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DE RHAM DECOMPOSITION THEOREMS FOR FOLIATED MANIFOLDS

by R.A. BLUMENTHAL and J.J. HEBDA

1. Introduction.

Let $\mathcal{F}$ be a smooth foliation of a smooth manifold M. We study the influence of the tangential and transverse geometry of $\mathcal{F}$ on the global structure of the foliated manifold $(M,\mathcal{F})$.

THEOREM A. – *Let* M *be a complete Riemannian manifold and let* $\mathcal{F}$ *be a totally geodesic foliation of* M *with integrable normal bundle. Then the universal cover* $\tilde{M}$ *of* M *is topologically a product* $L \times H$ *where*

i) L *is the universal cover of the leaves of* $\mathcal{F}$,

ii) H *is the universal cover of the leaves of the foliation* $\mathcal{G}$ *determined by the normal bundle of* $\mathcal{F}$,

iii) *the lift of* $\mathcal{F}$ *to* $\tilde{M}$ *is the foliation by leaves of the form* $L \times \{p\}$, $p \in H$,

iv) *the lift of* $\mathcal{G}$ *to* $\tilde{M}$ *is the foliation by leaves of the form* $\{p\} \times H$, $p \in L$, *and*

v) *the projection* $\tilde{M} \to L$ *onto the first factor is a Riemannian submersion.*

From Theorem A we obtain the following corollary, originally proved in [5].

COROLLARY B. – *Let* $\mathcal{F}$ *be a codimension-one totally geodesic foliation of a complete Riemannian manifold* M. *Then the universal cover of* M *is a product* $L \times \mathbf{R}$ *and the lift of* $\mathcal{F}$ *is the product foliation.*

In [9] it is shown that if M is a compact 3-manifold admitting a codimension-1 totally geodesic foliation, then $\pi_1(M)$ is infinite. From Corollary B we obtain.

COROLLARY C. – *If* M *is compact with finite fundamental group, then no codimension-*1 *foliation of* M *is geodesible.*

Theorem A is closely related to the decomposition theorem of De Rham [17], [11]. Indeed, if $\mathcal{G}$ is also totally geodesic then Theorem A shows that $\tilde{M}$ is a Riemannian product $L \times H$ from which De Rham's theorem follows.

In [17] De Rham studies a Riemannian manifold M by considering the subspaces of the tangent space $T_x(M)$ invariant under the action of the linear holonomy group with reference point $x \in M$. We apply similar considerations to the study of Riemannian foliations. Let M be a smooth manifold and let $\mathcal{F}$ be a smooth Riemannian foliation of M. Let Q be the normal bundle of $\mathcal{F}$ and let g be a smooth metric on Q invariant under the natural parallel transport along curves lying in a leaf of $\mathcal{F}$. Let ∇ be the unique torsion-free metric-preserving basic connection on Q and let $\Psi(x)$ be the holonomy group of ∇ with reference point $x \in M$. We say $\mathcal{F}$ is irreducible (reducible) if the action of $\Psi(x)$ on Q_x is irreducible (reducible).

THEOREM D. – *Let* $\mathcal{F}$ *be a smooth codimension-q Riemannian foliation of a smooth manifold* M. *There is a family* $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k$ *of foliations of* M *such that* $\mathcal{F} = \bigcap_{i=0}^{k} \mathcal{F}_i$ *where* $\mathcal{F}_0$ *is a Lie foliation (possibly of codimension-*0*) modeled on an abelian Lie group and* $\mathcal{F}_1, \ldots, \mathcal{F}_k$ *are all irreducible Riemannian foliations. The partition of* q *given by* $q = \sum_{i=0}^{k} \operatorname{codim}(\mathcal{F}_i)$ *is unique up to order and depends only on* $(\mathcal{F}, g)$.

Let R be the curvature of ∇. We say $\mathcal{F}$ has recurrent curvature if there exists a base-like one-form α on M such that $\nabla R = R \otimes \alpha$.

THEOREM E. – *Let* M *be a compact simply connected analytic manifold and let* $\mathcal{F}$ *be an irreducible analytic Riemannian foliation of* M *with recurrent curvature and* $\operatorname{codim}(\mathcal{F}) \geqslant 3$. *Then* M *fibers over a compact simply connected irreducible Riemannian symmetric space with the leaves of* $\mathcal{F}$ *as fibers.*

2. Totally geodesic foliations.

We begin with a few remarks concerning the sheaf $\mathscr{I}$ of germs of isometries between two Riemannian manifolds L_0 and L_1. Recall that every germ of an isometry at $x_0 \in L_0$ is represented by an isometry from an open neighborhood of x_0 in L_0 onto some open subset of L_1. Furthermore, there exist two local homeomorphisms, the source map $\pi_0 : \mathscr{I} \to L_0$, and the evaluation map $\pi_1 : \mathscr{I} \to L_1$. Finally, every germ $f \in \mathscr{I}$ defines a linear isometry

$$f_* : T_{\pi_0(f)}(L_0) \to T_{\pi_1(f)}(L_1)$$

via the differential at $\pi_0(f)$ of any isometry representing the germ f.

Let $\tau : [a,b] \to L_0$ be a piecewise smooth path. A *lift* of τ is a path $\tilde{\tau} : [a,b] \to \mathscr{I}$ such that $\pi_0 \circ \tilde{\tau} = \tau$.

LEMMA 2.1. — *If* τ *has a lift* $\tilde{\tau}$ *and* X *is a vector field parallel along* τ, *then* $(\tilde{\tau}(s))_*(X_s)$ *for* $a \leqslant s \leqslant b$ *is a vector field parallel along the curve* $\pi_1 \circ \tilde{\tau}$ *in* L_1.

Proof. — Fix $s_0 \in [a,b]$. Let f be an isometry from a neighborhood V of $\tau(s_0)$ in L_0 onto an open subset of L_1 that represents the germ $\tilde{\tau}(s_0)$. Then for all s near s_0, $\tau(s) \in V$,

$$\pi_1(\tilde{\tau}(s)) = f(\tau(s)) \qquad \text{and} \qquad f_*(X_s) = (\tilde{\tau}(s))_*(X_s).$$

The result is immediate since f is an isometry on V.

COROLLARY 2.2. — *Suppose* τ *has a lift* $\tilde{\tau}$ *and fix* $s_0 \in [a,b]$. *If* C_s *denotes the development of* τ *into* $T_{\tau(s_0)}(L_0)$ *and* $\hat{C}_s$ *denotes the development of* $\pi_1 \circ \tilde{\tau}$ *into* $T_{\pi_1(\tilde{\tau}(s_0))}(L_1)$, *then* $(\tilde{\tau}(s_0))_* C_s = \hat{C}_s$.

