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EQUIVALENCE OF DIFFERENTIABLE FUNCTIONS, RATIONAL FUNCTIONS AND POLYNOMIALS

by **Masahiro SHIOTA**

1. Introduction.

We consider in this paper when a differentiable function on $\mathbf{R}^n$ can be transformed to a (« equivalent ») polynomial or a rational function by a diffeomorphism. Assume $n = 1$. Then a non-constant C^∞ function is equivalent to a polynomial if and only if it is proper, the number of critical points is finite and the derivative is nowhere flat (R. Thom [9]). We want to generalize the dimension. We see in [3], [4] a generalization in another direction to C-polynomials.

In Section 2 we treat functions on $\mathbf{R}^n$ with isolated critical points.

THEOREM 1. — *A C^∞ function on $\mathbf{R}^n (n \neq 4,5)$ is equivalent to a polynomial if it is proper, the number of critical points is finite and the Milnor number of the germ at each critical point is finite.*

THEOREM 2. — *In the above theorem, if we replace the condition on the Milnor number by one that the germ at each critical point is locally equivalent to a germ of a rational function, then the function is equivalent to a rational function.*

In the case $n = 3$, we can change the properness condition in these results to $(*)$ that the absolute value of the differential is larger than a positive constant outside a bounded set (theorem 3,4 in § 3). This is impossible for general n. We have a counter-example.

Section 4 deals with the case of dimension 2.

THEOREM 5. — *An analytic function on $\mathbf{R}^2$ is equivalent to a polynomial, if it is proper and the number of critical values is finite, or if the above condition $(*)$ is satisfied.*

We will consider also C^∞ functions on an affine smooth algebraic varieties, and we obtain similar results to Theorem 1, 3, 5. For example, any analytic function on an algebraic variety homeomorphic to S^2 or $P^2(\mathbf{R})$ is equivalent to the restriction of a polynomial (theorem 5‴).

In § 5, these results are modified to the problem of equivalence to Nash functions.

The restriction of a polynomial or of a rational function on an algebraic subset is called briefly *a polynomial* or *a rational function*. Any affine smooth algebraic variety is diffeomorphic to the interior of a compact C^∞ manifold with boundary. We call the boundary as *the boundary of the algebraic variety*. We remark that the boundary is not unique. For f a C^∞ function on a manifold and x a point, f_x denotes the germ of f at x.

We remark that the diffeomorphisms of equivalence in the theorems are chosen analytic, if the given functions are all analytic (see [4]).

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2. C^∞ functions with isolated critical points.

DEFINITION 1. — *C^∞ functions f_1, f_2 on a C^∞ manifold M are equivalent if there exists a C^∞ diffeomorphism τ of M such that $f_1 \circ \tau = f_2$. C^∞ function germs φ_1, φ_2 at a point a in M are equivalent if there exists a C^∞ local diffeomorphism π of M at a such that $\varphi_1 \circ \pi = \varphi_2$.*

DEFINITION 2. — *The Milnor number of a germ φ of a C^∞ function at 0 in $\mathbf{R}^n$ is the dimension of the real vector space $\mathcal{E}_n/(\partial\varphi/\partial x_1, \ldots, \partial\varphi/\partial x_n)$. Here $\mathcal{E}_n$ is the ring of C^∞ function germs at 0 in $\mathbf{R}^n$.*

The proofs of the results of this paper are based on the following lemmas (see [3], [4], [11]). The first one is essentially due to J. N. Mather. Let M be a C^∞ manifold, and let $X_1, \ldots, X_k$ be C^∞ vector fields on M.

LEMMA 3. — *Let f, g be C^∞ functions on M and let $a_i(x,t)$ be C^∞ functions on $M \times [0,1]$, $i = 1, \ldots, k$. Assume that*

$$f(x) - g(x) = \sum_{i=1}^{k} a_i(x,t)(tX_if + (1-t)X_ig) \qquad \text{on} \qquad M \times [0,1]$$

and that a_i are near to the zero function in the Whitney topology. Then f and g are equivalent, and the diffeomorphism can be chosen near to the identity.

