UNFOLDINGS OF FOLIATIONS
WITH MULTIFORM FIRST INTEGRALS

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In this note we study unfoldings of codim 1 local foliations $F = (\omega)$ generated by germs $\omega$ of the form

$$\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$$

for some germs $f_i$ of holomorphic functions and complex numbers $\lambda_i$, generalizing the situation considered in [10].

For such a foliation $F$ satisfying some side conditions, we determine the set $U(F)$ of equivalence classes of first order unfoldings ((1.7) Proposition) and give explicitly a universal unfolding of $F$ ((1.11) Theorem) as an application of the versality theorem in [7]. In section 2, it is shown that the unfolding theory for $F = (\omega)$, $\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$ is equivalent to the unfolding theory for the "multiform function" $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$. In section 3, we consider foliations with holomorphic or meromorphic first integrals. In either case, it turns out that the given generator $\omega$ is of the form considered in section 1. Thus, under the conditions of (1.11) Theorem, such a foliation has a universal unfolding (Theorems (3.4) and (3.10)). If the conditions are not satisfied, then the space $U(F)$ may have obstructed elements ((3.6) Example).

This work was inspired by the extension theory of Cerveau and Moussu for forms with holomorphic integrating factors [1,4]. An unfolding is certainly an extension and, by the implicit function

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theorem, an extension can be thought of as an unfolding. Also a
morphism in the unfolding theory is a morphism in the extension
theory. However, the converse is not true in general. Thus a versal
unfolding is a versal extension but not vice versa. In [1] and [4],
it is proved that a germ $\omega$ of the form in section 1 of this note
(or more, generally, $\omega$ with holomorphic integrating factor $f$,
i.e., $d\left(\frac{\omega}{f}\right) = 0$ for some $f$ in $\Theta$) has a mini-versal extension.

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1. Unfoldings of $\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$.

Let $\mathcal{O}$ (or simply $\Theta$) denote the ring of germs of holomorphic
functions at the origin 0 in $\mathbf{C}^n = \{(z_1, \ldots, z_n)\}$ and let $\mathcal{O}$ (or
simply $\Omega$) denote the $\mathcal{O}$-module of germs of holomorphic 1-forms
at 0. For an element $\omega$ in $\Omega$, we denote by $S(\omega)$ (the germ at
0 of) the set of zeros of $\omega$ and call it the singular set of $\omega$.

Let $\omega$ be an element in $\mathcal{O}$ of the form

$$\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i},$$

where $f_i$ are germs in $\Theta$ and $\lambda_i$ are complex numbers. If we set
$F_i = f_1 \ldots \hat{f_i} \ldots f_p$ (omit $f_i$) for each $i = 1, \ldots, p$, we may
write $\omega = \sum_{i=1}^{p} \lambda_i F_i df_i$. Note that $\omega$ is integrable; $d\omega \wedge \omega = 0$.

By regrouping the $f_i$'s, if necessary, we may always assume that

$$(1.1) \quad \lambda_i \neq \lambda_j (\neq 0), \quad \text{if } i \neq j.$$  

In what follows we also assume that codim $S(\omega) \geq 2$, which implies that

$$(1.2) \quad \text{each } f_i \text{ is reduced, i.e., for any non-unit } g \text{ in } \Theta, f_i \text{ is}
\text{not divisible by } g^2,$$
and that

$$(1.3) \quad f_i \text{ and } f_j \text{ are relatively prime, if } i \neq j.$$  

Let $F$ be the codim 1 local foliation at 0 in $\mathbf{C}^n$ generated
by \( \omega \) as above ([6] 4, [7] 1, [8]). The set \( U(F) \) of equivalence classes of first order unfoldings of \( F \) is given by ([6] 6, [7] 1).

\[
U(F) = I(\omega)/\left( \sum_{i=1}^{p} \lambda_i F_i \partial f_i \right),
\]

where \( I(\omega) \) is an ideal in \( \mathcal{O} \) defined by

\[
I(\omega) = \{ h \in \mathcal{O} \mid h \omega = \eta \wedge \omega \text{ for some } \eta \in \Omega \}
\]

and \( \left( \sum_{i=1}^{p} \lambda_i F_i \partial f_i \right) \) is the ideal generated by

\[
\sum_{i=1}^{p} \lambda_i F_i \frac{\partial f_i}{\partial z_1}, \ldots, \sum_{i=1}^{p} \lambda_i F_i \frac{\partial f_i}{\partial z_n}.
\]

For a \( q \)-tuple of integers \( i_1, \ldots, i_q \) with \( 1 \leq i_1 < \ldots < i_q \leq p \), let \( I(i_1, \ldots, i_q) \) denote the ideal in \( \mathcal{O} \) generated by

\[
f_{i_2} \ldots f_{i_q}, \ldots, f_{i_1} \ldots \hat{f}_{i_j} \ldots f_{i_q} \text{ (omit } f_{i_j}).\]

Note that \( I(1, \ldots, p) = (F_1, \ldots, F_p) \) (the ideal generated by \( F_1, \ldots, F_p \)). We denote by \( \text{ht}_I \) the height of an ideal \( I \) in \( \mathcal{O} \).