Proof. — C_s is the curve in $T_{\tau(s_0)}(L_0)$ obtained by parallel translating the tangent vector to τ at $\tau(s)$ along τ back to $\tau(s_0)$. Likewise, $\hat{C}_s$ is obtained by parallel translating the tangent vector to $\pi_1 \circ \tilde{\tau}$ at $\pi_1(\tilde{\tau}(s))$ along $\pi_1 \circ \tilde{\tau}$ back to $\pi_1(\tilde{\tau}(s_0))$. Since $(\tilde{\tau}(s))_*$ sends the tangent vector of τ to that of $\pi_1 \circ \tilde{\tau}$, lemma 2.1 gives the conclusion.

The next lemma is a standard result.

LEMMA 2.3. – *Fix a germ* $\mathscr{f} \in \mathscr{I}$. *If every path* $\tau : [a,b] \to \mathrm{L}_0$ *with* $\tau(a) = \pi_0(\mathscr{f})$ *has a lift* $\tilde{\tau}$ *with* $\tilde{\tau}(a) = \mathscr{f}$, *then the connected component of* $\mathscr{I}$ *containing* $\mathscr{f}$ *is a covering space of* L_0.

Let $p_0 : \tilde{\mathrm{L}}_0 \to \mathrm{L}_0$ and $p_1 : \tilde{\mathrm{L}}_1 \to \mathrm{L}_1$ be the universal covering spaces of L_0 and L_1 respectively. To every germ $\mathscr{f} \in \mathscr{I}$ and every $\tilde{x}_0 \in \tilde{\mathrm{L}}_0$ and $\tilde{x}_1 \in \tilde{\mathrm{L}}_1$ such that $p_0(\tilde{x}_0) = \pi_0(\mathscr{f})$ and $p_1(\tilde{x}_1) = \pi_1(\mathscr{f})$, one associates a germ $\tilde{\mathscr{f}}$ of an isometry from $\tilde{\mathrm{L}}_0$ to $\tilde{\mathrm{L}}_1$ by taking the germ at $\tilde{x}_0$ of the map $p_1^{-1} \circ f \circ p_0$ which is defined in a neighborhood of $\tilde{x}_0$ where f represents $\mathscr{f}$ and p_1^{-1} denotes the inverse of the local isometry defined by restricting p_1 to a small neighborhood of $\tilde{x}_1$. Clearly, if $\mathscr{f} \in \mathscr{I}$ has the path lifting property described in lemma 2.3, then so does $\tilde{\mathscr{f}}$ in the sheaf $\tilde{\mathscr{I}}$ of germs of isometries from $\tilde{\mathrm{L}}_0$ to $\tilde{\mathrm{L}}_1$.

LEMMA 2.4. – *Suppose* L_0 *and* L_1 *are complete Riemannian manifolds and* $\mathscr{f} \in \mathscr{I}$ *is as in lemma* 2.3. *Then* $\tilde{\mathscr{f}}$ *defines an isometry from* $\tilde{\mathrm{L}}_0$ *onto* $\tilde{\mathrm{L}}_1$.

Proof. – Let π_0 and π_1 denote the source and evaluation maps of $\tilde{\mathscr{I}}$ restricted to the connected component containing $\tilde{\mathscr{f}}$. Since $\tilde{\mathrm{L}}_0$ is simply connected, the above discussion and lemma 2.3 imply that π_0 is a homeomorphism. Thus $\pi_1 \circ \pi_0^{-1}$ is a local isometry from the complete manifold $\tilde{\mathrm{L}}_0$ into $\tilde{\mathrm{L}}_1$. Hence it is an isometry since it is a covering map (p. 176 [11]) and $\tilde{\mathrm{L}}_1$ is simply connected.

Throughout the remainder of this section, $\mathscr{F}$ is a smooth codimension k totally geodesic foliation of the connected Riemannian manifold (M,g). An $\mathscr{H}$-curve $\sigma : [c,d] \to \mathrm{M}$ is a piecewise smooth curve all of whose tangent vectors are perpendicular to the leaves of $\mathscr{F}$.

Let $f : \mathrm{U} \to \mathbf{R}^k$ be a submersion constant on the leaves of $\mathscr{F}$ restricted to the open set $\mathrm{U} \subset \mathrm{M}$. Given an $\mathscr{H}$-curve $\sigma : [c,d] \to \mathrm{U}$, let $\bar{\gamma} = f \circ \sigma$. For all $x \in \mathrm{f}^{-1}(\bar{\gamma}(c))$ near $\sigma(c)$ there is a unique $\mathscr{H}$-curve, γ_x, such that $f(\gamma_x(t)) = \bar{\gamma}(t)$ and $\gamma_x(c) = x$. According to the proof of proposition 1.4 of [10], this defines a family of isometries $\varphi_t : \mathrm{V}_c \to \mathrm{V}_t \; (c \leqslant t \leqslant d)$ where $\varphi_t(x) = \gamma_x(t)$ and V_t is a neighborhood of $\sigma(t)$ in the leaf of $\mathscr{F}$ through $\sigma(t)$.

These families of isometries can be pasted together along an $\mathscr{H}$-curve $\sigma : [0,1] \to \mathrm{M}$ in the following way. Let

$$0 = t_0 < t_1 < \cdots < t_r = 1$$

be a partition of $[0,1]$ so that $\sigma/[t_{i-1}, t_i] \subset \mathrm{U}_i$ where $f_i : \mathrm{U}_i \to \mathbf{R}^k$ is a submersion constant on the leaves of $\mathscr{F}/U_i (i=1, \dots, r)$. For each curve $\sigma/[t_{i-1}, t_i]$, the above construction gives a family of isometries. By cutting down the domains of these isometries and composing them in the proper order, one obtains a family of isometries

$$(*) \qquad \varphi_t : V_0 \to V_t \quad (0 \leqslant t \leqslant 1)$$

where

(1) V_t is a neighborhood of $\sigma(t)$ in the leaf of $\mathscr{F}$ through $\sigma(t)$,

(2) $\varphi_t(\sigma(0)) = \sigma(t)$ for all t,

(3) for each $x \in V_0$, the curve $\varphi_t(x)$ is an $\mathscr{H}$-curve, and

(4) φ_0 is the identity map of V_0.

