\in\mathbf{N}}$. Define $T_{\mathcal{U},\psi} : Z^p(\mathcal{U}, \mathcal{O}(E)) \longrightarrow Z^p(X, E)$ by

$$T_{\mathcal{U},\psi}(\xi) = \sum_{i_0 \ldots i_p} \xi_{i_0 \ldots i_p} \psi_{i_0} \bar\partial \psi_{i_1} \wedge \ldots \wedge \bar\partial \psi_{i_p}$$

$T_{\mathcal{U},\psi}$ is a continuous linear operator. The operator

$$T_{\mathcal{U}} : H^p(\mathcal{U}, \mathcal{O}(E)) \longrightarrow H^p_{\bar\partial}(X, E),$$

induced by $T_{\mathcal{U},\psi}$, does not depend on ψ. Furthermore if $\mathcal{U} = (U_i)_{i\in\mathbf{N}}$ is a Stein covering then $T_{\mathcal{U}}$ is an algebraic and topological isomorphism (cf. [3, pp. 225-227]).

Let now $X' \subset X$ be an open subset and $\mathcal{U}' = (U_i')_{i\in\mathbf{N}}$ the covering defined by $U_i' = U_i \cap X'$. Since $T_{\mathcal{U}}$ does not depend on ψ we get the following commutative diagram :

$$\begin{array}{ccc} H^p(\mathcal{U}, \mathcal{O}(E)) & \xrightarrow{T_{\mathcal{U}}} & H^p_{\bar\partial}(X, E) \\ \downarrow & & \downarrow \\ H^p(\mathcal{U}', \mathcal{O}(E)) & \xrightarrow{T_{\mathcal{U}'}} & H^p_{\bar\partial}(X', E) \end{array}$$

If $\mathcal{U} = (U_i)_{i\in\mathbf{N}}$ is a Stein covering of X we may define the isomorphism $H^p_{\bar\partial}(X, E) \longrightarrow H^p(X, \mathcal{O}(E))$ as the composed map $H^p_{\bar\partial}(X, E) \xrightarrow{T_{\mathcal{U}}^{-1}} H^p(\mathcal{U}, \mathcal{O}(E)) \xrightarrow{\Theta_{\mathcal{U}}} H^p(X, \mathcal{O}(E))$. One verifies immediately that the above isomorphism does not depend on $\mathcal{U}$ and denote this isomorphism by L_X. For any open subset $X' \subset X$ we have a commutative diagram :

$$\begin{array}{ccc} H^p_{\bar\partial}(X, E) & \xrightarrow{L_X} & H^p(X, \mathcal{O}(E)) \\ \downarrow & & \downarrow \\ H^p_{\bar\partial}(X', E) & \xrightarrow{L_{X'}} & H^p(X', \mathcal{O}(E)) \end{array}$$

δ) *Proof of Theorem 2*

We shall suppose $q > 0$ since the case $q = 0$ is well known. Let $\mathcal{U} = (U_i)_{i\in\mathbf{N}}$ be a locally finite Stein covering of Y and $D \subset X$ having properties a), b), c) from Corollary 2. Put $D' = D \cap Y$, $U_i' = U_i \cap D$, $\mathcal{U}' = (U_i')_{i\in\mathbf{N}} =$ a locally finite open covering of D'. Let $\psi = (\psi_i)_{i\in\mathbf{N}}$ be a partition of unity with respect to $\mathcal{U}'$ and let E be a holomorphic vector bundle over D such that $\mathcal{F}|_D \xrightarrow[\sim]{\sigma} \mathcal{O}(E)$.

