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# DEGREE OF THE FIBRES OF AN ELLIPTIC FIBRATION

by Alexandru BUIUM

## 1. Statement of the results.

Let $f: X \longrightarrow B$ be an elliptic fibration over the complex field i.e. a morphism from a smooth complex projective surface X to a smooth curve B such that the general fibre F of $f$ is a smooth elliptic curve and no fibre contains exceptional curves of the first kind. Consider the following subsets of Pic(X):

$$N_e = \{\mathcal{L} \in \text{Pic}(X),\ \mathcal{L} = \mathcal{O}_X(D) \text{ for some effective } D\}$$
$$N_s = \{\mathcal{L} \in \text{Pic}(X),\ \mathcal{L} \text{ is spanned by global sections}\}$$
$$N_a = \{\mathcal{L} \in \text{Pic}(X),\ \mathcal{L} \text{ is ample}\}$$
$$N_v = \{\mathcal{L} \in \text{Pic}(X),\ \mathcal{L} \text{ is very ample}\}$$

and let $n_e, n_s, n_a, n_v$ be the minima of the non-zero intersection numbers $(\mathcal{L}.F)$ when $\mathcal{L}$ runs through $N_e, N_s, N_a$ and $N_v$ respectively. In [3] p. 259, Enriques investigates the possibility of finding a birational model of X in the projective space $\mathbf{P}^3$ such that the fibres of $f$ have degree $n_e$. His analysis suggests the following problem: find the minimum possible degree of the fibres of $f$ in an embedding of X in a projective space. In other words: find $n_v$. There obviously exist inequalities: $n_e \leqslant n_s \leqslant n_v$ and $n_a \leqslant n_v$.

Let $m$ denote the maximum of the multiplicities of the fibres of $f$. The aim of this paper is to prove the following propositions:

PROPOSITION 1. – *Equality* $n_e = n_s$ *holds if and only if* $n_e \geqslant 2m$.

PROPOSITION 2. — *Equality* $n_a = n_v$ *holds if and only if* $n_a \geq 3m$.

The statements above are consequences of the following more precise results:

THEOREM 1. — *There exists a constant* $C_1$ *depending only of the fibration such that for any effective divisor* D *on* X *which does not contain in its support any component of any reducible fibre and such that* D *is either reduced dominating* B, *or ample, the following conditions are equivalent:*

1) $(D.F) \geq 2m$.

2) $\mathcal{O}_X(D) \otimes f^*L$ *is spanned by global sections for any* $L \in \mathrm{Pic}(B)$ *with* $\deg(L) \geq C_1$.

3) $\mathcal{O}_X(D) \otimes f^*L$ *is spanned by global sections for some* $L \in \mathrm{Pic}(B)$.

THEOREM 2. — *There exists a constant* $C_2$ *depending only on the fibration such that for any ample sheaf* $\mathcal{L} \in \mathrm{Pic}(X)$ *the following conditions are equivalent:*

1) $(\mathcal{L}.F) \geq 3m$.

2) $\mathcal{L} \otimes f^*L$ *is very ample for any* $L \in \mathrm{Pic}(B)$ *with* $\deg(L) \geq C_2$.

3) $\mathcal{L} \otimes f^*L$ *is very ample for some* $L \in \mathrm{Pic}(B)$.

Our proofs are based on Bombieri's technique from [2]. Therefore the main point will be to prove that certain divisors on X are numerically connected.

## 2. Two lemmas.

LEMMA 1. — *Let* D *be an effective divisor on* X *which does not contain in its support any component of any reducible fibre. Suppose* D *is either reduced or ample and put* $T = D + a_1F_1 + \ldots + a_pF_p$ *where* $F_i$ *are distinct fibres and* $a_i \in \mathbf{Q}$, $a_i > 0$ *for* $1 \leq i \leq p$. *Suppose furthermore that* $a_1 + \ldots + a_p \geq 2$. *Then we have:*

1) *If* $(D.F) \geq 2m$ *then* T *is 2-connected.*

2) *If* $(D.F) \geq 3m$ *and* D *is integral and ample then* T *is 3-connected.*

*Proof.* – Suppose $T = T_1 + T_2$ where $T_k > 0$ and

$$T_k = D_k + A_k$$
$$D_1 + D_2 = D$$
$$A_1 + A_2 = A = a_1 F_1 + \ldots + a_p F_p .$$