We will call a family $(*)$ satisfying (1)-(4) an *element of holonomy along the $\mathscr{H}$-curve* σ.

LEMMA 2.5. — *Let* $\sigma : [0,1] \to M$ *be an $\mathscr{H}$-curve. Then there exists an element of holonomy along* σ. *Furthermore, if* φ_t^1 *and* $\varphi_t^2 (0 \leqslant t \leqslant 1)$ *are two elements of holonomy along* σ, *then* $\varphi_t^1(x) = \varphi_t^2(x) (0 \leqslant t \leqslant 1)$ *for all* x *sufficiently near* $\sigma(0)$.

Proof. — Existence has been shown already. To obtain uniqueness, take a partition $0 = t_0 < t_1 < \cdots < t_r = 1$ of $[0,1]$ and submersions $f_i : U_i \to \mathbf{R}^k$ constant on the leaves of $\mathscr{F}/U_i$ with $\sigma/[t_{i-1}, t_i] \subset U_i$. For every i and x sufficiently near $\sigma(0)$,

$$f_i(\varphi_t^1(x)) = f_i(\sigma(t)) = f_i(\varphi_t^2(x)) \qquad \text{for} \qquad t \in [t_{i-1}, t_i]$$

by properties (1) and (2) of an element of holonomy. Moreover, both $\varphi_t^1(x)$ and $\varphi_t^2(x)$ are $\mathscr{H}$-curves. Thus, by the uniqueness of $\mathscr{H}$-curve lifts of $f_i \circ \sigma$, if $\varphi_{t_{i-1}}^1(x) = \varphi_{t_{i-1}}^2(x)$ then $\varphi_t^1(x) = \varphi_t^2(x)$ for all $t \in [t_{i-1}, t_i]$. Finally, this holds for all $t \in [0,1]$ by induction on i since $\varphi_0^1(x) = x = \varphi_0^2(x)$ for all x near $\sigma(0)$ by property (4).

Let $\sigma : [0,1] \to M$ be a fixed $\mathscr{H}$-curve. For each $t \in [0,1]$, L_t denotes the leaf of $\mathscr{F}$ through $\sigma(t)$ with the induced metric and $\mathscr{I}_t$ denotes the sheaf of germs of isometries from L_0 into L_t. Now, let $\tau : [a,b] \to L_0$ be a piecewise smooth curve with $\tau(a) = \sigma(0)$.

DEFINITION. — *A continuation* Φ *of* σ *along* τ *is a finite sequence* φ_t^i *of elements of holonomy along $\mathscr{H}$-curves defined on open sets* $V_0^i \subset L_0$ *for*

$i = 1, \ldots, r$ and a partition $a = s_0 < s_1 < \cdots < s_r = b$ of $[a,b]$ such that

(1) *φ_t^1 is an element of holonomy along σ,*

(2) *$\varphi_t^{i-1} = \varphi_t^i$ on $V_0^{i-1} \cap V_0^i$ for all i, and*

(3) *$\tau/[s_{i-1}, s_i] \subset V_0^i$ for all i.*

Clearly, a continuation Φ of σ along τ gives a lift Φ_t of τ to $\mathscr{I}_t$ for all $t \in [0,1]$ by defining $\Phi_t(s)$ to be the germ of the isometry φ_t^i at $\tau(s)$ when $s \in [s_{i-1}, s_i]$.

LEMMA 2.6. — *Suppose (M,g) is complete. Then for every $\mathscr{H}$-curve $\sigma : [0,1] \to M$ and every piecewise smooth curve $\tau : [a,b] \to L_0$ with $\tau(a) = \sigma(0)$, there exists a continuation of σ along τ.*

Proof. — Set $s_0 = \sup \{s \in [a,b] :$ there exists a continuation of σ along $\tau/[a,s]\}$. By lemma 2.5, $s_0 > a$. We must show $s_0 = b$. Hence, suppose $s_0 \neq b$. Let φ_t be an element of holonomy along σ and let C_s be the development of τ into $T_{\sigma(0)}(L_0)$. Now, since M is a complete Riemannian manifold, so is every leaf of $\mathscr{F}$. Hence, for every $t \in [0,1]$, there exists a curve $\Psi_t : [a,b] \to L_t$ with $\Psi_t(a) = \sigma(t)$ whose development in $T_{\sigma(t)}(L_t)$ is $\varphi_{t*}(C_s)$ (see [11], p. 172).

On the other hand, for each $s < s_0$, there exists a continuation Φ of σ along $\tau/[a,s]$. This gives rise to a family Φ_t of lifts of $\tau/[a,s]$ to $\mathscr{I}_t$. Letting $\pi_1 : \mathscr{I}_t \to L_t$ be the evaluation map, it follows that for each $t \in [0,1]$, $\pi_1(\Phi_t(s)) = \Psi_t(s)$ for all $s < s_0$ since these two curves have the same developments by corollary 2.2. Now, by construction, for each fixed $s < s_0$, $\pi_1(\Phi_t(s))$ is an $\mathscr{H}$-curve in t. Hence for each $s < s_0$, $\Psi_t(s)$ is an $\mathscr{H}$-curve in t. By continuity, so is $\Psi_t(s_0)$. Therefore, there exists an element of holonomy along $\Psi_t(s_0)$ by lemma 2.5. Furthermore, since $\Psi_t(s) = \pi_1(\Phi_t(s))$ for $s < s_0$ with s near s_0, the uniqueness part of lemma 2.5 implies that the element of holonomy along $\Psi_t(s_0)$ agrees with that along $\pi_1(\Phi_t(s))$ on the overlap of their domains. Hence there exists a continuation of σ beyond s_0. This contradiction implies $s_0 = b$.

Taking $\delta(t,s) = \Psi_t(s)$, one has the following result.

COROLLARY 2.7. — *If (M,g) is complete, then for every $\mathscr{H}$-curve $\sigma : [0,1] \to M$ and every piecewise smooth curve $\tau : [a,b] \to L_0$ with*

$\tau(a) = \sigma(0)$, *there exists a homotopy* $\delta : [0,1] \times [a,b] \to M$ *such that*

(i) $\delta(t,a) = \sigma(t)$ *for all* t,

(ii) $\delta(0,s) = \tau(s)$ *for all* s,

(iii) $\delta(t,s) \in L_t$ *for all* t *and* s, *and*

(iv) $t \to \delta(t,s)$ *is an* $\mathscr{H}$*-curve for every fixed* s. *In particular,* $t \to \delta(t,b)$ *is an* $\mathscr{H}$*-curve starting at* $\tau(b)$ *and ending in* L_1.

