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CLASSIFICATION OF NASH MANIFOLDS

by Masahiro SHIOTA

1. Introduction.

In this paper we show when two Nash manifolds are Nash diffeomorphic. A semi-algebraic set in a Euclidean space is called a *Nash manifold* if it is an analytic manifold, and an analytic function on a Nash manifold is called a *Nash function* if the graph is semi-algebraic. We define similarly a Nash mapping, a Nash diffeomorphism, a Nash manifold with boundary, etc. It is natural to ask a question whether any two C^∞ diffeomorphic Nash manifolds are Nash diffeomorphic. The answer is negative. We give a counter-example in Section 5. The reason is that Nash manifolds determine uniquely their "boundary". In consideration of the boundaries, we can classify Nash manifolds by Nash diffeomorphisms as follows. Let M, M_1, M_2 denote Nash manifolds.

THEOREM 1. – *There exist a compact real non-singular affine algebraic set* X, *a non-singular algebraic subset* Y *of* X *of codimension* 1, *and a union* M' *of connected components of* X-Y *such that* M *is Nash diffeomorphic to* M' *and that the closure* $\overline{M}'$ *of* M' *is a Nash manifold with boundary* Y. *Here* Y *is empty if* M *is compact.*

In the above we call $\overline{M}'$ a *compactification* of M.

THEOREM 2. – *Let* N_1, N_2 *be any respective compactifications of* M_1, M_2. *Then the following are equivalent.*

(i) M_1 *and* M_2 *are Nash diffeomorphic.*

(ii) N_1 *and* N_2 *are Nash diffeomorphic.*

(iii) N_1 *and* N_2 *are* C^∞ *diffeomorphic.*

By the h-cobordism theorem [5] we have

COROLLARY 3. – *Assume that* M_1 *and* M_2 *are* C^∞ *diffeomorphic, that the dimension of* M_1 *is not* 3,4 *nor* 5, *and that if* $\dim M_1 \geqslant 6$, *for any compact subset* A *of* M_1 *there exists a compact subset* $A' \supset A$ *of* M_1 *such that* $M_1 - A'$ *is simply connected. Then* M_1 *and* M_2 *are Nash diffeomorphic.*

The correspondence $M \longrightarrow$ the compactification of M shows the following.

COROLLARY 4. – *The Nash diffeomorphism classes of all Nash manifolds are in* (1-1)*-correspondence with the* C^∞ *diffeomorphism classes of all* C^∞ *compact manifolds with or without boundary.*

The next corollaries may be useful when we consider Nash manifolds and Nash functions.

COROLLARY 5. – *Let* $M_1 \supset M'_1$, M_2 *be Nash manifolds and a compact Nash submanifold. Let* $f : M_1 \longrightarrow M_2$ *be a* C^∞ *mapping such that* $f|_{M'_1}$ *is a Nash mapping. Then we can approximate* f *by Nash mappings fixing on* M'_1 *in the compact-open* C^∞ *topology.*

COROLLARY 6. – *Assume that* M *is compact and contained in* $\mathbf{R}^n$. *Then there exist Nash functions* $f_1, \ldots, f_p$ *on* $\mathbf{R}^n$ *such that the common zero point set of* $f_1, \ldots, f_p$ *is* M *and that* $\operatorname{grad} f_1, \ldots, \operatorname{grad} f_p$ *on* M *span the normal bundle of* M *in* $\mathbf{R}^n$.

2. Preparation.

See [3] for the fundamental properties of semi-algebraic sets.

LEMMA 7. – *Let* $M \subset \mathbf{R}^n$ *be a Nash manifold. Then there exists a Nash tubular neighborhood* U *of* M *in* $\mathbf{R}^n$, (i.e. U *is a Nash* *manifold and the orthogonal projection* $p : U \longrightarrow M$ *is a Nash mapping).*

Proof. – Let $\overline{\overline{M}}$ be the Zariski closure of M in $\mathbf{R}^n$. Let $\mathrm{Sing}(\overline{\overline{M}})$ denote the set of singular points of $\overline{\overline{M}}$. Then $M\text{-}\mathrm{Sing}(\overline{\overline{M}})$ is open and dense in M. Consider the normal bundle

$$N = \{(x, y) \in M \times \mathbf{R}^n \mid y \text{ is a normal vector of } M \text{ at } x \text{ in } \mathbf{R}^n\}.$$

Then clearly N is an analytic manifold. Moreover N is semi-algebraic. The reason is the following. We define the normal bundle $\widetilde{N}$ of $\overline{\overline{M}}\text{-}\mathrm{Sing}(\overline{\overline{M}})$ in the same way. Since $\widetilde{N}$ is an algebraic subset of $(\overline{\overline{M}}\text{-}\mathrm{Sing}(\overline{\overline{M}})) \times \mathbf{R}^n$, $\widetilde{N} \cap (M \times \mathbf{R}^n)$ is semi-algebraic. The equality

$$\widetilde{N} \cap (M \times \mathbf{R}^n) = N \cap ((M\text{-}\mathrm{Sing}(\overline{\overline{M}})) \times \mathbf{R}^n)$$

and the dense property of $M\text{-}\mathrm{Sing}(\overline{\overline{M}})$ in M imply that N is the topological closure of $\widetilde{N} \cap (M \times \mathbf{R}^n)$ in $M \times \mathbf{R}^n$. Hence N is semi-algebraic.

The mapping $q : M \ni (x, y) \longrightarrow x + y \in \mathbf{R}^n$ is obviously of Nash class. Let E_1 be the set of critical points of the mapping $q \times q : N \times N \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$. Then $(N \times N) - E_1$ contains

$$\Delta_1 = \{(z_1, z_2) \in N \times N \mid z_1 = z_2 = (x, 0)\}.$$

Let E_2 be the set of all points $(z_1, z_2) \in N \times N$ such that $q(z_1) = q(z_2)$. Then E_2 is a closed semi-algebraic subset of $N \times N$ and contains the diagonal

$$\Delta_2 = \{(z_1, z_2) \in N \times N \mid z_1 = z_2\}.$$

Moreover the topological closure $\overline{E_2 - \Delta_2}$ does not intersect with Δ_1 because of the existence of C^∞ tubular neighborhoods of M. Hence $E_1 \cup (\overline{E_2 - \Delta_2})$ is a closed semi-algebraic subset of $N \times N$ which does not intersect with Δ_1.

Let φ be a positive continuous function on M defined by

$$\varphi(x) = \mathrm{dist}((x, 0, x, 0), E_1 \cup (\overline{E_2 - \Delta_2})).$$

It is easy to see that any distance function from a semi-algebraic set is semi-algebraic (i.e. the graph is semi-algebraic). Hence φ is semi-algebraic. Put

$$N' = \{(x, y) \in N \mid 2\,|y| < \varphi(x)\}.$$

Then N' is an open semi-algebraic subset of N. We want to see that the restriction of q to N' is a Nash diffeomorphism into $\mathbf{R}^n$. It is trivial that the restriction is an immersion. Assume the existence of points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in N' such that $q(z_1) = q(z_2)$, $z_1 \neq z_2$. Then we have

$$x_1 + y_1 = x_2 + y_2 ,$$

$$\left.\begin{array}{l}\mathrm{dist}^2((x_1, 0, x_1, 0), (z_1, z_2)) \\ \mathrm{dist}^2((x_2, 0, x_2, 0), (z_1, z_2))\end{array}\right\} = |x_1 - x_2|^2 + y_1^2 + y_1^2 + y_2^2$$
$$\geqslant \varphi(x_1)^2, \varphi(x_2)^2 ,$$

and $$2|y_1| < \varphi(x_1), \ 2|y_2| < \varphi(x_2) .$$

It follows that $|x_1 - x_2|^2 = |y_1 - y_2|^2$ and

$$|x_1 - x_2|^2 + y_1^2 + y_2^2 > 4y_1^2, 4y_2^2 .$$

Hence $|y_1 - y_2|^2 > y_1^2 + y_2^2$. This is a contradiction. Therefore $q(N')$ is a Nash tubular neighborhood of M in $\mathbf{R}^n$. The proof is complete.

The following lemma will be used in the proof of Theorem 2, but this may be interesting itself. The case of polynomials on a Euclidean space was treated in Remark 6 in [11].