We get

$$(T_1 . T_2) = (D_1 . D_2) + (D_1 . A_2) + (D_2 . A_1) + (A_1 . A_2) .$$

If in addition $D$ is integral we may suppose $D_2 = 0$. Since by [6] ample divisors are 1-connected it follows that in any case $(D_1 . D_2) \geqslant 0$. On the other hand we have $(D_1 . A_2) \geqslant 0$ and $(D_2 . A_1) \geqslant 0$ because any common component of $D$ and $A$ must be a rational multiple of a fibre. We may write $A_2 = Z_1 + \ldots + Z_p$ where $Z_i \leqslant a_i F_i$ for $1 \leqslant i \leqslant p$. We get

$$(A_1 . A_2) = (A - A_2 . A_2) = -(A_2^2) = -(Z_1^2) - \ldots - (Z_p^2) .$$

By [1] p. 123 we have $(Z_i^2) \leqslant 0$ for any $i$. Suppose first that there exists an index $i$ such that $(Z_i^2) < 0$. By [5], $(Z_i^2) = -2$, consequently $(T_1 . T_2) \geqslant 2$. If an addition $D$ is integral and ample then $A_2 \neq 0$ (because otherwise $T_2 = 0$) hence $(D_1 . A_2) \geqslant 1$ and we get $(T_1 . T_2) \geqslant 3$.

Now suppose $(Z_i^2) = 0$ for any $i$. Then by [1] p.123, we must have $Z_i = c_{i2} F_i$ where $c_{i2} \in \mathbf{Q}$, $0 \leqslant c_{i2} \leqslant a_i$, hence

$$A_1 = c_{11} F_1 + \ldots + c_{p1} F_p$$

where $c_{i1} + c_{i2} = a_i$. If both $D_1$ and $D_2$ dominate $B$ we get $(D_k . F) \geqslant 1$ for $k = 1, 2$ hence

$$(T_1 . T_2) \geqslant (D_1 . A_2) + (D_2 . A_1) \geqslant c_{12} + \ldots + c_{p2} + c_{11} + \ldots + c_{p1}$$
$$= a_1 + \ldots + a_p \geqslant 2$$

and we are done. If $D_k = 0$ for $k = 1$ or $k = 2$ then $A_k \neq 0$ hence there exists an index $i_0$ such that $c_{i_0 k} > 0$. Now if $m_0$ denotes the multiplicity of $F_{i_0}$ we have $c_{i_0 k} \geqslant 1/m_0 \geqslant 1/m$. Consequently we get $(T_1 . T_2) = (A_k . D) \geqslant c_{i_0 k}(D . F) \geqslant (D . F)/m$ and we are done again. Finally if $D_k \neq 0$ and $D_k$ does not dominate $B$ we get $(T_1 . T_2) \geqslant (D_1 . D_2) = (D . D_k) \geqslant (D . F)/m$ and the lemma is proved.

LEMMA 2. – *Let* $m_1, \ldots, m_r$ *denote the multiplicities of the multiple fibres of* $f$. *Then for any reduced effective divisor* $D$ *not* *containing in its support any component of any reducible fibre we have* $(D^2) \geqslant -(D.F)\left(\chi(\mathcal{O}_X) + \sum_{j=1}^{r} (m_j - 1)/m_j\right)$.

*Proof.* – We may suppose $D = D_1 + \ldots + D_t$ where $D_i$ are integral, distinct, dominating $B$. For any $i = 1, \ldots, t$ let $E_i$ be the normalization of $D_i$. By adjuction formula and by Hurwitz formula we get:

$$(D_i^2) + (D_i.K) = 2p_a(D_i) - 2 \geqslant 2p_a(E_i) - 2 \geqslant [E_i : B]\,(2p_a(B) - 2).$$

Consequently:

$$\begin{aligned}(D^2) &\geqslant \sum_{i=1}^{t} (D_i^2) \geqslant \Big(\sum_{i=1}^{t} [E_i : B]\Big)(2p_a(B) - 2) - (D.K)\\ &= (D.F)\,(2p_a(B) - 2) - (D.F)\,\Big(2p_a(B) - 2 + \chi(\mathcal{O}_X)\\ &\qquad + \sum_{j=1}^{r} (m_j - 1)/m_j\Big)\end{aligned}$$

because of the formula for the canonical divisor $K$ (see [4] p. 572) and we are done.