Corollary 2.8. – *If* (M,g) *is complete, then any two leaves of* $\mathscr{F}$ *are connected by an* $\mathscr{H}$*-curve.*

Proof. – Define an equivalence relation on the leaves of $\mathscr{F}$ by saying $L \sim L'$ if they are connected by an $\mathscr{H}$-curve. This relation is clearly reflexive and symmetric. To show it is transitive suppose that $\sigma_0 : [0,1] \to M$ and $\sigma_1 : [1,2] \to M$ are $\mathscr{H}$-curves with

$$\sigma_0(0) \in L_0, \qquad \sigma_0(1) \in L_1, \qquad \sigma_1(1) \in L_1 \qquad \text{and} \qquad \sigma_1(2) \in L_2.$$

Let τ be any curve in L_1 joining $\sigma_0(1)$ to $\sigma_1(1)$. The homotopy of corollary 2.7 applied to σ_1 and τ gives an $\mathscr{H}$-curve $\sigma_2 : [1,2] \to M$ with $\sigma_2(1) = \sigma_0(1)$ and $\sigma_2(2) \in L_2$. The union of σ_0 and σ_2 is an $\mathscr{H}$-curve connecting L_0 to L_2. Now, since equivalence classes are clearly open saturated sets and M is connected, there is only one equivalence class.

Remark. – In the terminology of Hermann [7], corollary 2.7 proves that every $\mathscr{H}$-curve is regular, while corollary 2.8 shows that every pair of leaves of $\mathscr{F}$ are regularly connected. Hence, by theorem 2.1 of [7], any two leaves of $\mathscr{F}$ have diffeomorphic universal covers. In fact, more is true.

Corollary 2.9. – *If* (M,g) *is complete, then any two leaves of* $\mathscr{F}$ *have isometric universal covering spaces.*

Proof. – By 2.8, any two leaves are connected by an $\mathscr{H}$-curve. This defines a germ of an isometry between them by lemma 2.5. By lemma 2.6 this germ has the path lifting property described in lemma 2.3. Hence the universal covering spaces of the two leaves are isometric by lemma 2.4.

From this point on, we assume that the normal distribution to $\mathscr{F}$ is integrable and thus defines a foliation $\mathscr{G}$ of codimension $n-k$ $(n = \dim M)$ orthogonal to $\mathscr{F}$.

Another consequence of corollary 2.7 is the following theorem first proved by Johnson and Whitt using a different method.

THEOREM 2.10 [10]. – *If* (M,g) *is complete, every leaf of* $\mathcal{G}$ *meets every leaf of* $\mathcal{F}$.

Proof. – Let H be a leaf of $\mathcal{G}$ and L a leaf of $\mathcal{F}$. Let L_0 be a leaf of $\mathcal{F}$ that meets H at x. By corollary 2.8 there exists an $\mathcal{H}$-curve σ joining L_0 to L. Let τ be a curve in L_0 joining $\sigma(0)$ to x. The homotopy of corollary 2.7 gives an $\mathcal{H}$-curve joining x to a point in L. This curve must lie in H. Thus there is a point in the intersection of H and L.

We now begin the proof of theorem A.

Fix a leaf L_0 of $\mathcal{F}$. If $x \in M$, let L_x denote the leaf of $\mathcal{F}$ through x. There is a neighborhood U of x in M and a Riemannian submersion $f_U : U \to L_x \cap U$ constant along the leaves of $\mathcal{G}/U$ [10]. Now, since (M,g) is complete, theorem 2.10 implies the leaf H_x of $\mathcal{G}$ through x meets L_0. Thus there exist $\mathcal{H}$-curves $\sigma : [0,1] \to M$ with $\sigma(0) = x$ and $\sigma(1) \in L_0$. If φ_1 is the isometry from a neighborhood of x in L_x to a neighborhood of $\sigma(1)$ in L_0 defined by an element of holonomy along σ, then $f = \varphi_1 \circ f_U : U \to L_0$ is a Riemannian submersion which is constant along the leaves of $\mathcal{G}/U$. Thus we can find an L_0-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta\in A}$ on M where

(i) $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M,

(ii) $f_\alpha : U_\alpha \to L_0$ is a Riemannian submersion whose level sets are the leaves of $\mathcal{G}/U_\alpha$,

(iii) $g_{\alpha\beta} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ is an isometry satisfying $f_\alpha = g_{\alpha\beta} \circ f_\beta$ on $U_\alpha \cap U_\beta$.

Furthermore, by construction each $g_{\alpha\beta}$ is an isometry defined by an element of holonomy along an $\mathcal{H}$-curve $\sigma : [0,1] \to M$ with both $\sigma(0)$ and $\sigma(1)$ lying in L_0. Hence, by lemma 2.6 the germs of the isometry $g_{\alpha\beta}$ have the path lifting property described in lemma 2.3. By cutting down the U_α, we may suppose that $f_\alpha(U_\alpha)$ is contained in a neighborhood over which the universal cover $\tilde{L}_0$ of L_0 is trivial. Thus we may lift the f_α to obtain Riemannian submersions $\tilde{f}_\alpha$ into $\tilde{L}_0$ and an $\tilde{L}_0$-cocycle $\{(U_\alpha, \tilde{f}_\alpha, \tilde{g}_{\alpha\beta})\}_{\alpha,\beta\in A}$ where

(i) $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M,

(ii) $\tilde{f}_\alpha : U_\alpha \to \tilde{L}_0$ is a Riemannian submersion whose level sets are the leaves of $\mathcal{G}/U_\alpha$,

(iii) $\tilde{g}_{\alpha\beta} : \tilde{f}_\alpha(U_\alpha \cap U_\beta) \to \tilde{f}_\beta(U_\alpha \cap U_\beta)$ is an isometry satisfying $\tilde{f}_\alpha = \tilde{g}_{\alpha\beta} \circ \tilde{f}_\beta$.

Without loss of generality, we may assume $U_\alpha \cap U_\beta$ is connected whenever it is non-empty. Hence, by construction and lemma 2.4, each $\tilde{g}_{\alpha\beta}$ extends to an isometry of $\tilde{L}_0$.