Proof. – Put

$$F(x,t) = tf(x) + (1-t)g(x), \qquad (x,t) \in M \times [0,1].$$

We regard $Y = \frac{\partial}{\partial t} - \sum_{i=1}^{k} a_i X_i$ as a vector field on $M \times [0,1]$. Then we have $YF \equiv 0$. Consider the integral curve of Y passing each point $(x,0) \in M \times 0$. Since $\sum_{i=1}^{k} a_i X_i$ is near to the zero vector field, the curve passes the unique point $(y,1)$ in $M \times 1$. Hence it follows that

$$g(x) = F(x,0) = F(y,1) = f(y).$$

Let the correspondance $x \to y$ be denoted by π. Then π is a diffeomorphism of M near to the identity and satisfies $g = f \circ \pi$. Here we remark that if $a_i = 0$ on $x \times [0,1]$ then $\pi(x) = x$ and that if we assume only that a_i are near to the zero function in the C^0 Whitney topology, then π is near to the identify in the C^0 Whitney topology.

LEMMA 4. – *Let f be a C^∞ function, and let $g_1, \ldots, g_{k'}$, be linear combinations of $X_i f$ for $i = 1, \ldots, k$ with C^∞ functions as coefficients such that also $X_i g_j$ for all i, j are linear combinations. Then f is equivalent to $f + \sum_{j=1}^{k'} a_j g_j$ for any small C^∞ functions a_j in the Whitney topplogy. Particularly f is equivalent to $f + \sum_{ij=1}^{k} b_{ij} X_i f X_j f$ for small C^∞ functions b_{ij}. Here the diffeomorphism can be chosen near to the identify.*

Proof. – By the above lemma, we only need to find small C^∞ functions $c_1(x,t), \ldots, c_k(x,t)$ such that

$$\text{(1)} \qquad \sum_{j=1}^{k'} a_j g_j = \sum_{i=1}^{k} c_i \left(X_i f + (1-t) X_i \left(\sum_{j=1}^{k'} a_j g_j \right) \right)$$

on $M \times [0,1]$.

By the assumption there exist small C^∞ functions $d_i(x)$, $d'_{ij}(x)$,

$i, j = 1, \ldots, k$ such that

$$\sum_{j=1}^{k'} a_j g_j = \sum_{j=1}^{k} d_i X_i f, \quad X_i\left(\sum_{j=1}^{k'} a_j g_j\right) = \sum_{j=1}^{k} d'_{ij} X_j f.$$

Hence (1) is equivalent to

$$\sum_{i=1}^{k} d_i X_i f = \sum_{i=1}^{k} c_i \left(X_i f + (1-t) \sum_{j=1}^{k} d'_{ij} X_j f\right). \tag{2}$$

Denote by C, D, D′ the $1 \times k$ matrices $(c_1, \ldots, c_k)$, $(d_1, \ldots, d_k)$, the $k \times k$ matrix (d'_{ij}) respectively. Then (2) is written as follows:

$$D \begin{pmatrix} X_1 f \\ \vdots \\ X_k f \end{pmatrix} = (C + (1-t)CD') \begin{pmatrix} X_1 f \\ \vdots \\ X_k f \end{pmatrix}.$$

Hence it is sufficient to choose a matrix C so that

$$D = C(I + (1-t)D')$$

where I is the $k \times k$ unit matrix. Since $I + (1-t)D'$ is invertible, $C = D(I + (1-t)D')^{-1}$ exists and satisfies this equality. Clearly all elements of C are small C^∞ functions. Hence Lemma 4 is proved.

We use this in the following form.

Lemma 5. — *With the same f and g_j $j = 1, \ldots, k'$, let $U \subset M$ be a compact set. Then, for C^∞ functions a_j small in a closed neighborhood of* U *there exists a C^∞ diffeomorphism τ which is close to the identity in Whitney topology such that*

$$f \circ \tau = f + \sum_{j=1}^{k'} a_j g_j \quad \textit{on } U.$$

We can treat the local case in the same way. For example, the next lemma follows from the remark at the end of Proof of Lemma 3.

Lemma 4′. — *Let f be a germ of a C^∞ function at 0 in $\mathbf{R}^n$ critical at 0. Then f is equivalent to $f + \sum_{ij=1}^{n} b_{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$ for any germs of C^∞ functions b_{ij}, with small $|b_{ij}(0)|$. Here the Jacobian matrix of the local diffeomorphism at 0 can be chosen near to the unit.*

The following remark and lemma show the behaviours of proper polynomials and of some C^∞ functions at infinity.