## 3. Proofs of Theorems 1 and 2.

Suppose $m_1 Y_1, \ldots, m_r Y_r$ are all the multiple fibres of $f$ each having multiplicity $m_j$, $1 \leqslant j \leqslant r$ and take $b_j \in B$ such that $m_j Y_j = f^*(b_j)$. By the formula for the canonical divisor $K$ we may write

$$\mathcal{O}_X(K) = f^*M \otimes \mathcal{O}_X\Big(\sum_{j=1}^{r} (m_j - 1)\, Y_j\Big)$$

where $M \in \mathrm{Pic}(B)$, $\deg(M) = 2p_a(B) - 2 + \chi(\mathcal{O}_X)$.

Furthermore for any points $x, x_1, x_2$ on $X$ denote by $p : \widetilde{X} \longrightarrow X$ and $q : \hat{X} \longrightarrow X$ the blowing ups of $X$ at $x$ and $\{x_1, x_2\}$ respectively and let $W, W_1, W_2$ be the corresponding exceptional curves. Put $y = f(x)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$.

*Proof of Theorem 1.* – To prove 1) $\Longrightarrow$ 2) it is sufficient by [2] to prove that $H^1(\widetilde{X}, p^*\mathcal{O}_X(D) \otimes p^*f^*L \otimes \mathcal{O}_{\widetilde{X}}(-W)) = 0$ for any $x \in X$ hence by Bombieri-Ramanujam vanishing theorem [2] to prove that the linear system

$$\Lambda = | p^* \mathcal{O}_X(D - K) \otimes p^* f^* L \otimes \mathcal{O}_{\tilde{X}}(-2W) |$$

contains an 1-connected divisor with selfintersection $> 0$. Now by Lemma 2 the selfintersection of $\Lambda$ is

$$(D^2) - 2(D.K) + 2(D.F)\deg(L) - 4 > 0$$

provided $\deg(L) \geqslant \alpha_1$ where $\alpha_1$ is a constant depending only on the fibration. Now by Riemann-Roch on B we get that

$$| L \otimes M^{-1} \otimes \mathcal{O}_B(-b_1 - \ldots - b_r - 2y) | \neq \phi$$

provided $\deg(L) - \deg(M) - r - 2 \geqslant p_a(B)$. Hence there exists a constant $\alpha_2$ depending only on $f$ such that for $\deg(L) \geqslant \alpha_2$ we may find a divisor $\underline{b} \in | L \otimes M^{-1} |$ with $b_1 + \ldots + b_r + 2y \leqslant \underline{b}$. It follows that

$$G = p^*(D + f^*\underline{b} - \sum_{j=1}^{r} (m_j - 1) Y_j) - 2W \in \Lambda .$$

Now for $\deg(L) - \deg(M) - \sum_{j=1}^{r} (m_j - 1)/m_j \geqslant 2$ the divisor $D + f^*\underline{b} - \sum_{j=1}^{r} (m_j - 1) Y_j$ must be 2-connected by Lemma 1. It follows by a standard computation that in this case G is 1-connected. Hence we may choose $C_1 = \max\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_3 = \deg(M) + \sum_{j=1}^{r} (m_j - 1)/m_j + 2$ and we are done.

2) $\Longrightarrow$ 3) is obvious.