Let $I(\tilde{L}_0)$ be the isometry group of $\tilde{L}_0$. Define

$$P = \{[g \circ f_\alpha]_x : x \in U_\alpha,\ \alpha \in A,\ g \in I(\tilde{L}_0)\}$$

where $[g \circ f_\alpha]_x$ denotes the germ of $g \circ f_\alpha$ at x. Let $\pi : P \to M$ be the source map. Then $\pi : P \to M$ is a smooth principal $I(\tilde{L}_0)$-bundle where $I(\tilde{L}_0)$ has the discrete topology. Let P_0 be a connected component of P. Then P_0 is a regular covering of M and the evaluation map $F : P_0 \to \tilde{L}_0$ is a Riemannian submersion constant along the leaves of $\pi^{-1}(\mathscr{G})$. Since the metric on P_0 is complete and bundle-like for $\pi^{-1}(\mathscr{G})$ we have that $F : P_0 \to \tilde{L}_0$ is a locally trivial fiber space [8].

Let $L \in \pi^{-1}(\mathscr{F})$. Then L is a complete Riemannian manifold and $F/L : L \to \tilde{L}_0$ is an isometric immersion. Hence L is a covering space of $\tilde{L}_0$ with projection F/L [11]. Hence $F/L : L \to \tilde{L}_0$ is an isometry.

Fix $H_0 \in \pi^{-1}(\mathscr{G})$. Let $p \in P_0$. The leaf L_p of $\pi^{-1}(\mathscr{F})$ through p meets H_0 (Theorem 2.10). Suppose $z_1, z_2 \in L_p \cap H_0$. Then $F(z_1) = F(z_2)$. Since $F/L_p : L_p \to \tilde{L}_0$ is injective we have $z_1 = z_2$. Thus $L_p \cap H_0$ consists of a single point $\varphi(p)$. Define

$$\Phi : P_0 \to H_0 \times \tilde{L}_0$$

by

$$\Phi(p) = (\varphi(p),\ F(p)).$$

Suppose $\Phi(p_1) = \Phi(p_2)$. Then $\varphi(p_1) = \varphi(p_2)$ and so $L_{p_1} = L_{p_2}$. Since $F(p_1) = F(p_2)$ we have that $p_1 = p_2$. Let $(a,b) \in H_0 \times \tilde{L}_0$. Then

$$L_a \cap F^{-1}\{b\} = \{p\} \qquad \text{and} \qquad \Phi(p) = (\varphi(p),\ F(p)) = (a,b).$$

Thus Φ is a diffeomorphism and the following diagrams commute :

$$\begin{tikzcd}
P_0 \arrow[rr] \arrow[dr, "\varphi"'] & & H_0 \times \tilde{L}_0 \arrow[dl, "\pi_1"] \\
& H_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
P_0 \arrow[rr] \arrow[dr, "F"'] & & H_0 \times \tilde{L}_0 \arrow[dl, "\pi_2"] \\
& \tilde{L}_0 &
\end{tikzcd}$$

This completes the proof of Theorem A.

We now consider totally geodesic foliations for which every leaf is flat.

THEOREM 2.11. – *If* M *is a compact Riemannian manifold with a codimension-one totally geodesic transversely oriented foliation* $\mathscr{F}$ *by flat leaves, then* M *fibers over* S^1 *and the universal cover of* M *is* $\mathbf{R}^n$.

Proof. – Recall one of Plante's [14] characterizations of the growth of a leaf L of $\mathscr{F}$. Let $p \in L$ and define the growth function of L at p by $g_p(r) = \text{vol}\,(B_p(r))$ where $B_p(r)$ denotes the open ball in L of radius r centered at p. The growth type of L is then the growth type of the function $g_p : \mathbf{R}^+ \to \mathbf{R}^+$ and depends only on L. Since each leaf of $\mathscr{F}$ is a complete flat Riemannian manifold, it follows that the universal cover of each leaf is $\mathbf{R}^{n-1}$ with its standard metric. Hence each leaf of $\mathscr{F}$ has polynomial growth of degree $\leqslant n-1$. In particular, all the leaves of $\mathscr{F}$ have non-exponential growth. Hence either $\mathscr{F}$ has a compact leaf or else $\mathscr{F}$ is without holonomy [15]. If $\mathscr{F}$ has a compact leaf, then M fibers over S^1 [10]. If $\mathscr{F}$ is without holonomy, then $\mathscr{F}$ is topologically conjugate to a foliation defined by a non-vanishing closed one-form [18]. Hence M fibers over S^1 by Tischler's Theorem [19]. Finally $\tilde{M} \cong \mathbf{R}^{n-1} \times \mathbf{R} = \mathbf{R}^n$.

COROLLARY 2.12. – *Let* M *be a compact, orientable,* 3*-dimensional Riemannian manifold with a codimension-*1 *totally geodesic transversely oriented foliation* $\mathscr{F}$ *by flat leaves. Then*

1) M *fibers over* S^1,

2) *the universal cover of* M *is* $\mathbf{R}^3$,

3) $\pi_1(M)$ *is solvable,*

4) $H_1(M,Z) \neq 0$, *and*

5) *if* $\mathscr{G}$ *has no closed orbits, then* $\pi_1(M)$ *is abelian and* M *fibers over* T^2.

Proof. – (1) and (2) follow from the previous Theorem. Since $\mathscr{G}$ is a codimension-2 Euclidean foliation, it follows that $\pi_1(M)$ is solvable and $H_1(M,Z) \neq 0$ [1]. Suppose $\mathscr{G}$ has no closed orbit. Then all the leaves of $\mathscr{G}$ are simply connected and hence $\pi_1(M)$ is abelian [1]. Moreover, since $\mathscr{G}$ is without holonomy, M fibers over T^2 [2].

PROPOSITION 2.13. – *Let* M *be a compact* 3*-dimensional Riemannian manifold with a codimension-*1 *totally geodesic foliation* $\mathscr{F}$ *by leaves of constant negative curvature. Then* $\pi_1(M)$ *has exponential growth.*

Proof. – In this case $\mathscr{G}$ is a codimension-two hyperbolic foliation and hence $\pi_1(M)$ has exponential growth [1].

See [5] for a more complete description of codimension-1 totally geodesic foliations of 3-dimensional manifolds.