Remark 6. – Let f_1, f_2 be positive proper polynomials on $\mathbf{R}^n$. Then there exists τ a C^∞ diffeomorphism of $\mathbf{R}^n$ such that τ is the identity on a given bounded subset and that $f_1 \circ \tau$ and f_2 are equal outside a bounded subset.

Proof. – We can assume $f_2(x) = |x|^2$. Put

$$B = \{x \in \mathbf{R}^n | \langle x, \operatorname{grad} f_1(x) \rangle = -|x| |\operatorname{grad} f_1(x)|\}.$$

Here $\langle \ , \ \rangle$ means the inner product of vectors. Then B is semi-algebraic. Obviously B is the set of points x where $\operatorname{grad} f_1$ is zero or $\operatorname{grad} f_2$ is a product of $-\operatorname{grad} f_1$ and a non-negative number. Moreover B is bounded. We will prove this fact in a more general form in Proof of Proposition 8, hence here we assume this.

Let K be a sufficiently large number, let g be a C^∞ function on $\mathbf{R}^n$ such that

$$0 \leqslant g \leqslant 1, \qquad g(x) = \begin{cases} 0 & \text{for} \quad |x| \leqslant K \\ 1 & \text{for} \quad |x| \geqslant 2K. \end{cases}$$

Put

$$D = \{|x| = K\}, \qquad D' = \{|x| \geqslant K\}, \qquad D'' = \{|x| \geqslant 2K\},$$

$$v = g \operatorname{grad} f_1 / |\operatorname{grad} f_1| + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} / |x| \qquad \text{on } D'.$$

Then v is a non-singular vector field on D'. Moreover vf_1, vf_2 are positive on D'', D' respectively. The integral curves of v define a diffeomorphism $\pi : D \times [0,\infty) \to D'$ such that

$$f_2 \circ \pi(z,t) = t + K \qquad \text{for} \qquad (z,t) \in D \times [0,\infty)$$

and

$$\frac{\partial f_1 \circ \pi}{\partial t}(z,t) > 0 \qquad \text{for large } t.$$

As $f_1 \circ \pi$ is proper, there exists also a diffeomorphism π' of $D \times [0,\infty)$ such that

$$\pi'(z,t) = (z, s(z,t)) \qquad \text{for any } (z,t),$$
$$f_1 \circ \pi \circ \pi'(z,t) = t + K \qquad \text{for large } t$$

and that π' is the identity near $D \times D$, where s is a C^∞ function. Let τ be the extension of $\pi \circ \pi'^{-1} \circ \pi^{-1}$ onto $\mathbf{R}^n$ which is the identity on $\mathbf{R}^n - D'$. Then the equation $f_2 \circ \tau = f_1$ holds true outside a bounded set. Hence Remark 6 follows.

LEMMA 7. — *Let f_1, f_2 be positive proper C^∞ functions on $\mathbf{R}^n$. Assume $n \neq 4,5$ and that the sets of critical points are bounded. Then the same result as in Remark 6 holds true.*

Proof. — Let a be a larger number than any critical values of f_1, f_2. Put

$$S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}, \qquad B = \{|x| \leqslant 1\},$$
$$W_i = \{x \in \overline{\mathbf{R}^n - B} \mid f_i(x) \leqslant a\}, \qquad i = 1, 2.$$

Assume that the given bounded subsets are contained in B and that $|f_i| < a$ on B. By the assumption, $f_i^{-1}(a) \times [0, \infty)$ is diffeomorphic to $\{f_i \geqslant a\}$ for $i = 1, 2$. Hence we only have to prove that W_i are diffeomorphic to $S^{n-1} \times [0,1]$. It follows from the same reason as above that $(W_i, S^{n-1}, f_i^{-1}(a))$ are h-cobordisms (i.e. $\partial W_i = S^{n-1} \cup f_i^{-1}(a)$, and $S^{n-1} \subset W_i$ and $f_i^{-1}(a) \subset W_i$ are homotopy equivalences). Hence, from [2], the assertion for $n \geqslant 6$ follows. This holds true trivially for $n = 1, 2$, and for $n = 3$ because W_i can be imbedded in $S^{n-1} \times [0,\infty)$ ([13] or [8]).