(1.4) Lemma. — Suppose \( \text{ht}(f_i, f_j, f_k) = 3 \) if \( i, j, k \) are distinct and \( f_i, f_j, f_k \) are non-units. Then we have

\[
I(i_1, \ldots, i_q) = \bigcap_{\{i_1, \ldots, i_{q-1}\} \subseteq \{i_1, \ldots, i_q\}} I(f_1, \ldots, i_{q-1})
\]

for \( q \geq 3 \).

Proof. — Without loss of generality, we may assume that \( (i_1, \ldots, i_q) = (1, \ldots, q) \). Obviously, the left hand side in the above equality is in the right hand side. Take any element \( h \) in the right hand side. We set \( F'_{ij} = f_1 \ldots \hat{f}_i \ldots f_j \ldots f_q \) (omit \( f_i \) and \( f_j \)) for each pair of distinct indexes \( i, j \) and

\[
F'_{ijk} = f_1 \ldots \hat{f}_i \ldots \hat{f}_j \ldots \hat{f}_k \ldots f_q
\]

for each triple of distinct indexes \( i, j, k \). Then we may write

\[
h = \sum_{i \neq j} a_{ij} F'_{ij}, \quad a_{ij} \in \mathcal{O},
\]

for each \( j = 1, \ldots, q \). Now we show that \( a_{ij} \) is in the ideal \( (f_i, f_j) \) for each \( i, j \) with \( i \neq j \), which would imply that \( h \) is in \( I(1, \ldots, q) \).
This is obviously true if \( f_i \) or \( f_j \) is a unit. Thus we assume that \( f_i \) and \( f_j \) are non-units. If \( k \) is an index different from \( i \) or \( j \), we have, from (1.5),

\[
F'_{ijk}(a_{ij}f_k - a_{ik}f_j) = \left( \sum_{k \neq i,k} a_{kk} F'_{i\neq k} \right. - \left. \sum_{m \neq i,j} a_{mj} F'_{imj} \right) f_i.
\]

By our assumption, \( f_i \) and \( F'_{ijk} \) are relatively prime. Hence

\[
a_{ij}f_k - a_{ik}f_j = a f_i
\]

for some \( a \) in \( \mathbb{C} \). Thus \( a_{ij}f_k \) is in \( (f_i, f_j) \). If \( f_k \) is a unit, then \( a_{ij} \) is in \( (f_i, f_j) \). If \( f_k \) is a non-unit, then by our assumption \( ht(f_i, f_j, f_k) = 3 \). Hence \( a_{ij} \) is in \( (f_i, f_j) \). Q.E.D.

(1.6) COROLLARY. – Under the assumption of (1.4) Lemma,

\[
(F_1, \ldots, F_p) = \bigcap_{i \neq j} (f_i, f_j).
\]

(1.7) PROPOSITION. – If the assumption of (1.4) Lemma is satisfied and if \( df_1 \wedge \ldots \wedge df_p \neq 0 \), then we have \( I(\omega) = (F_1, \ldots, F_p) \), thus

\[
U(F) = (F_1, \ldots, F_p) / \left( \sum_{i=1}^p \lambda_i F_i \delta f_i \right).
\]

Proof. – If we set \( F_{ij} = f_1 \ldots \hat{f}_i \ldots \hat{f}_j \ldots f_p \) for \( i \neq j \), we have

\[
d\omega = \sum_{1 < i < j < p} (\lambda_i - \lambda_j) F_{ij} df_i \wedge df_j.
\]

From this we see easily that

\[
\lambda_i F_i d\omega = \sum_{i \neq j} (\lambda_i - \lambda_j) F_{ij} df_j \wedge \omega,
\]

which shows that \( (F_1, \ldots, F_p) \subset I(\omega) \). Conversely, take any element \( h \) in \( I(\omega) \). Thus

\[
hd\omega = \eta \wedge \omega
\]

for some \( \eta \) in \( \Omega \). Let \( U \) be a small neighborhood of 0 on which the germs \( f_1, \ldots, f_p, h \) and \( \eta \) have representatives and let \( S \) be the set of zeros of \( df_1 \wedge \ldots \wedge df_p \) in \( U \). By our assumption, the set \( S \) is an analytic set of codim \( \geq 1 \). As in the proof of [10] (2.1) Lemma, from (1.8), we may write
for some holomorphic functions $\phi_1, \ldots, \phi_p$ on $U - S$. Now we show that $\phi_i$ can be extended to holomorphic functions on $U$. From (1.9) and (1.10), we have

$$\phi_i \omega = \lambda_i \eta + \sum_{j \neq i} (\lambda_j - \lambda_i) F_{ij} df_j$$

for each $i = 1, \ldots, p$. Since the right hand side is holomorphic in $U$, this shows that $\phi_i$ is holomorphic in $U - S(\omega)$. Therefore, by the assumption that $\text{codim} S(\omega) \geq 2$, $\phi_i$ can be extended to a holomorphic function on $U$. Thus from (1.10) and (1.6) Corollary, we see that $h$ is in $(F_1, \ldots, F_p)$. Q.E.D.