Example. – This example uses the method of suspension of [6]. Let L be a compact manifold and let $\pi : \tilde{L} \to L$ be the universal cover of L. Let H be a manifold and let $\varphi : \pi_1(L) \to \mathrm{Diff}(H)$ be a homomorphism. The foliation of $\tilde{L} \times H$ by leaves of the form $\tilde{L} \times \{\mathrm{pt.}\}$ passes to a foliation $\mathscr{F}$ of the associated fiber bundle $M = \tilde{L} \times_{\pi_1(L)} H$ transverse to the fibers. Let $\mathscr{G}$ be the foliation of M by the fibers of the bundle. Let $T(\mathscr{F})$ and $T(\mathscr{G})$ be the subbundles of $T(M)$ tangent to $\mathscr{F}$ and $\mathscr{G}$, respectively. Then $T(M) = T(\mathscr{F}) \oplus T(\mathscr{G})$. Put any Riemannian metric on L. This induces a metric on $T(\mathscr{F})$. Put any metric on $T(\mathscr{G})$. By decreeing $T(\mathscr{F})$ and $T(\mathscr{G})$ to be orthogonal, we obtain a Riemannian metric on M. This metric is bundle-like for $\mathscr{G}$. Hence $\mathscr{F}$ is totally geodesic [10].

e.g. Let $L = T_2$, the two-holed torus, and let $H = S^1$. Then $\pi_1(L)$ is a subgroup of $SL(2,\mathbf{R})$ and hence acts in a natural way on S^1. Endowing L with the hyperbolic metric, we obtain a codimension-1 foliation of M with totally geodesic leaves of constant negative curvature.

Example. – Let G be a connected Lie group admitting a bi-invariant metric $\langle -,- \rangle$. Since the 1-parameter subgroups of G are geodesics, the left cosets of a connected subgroup H form a totally geodesic foliation of G. Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on G, let $\mathfrak{h}$ be the subalgebra associated to H, and let $\mathfrak{h}^\perp$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Since $\langle [X,Y],Z \rangle = \langle X,[Y,Z] \rangle$ for every $X, Y, Z \in \mathfrak{g}$, on taking $X, Y \in \mathfrak{h}^\perp$ and $Z \in \mathfrak{h}$ it follows that $\mathfrak{h}^\perp$ is integrable if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, i.e. H is a normal subgroup of G. If this is the case, the same argument shows $\mathfrak{h}^\perp$ to be an ideal which thus defines a connected normal subgroup K of G that is orthogonal to H. Therefore on passing to the universal covers of G, H and K we have $\tilde{G} = \tilde{H} \times \tilde{K}$ as a group product. An irrational flow on the torus is a particular case of this example.

Example. – Let (L,g) and (H,h) be two Riemannian manifolds and let $p : \tilde{H} \to H$ be the universal cover of H and $\tilde{h}$ the lifted metric on $\tilde{H}$. Let $\rho : \pi_1(H) \to \mathrm{Iso}\ (L)$ be a representation. Finally let $\lambda : L \to (0,\infty)$ be a smooth positive function invariant under $\pi_1(H)$,

$$\text{i.e.}\quad \lambda(\rho(\gamma)(x)) = \lambda(x) \text{ for all } \gamma \in \pi_1(H) \text{ and } x \in L.$$

Define an action of $\pi_1(H)$ on $L \times \tilde{H}$ by

$$\gamma(x,y) = (\rho(\gamma)(x), \gamma(y)) \text{ for } \gamma \in \pi_1(H) \text{ and } (x,y) \in L \times \tilde{H}.$$

This action is properly discontinuous. The foliation of $L \times \tilde{H}$ by leaves of the form $L \times \{\text{pt.}\}$ passes to a foliation $\mathscr{F}$ of $(L \times \tilde{H})/\pi_1(H)$, while that by leaves of the form $\{\text{pt.}\} \times \tilde{H}$ passes to a foliation $\mathscr{G}$. Furthermore this action operates as isometries when $L \times \tilde{H}$ has the warped product metric $\tilde{g} = g \oplus \lambda\tilde{h}$. In this metric the leaves $L \times \{\text{pt.}\}$ are totally geodesic and are othogonal to the leaves $\{\text{pt.}\} \times \tilde{H}$. Thus $\mathscr{F}$ is a totally geodesic foliation with the orthogonal complementary foliation $\mathscr{G}$. (Choosing λ to be non-constant prevents the foliation $\mathscr{G}$ from being totally geodesic.)

e.g. Let $L = S^2$ with the canonical metric and let $H = S^1$. Let $\rho : \pi_1(H) \to \text{Iso}\ (L) = 0(3)$ be defined by letting the generator of $\pi_1(H)$ go to some (perhaps irrational) rotation of S^2 around the north-south polar axis. Finally, λ can be any smooth positive function constant on the lines of latitude.

3. Riemannian foliations.

Let M be a smooth manifold and let $\mathscr{F}$ be a smooth codimension-q Riemannian foliation of M. Let $T(M)$ be the tangent bundle of M and let $E \subset T(M)$ be the subbundle tangent to $\mathscr{F}$. Choose an imbedding of the normal bundle Q of $\mathscr{F}$ as a subbundle of $T(M)$ satisfying $T(M) = E \oplus Q$. Since $\mathscr{F}$ is Riemannian, there is a smooth metric g on Q invariant under the natural parallelism along the leaves. This is equivalent to the existence of a bundle-like metric in the sense of Reinhart [16]. Let $\Gamma(E)$, $\Gamma(Q)$, and $\mathscr{X}(M)$ denote the spaces of smooth sections of the vector bundles E, Q, and $T(M)$ respectively. Recall that a connection $\nabla : \mathscr{X}(M) \times \Gamma(Q) \to \Gamma(Q)$ is basic if it induces the natural parallelism along the leaves. Equivalently, $\nabla_X Y = [X,Y]_Q$ for all $X \in \Gamma(E)$, $Y \in \Gamma(Q)$ where $[X,Y]_Q$ denotes the Q-component of the Lie bracket of X and Y [4]. Let ∇ be the unique metric-preserving basic connection on Q with zero torsion ($\nabla_X Y_Q - \nabla_Y X_Q = [X,Y]_Q$ for all

$X, Y \in \mathscr{X}(M)$) [12], [13]. Let $x \in M$ and let $C(x)$ be the loop space at x. For each $\tau \in C(x)$, the parallel transport along τ is an isometry of Q_x. The set of all such isometries of Q_x is the holonomy group $\Psi(x)$ of ∇ with reference point x.

We now prove Theorem D. If $\mathscr{F}$ is irreducible, we are done. Assume $\mathscr{F}$ is reducible. Let Q'_x be a non-trivial subspace of Q_x invariant by $\Psi(x)$. Let $y \in M$. Choose a curve τ from x to y and let $Q'_y \subset Q_y$ be the image of Q'_x by the parallel translation along τ. Then Q'_y depends only on the point y and so we obtain a smooth distribution $Q' \subset Q \subset T(M)$.