Consider the linear continuous map

$$R : Z^q(\mathcal{U}, \mathcal{F}) \longrightarrow Z^q(D', E)$$

obtained by composition of the maps

$$Z^q(\mathcal{U}, \mathcal{F}) \longrightarrow Z^q(\mathcal{U}', \mathcal{F}) \xrightarrow[\sim]{\sigma} Z^q(\mathcal{U}', \mathcal{O}(E)) \xrightarrow{T_{\mathcal{U}',\psi}} Z^q(D', E).$$

Let V be an open neighbourhood of ∂Y and let $\varphi \in C^\infty(V, \mathbf{R})$ be a strongly q-pseudoconvex function such that

$$V \cap Y = \{x \in V \mid \varphi(x) < 0\}.$$

Let $(p_j)_{j \in \mathbf{N}} \subset \partial Y \cap D$ be a dense subset of points of $\partial Y \cap D$, $p_i \neq p_j$ for $i \neq j$.

For each $j \in \mathbf{N}$ we may find a neighbourhood $V_j \subset\subset V \cap D$ of p_j and we may find in V_j:

– q-discs $D_{\nu,j}(r)$ $0 < r \leqslant r_j$ $\nu \in \mathbf{N}^*$ having the properties from the proof of [4, Théorème 3]

– $L_j \subset V_j$ closed submanifolds such that $L_j \cap \overline{Y} = \{p_j\}$ (here L_j corresponds to the set A in the proof of [4, Proposition 6])

– differential forms $t^j_\alpha \in Z^q(V_j \backslash L_j, E)$ $(\alpha \in \mathbf{N}^{q+1})$ such that the following holds :

for any element of the form $t_j = \sum_\alpha c_\alpha t^j_{\alpha+1}$ $c_\alpha \in \mathbf{C}$ (the sum being finite and not all of the c_α's being zero) there exists an E^*-valued $(q, 0)$ holomorphic form γ_j on V_j (E^* is the dual of E) such that $\lim_{\nu \to \infty} \left| \int_{D_{\nu,j}(r_j)} \gamma_j \wedge t_j \right| = \infty$.

Let $\rho_j \in C_0^\infty(V, \mathbf{R})$, $\rho_j \geqslant 0$, $\rho_j | L_j = 0$, $\rho_j > 0$ on $\partial Y \backslash \{p_j\}$ and choose $\epsilon_j > 0$ such that $\varphi - \epsilon_j \rho_j$ is strongly q–pseudoconvex on V. Putting $Y_j = Y \cup \{x \in V \mid \varphi(x) - \epsilon_j \rho_j(x) < 0\}$ we get $\overline{Y} \backslash \{p_j\} \subset Y_j$, $p_j \in \partial Y \cap \partial Y_j$ and $Y_j \cap L_j = \emptyset$.

Take now $h_i \in C_0^\infty(V_j, \mathbf{R})$, $h_j \geqslant 0$, $h_j(p_j) > 0$ and $\epsilon'_j > 0$ such that $\varphi - \epsilon_j \rho_j - \epsilon'_j h_j$ is strongly q-pseudoconvex on V and put $V'_j = \{x \in V_j \mid \varphi(x) - \epsilon_j \rho_j(x) - \epsilon'_j h_j(x) < 0\}$ and $Y'_j = Y_j \cup V'_j$. Then V'_j is an open neighbourhood of p_j, $Y_j \cap V'_j = Y_j \cap V_j$ and Y'_j is strongly q-pseudoconvex, hence $\dim_{\mathbf{C}} H^{q+1}(Y'_j, \mathcal{F}) < \infty$.

Let $S_j \subset Z^q(Y_j \cap V_j, E)$ be the linear span of the elements of the form $t^j_{\alpha+1}$ $(\alpha \in \mathbf{N}^{q+1})$ and let $K_j \subset H^q(Y_j \cap V_j, \mathcal{F})$ be the image of S_j by the map

$$\delta_j : Z^q(Y_j \cap V_j, E) \longrightarrow H^q(Y_j \cap V_j, \mathcal{F})$$

obtained by composing the maps

$$Z^q(Y_j \cap V_j, E) \longrightarrow H^q_{\bar\partial}(Y_j \cap V_j, E) \xrightarrow[\sim]{L_{Y_j \cap V_j}} H^q(Y_j \cap V_j, \mathcal{O}(E)) \xrightarrow[\sim]{\sigma} H^q(Y_j \cap V_j, \mathcal{F}).$$