LEMMA 8. – *Let* $M \subset \mathbf{R}^n$ *be a Nash manifold closed in* $\mathbf{R}^n$. *Let* f_1, f_2 *be positive proper Nash functions on* M. *Then there exists a* C^∞ *diffeomorphism* τ *of* M *such that* $f_1 \circ \tau$ *and* f_2 *are equal outside a bounded subset of* M.

Proof. – The case where M is compact is trivial. Hence we assume M to be not compact. Let $\widetilde{f}_1$, $\widetilde{f}_2$ be the extension of f_1, f_2 respectively onto a Nash tubular neighborhood U of M defined by $\widetilde{f}_i = f_i \circ p$, $i = 1,2$, where p is the orthogonal projection. Then $\widetilde{f}_i$ are Nash functions, since any composition of Nash mappings is of Nash class. We regard $\operatorname{grad} \widetilde{f}_i$, $i = 1,2$ as Nash mappings from U to $\mathbf{R}^n$ also. The restrictions of $\operatorname{grad} \widetilde{f}_1$ and $\operatorname{grad} \widetilde{f}_2$ to M are vector fields of M. Let the restrictions be denoted by w_1, w_2 respectively. Put

$$B = \{x \in M \mid \langle w_{1x}, w_{2x} \rangle = - |w_{1x}| \, |w_{2x}|\} .$$

Here $\langle\,,\rangle$ means the inner product as vectors. Then B is semi-algebraic because of

$$B = M \cap \{x \in U \mid \langle \operatorname{grad} \widetilde{f}_1(x), \operatorname{grad} \widetilde{f}_2(x)\rangle = -|\operatorname{grad} \widetilde{f}_1(x)|\,|\operatorname{grad} \widetilde{f}_2(x)|\}.$$

Obviously B is the set of points x where w_{1x} is zero or w_{2x} is a multiple of $-w_{1x}$ by a real non-negative number.

We will prove by reduction to absurdity that B is bounded. Assume it to be unbounded. As $\mathbf{R}^n$ is Nash diffeomorphic to $S^n - \{\text{a point } a\}$ by the stereographic projection, we identify them. The germ of B at a is not empty. Hence, considering the germ, we obtain easily an unbounded one-dimensional semi-algebraic set $B' \subset B$ (see [3]). We can assume that B' is a Nash manifold with boundary and Nash diffeomorphic to $[0,\infty)$, because the set of singular points of one-dimensional semi-algebraic set is a semi-algebraic set of dimension 0. Let v be a C^∞ non-singular vector field on B'. Then, by the definition of B, we have

$$vf_1(x) \times vf_2(x) \leqslant 0 \quad \text{for} \quad x \in B'.$$

On the other hand, any non-constant Nash function defined on $[0,\infty)$ is monotone outside a bounded subset, because the set of critical points is a semi-algebraic set of dimension 0. Hence one of the functions $f_1|_{B'}$ and $f_2|_{B'}$ is monotone decreasing outside a bounded subset. This contradicts the fact that f_1, f_2 are proper and positive.

Let K be a large real number, let φ be a C^∞ function on M such that

$$0 \leqslant \varphi \leqslant 1, \quad \varphi = \begin{cases} 0 & \text{for} \quad |x| \leqslant K^{1/2} \\ 1 & \text{for} \quad |x| \geqslant (2K)^{1/2}. \end{cases}$$

Put $L = M \cap \{|x| = K^{1/2}\}$, $L' = M \cap \{|x| \geqslant K^{1/2}\}$,

$$L'' = M \cap \{|x| \geqslant (2K)^{1/2}\}.$$

For any real c_1, $c_2 \geqslant 0$ with $c_1 + c_2 > 0$, the vector field $w' = c_1 w_1 + c_2 w_2$ is non-singular outside B and satisfies $w'f_1$, $w'f_2 > 0$ at any point $x \notin B$ such that $c_1|w_{1x}| = c_2|w_{2x}|$. Choose K so that $L' \cap B = \emptyset$. Put

$$w = w_1/|w_1| + \varphi w_2/|w_2| \quad \text{on} \quad L'.$$

Then w, w_1 and w_2 are non-singular vector fields on L'. Moreover wf_1, wf_2 are positive on L', L'' respectively. It is sufficient to consider the case

$$f_1(x) = x_1^2 + \cdots + x_n^2 \quad \text{for} \quad x = (x_1, \ldots, x_n) \in \mathbf{R}^n .$$

Since L is a level of f_1, L is smooth, and w_1 is transversal to L.

On any maximal integral curve of w, f_1 is non-singular and monotone, and the set of values is $[K, \infty)$. Let ψ_t be the local 1 parameter group of transformations of L' defined by w. Then ψ_t is well-defined for $0 \leqslant t < \infty$. Put

$$\pi'(z, t) = \psi_t(z) \quad \text{for} \quad (z, t) \in L \times [0, \infty) .$$

It follows that π' is a diffeomorphism onto L'. The mapping

$$(z, t) \longrightarrow (z, f_1 \circ \pi'(z, t) - K)$$

is a diffeomorphism of $L \times [0, \infty)$. Let $(z, t) \longrightarrow (z, s(z, t))$ be the inverse diffeomorphism. Put

$$\pi(z, t) = \pi'(z, s(z, t)) \quad \text{for} \quad (z, t) \in L \times [0, \infty) .$$

Then π is a diffeomorphism from $L \times [0, \infty)$ to L' such that

$$f_1 \circ \pi(z, t) = t + K \quad \text{for} \quad (z, t) \in L \times [0, \infty) .$$

By the definition of π and π' we have a positive C^∞ function ρ on L' such that $\pi_*\left(\frac{\partial}{\partial t}\right) = \rho w$.

It follows from $\pi(L \times \{t \geqslant K\}) = L''$ that

$$\frac{\partial f_2 \circ \pi}{\partial t}(z, t) > 0 \quad \text{for} \quad t \geqslant K .$$

Hence, for each $x \in L$, the t-function $f_2 \circ \pi(x, t)$ on $[K, \infty)$ is proper and non-singular. Choose real $K' (\geqslant K)$ so that

$$f_2 \circ \pi(x, t) > K \quad \text{for} \quad (x, t) \in L \times [K', \infty) .$$

Then we have a C^∞ function f_3 on $L \times [0, \infty)$ such that $f_3(x, t)$, $0 \leqslant t < \infty$, is C^∞ regular for each fixed $x \in L$, that $f_3(x, t) = t + K$ in a neighborhood of $L \times 0$ and that $f_3(x, t) = f_2 \circ \pi(x, t)$ for $(x, t) \in L \times [K', \infty)$. It follows that $(x, t) \longrightarrow (x, f_3(x, t) - K)$ is a diffeomorphism of $L \times [0, \infty)$. Let $\pi'' : (x, t) \longrightarrow (x, s'(x, t))$ be the inverse. Then we see that

$$f_2 \circ \pi \circ \pi''(x, t) = t + K \quad \text{if} \quad s'(x, t) \geqslant K' .$$

Hence

$$f_1 \circ \pi = f_2 \circ \pi \circ \pi'' \quad \text{if} \quad s'(x, t) \geqslant K' .$$

Since $s'(x, t) = t$ in a neighborhood of $L \times 0$, we can extend $\pi \circ \pi''^{-1} \circ \pi^{-1}$ onto M so that the extension τ is the identity on $M - L'$. Then $f_1 \circ \tau = f_2$ outside a bounded set. Hence Lemma is proved.

3. Proofs of Theorems 1, 2.

For the sake of brevity we assume that M, M_1 and M_2 are connected. We also assume that the manifolds are not compact, because the other case is well-known. Let n' be the dimension of the manifolds. Let $G_{m,m'}$ denote the Grassmann manifold of m-linear subspaces in $\mathbf{R}^{m+m'}$. Put

$$E_{m,m'} = \{(\lambda, x) \in G_{m,m'} \times \mathbf{R}^{m+m'} \mid x \in \lambda\} .$$

Then $G_{m,m'}$ has naturally affine non-singular algebraic structure [7].

Let $\partial M'$ denote $\overline{M}' - M'$ if M' is a manifold contained in $\mathbf{R}^n$ and the usual boundary if M' is a compact manifold with boundary.

Proof of Theorem 1. – (1) First we reduce the problem to the case in which there exist a real compact non-singular algebraic set $X \subset \mathbf{R}^n$ and an algebraic subset Z of X satisfying the following conditions, (this was shown in the proof of Proposition 1 in [9]).

(i) M is a connected component of $X - Z$.