For an element $h$ in $\mathcal{O}$, we denote the corresponding element in $\mathcal{O}/\left( \sum_{i=1}^{p} \lambda_i F_i \partial f_i \right)$ by $[h]$. The following result follows from (1.7) Proposition and the versality theorem in [7] (cf. the proof of [10] (2.4) Theorem).

(1.11) **Theorem.** - Let $F = (\omega)$ be a codim 1 local foliation at 0 in $\mathbb{C}^n$ generated by a germ $\omega$ of the form

$$\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$$

for some $f_i$ in $\mathcal{O}$ and $\lambda_i$ in $\mathbb{C}$. Suppose the conditions (a) $\lambda_i \neq \lambda_j$ ($\neq 0$) for $i \neq j$, (b) $\text{codim} S(\omega) \geq 2$, (c) $\text{ht}(f_i, f_j, f_k) = 3$ for $i \neq j \neq k \neq i$ such that $f_i, f_j, f_k$ are non-units, and (d) $df_1 \wedge \ldots \wedge df_p \neq 0$ are satisfied. If the dimension of the $\mathbb{C}$-vector space $(F_1, \ldots, F_p)/\left( \sum_{i=1}^{p} \lambda_i F_i \partial f_i \right)$, $F_i = f_1 \ldots \hat{f}_i \ldots \ldots f_p$, is finite, then $F$ has a universal unfolding. In fact, if

$$\left[ \sum_{i=1}^{p} \lambda_i u_i^{(1)} F_i \right], \ldots, \left[ \sum_{i=1}^{p} \lambda_i u_i^{(m)} F_i \right], \ u_i^{(j)} \in \mathcal{O},$$

is a $\mathbb{C}$-basis of $(F_1, \ldots, F_p)/\left( \sum_{i=1}^{p} \lambda_i F_i \partial f_i \right)$, then the unfolding...
$\Phi = (\tilde{\omega})$ of $F$ with parameter space $C^m = \{(t_1, \ldots, t_m)\}$ generated by $\tilde{\omega} = \tilde{f}_1 \ldots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$, where $\tilde{f}_i$ are germs in $n+m$ given by $\tilde{f}_i = f_i + \sum_{k=1}^m u_i^{(k)} t_k$, is universal.

(1.12) Corollary (Cerveau-Lins Neto [1] Th. E5, [2] Prop. 6, see also [9] (3.2) Th.). — If $F = (\omega)$ is the codim 1 local foliation at $0$ in $C^n = \{(z_1, \ldots, z_n)\}$ generated by $\omega = z_1 \ldots z_n \sum_{i=1}^n \lambda_i \frac{dz_i}{z_i}$ for some $\lambda_i$ in $C$ with $\lambda_i \neq \lambda_j \neq 0$ $(i \neq j)$, then every unfolding of $F$ is trivial, in fact $U(F) = 0$.

Proof. — We have

$$(F_1, \ldots, F_n) = \left( \sum_{i=1}^n \lambda_i F_i \right) = (z_1 \ldots \tilde{z}_i \ldots z_n).$$

Hence $U(F) = 0$.

(1.13) Remark. — The universal unfolding given in (1.11) Theorem is infinitesimally versal. However, if the conditions in (1.11) are not satisfied, $U(F)$ may have obstructed elements (see (3.6) Example).

(1.14) Remark. — Let $F = (\omega)$ be a codim 1 local foliation at $0$ in $C^n$ generated by a germ $\omega$ of the form

$$\omega = f_1 \ldots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}, \quad \lambda_i \neq \lambda_j \neq 0 \ (i \neq j),$$

with codim $S(\omega) \geq 2$ and let $\Phi$ be an unfolding of $F$ with parameter space $C^k$. Then by a result of Cerveau and Moussu ([1] 4e Partie, Th. C4, [4]), we have that

(1.15) $\Phi$ has a generator $\tilde{\omega}$ of the form

$$\tilde{\omega} = \tilde{f}_1 \ldots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}, \quad \tilde{f}_i \in n+k \Theta.$$

Moreover, if $\omega$ has no meromorphic first integrals (Sec. 3), then we may assume that ([1] 2e Partie, Ch. I, Prop. 1.5, [3])

(1.16) $\tilde{f}_i(z, 0) = f_i(z), \quad i = 1, \ldots, p$. 
The facts (1.15) and (1.16) also follow from (1.11) Theorem in case the conditions in (1.11) are satisfied.