By [4, Proposition 6] we have $\dim_{\mathbf{C}} K_j = \infty$. By Mayer-Vietoris exact sequence

$$H^q(Y_j, \mathcal{F}) \oplus H^q(V_j', \mathcal{F}) \xrightarrow{\alpha_j} H^q(Y_j \cap V_j, \mathcal{F}) \xrightarrow{\beta_j} H^{q+1}(Y_j', \mathcal{F})$$

and by the conditions $\dim_{\mathbf{C}} K_j = \infty$, $\dim_{\mathbf{C}} H^{q+1}(Y_j', \mathcal{F}) < \infty$ there exists $d_j \in K_j \setminus \{0\}$ such that $\beta_j(d_j) = 0$. Let $t_j \in S_j$ such that $\delta_j(t_j) = d_j$ and let $\xi_j \in H^q(Y_j, \mathcal{F}), v_j \in H^q(V_j', \mathcal{F})$ such that $\xi_j|_{Y_j \cap V_j} - v_j|_{Y_j \cap V_j} = d_j$.

If $V_j'' \subset V_j'$ is a Stein neighbourhood of p_j we have $\xi_j | Y_j \cap V_j'' = d_j$. Put $\xi_j' = \xi_j|_Y$ and let $\tau_j \in Z^q(\mathcal{U}, \mathcal{F})$ be such that ξ_j' is the image of τ_j by the map

$$Z^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(\mathcal{U}, \mathcal{F}) \xrightarrow[\sim]{\Theta_{\mathcal{U}}} H^q(Y, \mathcal{F}).$$

Let η_j be the restriction of τ_j on D', i.e. $\eta_j = R(\tau_j)$.

We claim that for any point p_s and for any $j \in \mathbf{N}$ there exist a Stein neighbourhood U_s^j of p_s, $U_s^j \subset D$, and an E-valued C^∞ form λ_s^j of type $(0, q-1)$ on $V_s^j = Y \cap U_s^j$ such that

a) $\eta_j|_{V_s^j} = \bar\partial \lambda_s^j$ for $j \neq s$

b) $\eta_j|_{V_j^j} = t_j + \bar\partial \lambda_j^j$ for $j = s$.

The claim can be proved like this : for any $s \neq j$ take U_s^j a Stein neighbourhood of p_s contained in $Y_j \cap D$ and for $s = j$ take $U_j^j = V_j''$.

Let $\mathcal{M}_s^j$ be the Stein covering of V_s^j given by $\{U_i \cap V_s^j | i \in \mathbf{N}\}$. We have a commutative diagram

$$\begin{array}{ccccccc}
H^q(\mathcal{U}, \mathcal{F}) & \longrightarrow & H^q(\mathcal{M}_s^j, \mathcal{F}) & \xrightarrow[\sim]{\sigma} & H^q(\mathcal{M}_s^j, \mathcal{O}(E)) & \xrightarrow[\sim]{T_{\mathcal{M}_s^j}} & H^q_{\bar\partial}(V_s^j, E) \\
\wr\downarrow \Theta_{\mathcal{U}} & & \wr\downarrow \Theta_{\mathcal{M}_s^j} & & \wr\downarrow \Theta_{\mathcal{M}_s^j} & & \wr\downarrow L_{V_s^j} \\
H^q(Y, \mathcal{F}) & \longrightarrow & H^q(V_s^j, \mathcal{F}) & \xrightarrow[\sim]{\sigma} & H^q(V_s^j, \mathcal{O}(E)) & \xrightarrow[\sim]{\mathrm{id}} & H^q(V_s^j, \mathcal{O}(E))
\end{array}$$