(ii) For every point $a \in Z$, there exists a smooth rational mapping ζ from X to $\mathbf{R}^{n''}$ for some integer $n'' \leqslant n'$ such that $\zeta(a) = 0$, that

$$Z \begin{Bmatrix} = \\ \subset \end{Bmatrix} \zeta^{-1}(\{(x_1, \ldots, x_{n''}) \in \mathbf{R}^{n''} \mid x_1 \ \ldots \ x_{n''} = 0\}) \begin{Bmatrix} \text{on } U \\ \text{on } X \end{Bmatrix}$$

where U is a neighborhood of a in X, and that ζ is a submersion on U. In this case we say that Z has only *normal crossings* at a in X.

Proof. – The boundary ∂M is a closed semi-algebraic set in $\mathbf{R}^n$. By Lemma 6 in [6], there exists a continuous function η on $\mathbf{R}^n$ such that $\eta^{-1}(0) = \partial M$ and that the restriction of η to $\mathbf{R}^n - \partial M$ is of Nash class, (see the remark after Proposition 1 in [9]). Consider the graph of the restriction of $1/\eta$ to M. Then the graph is closed in $\mathbf{R}^n \times \mathbf{R}$ and Nash diffeomorphic to M. Since $\mathbf{R}^{n+1}$ is Nash diffeomorphic to S^{n+1} – *a point* by the Stereographic projection, we can assume that the Zariski closure $\overline{\overline{M}}$ in $\mathbf{R}^n$ is compact and that ∂M is a point. Let $\lambda : M' \longrightarrow \overline{\overline{M}}$ be the normalization of $\overline{\overline{M}}$ (see [7]). Then there exists a Nash manifold M'' open in M' such that the restriction of λ to M'' is Nash diffeomorphic onto M and that M'' is a set of non-singular points of M'. It follows that $\partial M'' \subset \lambda^{-1}(\partial M)$ and that M' is compact because so is $\overline{\overline{M}}$. Apply Hironaka's desingularization theorem [2] to M'. Then we have a compact non-singular affine algebraic set X of dimension n' and a smooth rational mapping $\mu : X \longrightarrow M'$ such that the restriction of μ to $\mu^{-1}(M'')$ is diffeomorphic onto M''. Moreover we can suppose that $Z = \mu^{-1}(\lambda^{-1}(\partial M))$ has only normal crossings (Main Theorem II in [2]). This means (ii). As $\partial\mu^{-1}(M'') \subset Z$, $\mu^{-1}(M'')$ is a connected component of $X - Z$. Hence we can assume (i).

(2) Let $p : V \longrightarrow X$ be the orthogonal projection of a Nash tubular neighborhood V of X in $\mathbf{R}^n$. Put

$$Z' = Z \cap \bar{M},$$

$$F = \{(x, y) \in X \times \mathbf{R}^n \mid y \text{ is a normal vector of } X \text{ at } x \text{ in } \mathbf{R}^n\}.$$

Then the projection $F \longrightarrow X$ shows that F is the normal bundle of X in $\mathbf{R}^n$. It is easy to see that F is a non-singular algebraic set. Let $F|_Y$ denote $F \cap Y \times \mathbf{R}^n$, the restriction of the bundle to Y, for any subset Y of X.

We want to show the following. There exist a compact non-singular algebraic set Y in M of codimension 1, a connected component M' of $M - Y$, a polynomial mapping $q : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$ and open neighborhoods U_1, U_2 of $Y \times 0 \times 0$ in $F|_Y \times \mathbf{R}$ such that,

(i) $q(x, 0, 0) = x$ for $x \in Y$,

(ii) $\bar{U}_1 \subset U_2$,

(iii) $q|_{U_1}$ is a diffeomorphism into $\mathbf{R}^n$ whose image contains $M - M'$,

(iv) $q|_{U_2}$ is an immersion whose image contains $\bar{M} - M'$. Hence we can say that $F|_Y \times \mathbf{R}$ and $q(U_1)$ are the normal bundle of Y in $\mathbf{R}^n$ and a « bent » tubular neighborhood respectively.

Proof. – Let $\mathfrak{a}$ be the ideal of the smooth rational function ring on X consisting of functions which vanish on Z. Let ξ be the square sum of finite generators of $\mathfrak{a}$. Then for every point a of Z, there exists an analytic local coordinate system $(x_1, \dots, x_{n'})$ for X centered at a such that $\xi = x_1^2 \dots x_{n''}^2$ in a neighborhood of a for some n''. Put $Y = \xi^{-1}(\epsilon) \cap M$ for sufficiently small $\epsilon > 0$. Here Y is not necessarily algebraic, so we approximate later it by an algebraic set.

For any point $a \in Z'$, consider the set of all connected components of $M \cap$ (a small ball with center at a). Let T be the disjoint union of the set as a runs on Z'. Hence an element c of T means a pair of a point $\sigma_1(c)$ of Z' and a connected set $\sigma_2(c)$ contained in M. Then T has a topological manifold structure such that $\sigma_1 : T \longrightarrow X$ is a topological immersion and that $\sigma_2(c) \cap \sigma_2(c') \neq \phi$ for close c, $c' \in T$. Let $v_1 : T \longrightarrow \mathbf{R}^n$ be a continuous mapping which satisfies the following conditions. For every point c of T, let $(x_1, \dots, x_{n'})$ be an analytic local coordinate system for X centered at $a = \sigma_1(c)$ such that $\sigma_2(c) = \{x_1 > 0, \dots, x_{n''} > 0\}$, $n'' \leqslant n'$, in a neighborhood of a. Then $v_1(c)$ is a vector tangent to X at a and satisfies

$$v_1(c)\, x_i > 0 \quad \text{for} \quad 1 \leqslant i \leqslant n'',$$

here we regard $v_1(c)$ as a tangent vector of X at a. This means that $v_1(c)$ points at a point of $\sigma_2(c)$. The existence of v_1 is trivial. Moreover we can assume the following, using a C^∞ partition of unity. For every $c \in T$, there exists a C^∞ vector field $v_2(c)$ on a small neighborhood of $a = \sigma_1(c)$ in $\mathbf{R}^n$ such that $v_2(c)_a = v_1(c)$ and that $v_2(c') = v_2(c'')$ on the common domain of definition for any close c', $c'' \in T$.

Put

$$\sigma_2'(c) = p^{-1}\, \sigma_2(c) \quad \text{for} \quad c \in T.$$

We remark that $p^{-1}(\mathrm{Z})$ has only analytic normal crossings in V (see [2] for the definition) and that $\sigma_2'(c)$ can be regarded as a connected component of $p^{-1}(\mathrm{M}) \cap$ (a small ball with center at $\sigma_1(c)$), because we are concerned with only an arbitrarily small neighborhood of Z'. Consider the restrictions of $v_2(c)$ to $\sigma_2'(c)$ for all $c \in \mathrm{T}$. Then the restrictions of $v_2(c)$ and $v_2(c')$ to $\sigma_2'(c) \cap \sigma_2'(c')$ are equal for c, $c' \in \mathrm{T}$. Hence we have a C^∞ vector field v_3 on (a neighborhood of Z' in $\mathbf{R}^n) \cap p^{-1}(\mathrm{M})$ such that $v_3 = v_2(c)$ on $\sigma_2'(c)$. By the property of v_1, v_3 is transversal to $p^{-1}(\mathrm{Y})$ for any small $\epsilon > 0$ $(\mathrm{Y} = \xi^{-1}(\epsilon) \cap \mathrm{M})$.

Fix ϵ. Using the integral curves of v_3, we obtain a C^∞ imbedding q_1 of a neighborhood U_1 of $\mathrm{Y} \times 0 \times 0$ in $\mathrm{F}|_1 \times \mathbf{R}$ into $\mathbf{R}^n$ such that $q_1(x, y, 0) = x + y$,

$$\frac{\partial q_1}{\partial t}(x, y, t) = v_{3q_1}(x, y, t) \quad \text{for} \quad (x, y, 0),\ (x, y, t) \in \mathrm{U},$$

and that $q_1(\mathrm{U}_1)$ is equal to (a neighborhood of Z' in $\mathbf{R}^n) \cap p^{-1}(\mathrm{M})$. Here U_1 is chosen so that $(\mathrm{U}_1, \mathrm{Y} \times 0 \times 0)$ is C^∞ diffeomorphic to $(\mathrm{F}|_\mathrm{Y} \times \mathbf{R}, \mathrm{Y} \times 0 \times 0)$. From these arguments it follows that $\mathrm{M} - \mathrm{Y}$ has two connected components the closure of one of which does not intersect with $\partial\mathrm{M}$. Let the component be written as M'. Then we can assume that $q_1(\mathrm{U}_1)$ contains $\mathrm{M} - \mathrm{M}'$ and hence that (iii). Let q_2 be a C^∞ extension of q_1 to $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$. Then there exists an open neighborhood U_2 of $\mathrm{Y} \times 0 \times 0$ in $\mathrm{F}|_\mathrm{Y} \times \mathbf{R}$ such that (ii) and (iv) are satisfied.