LEMMA 3.1. – *The distribution* $E \oplus Q'$ *is involutive.*

Proof. – If $X, Y \in \Gamma(E)$, then $[X,Y] \in \Gamma(E)$ since E is involutive. Suppose $X \in \Gamma(E)$, $Y \in \Gamma(Q')$. Then $[X,Y]_Q = \nabla_X Y \in \Gamma(Q')$ and hence $[X,Y] \in \Gamma(E \oplus Q')$. Finally, suppose $X, Y \in \Gamma(Q')$. Then $[X,Y]_Q = \nabla_X Y - \nabla_Y X \in \Gamma(Q')$ and so $[X,Y] \in \Gamma(E \oplus Q')$.

LEMMA 3.2. – *Let* $\mathscr{F}'$ *be the foliation integral to* $E \oplus Q'$. *Then* $\mathscr{F}'$ *is a Riemannian foliation and the restriction of* ∇ *to the normal bundle of* $\mathscr{F}'$ *is the unique torsion-free metric-preserving basic connection for* $\mathscr{F}'$.

Proof. – Let Q'' be the orthogonal complement of Q' in Q. Then Q'' is the normal bundle of $\mathscr{F}'$. If $X \in \mathscr{X}(M)$ and $Y \in \Gamma(Q'')$ then, since Q'' is holonomy invariant, $\nabla_X Y \in \Gamma(Q'')$ and ∇ determines a connection on Q''. Let $X \in \Gamma(E \oplus Q')$, $Y \in \Gamma(Q'')$. Then

$$\nabla_X Y = \nabla_{X_E} Y + \nabla_{X_{Q'}} Y = [X_E, Y]_{Q''} + \nabla_{X_{Q'}} Y .$$

But $[X_{Q'}, Y]_Q = \nabla_{X_{Q'}} Y - \nabla_Y X_{Q'}$ and so $\nabla_{X_{Q'}} Y = [X_{Q'}, Y]_{Q''}$. Hence

$$\nabla_X Y = [X_E, Y]_{Q''} + [X_{Q'}, Y]_{Q''} = [X,Y]_{Q''}$$

which shows that ∇ is a basic connection for $\mathscr{F}'$. Since ∇ is metric-preserving and induces the natural parallelism along the leaves of $\mathscr{F}'$, it follows that the restriction of g to Q'' is invariant under the natural parallel transport along $\mathscr{F}'$ and so $\mathscr{F}'$ is a Riemannian foliation completing the proof of the lemma.

We may decompose Q_x as a direct sum $Q_x = Q_x^0 \oplus Q_x^1 \oplus \cdots \oplus Q_x^k$ of mutually orthogonal subspaces invariant under $\Psi(x)$ where Q_x^0 is the set of vectors in Q_x which are fixed by $\Psi(x)$ and $Q_x^1, \dots, Q_x^k$ are all irreducible. For each $i = 0, 1, \dots, k$ let $\mathscr{F}_i$ be the foliation of M which is integral to the distribution $E \oplus Q^0 \oplus \cdots \oplus \hat{Q}^i \oplus \cdots \oplus Q^k$ where $\hat{Q}^i$ indicates that Q^i is omitted. By lemma 3.2 each $\mathscr{F}_i$ is a Riemannian foliation and clearly $\mathscr{F}_1, \dots, \mathscr{F}_k$ are irreducible. If $\operatorname{codim}(\mathscr{F}_0) = 0$, we are done. Assume $\operatorname{codim}(\mathscr{F}_0) = m > 0$. Let $Y_{1_x}, \dots, Y_{m_x}$ be a basis of Q_x^0. Since $Y_{1_x}, \dots, Y_{m_x}$ are fixed by $\Psi(x)$, these vectors can be extended to vector fields $Y_1, \dots, Y_m \in \Gamma(Q^0)$ which are parallel with respect to ∇. In particular, $Y_1, \dots, Y_m$ are parallel along the leaves of $\mathscr{F}_0$. For $i, j = 1, \dots, m$ we have

$$[Y_i, Y_j]_Q = \nabla_{Y_i} Y_j - \nabla_{Y_j} Y_i = 0 \qquad \text{and so} \qquad [Y_i, Y_j]_{Q^0} = 0 .$$

Hence $\mathscr{F}_0$ can be defined by local submersions to $\mathbf{R}^m$ which on overlaps differ by translations thus showing that $\mathscr{F}_0$ is a Lie foliation modeled on $\mathbf{R}^m$. Clearly $\mathscr{F} = \bigcap_{i=0}^{k} \mathscr{F}_i$ and the proof of Theorem D is complete.

COROLLARY 3.3. – *Let* $\mathscr{F}$ *be a Riemannian foliation of a compact manifold* M. *Let* $m = \operatorname{codim}(\mathscr{F}_0)$ *(possibly* $m=0$*). Then*

i) M *fibers over the m-dimensional torus* T^m.

ii) *The universal cover of* M *is a product* $L \times \mathbf{R}^m$ *where* L *is the universal cover of the leaves of* $\mathscr{F}_0$ *and the lift of* $\mathscr{F}_0$ *is the product foliation.*

Proof. – Let $Y_1, \dots, Y_m$ be as in the proof of Theorem D. Let $\omega_1, \dots, \omega_m$ be smooth one-forms on M which vanish on vectors tangent to $\mathscr{F}_0$ and satisfy $\omega_i(Y_j) = \delta_{ij}$. Fix $1 \leqslant i \leqslant m$. If

$$X, Z \in \Gamma(E \oplus Q^1 \oplus \cdots \oplus Q^k),$$

then $d\omega_i(X,Z) = -\omega_i[X,Z] = 0$ since $E \oplus Q^1 \oplus \cdots \oplus Q^k$ is involutive. If $X \in \Gamma(E \oplus Q^1 \oplus \cdots \oplus Q^k)$ and $j \in \{1, \dots, m\}$, then $d\omega_i(X,Y_j) = -\omega_i[X,Y_j] = 0$ since Y_j is parallel along the leaves of $\mathscr{F}_0$. If $j_1, j_2 \in \{1, \dots, m\}$, then $d\omega_i(Y_{j_1}, Y_{j_2}) = -\omega_i[Y_{j_1}, Y_{j_2}] = 0$ since $[Y_{j_1}, Y_{j_2}]_{Q^0} = 0$. Thus $d\omega_i = 0$ and so $\omega_1, \dots, \omega_m$ are closed linearly independent one-forms. By Tischler's theorem [19], M fibers over T^m. The second statement follows from Corollary 2 in [1].