(1.17) Remark. – If a foliation $F$ is generated by a germ $\omega$ of the form $\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$, then $F$ has a generator of a similar form such that each function germ involved in the expression is a non-unit.

2. Multiform functions.

A germ of multiform function at 0 in $C^n$ is an expression $f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ for some germs $f_i$ in $n^\odot$ and non-zero complex numbers $\lambda_i$. Two multiform functions $f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ and $g_1^{\mu_1} \ldots g_q^{\mu_q}$ are equal if they are equal as germs of multivalued functions, i.e., $f_1^{\lambda_1} \ldots f_p^{\lambda_p} g_1^{-\mu_1} \ldots g_q^{-\mu_q} = 1$. Let $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ be a multiform function. By regrouping the factors of the $f_i$'s, if necessary, we may always assume that the conditions (1.1), (1.2) and (1.3) are satisfied. Then the expression $f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ is uniquely determined up to the order of the $f_i$'s and units of $\odot$. The critical set $C(f)$ of $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ is defined to be the singular set $S(\omega)$ of the 1-form $\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i}$. In this section, we consider only multiform functions $f$ with $\text{codim } C(f) \geq 2$.

An unfolding of $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ is a germ $\overline{f}$ of multiform function at 0 in $C^n \times C^m = \{(z, t)\}$ which can be written as $\overline{f} = \overline{f}_1^{\lambda_1} \ldots \overline{f}_p^{\lambda_p}$ for $\overline{f}_i$ in $n+m^\odot$ with $\overline{f}_i(z, 0) = f_i(z)$, $i = 1, \ldots, p$. We call $C^m$ the parameter space of $\overline{f}$.

(2.1) Definition. – Let $\overline{f} = \overline{f}_1^{\lambda_1} \ldots \overline{f}_p^{\lambda_p}$ and $g = g_1^{\lambda_1} \ldots g_p^{\lambda_p}$ be two unfoldings of $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ with parameter spaces $C^m$ and $C^\ell$, respectively. A morphism from $g$ to $\overline{f}$ consists of germs of holomorphic maps $\Phi : (C^n \times C^\ell, 0) \longrightarrow (C^n \times C^m, 0)$ and $\phi : (C^\ell, 0) \longrightarrow (C^m, 0)$ such that

(a) the diagram

$$(C^n \times C^\ell, 0) \xrightarrow{\Phi} (C^n \times C^m, 0)$$

$$(C^\ell, 0) \xrightarrow{\phi} (C^m, 0)$$
is commutative, where the vertical maps are the projections,

(b) \( \Phi(z, 0) = (z, 0) \) and

(c) \( g = \Phi^* \tilde{f} \), i.e., \( g_1^{\lambda_1} \ldots g_p^{\lambda_p} = (\Phi^* f_1)^{\lambda_1} \ldots (\Phi^* f_p)^{\lambda_p} \).

(2.3) Definition. – An unfolding \( \tilde{f} \) of \( f \) is versal if for any unfolding \( g \) of \( f \), there is a morphism from \( g \) to \( \tilde{f} \).

Note that if \( \tilde{f} = \tilde{f}_1^{\lambda_1} \ldots \tilde{f}_p^{\lambda_p} \) is an unfolding of \( f = f_1^{\lambda_1} \ldots f_p^{\lambda_p} \), then \( \mathcal{S} = (\tilde{\omega}) \), \( \tilde{\omega} = \tilde{f}_1 \ldots \tilde{f}_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i} \), is an unfolding of \( F = (\omega) \), \( \omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i} \), with the same parameter space as that of \( \tilde{f} \). For the definition of morphisms for unfoldings of foliations, see [10] (1.2) Definition.

(2.4) Lemma. – Let \( \tilde{f} = \tilde{f}_1^{\lambda_1} \ldots \tilde{f}_p^{\lambda_p} \) and \( g = g_1^{\lambda_1} \ldots g_p^{\lambda_p} \) be two unfoldings of \( f = f_1^{\lambda_1} \ldots f_p^{\lambda_p} \) with parameter spaces \( \mathcal{C}^m \) and \( \mathcal{C}^k \), respectively. A pair \( (\Phi, \phi) \) of germs of holomorphic maps \( \Phi: (\mathcal{C}^m \times \mathcal{C}^k, 0) \rightarrow (\mathcal{C}^m \times \mathcal{C}^m, 0) \) and \( \phi: (\mathcal{C}^k, 0) \rightarrow (\mathcal{C}^m, 0) \) is a morphism from \( g \) to \( f \) if and only if it is a morphism from \( \mathcal{S} = (\tilde{\omega}) \), \( \tilde{\omega} = f_1^{\lambda_1} \ldots f_p^{\lambda_p} \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i} \), to
\[
\mathcal{S} = (\tilde{\omega}) \), \( \tilde{\omega} = \tilde{f}_1^{\lambda_1} \ldots \tilde{f}_p^{\lambda_p} \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i} \).