We need to approximate Y and q_2 by an algebraic set and a polynomial mapping. Since $\overline{\mathrm{M}}'$ is a C^∞ manifold with boundary, we have a C^∞ function χ on X such that χ is C^∞ regular on Y and that the zero set of χ is Y. Approximate χ by a smooth rational function in the C^∞ topology, and consider the zero set. If we use the same notation Y for the set, Y is a compact non-singular algebraic set in M of codimension 1. We have no problem to apply the above argument to this Y, because the old Y can be transformed to the new one by a C^∞ diffeomorphism of $\mathbf{R}^n$ arbitrarily close to the identity. By the equality

$$q_2(x, 0, 0) = x \quad \text{for} \quad x \in \mathrm{Y},$$

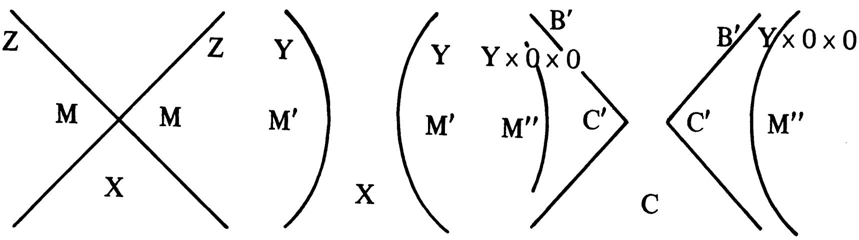
we have polynomial functions $\nu_1, \ldots, \nu_k$ on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ and C^∞ mapping $\rho_1, \ldots, \rho_k : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$ such that

$$q_2 = \sum_{i=1}^{k} \nu_i \rho_i + \text{the projection onto the first factor,}$$

and that $\nu_i = 0$ on $Y \times 0 \times 0$. Approximate ρ_i by polynomial mappings ρ_i' in the compact-open C^∞ topology. Then

$$q = \sum_{i=1}^{k} \nu_i \rho_i' + \text{the projection}$$

is what we wanted. We have to modify U_1, U_2 so that (iii), (iv) remain valid. But this is easy to see, hence we omit it.



(3) By (iii) in (2), q maps diffeomorphically $(q^{-1}(M-M') \cap U_1, Y \times 0 \times 0)$ onto $(M-M', Y)$. The construction of Y and q in (2) shows that $(q^{-1}(M-M') \cap U_1, Y \times 0 \times 0)$ is C^∞ diffeomorphic to $(Y \times (-1, 0], Y \times 0)$. Hence M and M' are C^∞ diffeomorphic. We want to prove that they are Nash diffeomorphic. As it is not easy to prove directly this, we will use an intermediary Nash manifold N which shall be Nash diffeomorphic to M and M'. In (3) we will define a C^∞ manifold M'' whose approximation shall be N.

Let $q' : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$ be the projection to the first factor. Put $A = F|_Y \times \mathbf{R}$, $S =$ the critical point set of $q|_{F|_Y \times \mathbf{R}}$,

$B = \overline{\overline{(A \cap q^{-1}(Z)) - S}}$ (where $\overline{\overline{}}$ means the Zariski closure),

$$C = \overline{\overline{(A \cap q^{-1}(X)) - S}} \quad \text{and} \quad B' = B \cap \overline{U}_1 .$$

Then A is a non-singular algebraic set, B and C are algebraic sets of dimension $n'-1$, n' respectively, and B' is a semi-algebraic set of dimension $n'-1$. Moreover B has only normal crossings in C at every point of $B \cap U_2$ (see (ii) in (1)), C is non-singular at every point of $C \cap U_2$, and for every point a of B' there exists an algebraic local coordinate system $(x_1, \ldots, x_{n'})$ for C centered at a such that

$$B' = \{x_1 = 0, x_2 \geqslant 0, \ldots, x_{n''} \geqslant 0\} \cup \ldots$$
$$\cup \{x_1 \geqslant 0, \ldots, x_{n''-1} \geqslant 0, x_{n''} = 0\}$$

in a neighborhood of a for some $n'' \leqslant n'$ and that

(i) q' maps diffeomorphically $\{x_1 = 0\}, \ldots, \{x_{n''} = 0\}$ into Y. We remark that B$'$ is naturally homeomorphic to T in (2). Put

$$C' = q^{-1}(M - \overline{M}') \cap U_1 .$$

Then C$'$ is the subdomain of C sandwiched in between B$'$ and $Y \times 0 \times 0$.

We want to find a C^∞ manifold M$''$ in $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ and a C^∞ diffeomorphism $\varphi : M'' \longrightarrow M$ such that

(ii) $M'' \supset C'$, $\varphi = q$ on C', $\overline{M}'' \cap B = \partial M'' = B'$ and $M'' \cap C = C'$.

Proof. – Since q maps $(C' \cup Y \times 0 \times 0, Y \times 0 \times 0)$ diffeomorphically to $(M - M', Y)$, we only have to find a compact C^∞ manifold $M^{(3)}$ with boundary in $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ and a diffeomorphism $\varphi' : M^{(3)} \longrightarrow \overline{M}'$ such that

(iii) $\partial M^{(3)} = Y \times 0 \times 0$,

(iv) $q = \varphi'$ on $Y \times 0 \times 0$,

(v) $M^{(3)} \cap C = Y \times 0 \times 0$, and

(vi) $M^{(3)} \cup C'$ is a C^∞ manifold.

Let O_ϵ denote the ϵ-neighborhood of $Y \times 0 \times 0$ in $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ for small $\epsilon > 0$. Let χ_i, $i = 1,2$, be a C^∞ function on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ such that

$$0 \leqslant \chi_i \leqslant 1, \quad \chi_i = \begin{cases} 1 & \text{outside} \quad O_{2\epsilon} \\ 0 & \text{in} \quad O_\epsilon \end{cases}$$

and that if $\chi_1(x) \neq 1$ then $\chi_2(x) = 0$. Consider the mapping

$$\varphi'' : O_{3\epsilon} \cap (C - C') \longrightarrow \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$$

defined by

$$\varphi''(z) = (1 - \chi_2(z))\,(0, z_2, z_3) + \chi_1(z)\,(q(z) - z_1, 0, 0) + (z_1, 0, 0),$$
$$z = (z_1, z_2, z_3).$$

Take sufficiently small ϵ. Then, choosing χ_i suitably we see that φ'' is a C^∞ diffeomorphism. It follows that

$$\varphi''((O_{3\epsilon} - O_{2\epsilon}) \cap (C - C')) \subset M' \times 0 \times 0 .$$

Put

$$\mathrm{M}^{(3)} = (\mathrm{M}' - q(\mathrm{O}_{3\epsilon} \cap (\mathrm{C} - \mathrm{C}'))) \times 0 \times 0 \cup \varphi''(\mathrm{O}_{3\epsilon} \cap (\mathrm{C} - \mathrm{C}')) .$$

Then $\mathrm{M}^{(3)}$ is a compact C^∞ manifold with boundary $\mathrm{Y} \times 0 \times 0$ (iii). Let $\varphi'^{-1} : \overline{\mathrm{M}}' \longrightarrow \mathrm{M}^{(3)}$ be defined by

$$\varphi'^{-1}(x) = \begin{cases} \varphi''(q^{-1}(x) \cap \mathrm{O}_{3\epsilon} \cap (\mathrm{C} - \mathrm{C}')) \text{ if } x \in q(\mathrm{O}_{3\epsilon} \cap (\mathrm{C} - \mathrm{C}')) \\ (x, 0, 0) \qquad \text{otherwise} . \end{cases}$$

Then φ'^{-1} is a C^∞ diffeomorphism such that $\varphi' = q$ in a neighborhood of $\mathrm{Y} \times 0 \times 0$ (iv). From $\varphi''(\mathrm{O}_\epsilon \cap (\mathrm{C} - \mathrm{C}')) = \mathrm{O}_\epsilon \cap (\mathrm{C} - \mathrm{C}')$, (vi) follows. For (v), we modify $\mathrm{M}^{(3)}$ as follows. Increasing the dimension n if necessary, we can assume that

$$\mathrm{X} \subset \mathbf{R}^{n-1} \times 0 , \quad \text{and hence} \quad \mathrm{C},\ \mathrm{M} \subset \mathbf{R}^{n-1} \times 0 \times \mathbf{R}^n \times \mathbf{R} .$$

Let χ_3 be a C^∞ function on $\mathrm{M}^{(3)} \cup \mathrm{C}'$ such that $\chi_3 = 0$ on $\mathrm{Y} \times 0 \times 0 \cup \mathrm{C}'$ and > 0 on $\mathrm{M}^{(3)} - \mathrm{Y} \times 0 \times 0$. Consider

$$\{(x_1, \chi_3(x_1, 0, y, t), y, t) \mid (x_1, 0, y, t) \in \mathrm{M}^{(3)}\}$$

in place of $\mathrm{M}^{(3)}$. Then (v) is satisfied.