We now prove Theorem E. Let $R : \mathscr{X}(M) \times \mathscr{X}(M) \times \Gamma(Q) \to \Gamma(Q)$ be the curvature of ∇, that is, $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. Recall that a differential r-form ω on M is base-like [16] if $i_X\omega = i_X d\omega = 0$ for all $X \in \Gamma(E)$ where i_X denotes the interior product. Since $\mathcal{F}$ has recurrent curvature, there is a base-like one-form α on M such that $\nabla R = R \otimes \alpha$. Let $x \in M$. Let $f : U \to V$ be a submersion whose level sets are the leaves of $\mathcal{F}/U$ where U is a neighborhood of x in M and V is an open set in $\mathbf{R}^q$, $q = \operatorname{codim}(\mathcal{F})$. There is a unique Riemannian metric $\bar{g}$ on V such that $f^*(\bar{g}) = g$. Let $\bar{\nabla}$ be the Riemannian connection on V. Then $f^{-1}(\bar{\nabla}) = \nabla/U$. Let $\bar{R}$ be the curvature of $\bar{\nabla}$. Since α is base-like, there is a unique one-form $\bar{\alpha}$ on V such that $f^*\bar{\alpha} = \alpha$. Then we have $\bar{\nabla} = \bar{R} \otimes \bar{\alpha}$ and so V is a Riemannian manifold with recurrent curvature tensor.

Let $\pi : O(Q) \to M$ be the orthonormal frame bundle of Q, a principal $O(q)$-bundle. Let Γ be the connection in $O(Q)$ corresponding to ∇ and let Γ_U be the connection in $O(Q)/U = \pi^{-1}(U)$ induced by Γ. Let $u \in \pi^{-1}(x)$. Let $\Psi(u)$ (respectively, $\Psi^0(u)$) be the holonomy group (respectively, restricted holonomy group) of Γ with reference point u. Let $\Psi(u,U)$ (respectively, $\Psi^0(u,U)$) be the holonomy group (respectively, restricted holonomy group) of Γ_U with reference point u. Let $\bar{\pi} : O(V) \to V$ be the orthonormal frame bundle of V and let $\bar{\Gamma}$ be the connection in $O(V)$ corresponding to $\bar{\nabla}$. Let $\bar{u} = f_*(u)$ and let $\bar{\Psi}(\bar{u})$ (respectively, $\bar{\Psi}^0(\bar{u})$) be the holonomy group (respectively, restricted holonomy group) of $\bar{\Gamma}$ with reference point $\bar{u}$. Since M is simply connected, we have $\Psi^0(u) = \Psi(u)$. By choosing U and V to be simply connected, we have $\Psi^0(u,U) = \Psi(u,U)$ and $\bar{\Psi}^0(\bar{u}) = \bar{\Psi}(\bar{u})$. Since Γ is a real analytic connection in the real analytic principal fiber bundle $O(Q)$, we have (shrinking U if necessary) $\Psi^0(u) = \Psi^0(u,U)$ [11]. Since

$$O(Q)/U = f^{-1}(O(V)) \qquad \text{and} \qquad \Gamma_U = f^{-1}(\bar{\Gamma}),$$

we have $\Psi(u,U) \subset \bar{\Psi}(\bar{u})$ and hence $\Psi(u) \subset \bar{\Psi}^0(\bar{u})$. Since $\mathcal{F}$ is irreducible, it follows that the restricted linear holonomy group of V is irreducible. Since V has recurrent curvature tensor and $\dim(V) \geqslant 3$, its curvature tensor is parallel [11]; i.e., $\bar{\nabla}\bar{R} = 0$. Thus $\nabla R = 0$. Hence M fibers over a simply connected Riemannian symmetric space N with the leaves of $\mathcal{F}$ as fibers [3]. Clearly N is compact and is necessarily irreducible.

BIBLIOGRAPHY

[1] R. A. Blumenthal, Transversely homogeneous foliations, *Annales Inst. Fourier*, 29 (1979), 143-158.

[2] R. A. Blumenthal, Riemannian homogeneous foliations without holonomy, *Nagoya Math. J.*, 83 (1981), 197-201.

[3] R. A. Blumenthal, Riemannian foliations with parallel curvature, *Nagoya Math. J.* (to appear).

[4] R. Bott, Lectures on characteristic classes and foliations (notes by L. Conlon), *Lecture Notes in Math.*, no. 279, Springer-Verlag, New York, 1972, 1-80.

[5] Y. Carriere and E. Ghys, Feuilletages totalement géodésiques, *An. Acad. Brasil, Ciênc.*, 53 (1981), 427-432.

[6] A. Haefliger, Variétés feuilletées, *Ann. Scuola Norm. Pisa*, 16 (1962), 367-397.

[7] R. Hermann, On the differential geometry of foliations, *Annals of Math.*, 72 (1960), 445-457.

[8] R. Hermann, A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle, *Proc. Amer. Math. Soc.*, 11 (1960), 236-242.

[9] D. Johnson and L. Whitt, Totally geodesic foliations on 3-manifolds, *Proc. Amer. Math. Soc.*, 76 (1979), 355-357.

[10] D. Johnson and L. Whitt, Totally geodesic foliations, *Journal of Diff. Geom.*, 15 (1980), 225-235.

[11] S. Kobayashi and K. Nomizu, Foundations of differential geometry, vol. I, *Interscience Tracts in Pure and Appl. Math.*, 15, Interscience, New York, 1963.

[12] C. Lazarov and J. Pasternak, Secondary characteristic classes for Riemannian foliations, *Journal of Diff. Geom.*, 11 (1976), 365-385.

[13] P. Molino, Etude des feuilletages transversalement complets et applications, *Ann. Scient. Ec. Norm. Sup.*, 10 (1977), 289-307.

[14] J. F. Plante, Foliations with measure preserving holonomy, *Annals of Math.*, 102 (1975), 327-361.

[15] J. F. Plante, Measure preserving pseudogroups and a theorem of Sacksteder, *Annales Inst. Fourier*, 25, 1 (1975), 237-249.

[16] B. Reinhart, Foliated manifolds with bundle-like metrics, *Annals of Math.*, 69 (1959), 119-132.

[17] G. De Rham, Sur la réductibilité d'un espace de Riemann, *Comm. Math. Helv.*, 26 (1952), 328-344.

[18] R. Sacksteder, Foliations and pseudogroups, *Amer. J. of Math.*, 87 (1965), 79-102.

[19] D. Tischler, On fibering certain foliated manifolds over S^1, *Topology*, 9 (1970), 153-154.

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