To prove 3) $\Longrightarrow$ 1) we may suppose that L is trivial and that D has no common components with Y, where $mY$ is some fibre of multiplicity $m$. We only have to prove that $(D.Y) \geqslant 2$. Suppose $(D.Y) = 1$. By Riemann-Roch on the (possibly singular) curve Y we get

$$\begin{aligned} h^0(\mathcal{O}_Y(D)) &= h^0(\omega_Y(-D)) + \deg(\mathcal{O}_Y(D)) + \chi(\mathcal{O}_Y) \\ &= h^0(\mathcal{O}_Y(-D)) + 1 \end{aligned}$$

because the dualizing sheaf $\omega_Y$ is trivial. Now since $\mathcal{O}_Y(-D) \subset \mathcal{O}_Y$ we get $H^0(\mathcal{O}_Y(-D)) \subset H^0(\mathcal{O}_Y)$. Since by [5], $H^0(\mathcal{O}_Y)$ consists only of constants and since $\mathcal{O}_Y(-D)$ is not trivial we get $h^0(\mathcal{O}_Y(-D)) = 0$ hence $h^0(\mathcal{O}_Y(D)) = 1$. Since $\mathcal{O}_Y(D)$ is not trivial, it follows that $\mathcal{O}_Y(D)$ cannot be spanned by global sections, contradiction.

*Proof of Theorem 2.* – Note that 2) $\Longrightarrow$ 3) is obvious and that 3) $\Longrightarrow$ 1) follows easily considering as above a multiple fibre of the form $m$Y and noting that Y must have degree at least 3 with respect to any very ample divisor because $p_a(\mathrm{Y}) = 1$.

Let us prove 1) $\Longrightarrow$ 2). Start with an ample $\mathcal{L} \in \mathrm{Pic(X)}$ with $(\mathcal{L}.\mathrm{F}) \geqslant 3m$, put $\mathcal{M} = \mathcal{L} \otimes f^*\mathrm{L}$ for $\mathrm{L} \in \mathrm{Pic(B)}$ and let us prove first that $|\mathcal{M}|$ has no fixed components among the components of the reducible fibres of $f$ provided $\deg(\mathrm{L}) \geqslant \beta_1$ for some constant $\beta_1$. Let $\mathrm{Z}_1$ be a component of a reducible fibre F and look for a divisor in $|\mathcal{M}|$ not containing $\mathrm{Z}_1$ in its support. Note that by [5], $\mathrm{Z}_1$ is smooth rational with selfintersection $(\mathrm{Z}_1^2) = -2$. According to [5] there are two cases which may occur: either $(\mathrm{Z}_1 . \mathrm{Z}_2) \leqslant 1$ for any other component $\mathrm{Z}_2$ of F, or $\mathrm{F} = b(\mathrm{Z}_1 + \mathrm{Z}_2)$ for some natural $b$ where $\mathrm{Z}_2$ is smooth rational with $(\mathrm{Z}_2^2) = -2$ and $(\mathrm{Z}_1 . \mathrm{Z}_2) = 2$. In the first case put $\mathrm{Z} = \mathrm{Z}_1$ and choose a point $p \in \mathrm{Z}$. In the second case, since $b(\mathcal{L} . \mathrm{Z}_1) + b(\mathcal{L}.\mathrm{Z}_2) = (\mathcal{L}.\mathrm{F}) \geqslant 3m \geqslant 3b$ we must have $(\mathcal{L} . \mathrm{Z}_k) \geqslant 2$ for $k = 0$ or $k = 1$. Put in this case $\mathrm{Z} = \mathrm{Z}_1 + \mathrm{Z}_2 - \mathrm{Z}_k$ and take $p \in \mathrm{Z}_1 \cap \mathrm{Z}_2$. It will be sufficient to find a divisor in $|\mathcal{M}|$ not passing through $p$. We have the following exact sequence:

$$0 \longrightarrow \mathrm{H}^0(\mathcal{M}(-\mathrm{Z})) \longrightarrow \mathrm{H}^0(\mathcal{M}) \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathrm{P}^1}(c)) \longrightarrow \mathrm{H}^1(\mathcal{M}(-\mathrm{Z}))$$

where $c = (\mathcal{L}.\mathrm{Z}) \geqslant 1$. It is sufficient to prove that $\mathrm{H}^1(\mathcal{M}(-\mathrm{Z})) = 0$. We use Ramanujam's vanishing theorem [6]. By Serre duality it is sufficient to prove that

$$(\mathcal{M}(-\mathrm{Z}-\mathrm{K})^2) > 0 \quad \text{and} \quad (\mathcal{M}(-\mathrm{Z}-\mathrm{K}).\mathrm{R}) \geqslant 0$$