Proof. – We first note that if \( f = f_1^{\lambda_1} \ldots f_p^{\lambda_p} \) and
\[
\omega = f_1 \ldots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i} ,
\]
we may write \( d \log f = \frac{df}{f} = \frac{1}{f_1 \ldots f_p} \omega \). Suppose \( (\Phi, \phi) \) is a morphism from \( g \) to \( \tilde{f} \). Then we have
\[
(2.5) \quad \chi \cdot \theta = \Phi^* \tilde{\omega} ,
\]
where \( \chi = \frac{\Phi^* f_1^{\lambda_1} \ldots \Phi^* f_p^{\lambda_p}}{g_1^{\lambda_1} \ldots g_p^{\lambda_p}} \). Since the right hand side of (2.5) is holomorphic and \( \text{codim } S(\theta) \geq 2 \), we see that \( \chi \) is in \( n + 2 \mathcal{O} \).
Moreover, since \( \tilde{f}_i(z, 0) = g_i(z, 0) = f_i(z) \) and \( \Phi(z, 0) = (z, 0) \), we have \( \chi(z, 0) = 1 \). Hence \((\Phi, \phi)\) is a morphism from \( \mathcal{G}' \) to \( \mathcal{G} \).

Conversely, suppose \((\Phi, \phi)\) is a morphism from \( \mathcal{G}' \) to \( \mathcal{G} \). Then there is a germ \( \chi \) in \( _n^+ \) with \( \chi(z, 0) = 1 \) satisfying \( \chi \cdot \theta = \Phi^* \tilde{\omega} \). Now we prove that \( \chi \) is equal to \( \frac{\Phi^* \tilde{f}_1 \ldots \Phi^* \tilde{f}_p}{g_1 \ldots g_p} \). Once this is done, we have \( d \log g = d \log \Phi^* f \). Since the restrictions of \( g \) and \( \Phi^* \tilde{f} \) to \( \mathbb{C}^n \times \{0\} \) are both equal to \( f \), we get \( g = \Phi^* \tilde{f} \), which shows that \((\Phi, \phi)\) is a morphism from \( g \) to \( f \). Let \( s = (s_1, \ldots, s_q) \) be coordinates on \( \mathbb{C}^q \). In general, for an element \( \tilde{h} \) in \( _n^+ \), consider the power series expansion of \( \tilde{h} \) in \( s \); \( \tilde{h}(z, s) = \sum_{|\nu| \geq 0} h^{(\nu)}(z) s^\nu \), where \( \nu \) denotes an \( \ell \)-tuple \((\nu_1, \ldots, \nu_\ell)\) of non-negative integers, \( |\nu| = \nu_1 + \cdots + \nu_\ell \), \( s^\nu = s_1^{\nu_1} \ldots s_q^{\nu_q} \) and \( h^{(\nu)} \) are germs in \( _n^+ \). If \( h^{(0)} \neq 0 \), \( (0) = (0, \ldots, 0) \), then for each \( \nu \), there is a germ \( \phi^{(\nu)} \) of meromorphic function at \( 0 \) in \( \mathbb{C}^n \) such that

\[
\sum_{\lambda + \mu = \nu} h^{(\lambda)} \phi^{(\mu)} = \begin{cases} 1 \ldots |\lambda| = 0, \\ 0 \ldots |\lambda| > 0. \end{cases}
\]

Thus we have an expression \( \frac{1}{\tilde{h}} = \sum_{|\nu| \geq 0} \phi^{(\nu)} s^\nu \). If we set

\[
\rho = \chi \cdot \frac{g_1 \ldots g_p}{\Phi^* \tilde{f}_1 \ldots \Phi^* \tilde{f}_p},
\]

we may write

\[
\rho(z, s) = \sum_{|\nu| \geq 0} \rho^{(\nu)}(z) s^\nu,
\]

where \( \rho^{(\nu)} \) are germs of meromorphic functions at \( 0 \) in \( \mathbb{C}^n \) with \( \rho^{(0)} = 1 \). For our purpose, it suffices to show that \( \rho^{(\nu)} = 0 \) if \( |\nu| > 0 \). We may also write

\[
d \log \Phi^* \tilde{f} = \sum_{|\nu| \geq 0} \alpha^{(\nu)} s^\nu + \sum_{k=1}^{\ell} \sum_{|\nu| \geq 0} \nu_k F^{(\nu)}(z) s^{\nu-1} d s_k,
\]

\[
d \log g = \sum_{|\nu| \geq 0} \beta^{(\nu)} s^\nu + \sum_{k=1}^{g} \sum_{|\nu| \geq 0} \nu_k G^{(\nu)}(z) s^{\nu-1} d s_k,
\]