(4) Here we will approximate M'' by a Nash manifold N fixing the "boundary". Let L' be a small open semi-algebraic neighborhood of B' in C, and L be the union of M'' and L' such that $\overline{\mathrm{L}}$ is a C^∞ manifold with boundary. This is possible since C is non-singular at every point of B'. Let D' be an open tubular neighborhood of L in $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, and D be an open semi-algebraic subset of D' containing L. We can choose M'', L' and D so that $\mathrm{D} \cap \mathrm{C}$ is a small neighborhood of $\overline{\mathrm{C}}'$ in C and that $\mathrm{D} \cap \mathrm{B}$ is equal to $\mathrm{L}' \cap \mathrm{B}$ and that B has only normal crossings in C at every point of $\mathrm{D} \cap \mathrm{B}$. Let $r : \mathrm{D} \longrightarrow \mathrm{L}$ denote the orthogonal projection. Let $h : \mathrm{D} \longrightarrow \mathrm{E}_{m, n'}$, $m = 2n - n' + 1$, be defined by

$$h(z) = (h_1(z), h_2(z)) =$$

$$\text{(the normal vector space of L at } r(z) \text{ in } \mathbf{R}^{2n+1},\ z - r(z))$$

for $z \in \mathrm{D}$. Then h is a Nash map on $r^{-1}(\mathrm{L}')$, and $h_2^{-1}(0) = \mathrm{L}$.

Remark 9. – Let $f : \mathrm{M}_1 \longrightarrow \mathrm{M}_2$ be a C^∞ mapping of Nash manifolds. Then we can approximate f by Nash mappings in the compact-open C^∞ topology (this is announced in [9]).

Proof. – By Proposition 1 in [9] there exist a compact non-singular algebraic set $X_1 \subset \mathbf{R}^{n1}$, a closed semi-algebraic subset B'_1 of X_1 and a union M'_1 of connected components of $X_1 - B'_1$ such that

(i) M_1 is Nash diffeomorphic to M'_1,

(ii) for every point x of B'_1, there exists an analytic local coordinate system $(x_1, \ldots, x_{n'_1})$ for X_1 centered at x such that

$$(M'_1, B'_1) = (\{x_1 > 0, \ldots, x_{n''_1} > 0\},$$
$$\{x_1 = 0, x_2 \geqslant 0, \ldots, x_{n''_1} \geqslant 0\} \cup \ldots$$
$$\cup \{x_1 \geqslant 0, \ldots, x_{n''_1 - 1} \geqslant 0, x_{n''_1} = 0\})$$

in a neighborhood of x, for some $n''_1 \leqslant n'_1$. Hence we can say that $\overline{M}'_1$ is a compact analytic manifold with cornered boundary. We assume $M_1 = M'_1$. It follows that $\partial M_1 = B'_1$.

In the same way as (2), we can construct a compact non-singular algebraic set Y_1 in $M_1 \cap$ (an arbitrarily small neighborhood of ∂M_1) and an analytic imbedding $q'_1 : Y_1 \times [-1, 0] \longrightarrow X_1$ such that $q'_1(Y_1 \times 0) = Y_1$ and that the image of q'_1 is an arbitrarily small neighborhood of B'_1. Put

$$M''_1 = q'_1(Y_1 \times [-1, 0]) \cup M_1 .$$

Then M''_1 is a compact analytic manifold with boundary containing $\overline{M}_1$, and there exists a C^∞ diffeomorphism π of X_1 arbitrarily close to the identity such that $\pi(M''_1) \subset M_1$.

Let M_2 be contained in $\mathbf{R}^{n2}$, and p be the orthogonal projection of a Nash tubular neighborhood of M_2 in $\mathbf{R}^{n2}$ (Lemma 7). Consider $f \circ \pi$ on M''_1. Then $f \circ \pi$ is extensible to X_1 and hence to $\mathbf{R}^{n1}$ as a C^∞ mapping to $\mathbf{R}^{n2}$. Let η be an extension, and η' be a polynomial approximation of η. Then $f' = \eta'|_{M_1} : M_1 \longrightarrow \mathbf{R}^{n2}$ is an approximation of $f : M_1 \longrightarrow \mathbf{R}^{n2}$. Since the closure of $\pi(M_1)$ in X_1 is compact, we can assume that $f'(M_1)$ is contained in the Nash tubular neighborhood of M_2. Hence $p \circ f' : M_1 \longrightarrow M_2$ is a Nash approximation of f. Thus Remark is proved.

In many cases we want Nash approximation to be fixed on a given semi-algebraic set. So the following are useful.

LEMMA 10. – *For any* C^∞ *function* g *on* D *vanishing on* $D \cap B$, *there exist* C^∞ *functions* $\alpha_1, \ldots, \alpha_\ell$ *and Nash functions* $\beta_1, \ldots, \beta_\ell$ *on* D *such that*

$$g = \alpha_1 \beta_1 + \cdots + \alpha_\ell \beta_\ell$$
$$\beta_1 = \cdots = \beta_\ell = 0 \quad \textit{on} \quad D \cap B .$$

Proof. – Let $\mathfrak{p}$ be the ideal of the smooth rational function ring on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ consisting of functions which vanish on B. Let $\beta_1, \ldots, \beta_\ell$ be a system of generators of $\mathfrak{p}$. We want to find $\alpha_1, \ldots, \alpha_\ell$ so that the equality in Lemma is satisfied for these β_i, α_i. By a C^∞ partition of unity we only need to see this locally. This is trivial in a neighborhood of any point of $D - B$.

For any point a of $D \cap B$, there exist smooth rational functions $\gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_{n''}$ with $k = 2n + 1 - n'$ and for some $n'' \leqslant n'$ such that $\gamma_1, \ldots, \gamma_k$ vanish on C, that $\delta_1 \ldots \delta_{n''}$ vanishes on B, that

$$B = \{\gamma_1 = \cdots = \gamma_k = \delta_1 \ldots \delta_{n''} = 0\}$$

in a neighborhood of a and that

$$\gamma_1 \times \cdots \times \gamma_k \times \delta_1 \times \cdots \times \delta_{n''} : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^{k+n''}$$

is a submersion in a neighborhood of a, since C is non-singular at a, and B has only normal crossings at a in C.

Hence it is sufficient to prove that if

$$D = \mathbf{R}^{k''} = \{(x_1, \ldots, x_{k''})\} \quad \text{and}$$
$$B = \{x_1 = \ldots = x_k = x_{k+1} \ldots x_{k'} = 0\}$$

with $k \leqslant k' \leqslant k''$, and if a C^∞ function g on $\mathbf{R}^{k''}$ vanishes on B, then $g = \alpha_1 x_1 + \ldots + \alpha_k x_k + \alpha_{k+1} x_{k+1} \ldots x_{k'}$ for some C^∞ functions $\alpha_1, \ldots, \alpha_{k+1}$. The case when $k = 0$ is trivial. Hence, considering $g(0, \ldots, 0, x_{k+1}, \ldots, x_{k''})$, we have a C^∞ function α_{k+1} on $\mathbf{R}^{k''}$ such that $g = \alpha_{k+1} x_{k+1} \ldots x_{k'}$ on $0 \times \mathbf{R}^{k''-k}$. This implies that $g - \alpha_{k+1} x_{k+1} \ldots x_{k'}$ vanishes on $0 \times \mathbf{R}^{k''-k}$. Then the existence of $\alpha_1, \ldots, \alpha_k$ which satisfy $g - \alpha_{k+1} x_{k+1} \ldots x_{k'} = \alpha_1 x_1 + \ldots + \alpha_k x_k$ is well-known. Hence Lemma follows.