for any integral curve R. Now

$$\begin{aligned}(\mathcal{M}(-\mathrm{Z}-\mathrm{K})^2) &= (\mathcal{L}^2) + 2(\mathcal{L}.\mathrm{F})\deg(\mathrm{L}) - 2 - 2(\mathcal{L}.\mathrm{Z}) - 2(\mathcal{L}.\mathrm{K}) \\ &> 2(\mathcal{L}.\mathrm{F})(\deg(\mathrm{L}) - 1 - d) - 2\end{aligned}$$

where $d \in \mathbf{Q}$, $\mathrm{K} \equiv d\mathrm{F}$. Consequently the selfintersection is $> 0$ for $\deg(\mathrm{L}) \geqslant d + 2$.

To check the second inequality suppose first that R is contained in a fibre F. We get $(\mathcal{M}(-\mathrm{Z}-\mathrm{K}).\mathrm{R}) = (\mathcal{L}.\mathrm{R}) - (\mathrm{Z}.\mathrm{R}) \geqslant 0$ because the only case when $(\mathrm{Z}.\mathrm{R}) = 2$ is $\mathrm{F} = b(\mathrm{Z}_1 + \mathrm{Z}_2)$ and $\mathrm{R} = \mathrm{Z}_k$. Now if R dominates B we get

$$\begin{aligned}(\mathcal{M}(-\mathrm{Z}-\mathrm{K}).\mathrm{R}) &= (\mathcal{L}.\mathrm{R}) + (\mathrm{F}.\mathrm{R})\deg(\mathrm{L}) - (\mathrm{Z}.\mathrm{R}) - (\mathrm{K}.\mathrm{R}) \\ &> (\mathrm{F}.\mathrm{R})\deg(\mathrm{L}) - (\mathrm{F}.\mathrm{R}) - d(\mathrm{F}.\mathrm{R}) \geqslant 0\end{aligned}$$

for $\deg(L) \geqslant d+1$, and we are done. Now if $\beta_1$ is chosen also such that $\beta_1 \geqslant 2p_a(B)$ it follows that $\mathcal{M}$ is still ample hence by Theorem 1 the linear system $|\mathcal{L} \otimes f^*L|$ is ample and base point free provided $\deg(L) \geqslant \beta_2 = \beta_1 + C_1$. By Bertini's theorem the above system contains an integral member D. To prove 1) $\Longrightarrow$ 2) it is sufficient by [2] to prove that

$$H^1(\widetilde{X}, p^*\mathcal{O}_X(D) \otimes p^*f^*L \otimes \mathcal{O}_{\widetilde{X}}(-2W)) = 0$$
$$H^1(\hat{X}, q^*\mathcal{O}_X(D) \otimes q^*f^*L \otimes \mathcal{O}_{\hat{X}}(-W_1 - W_2)) = 0$$

for any $x, x_1, x_2 \in X$, provided $\deg(L) \geqslant \beta_3$ for some constant $\beta_3$; in this case the constant $C_2 = \beta_2 + \beta_3$ will be convenient for our purpose.

Now exactly as in the proof of the Theorem 1 we may find a constant $\beta_3$ such that for $\deg(L) \geqslant \beta_3$ the linear systems

$$|p^*\mathcal{O}_X(D-K) \otimes p^*f^*L \otimes \mathcal{O}_{\widetilde{X}}(-3W)|$$

and

$$|q^*\mathcal{O}_X(D-K) \otimes q^*f^*L \otimes \mathcal{O}_{\hat{X}}(-2W_1 - 2W_2)|$$

have strictly positive selfintersections and contain divisors of the form

$$G_1 = p^*\Big(D + \sum_i a_i F_i\Big) - 3W$$

and

$$G_2 = q^*\Big(D + \sum_i b_i F_i\Big) - 2W_1 - 2W_2$$

with $a_i, b_i \in \mathbf{Q}$, $a_i \geqslant 0$, $b_i \geqslant 0$, $\sum_i a_i \geqslant 2$, $\sum_i b_i \geqslant 2$ and where $F_i$ are fibres. Then by Lemma 1 the divisors $D + \sum_i a_i F_i$ and $D + \sum_i b_i F_i$ are 3-connected hence by a standard computation, $G_1$ and $G_2$ are 1-connected and the Theorem is proved.

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