where \( \alpha^{(\nu)} \) and \( \beta^{(\nu)} \) are constants and \( \nu_k F^{(\nu)}(z) \) and \( \nu_k G^{(\nu)}(z) \) are terms involving \( F^{(\nu)}(z) \) and \( G^{(\nu)}(z) \).
where $1_k$ denotes the $k$-tuple with 1 in the $k$-th component and 0 in the others, the addition and subtraction of two $k$-tuples are done componentwise, $\alpha^{(v)}$ and $\beta^{(v)}$ are germs of meromorphic 1-forms and $F^{(v)}$ and $G^{(v)}$ are germs of meromorphic functions at 0 in $\mathbb{C}^n$. Note that $\alpha^{(0)} = \beta^{(0)}$. Since $d \log \Phi^* \tilde{f}$ and $d \log g$ are both closed forms, we have

$$dF^{(v)} = \alpha^{(v)} \quad \text{and} \quad dG^{(v)} = \beta^{(v)}.$$  

On the other hand, from $\rho \ d \log g = d \log \Phi^* \tilde{f}$, we have

$$\alpha^{(v)} = \sum_{\lambda + \mu = \nu} \rho^{(\lambda)} \beta^{(\mu)} \quad \text{and} \quad \nu_k F^{(v)} = \sum_{\lambda + \mu = \nu} \mu_k \rho^{(\lambda)} C^{(\mu)}$$

for all $\nu$. From (2.6) and (2.7), it is not difficult to show that $\rho^{(v)} = 0$ for $|\nu| > 0$. Q.E.D.

In view of (1.14) Remark and (2.4) Lemma, the unfolding theory for multiform functions $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ satisfying (1.1), (1.2), (1.3) and $\text{codim} \ C(f) \geq 2$ (as well as other conditions described in (1.14)) is equivalent to the unfolding theory for foliations $F = (\omega)$ with $\text{codim} \ S(F) \geq 2$ generated by germs $\omega$ of the form

$$\omega = f_1 \ldots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}, \quad \lambda_i \neq \lambda_j \neq 0 \ (i \neq j).$$

In particular, from (1.11) Theorem, we have the following

\textbf{(2.8) Theorem.} Let $f = f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ be a germ of multiform function at 0 in $\mathbb{C}^n$ satisfying (1.1), (1.2), (1.3), $\text{codim} \ C(f) \geq 2$ and the conditions (c) and (d) in (1.11) Theorem. If

$$\dim_{\mathbb{C}}(F_1, \ldots, F_p)/\left( \sum_{i=1}^p \lambda_i F_i \partial f_i \right), \ F_i = f_1 \ldots \hat{f_i} \ldots f_p,$$

is finite, then $f$ has a versal unfolding. In fact if $\tilde{f_i}$ are the germs in (1.11), then the unfolding $\tilde{f} = \tilde{f_1}^{\lambda_1} \ldots \tilde{f_p}^{\lambda_p}$ of $f$ is versal.

3. Foliations with holomorphic or meromorphic first integrals.

The following application of the results in section 1 was pointed out by K. Saito. First we observe the following
(3.1) Lemma. Let \( f \) be a germ in \( \mathcal{O} \) with \( f(0) = 0 \) and let \( g \) be a reduced germ in \( \mathcal{O} \). If \( df = g \theta \) for some \( \theta \) in \( \Omega \), then \( f \) is divisible by \( g^2 \).

Proof. From the condition, we see that \( f \) vanishes on the zero set of \( g \). Hence \( g \) divides \( f \); \( f = f' g \) for some \( f' \) in \( \mathcal{O} \). Then we have \( df = g df' + f' dg \). Thus \( f' \) must be also divisible by \( g \).

Q.E.D.

Similarly we have

(3.2) Lemma. Let \( f \) be a germ in \( \mathcal{O} \) with \( f(0) = 0 \) and let \( g \) be a germ in \( \mathcal{O} \) of the form \( g = f_1^{k_1} \cdots f_r^{k_r} \) for some germs \( f_i \) in \( \mathcal{O} \) and positive integers \( k_i \) such that (a) \( f_i \) are reduced, and (b) \( f_i \) and \( f_j \) are relatively prime if \( i \neq j \). If \( df = g \theta \) for some \( \theta \) in \( \Omega \), then \( f \) is divisible by \( f_1^{k_1+1} \cdots f_r^{k_r+1} \).