LEMMA 11. – *With the same g as in Lemma 10, there exists a Nash function g' on D arbitrarily close to g and vanishing on* $D \cap B$.

Proof. – Using Remark 9, we approximate α_i in Lemma 10 by Nash functions α'_i. Then $g' = \sum_{i=1}^{\ell} \alpha'_i \beta_i$ is a Nash approximation of g and vanishes on $B \cap D$.

We continue with the construction of N. By Remark 9 we have a Nash mapping $h'_1 : D \longrightarrow G_{m,n'}$ which is an approximation of h_1. Apply Lemma 11 to h_2. Then we have a Nash approximation $h'_2 : D \longrightarrow \mathbf{R}^{2n+1}$ of h_2 such that $h'_2 = 0$ on $D \cap B$. Let W be a Nash tubular neighborhood of $E_{m,n'}$ in $\mathbf{R}^{n''} \times \mathbf{R}^{2n+1}$ where $G_{m,n'}$ is naturally imbedded in $\mathbf{R}^{n''}$ for some n''. Let $s : W \longrightarrow E_{m,n'}$ be the orthogonal projection. Put

$$h'' = (h''_1, h''_2) = s \circ h' = s \circ (h'_1, h'_2) .$$

Then $h'' : D \longrightarrow E_{m,n'}$ is a Nash approximation of h, and h''_2 is identical to h_2 on $D \cap B$. Shrinking L and D if necessary, we take this approximation in the uniform C^∞ topology. Put

$$L'' = h''^{-1}(G_{m,n'} \times 0) = h''^{-1}_2(0) .$$

Then there exists a C^∞ diffeomorphism ψ from L'' to L close to the identity such that $\psi =$ identity on $D \cap B$, because h is transversal to $G_{m,n'} \times 0$ in $E_{m,n'}$. Put $\psi^{-1}(M'') = N$. It follows that L'' is a Nash manifold containing $D \cap B$ and that $(\bar{M}'', B')$ is C^∞ diffeomorphic to $(\bar{N}, B')$ identically on B'. Hence N is the required Nash manifold.

(5) We will prove that M and N are Nash diffeomorphic. Let $\Phi : L \longrightarrow X$ be the C^∞ extension of the diffeomorphism $\varphi : M'' \longrightarrow M$ to L such that $\Phi(z) = q(z)$ for $z \notin M''$. Let $\Psi : D \longrightarrow \mathbf{R}^n$ be a C^∞ extension of $\Phi \circ \psi : L'' \longrightarrow X$ to D. Then $\Psi = q$ on $D \cap B$, and $\Psi|_{L''}$ is an immersion. Apply Lemma 11 to $\Psi - q$. Then we obtain a Nash approximation Ψ' of Ψ such that $\Psi' = q$ on $D \cap B$. Compose $\Psi'|_{L''}$ with the orthogonal projection p of a Nash tubular neighborhood of X in $\mathbf{R}^n$. This is well-defined if we shrink L and D and if the approximation is chosen closely. Then the composed function $\Psi'' : L'' \longrightarrow X$ is an approximation of $\Psi|_{L''} = \Phi \circ \psi : L'' \longrightarrow X$

such that $\Psi'' = q$ on $\mathrm{D} \cap \mathrm{B} = \mathrm{L}'' \cap \mathrm{B}$. Moreover we see $\Psi''(\mathrm{N}) = \mathrm{M}$ as follows from the facts $\Psi''(\mathrm{B}') = q(\mathrm{B}') = \mathrm{Z}'$, that M is a connected component of $\mathrm{X} - \mathrm{Z}'$ and that $\Psi''|_{\mathrm{N}}$ is an immersion. It is trivial that $\mathrm{M} \cap \Psi''(\mathrm{N})$ is an open subset of M. Assume it to be not closed. Then there exists a convergent sequence of points $x_1, x_2, \ldots$ in $\Psi''(\mathrm{N})$ whose limit $x \in \mathrm{M}$ is not contained in $\Psi''(\mathrm{N})$. Let $z_1, z_2, \ldots$ be points of N such that $\Psi''(z_i) = x_i$, $i = 1, \ldots$. Choosing a subsequence, we can assume that $z_1, z_2, \ldots$ converges to $z \in \overline{\mathrm{N}}$. Then we have $\Psi''(z) = x$. This is a contradiction. Hence $\Psi''(\mathrm{N}) \supset \mathrm{M}$. In the same way as above, we see that the set

$$\{x \in \mathrm{M} \mid \# \, \Psi''|_{\mathrm{N}}^{-1}(x) \geqslant 2\}$$

is empty or equal to M. For any point $x \in \mathrm{M}$, if we choose the above approximation closely, this set does not contain x. Hence $\Psi''|_{\mathrm{N} \cap \Psi''^{-1}(\mathrm{M})}$ is diffeomorphic onto M. From the same reason it follows that any connected component of $\mathrm{X} - \mathrm{Z}' - \mathrm{M}$ does not contain any point of $\Psi''(\mathrm{N})$, namely that $\Psi''(\mathrm{N}) \subset \mathrm{M} \cup \mathrm{Z}'$. Then $\Psi''(\mathrm{N}) \cap \mathrm{Z}' = \emptyset$. Hence $\Psi''|_{\mathrm{N}}$ is a Nash diffeomorphism onto M.

(6) Finally we will prove that M' and N are Nash diffeomorphic. For any point $x \in \mathrm{L}'' \cap \mathrm{B} = \mathrm{D} \cap \mathrm{B}$, let $\mathrm{C}_x^\infty(\mathrm{L}'')$ denote the ring of C^∞ function germs at x in L''. Then the ideal $\mathfrak{q}_x \subset \mathrm{C}_x^\infty(\mathrm{L}'')$ of germs vanishing on $\mathrm{L}'' \cap \mathrm{B}$ is principal because of the normal crossings property of B in $\mathrm{C} \cap \mathrm{D}$. Moreover we have a polynomial function on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ which vanishes on $\mathrm{D} \cap \mathrm{B}$ and the germ of whose restriction to L'' is a generator of $\mathfrak{q}_x$. Choose D so small that the fundamental class of $\mathrm{L}'' \cap \mathrm{B}$ is mapped to the zero class in $\mathrm{H}_{n'-1}(\mathrm{L}''; \mathbf{Z}_2)$ by the inclusion map (this is possible since $\overline{\mathrm{N}}$ is a compact topological manifold with boundary B', here we use infinite chain). Then we see easily that the ideal $\mathfrak{q}$ of $\mathrm{C}^\infty(\mathrm{L}'')$, the ring of C^∞ functions on L'', of functions vanishing on $\mathrm{L}'' \cap \mathrm{B}$ is principal (see Lemma 1 in [12]). Approximate a generator of $\mathfrak{q}$ by a Nash function ρ by the method of Lemma 11 so that $\rho = 0$ on $\mathrm{L}'' \cap \mathrm{B}$. Then it follows that ρ generates q. Choose ρ so that $\rho > 0$ on N.

Recall $q' : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$, the projection to the first factor. The restriction of q' to B' is homeomorphic onto Y. Let us extend this to $\overline{\mathrm{N}}$ so that the extension maps diffeomorphically

N to M'. Let v be the unit normal vector field of Y in X pointing to the interior of M'. Choose small L'. Put

$$\theta(z) = p \circ (q' \circ \psi(z) + \rho(z) v \circ q' \circ \psi(z)) \quad \text{for} \quad z \in \psi^{-1}(L').$$

Here we regard v as a mapping from Y to $\mathbf{R}^n$, and p is the orthogonal projection of a Nash tubular neighborhood of X in $\mathbf{R}^n$. This is well-defined because $q'(z) \in Y$ for $z \in L'$. Clearly θ is a Nash mapping.