Proof of Theorem 1. — Let f be the function stated in the theorem. We assume it to be positive valued. Let $S = \{s_1, \ldots, s_k\}$ be the set of critical points. It is well-known [11] that by the assumption on the Milnor number, there exists an integer ℓ such that f_{s_i} is equivalent to f_{s_i} + any germ of a C^∞ function ℓ-flat at s_i for each i. Here the local diffeomorphism is chosen orientation preserving. Clearly there exists a polynomial g on $\mathbf{R}^n$ such that $g - f$ is ℓ-flat at each s_i. The local diffeomorphisms of the equivalences of g_{s_i} and f_{s_i} are extensible on the global $\mathbf{R}^n$. Transform f by the extended diffeomorphism. Then we can assume $f = g$ in a neighborhood of S. We put

$$h(x) = \prod_{i=1}^{k} |x - s_i|^{2\ell'}, \qquad g_1 = g + h$$

where ℓ' is an integer such that $2\ell' > \ell$ and $2k\ell' >$ the degree of g.

Then $g_{1s_i} - f_{s_i}$ is ℓ-flat for each i, and we have

$$g_1 = |x|^{2k\ell'} + \text{a polynomial of degree} < 2k\ell' .$$

Hence g_1 is proper. Apply Lemma 7 to f and g_1. Then, from the beginning we can assume $f = g$ in a neighborhood of S and on $\{|x| \geqslant K\}$ for a number K. The theorem follows from Proposition 8 below.

THEOREM 1′. — *Let* $M \subset \mathbf{R}^n$ *be an affine smooth algebraic variety of dimension* $n' \neq 4, 5$, *and* f *be a positive* C^∞ *function on* M *with the same conditions as in Theorem* 1. *Assume that the boundary of* M *is simply connected if* $n' \geqslant 6$ *and that any connected component of the boundary is not diffeomorphic to* $P^2(\mathbf{R})$ *if* $n = 3$. *Then* f *is equivalent to a polynomial.*

Sketch of the proof. — For the proof, we use polynomial vector fields on M (considering in $\mathbf{R}^n$) in place of $\partial/\partial x_i$, $i = 1, \ldots, n$ which span the tangent space of M at each point. We see the existence of such vector fields as follows. Let v_1, v_2 be polynomial vector fields on $\mathbf{R}^n$. Then $\langle v_1, v_1 \rangle v_2 - \langle v_1, v_2 \rangle v_1$ is a polynomial vector field orthogonal to v_1 and is non-singular at any point where v_1, v_2 are independent. We call this operation the orthogonalization. We orthogonalize any polynomial vector fields $v_1, v_2, \ldots$ in the same way. Let $g_1, \ldots, g_k$ be generators of the ideal of $\mathbf{R}[x_1 \cdot \ldots, x_n]$ defined by M. Since $\operatorname{grad} g_1, \ldots, \operatorname{grad} g_k$ on M span the normal vector bundle of M in $\mathbf{R}^n$, for any point x of M there exist a portion $h_1, \ldots, h_{n-n'}$, of $g_1, \ldots, g_k$ such that $\operatorname{grad} h_1, \ldots, \operatorname{grad} h_{n-n'}$, span the normal bundle of M at x in $\mathbf{R}^n$. Let $v_1, \ldots, v_{n-n'}, u_i$ be the orthogonalization of

$$\operatorname{grad} h_1, \ldots, \operatorname{grad} h_{n-n'}, \qquad \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n .$$

Then $u_1, \ldots, u_n$ span the tangent space of M at x. As there are only a finite number of selections of $h_1, \ldots, h_{n-n'}$ in $g_1, \ldots, g_k$, we obtain a finite number of polynomial vector fields which span the tangent space of M at each point.

We need also the fact that the set B of points x of M such that the angle of the vector x and the tangent space of M at x is larger than given $\varepsilon > 0$ is bounded.