Let \( F = (\omega) \) be a codim 1 local foliation at 0 in \( \mathbb{C}^n \) with codim \( S(\omega) \geq 2 \). Suppose \( \omega \) has a holomorphic first integral \( f \), i.e., \( \omega \wedge df = 0 \) for some \( f \) in \( \mathcal{O} \) ([5] p. 470). Without loss of generality, we may always assume that \( f(0) = 0 \). Since codim \( S(\omega) \geq 2 \), we may write \( df = g \omega \) for some \( g \) in \( \mathcal{O} \). If \( g \) is a unit in \( \mathcal{O} \), \( F = (\omega) = (df) \) is a Haefliger foliation and unfoldings of \( F \) are well understood [7,10]. We may write \( g = f_1^{k_1} \cdots f_r^{k_r} \), where \( k_i \) are positive integers with \( k_i \neq k_j \) for \( i \neq j \) and \( f_i \) are (non-constant) germs in \( \mathcal{O} \) satisfying the conditions (a) and (b) in (3.2) Lemma. Then, from (3.2) Lemma, we have \( f = f_1^{k_1+1} \cdots f_r^{k_r+1} f_{r+1} \) for some \( f_{r+1} \) in \( \mathcal{O} \). By computing \( df \), we have

\[
(3.3) \quad \omega = f_1 \cdots f_{r+1} \sum_{i=1}^{r+1} \lambda_i \frac{df_i}{f_i}, \quad \lambda_i = \begin{cases} k_i + 1 \cdots 1 \leq i \leq r, \\ 1 \cdots i = r + 1. \end{cases}
\]

Note that, since codim \( S(\omega) \geq 2 \), \( f_{r+1} \) is reduced and that \( f_{r+1} \) and \( f_i \) are relatively prime for \( i = 1, \ldots, r \). Let \( p = r \) and replace \( \lambda_i \) by \( f_{r+1} \lambda_i \) if \( f_{r+1} \) is a constant and let \( p = r + 1 \) otherwise. Then from (1.11) Theorem, we have

(3.4) Theorem. Let \( F = (\omega) \) be a codim 1 local foliation at 0 in \( \mathbb{C}^n \) with codim \( S(F) \geq 2 \). If \( \omega \wedge df = 0 \) for some \( f \) in \( \mathcal{O} \), then \( \omega \) can be written as (3.3). Moreover, if \( (a) \) \( \text{ht}(f_i, f_j, f_k) = 3 \)
for distinct indexes \( i, j, k = 1, \ldots, p \) such that \( f_i, f_j, f_k \) are non-units, (b) \( df_1 \wedge \ldots \wedge df_p \neq 0 \) and (c)

\[
\dim \mathbb{C}(F_1, \ldots, F_p) \left\langle \sum_{i=1}^{p} \lambda_i F_i \partial f_i \right\rangle, \quad F_i = f_1 \ldots f_i \ldots f_p,
\]

is finite, then \( F \) has a universal unfolding. In fact, a universal unfolding is constructed explicitly as in (1.11) Theorem.

(3.5) Example. — Let \( F = (\omega) \) be the foliation at 0 in \( \mathbb{C}^2 = \{(x, y)\} \) generated by

\[
\omega = y(3x + 2y^2) \, dx + 2x(x + 2y^2) \, dy.
\]

For \( f = x^2y^2(x + y^2) \) and \( g = xy \), we have \( df = g \omega \). Letting \( f_1 = F_2 = xy \), \( f_2 = F_1 = x + y^2 \), \( \lambda_1 = 2 \) and \( \lambda_2 = 1 \), we see that the complex vector space

\[
(F_1, F_2) \left\langle \sum_{i=1}^{2} \lambda_i F_i \partial f_i \right\rangle = (x + y^2, xy)/(y(3x + 2y^2), x(x + 2y^2))
\]

is three dimensional and we may choose \([x + y^2] = \left[ \frac{1}{2} \lambda_1 F_1 \right] \), \([xy] = [\lambda_2 F_2] \) and \([x^2] = \left[ \frac{1}{2} \lambda_1 x F_1 - \lambda_2 y F_2 \right] \) as its basis. Thus by (3.4) Theorem, we see that the unfolding \( \mathfrak{F} = (\tilde{\omega}) \) of \( F \) with parameter space \( \mathbb{C}^3 = \{(t_1, t_2, t_3)\} \) given by

\[
\tilde{\omega} = 2 \tilde{f}_2 \partial \tilde{f}_1 + \tilde{f}_1 \partial \tilde{f}_2,
\]

\[
\tilde{f}_1 = xy + \frac{1}{2} t_1 + \frac{1}{2} xt_3, \quad \tilde{f}_2 = x + y^2 + t_2 - yt_3
\]

is universal. Note that \( \partial \tilde{f} = \tilde{g} \tilde{\omega} \) for \( \tilde{f} = \tilde{f}_1 \tilde{f}_2 \) and \( \tilde{g} = \tilde{f}_1 \).

Here is an example of \( F = (\omega) \) with a holomorphic first integral which has obstructed elements in \( \text{U}(F) \).