which gives us a). Property b) can be deduced from the following diagram

$$\begin{tikzcd}[column sep=small]
\mathrm{H}^q(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d, "\Theta_{\mathfrak{U}}", "\wr"'] & \mathrm{H}^q(\mathfrak{N}_j^j,\mathscr{F}) \arrow[r, "\underset{\sim}{\sigma}"] \arrow[d, "\Theta_{\mathfrak{N}_j^j}", "\wr"'] & \mathrm{H}^q(\mathfrak{N}_j^j,\mathcal{O}(\mathrm{E})) \arrow[r, "\underset{\sim}{\mathrm{T}_{\mathfrak{N}_j^j}}"] \arrow[d, "\Theta_{\mathfrak{N}_j^j}", "\wr"'] & \mathrm{H}^q_{\partial}(\mathrm{V}_j^j,\mathrm{E}) \arrow[r] \arrow[d, "\mathrm{L}_{\mathrm{V}_j^j}", "\wr"'] & \mathrm{H}^q_{\partial}(\mathrm{Y}_j\cap \mathrm{V}_j,\mathrm{E}) \arrow[d, "\mathrm{L}_{\mathrm{Y}_j\cap \mathrm{V}_j}", "\wr"'] \\
\mathrm{H}^q(\mathrm{Y},\mathscr{F}) \arrow[r] & \mathrm{H}^q(\mathrm{V}_j^j,\mathscr{F}) \arrow[r, "\underset{\sim}{\sigma}"] & \mathrm{H}^q(\mathrm{V}_j^j,\mathcal{O}(\mathrm{E})) \arrow[r, "\underset{\sim}{id}"] & \mathrm{H}^q(\mathrm{V}_j^j,\mathcal{O}(\mathrm{E})) & \mathrm{H}^q(\mathrm{Y}_j\cap \mathrm{V}_j,\mathcal{O}(\mathrm{E})) \arrow[l] \arrow[dl, "\underset{\sim}{\sigma}"] \\
 & & & \mathrm{H}^q(\mathrm{Y}_j\cap \mathrm{V}_j,\mathscr{F}) \arrow[ull] &
\end{tikzcd}$$

Let now γ_j be an E^*-valued holomorphic $(q, 0)$ form on V_j such that

1) $\lim\limits_{\nu\to\infty} \left| \int_{\mathrm{D}_{\nu,j}(r_j)} \gamma_j \wedge t_j \right| = \infty .$

Using 1), relations a), b), Stokes' theorem and the fact that for any $0 < r \leqslant r_j$ we have $\bigcup\limits_{\nu=1}^{\infty} [\mathrm{D}_{\nu,j}(r_j) \backslash \mathrm{D}_{\nu,j}(r)] \subset\subset \mathrm{D}'$ it follows that

2) $\lim\limits_{\nu\to\infty} \left| \int_{\mathrm{D}_{\nu,j}(r_j)} \gamma_j \wedge \eta_j \right| = \infty$

and

3) $\left| \int_{\mathrm{D}_{\nu,j}(r_j)} \gamma_j \wedge \eta_s \right| \leqslant p_s^j$ if $j \neq s$

where $0 < p_s^j < \infty$.

Let $k_j > 0$ be sufficiently small real numbers such that for $|c_j| < k_j, c_j \in \mathbf{C}$, the series $\sum\limits_j c_j \tau_j$ converges in $\mathrm{Z}^q(\mathfrak{U}, \mathscr{F})$ and put $\eta = \mathrm{R}\left(\sum\limits_j c_j \tau_j\right) \in \mathrm{Z}^q(\mathrm{D}', \mathrm{E})$. If $c_j \neq 0$ are chosen sufficiently small then we get by 2) and 3) that

4) $\lim\limits_{\nu\to\infty} \left| \int_{\mathrm{D}_{\nu,j}(r_j)} \gamma_j \wedge \eta \right| = \infty .$

Since $\bigcup\limits_{\nu=1}^{\infty} [\mathrm{D}_{\nu,j}(r_j) \backslash \mathrm{D}_{\nu,j}(r)] \subset\subset \mathrm{D}'$ we get that 4) holds for any $0 < r \leqslant r_j$ from which we immediately deduce, via Stokes' theorem, that $\sum\limits_j c_j \tau_j$ defines an element in $\mathrm{H}^q(\mathrm{Y}, \mathscr{F})$ not extendable in any point of $\partial \mathrm{Y}$. Theorem 2 is completely proved.

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