We can assume that $L' \cap M'' = \{z \in M'' \mid \rho \circ \psi^{-1}(z) < \epsilon\}$ for some $\epsilon > 0$. Let v' be the unit C^∞ vector field on L' the family of whose maximal integral curves consists of $\{q'^{-1}(x) \cap L'\}_{x \in Y}$ and which points into M'' at every point of B'. For any point $a \in B'$, there exists a local analytic coordinate system $z = (z_1, \ldots, z_{n'})$ for L' centered at a such that $\rho \circ \psi^{-1}(z) = z_1 \ldots z_{n''}$ for some $n'' \leqslant n'$ and that $M'' = \{z_1 > 0, \ldots, z_{n''} > 0\}$ in a neighborhood of a. By (i) in (3), $v' z_i > 0$, $i = 1, \ldots, n''$ at a. It follows that $v' \rho \circ \psi^{-1} > 0$ on $L' \cap M''$. Hence v' is transversal to $\{z \in M'' \mid \rho \circ \psi^{-1}(z) = \epsilon'\}$ for some $\epsilon' > 0$. This implies that q' maps $\{z \in M'' \mid \rho \circ \psi^{-1}(z) = \epsilon'\}$ diffeomorphically onto Y. Therefore $\theta|_{N \cap \psi^{-1}(L')}$ is diffeomorphic onto (a C^∞ collar of $\bar{M}'$) $-$ Y. The transversality of v' shows also that $(M'' - L', \partial(M'' - L'))$ is diffeomorphic to $(M'' - C', \partial(M'' - C'))$ so that if the diffeomorphism maps a point $z \in \partial(M'' - L')$ to $z' \in \partial(M'' - C')$ we have $q(z) = q(z')$. Hence there exists a diffeomorphism from $(M'' - L', \partial(M'' - L'))$ to $(\bar{M}', Y)$ whose restriction to $\partial(M'' - L')$ is q'. Therefore we extend θ to $\Theta : L'' \longrightarrow X$ such that $\Theta|_N$ is a C^∞ diffeomorphism onto M'.

Apply Lemma 11 to $\Theta - q'$, and compose (the approximation mapping $+ q'$) with p. Then we have a Nash approximation $\Theta' : L'' \longrightarrow X$ of Θ such that $\Theta' = \Theta$ on $L'' \cap B$. To see that $\Theta'|_N$ is a Nash diffeomorphism onto M' we only need to show the following by the same reason as (5).

(i) $\Theta'(B') = q'(B') = Y$.

(ii) M' is a connected component of $X - Y$.

(iii) $\Theta'|_N$ is an immersion.

(i) and (ii) have been shown already. It is trivial that $\Theta'|_{N-\psi^{-1}(L')}$ is an immersion. Hence we only have to prove the following.

Statement. – Let $\theta_1 : \mathbf{R}^{n'} \longrightarrow \mathbf{R}^{n'-1}$ be a submersion, $K \subset \mathbf{R}^{n'}$ be a compact set. Let v_1 be a unit C^∞ vector field on $\mathbf{R}^{n'}$ the family of whose all maximal integral curves consists of $\{\theta_1^{-1}(y)\}_{y \in \mathbf{R}^{n'-1}}$. Put

$$\theta_2(x) = x_1 \ldots x_{n''} \quad \text{for} \quad x = (x_1, \ldots, x_{n'}) \in \mathbf{R}^{n'} .$$

Assume that $v_1 x_i > 0$, $i = 1, \ldots, n''(\leqslant n')$. Let (θ'_1, θ'_2) be a C^∞ close approximation of (θ_1, θ_2) such that $\theta_2'^{-1}(0) \supset \theta_2^{-1}(0)$. Then (θ'_1, θ'_2) is an immersion on $\{x_1 > 0, \ldots, x_{n''} > 0\} \cap K$.

Proof of Statement. – Since θ'_2 vanishes on

$$\{x_1 = 0\} \cup \ldots \cup \{x_{n''} = 0\} ,$$

there exists a C^∞ function η on $\mathbf{R}^{n'}$ such that $\theta'_2 = \eta\theta_2$. We see easily that η is close to the function 1 (see the statement at p. 268 in [10]). Replacing η by a C^∞ function which is equal to η in a neighborhood of K and is close to 1 in the Whitney C^∞ topology, we can assume that

$$\frac{\partial(\eta x_1)}{\partial x_1} > 0 \quad \text{on} \quad \mathbf{R}^{n'} .$$

Then $\pi : (x_1, \ldots, x_n) \longrightarrow (\eta x_1, x_2, \ldots, x_n)$ is a C^∞ diffeomorphism close to the identity. Since $\theta'_2 \circ \pi^{-1} = \theta_2$ and

$$\pi\{x_1 > 0, \ldots, x_{n''} > 0\} = \{x_1 > 0, \ldots, x_{n''} > 0\} ,$$

we need to treat only the case where $\theta'_2 = \theta_2$. By the same reason as above, we can assume that θ'_1 is sufficiently close to θ_1 in the Whitney C^∞ topology. Then θ'_1 is a submersion. Let v'_1 be the unit vector field on $\mathbf{R}^{n'}$ the family of whose all maximal integral curves consists of $\{\theta_1'^{-1}(y)\}_{y \in \mathbf{R}^{n'-1}}$ and which is close to v_1. Then we have $v'_1 x_i > 0$, $i = 1, \ldots, n''$. Hence

$$v'_1(x_1 \ldots x_{n''}) > 0 \quad \text{on} \quad \{x_1 > 0, \ldots, x_{n''} > 0\} .$$

This means that the Jacobian matrix of (θ'_1, θ_2) has the rank n' on $\{x_1 > 0, \ldots, x_{n''} > 0\}$. We complete the proofs of Statement and hence of Theorem 1.

Proof of Theorem 2. – Let N_1, N_2 be contained in non-singular algebraic sets X_1, $X_2 \subset \mathbf{R}^n$ respectively so that $\partial N_1 = Y_1$ and $\partial N_2 = Y_2$ are non-singular. The implication (ii) $\Longrightarrow$ (i) is trivial.

First we will prove (i) $\Longrightarrow$ (iii). Let φ_1, φ_2 be Nash functions on X_1, X_2 respectively such that $\varphi_i^{-1}(0) = Y_i$, $\{\varphi_i > 0\} = M_i$ and that φ_i are C^∞ regular at Y_i. The existence of such φ_i follows from the non-singular property of Y_i (see Lemma 1 in [12]) (in fact, we can choose as φ_i polynomial functions). Let Φ_1, Φ_2 be positive proper Nash functions on M_1 defined by

$$\Phi_1 = 1/(\varphi_1|_{M_1}), \quad \Phi_2 = (1/\varphi_2|_{M_2}) \circ \tau ,$$

where $\tau : M_1 \longrightarrow M_2$ be a Nash diffeomorphism. Apply Lemma 8 to Φ_1 and Φ_2. Then there exists a C^∞ diffeomorphism π of M_1 such that Φ_1 and $\Phi_2 \circ \pi$ are equal outside a compact subset of M_1. Hence we have $\varphi_1 = \varphi_2 \circ \tau \circ \pi$ on (a neighborhood of ∂M_1 in $\bar{M}_1) - \partial M_1$. This means that $\tau \circ \pi$ maps $\{\varphi_1 = \epsilon\}$ to $\{\varphi_2 = \epsilon\}$ for small $\epsilon > 0$. Hence the restriction of $\tau \circ \pi$ on $\{\varphi_1 \geqslant \epsilon\}$ for small $\epsilon > 0$ is a C^∞ diffeomorphism onto $\{\varphi_2 \geqslant \epsilon\}$. As $\{\varphi_1 \geqslant \epsilon\}$, $\{\varphi_2 \geqslant \epsilon\}$ are C^∞ diffeomorphic to N_1, N_2 respectively, N_1 and N_2 are C^∞ diffeomorphic.

We prove the inclusion (iii) $\Longrightarrow$ (ii) in the next general form.

LEMMA 12. – *Let $L_1 \supset L_2$, $L'_1 \supset L'_2$ be compact Nash manifolds with or without boundary and compact Nash submanifolds. Assume $L_2 = \partial L_1$ if $L_2 \cap \partial L_1 \neq \phi$. If there is a C^∞ diffeomorphism from (L_1, L_2) to (L'_1, L'_2), we can approximate it by Nash one. If the restriction of the given diffeomorphism to L_2 is of Nash class, the approximation can be chosen to take the same image as the diffeomorphism at each point of L_2.*

Proof. – The idea of the proof is the same as in Proof of Theorem 1, and this proof is easier than that since L_2 and L'_2 are smooth. Hence we give only the sketch. The case where L_1, L'_1 have the boundaries: Consider their doubles L_3, L'_3, and give them Nash structures [7]. We approximate the natural respective imbeddings of L_1, L'_1 into L_3, L'_3 by Nash mappings. Then we can regard L_1, L'_1 as contained in L_3, L'_3 respectively, and there is a C^∞ diffeomorphism from (L_3, L_1, L_2) to (L'_3, L'_1, L'_2). If we can approximate the induced diffeomorphism from $(L_3, \partial L_1 \cup L_2)$ to $(L'_3, \partial L'_1 \cup L'_2)$ by a Nash one, Lemma 12 follows. Hence we can assume that L_1, L'_1 have no boundary. Here we do not necessarily assume that L_2 has the global dimension, namely that the local dimension is constant.