(3.6) Example. — Let \( F = (\omega) \) be the foliation at 0 in \( \mathbb{C}^2 = \{(x, y)\} \) generated by

\[
\omega = y(3x + 2y^2) \, dx + x(3x + 4y) \, dy.
\]

For \( f = x^2y^3(x + y) \) and \( g = x^2y^3 \), we have \( df = g \omega \). Thus in the previous situation, we have \( f_1 = x \), \( f_2 = y \), \( f_3 = x + y \), \( \lambda_1 = 2 \), \( \lambda_2 = 3 \) and \( \lambda_3 = 1 \). Note that \( ht(f_1, f_2, f_3) = 2 \). If we set \( h = 3x + 4y \), then \( h d \omega = \eta \wedge \omega \) for \( \eta = 3dx \). Hence \( [h] \) is in \( \text{U}(F) \) and \( \mathfrak{F}^{(1)} = (\tilde{\omega}) \).
\[ \omega = y(3x + 2y)dx + (3x^2 + 4xy + t)dy + (3x + 4y)dt \]
is a first order unfolding of \( F \) corresponding to \([h]\). However, it is not difficult to show that there is no unfolding corresponding to \([h]\).

Next we consider a foliation \( F = (\omega) \) (codim \( S(\omega) \geq 2 \)) with a meromorphic first integral, i.e., we suppose that \( \omega \wedge d \left( \frac{f}{g} \right) = 0 \) for some relatively prime germs \( f \) and \( g \) in \( \mathfrak{g} \). In what follows we assume that \( g \) is reduced. Since \( \text{codim} S(\omega) \geq 2 \), we may write

\begin{equation}
(3.7) \quad gdf - fdg = h\omega
\end{equation}
or

\begin{equation}
(3.8) \quad d \left( \frac{f}{g} \right) = \frac{h}{g^2} \omega
\end{equation}

for some \( h \) in \( \mathfrak{g} \). Note that if \( h \) is a unit, \( F \) is generated by \( gdf - fdg \) and unfoldings of such an \( F \) are well understood [10]. Since \( f \) and \( g \) are relatively prime and \( g \) is reduced, from (3.7), we see that \( g \) and \( h \) are relatively prime. Thus by (3.8), \( \frac{f}{g} = c \) is a constant on the zero set of \( h \). If we write \( h = f_1^{k_1} \ldots f_r^{k_r} \), where \( k_i \) are positive integers with \( k_i \neq k_j \) for \( i \neq j \) and \( f_i \) are non-constant germs in \( \mathfrak{g} \) satisfying the conditions (a) and (b) in (3.2) Lemma, then we have \( f - gc = f_1^{k_1+1} \ldots f_r^{k_r+1} f_{r+2} \) for some \( f_{r+2} \) in \( \mathfrak{g} \). We set \( f_{r+1} = g \). By computing \( d \left( \frac{f}{g} \right) \), we have

\begin{equation}
(3.9) \quad \omega = f_1 \ldots f_{r+2} \sum_{i=1}^{r+2} \lambda_i \frac{df_i}{f_i}, \quad \lambda_i = \begin{cases} k_i + 1 \ldots 1 \leq i \leq r, \\ -1 \ldots i = r + 1, \\ 1 \ldots i = r + 2. \end{cases}
\end{equation}

Note that, since \( \text{codim} S(\omega) \geq 2 \), \( f_{r+2} \) is also reduced and that \( f_i \) and \( f_j \) are relatively prime for distinct indexes \( i, j \) with \( 1 \leq i, j \leq r + 2 \). Let \( p = r + 1 \) and replace \( \lambda_i \) by \( f_{r+2} \lambda_i \) if \( f_{r+2} \) is a constant and let \( p = r + 2 \) otherwise. Then from (1.11) Theorem, we have

(3.10) **Theorem.** - Let \( F = (\omega) \) be a codim 1 local foliation at 0 in \( \mathbb{C}^n \) with codim \( S(F) \geq 2 \). Suppose \( \omega \wedge d \left( \frac{f}{g} \right) = 0 \) for some \( f \) and \( g \) in \( \mathfrak{g} \) such that \( f \) and \( g \) are relatively prime and that \( g \) is reduced.
Then $\omega$ can be written as (3.9). If (a) $\text{ht}(f_i, f_j, f_k) = 3$ for distinct indexes $i, j, k = 1, \ldots, p$ such that $f_i, f_j, f_k$ are non-units, (b) $df_i \wedge \ldots \wedge df_p \neq 0$ and (c) $\dim_{\mathbb{C}}(F_1, \ldots, F_p) / \left( \sum_{i=1}^{p} \lambda_i F_i \partial f_i \right)$, $F_i = f_i \ldots \hat{f}_i \ldots f_p$, is finite, then $F$ has a universal unfolding.

In fact, a universal unfolding is constructed as in (1.11) Theorem.

**BIBLIOGRAPHY**


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