Assume that L_2 is connected for the sake of brevity. Let $\pi : (L_1, L_2) \longrightarrow (L'_1, L'_2)$ be a C^∞ diffeomorphism. If $\pi|_{L_2}$ is not of Nash class, by Remark 9 we approximate $\pi|_{L_2}$ by a Nash diffeomorphism $\pi' : L_2 \longrightarrow L'_2$. Choose π' very closely. Then we easily find a C^∞ extension $\pi'' : L_1 \longrightarrow L'_1$ of π' such that π'' is an approximation of π. Hence, from the beginning we can assume that $\pi|_{L_2}$ is of Nash class. Let L_1, L'_1 be contained in $\mathbf{R}^n$, $\mathbf{R}^{n'}$ respectively, and $p : \mathbf{R}^n \times \mathbf{R}^{n'} \longrightarrow \mathbf{R}^n$, $p' : \mathbf{R}^n \times \mathbf{R}^{n'} \longrightarrow \mathbf{R}^{n'}$ be the projections. Let $L''_2 \subset \mathbf{R}^{n+n'}$ denote the graph of $\pi|_{L_2}$. Then L''_2 is a Nash manifold such that $p|_{L''_2}$, $p'|_{L''_2}$ are Nash diffeomorphisms onto L_2, L'_2 respectively. By the normalization of the Zariski closure $\overline{L''_2}$ of L''_2, there exist a non-singular connected component S_2 of an algebraic set in $\mathbf{R}^{n''}$ and a linear mapping φ from $\mathbf{R}^{n''}$ to $\mathbf{R}^{n+n'}$ such that $\varphi|_{S_2}$ is diffeomorphic onto L''_2. Increasing n'' if necessary, we construct a C^∞ manifold S_1 in $\mathbf{R}^{n''}$ and C^∞ diffeomorphisms $\psi : S_1 \longrightarrow L_1$, $\psi' : S_1 \longrightarrow L'_1$ such that $S_2 \subset S_1$, $\psi = p \circ \varphi$ on S_2, $\psi' = p' \circ \varphi$ on S_2 and $\overline{\overline{S}}_2 \cap S_1 = S_2$. Using $E_{m,m'}$ in the same way as Proof (4) of Theorem 1, we reduce S_1 to a Nash manifold. Then we can find in the same way as Proofs (5), (6) Nash approximations $\Psi : S_1 \longrightarrow L_1$, $\Psi' : S_1 \longrightarrow L'_1$ of ψ, ψ' such that $\Psi = \psi$, $\Psi' = \psi'$ on S_2. Hence $\Psi' \circ \Psi^{-1} : L_1 \longrightarrow L'_1$ is a Nash approximation of π such that $\Psi' \circ \Psi^{-1} = \pi$ on L_2. Lemma is proved.

4. Proofs of Corollaries.

Proof of Corollary 3. – Let N_1, N_2 be the compactifications of M_1, M_2 respectively. By Theorem 2, we only have to prove that N_1 and N_2 are C^∞ diffeomorphic. Let L be a closed C^∞ collar of N_1. Put $L' = N_1 - L$, $L'' = \overline{L'} - L'$. Let $\pi : M_1 \longrightarrow M_2$ be a C^∞ diffeomorphism. We see easily that $(N_2 - \pi(L')\,;\ \partial N_2$,

$\pi(L''))$ is a C^∞ h-cobordism. On the other hand it follows from the assumption that ∂N_2 is simply connected for $\dim M_1 \geqslant 6$. Hence, by the h-cobordism theorem $N_2 - \pi(L')$ is diffeomorphic to $\partial N_2 \times [0,1]$. This means that there exists a homeomorphism $\tau : N_1 \longrightarrow N_2$ such that $\tau|_L$ and $\tau|_{L' \cup L''}$ are C^∞ diffeomorphic. It is easy to modify τ to be a C^∞ diffeomorphism. Hence Corollary 3 is proved.

Proof of Corollary 4. – The correspondence is trivially injective by Theorems 1,2.

Surjectivity: Let N be a compact C^∞ manifold with or without boundary. We need to give to N a Nash manifold structure. If N has the boundary, consider the double N', and regard N as naturally contained in N'. In the other case, put $N' = N$, $\partial N = \phi$. Then, by a Theorem in [1], $(N', \partial N)$ is C^∞ diffeomorphic to a pair (an affine non-singular algebraic set, a non-singular algebraic subset). By this diffeomorphism, we give to N' an algebraic structure. Then, since $N - \partial N$ is a union of connected components of $N' - \partial N$, $N - \partial N$ is a Nash manifold. Obviously N is the compactification of $N - \partial N$. Hence Corollary follows.

Proof of Corollary 5. – Let N_1 be the compactification of M_1. Obviously we can assume that f is extensible to N_1 and hence to the double of N_1. Consider a Nash manifold structure on the double and a Nash imbedding of M_1 into it. Then, from the beginning we can assume that M_1 is compact. Let M_2 be contained in $\mathbf{R}^n$, and q be the orthogonal projection of a Nash tubular neighborhood of M_2 in $\mathbf{R}^n$. Regard f as a mapping to $\mathbf{R}^n$. If we can approximate f by a Nash mapping $f' : M_1 \longrightarrow \mathbf{R}^n$ so that $f = f'$ on M'_1, then $q \circ f' : M_1 \longrightarrow M_2$ is a required Nash approximation of f. Hence it is sufficient to consider the case of $M_2 = \mathbf{R}$. We regard $\mathbf{R}$ as S^1-{a point a}. Let $L \subset M_1 \times S^1$ be the graph of f. Put $L' = M_1 \times \{b\}$ where b is a point of S^1. Then there exists a C^∞ diffeomorphism π of $M_1 \times S^1$ such that

$$\pi(x, b) = (x, f(x)) \quad \text{for} \quad x \in M_1 .$$

It follows that $\pi(L') = L$ and that $\pi|_{M'_1 \times \{b\}}$ is of Nash class. Apply Lemma 12 to

$$\pi : (M_1 \times S^1, M'_1 \times \{b\}) \longrightarrow (M_1 \times S^1, \pi(M'_1 \times \{b\})) .$$

Then we obtain a Nash approximation τ of π such that $\tau = \pi$ on $M'_1 \times \{b\}$. For every point $x \in M_1$, put $g(x) = p \circ \tau(x, b)$ where $p : M_1 \times S^1 \longrightarrow S^1$ is the projection onto the second factor. Then g is what we want.

Proof of Corollary 6. – This corollary follows from Lemma 12 and the fact that $\mathbf{R}^n$ is Nash diffeomorphic to S^n-{a point} and that (S^n, M) is C^∞ diffeomorphic to (an affine algebraic set, a non-singular algebraic subset) [1].

5. An example.

Let W, W′ be compact C^∞ manifold with boundary such that the interiors are C^∞ diffeomorphic, but W and W′ are not diffeomorphic (see Theorem 3 in [4]). Let X, X′ be the doubles of W, W′ respectively. We regard W, W′ as naturally contained in X, X′ respectively. By a theorem in [1] we can assume that X, X′, ∂W and $\partial W'$ are all non-singular algebraic sets in $\mathbf{R}^n$. Let P, P′ be polynomials on $\mathbf{R}^n$ such that

$$P^{-1}(0) = \partial W, \quad P'^{-1}(0) = \partial W'.$$

Put

$$Y = \{(x, y) \in X \times \mathbf{R} \mid yP(x) = 1\},$$
$$Y' = \{(x, y) \in X' \times \mathbf{R} \mid yP'(x) = 1\}.$$

Then Y and Y′ are C^∞ diffeomorphic non-singular affine algebraic sets, and their compactifications are the disjoint unions of 2 copies $W + W (\subset X + X)$ and $W' + W' (\subset X' + X')$ respectively. Hence, by Theorem 2, Y and Y′ are not Nash diffeomorphic. Here it is not essential that Y, Y′ are not connected. In fact we can find connected examples.

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