We prove this as follows. Let $v_1, \ldots, v_k$ be polynomial vector fields on M which span the tangent space of M at each point. Put

$$\mathrm{B}_{\varepsilon'} = \{(x,a_1,\ldots,a_k) \in \mathrm{M} \times \mathbf{R}^k | \sum_{i=1}^{k} a_i v_{ix} \neq 0 .$$
$$\langle x, \sum_{i=1}^{k} a_i v_{ix} \rangle \geqslant \varepsilon' |x| | \sum_{i=1}^{k} a_i v_{ix} | \}$$

for $\varepsilon' > 0$. Then $\mathrm{B}_{\varepsilon'}$ is semi-algebraic, and B is the complement of its image under the projection from $\mathrm{M} \times \mathbf{R}^k$ onto M for some ε'. Hence B is semi-algebraic. We will prove by reduction to absurdity that B is bounded. Assume it unbounded. As $\mathbf{R}^n$ is algebraically diffeomorphic to S^n-{a point a} where $\mathrm{S}^n = \{x \in \mathbf{R}^{n+1} | |x| = 1\}$, we identify $\mathbf{R}^n$ with its image. The germ of B at a is not empty. Hence, considering the germ, we obtain easily an unbounded one-dimensional semi-algebraic set $\mathrm{B}' \subset \mathrm{B}$ whose germ at a is connected. Since the subset of B′ of points where B′ is not C^ω smooth is a semi-algebraic set of dimension 0, we can assume that B′ is C^ω smooth. Let v be a vector field on B′ such that $|v| = 1$. Consider a mapping

$$x \in \mathrm{B}' \to (x, x/|x|, v_x) \in \mathrm{S}^n \times \mathbf{R}^n \times \mathbf{R}^n .$$

We see easily that the image is semi-algebraic and of dimension 1. Let B″ be its closure. Then $\mathrm{B}'' \cap a \times \mathbf{R}^n \times \mathbf{R}^n$ is of dimension 0, and hence this consists of one point (a,b,c). This means that $(x/|x|, v_x)$ tends to (b,c) as $x \in \mathrm{B}'$ tends to infinity. Assume $b = (1,\ldots,0)$. Then the germ of B′ at a is contained in $\{x_2^2 + \ldots + x_n^2 \leqslant \delta x_1^2\}$ for any $\delta > 0$. On the other hand, if $b \neq \pm c$, the germ would be outside $\{x_2^2 + \ldots + x_n^2 \leqslant \delta x_1^2\}$ for some $\delta > 0$. Hence $b = \pm c$. This implies that the angle of the vector x and the tangent space of B′ at x tends to 0 as x tends to infinity. This contradicts the definition of B. Therefore B is bounded.

We have to modify Lemma 7 as follows.

LEMMA 7′. — *Let* M′ *be a simply connected compact* C^∞ *manifold of dimension* $\geqslant 5$ *or a two dimensional connected compact* C^∞ *manifold not diffeomorphic to* $\mathrm{P}^2(\mathbf{R})$. *Let* f_1, f_2 *be positive proper* C^∞ *functions on* $\mathrm{M}' \times [0,\infty)$. *Assume that* f_1, f_2 *have no critical points. Then there exists* π *a* C^∞ *diffeomorphism of* $\mathrm{M}' \times [0,\infty)$ *such that* π *is the identity on a given bounded subset and that* $f_1 \circ \pi$ *and* f_2 *are equal outside a bounded subset.*

We can apply this result to our problem, because M-(a bounded subset) is diffeomorphic to $M' \times (0,\infty)$ (M' the boundary of M). The proofs of Lemma 7′ and Theorem 1′ proceed in the same manner as the correspondings. Hence we omit them.

Example. – The positiveness of f in Theorem 1′ is necessary. For example, put

$$M = \{(x,y,z) \in \mathbf{R}^3 | z(x^2+y^2-1)((x-1)^2+y^2-1)=1\}.$$

Then M is the graph of a rational function defined on

$$\{x^2+y^2 \neq 1\} \cup \{(x-1)^2+y^2 \neq 1\}.$$

Hence M has 4 connected components. Let M_2 be a connected component of M. Let f be a proper C^∞ function on M positive on M_2, negative on $M - M_2$. Then f is not equivalent to any rational function. The reason is the following. Assume that f is equivalent to a rational function. We can regard the equivalent rational function as defined on the (x,y)-plane. We write the function as $g(x,y) = g_1(x,y)/g_2(x,y)$ where g_1, g_2 are polynomials and have no common factor. Then, 1) the set of zero points of g_1 is a finite set contained in $S = \{x^2+y^2=1 \text{ or } (x-1)^2+y^2=1\}$, 2) g_2 is divisible by $(x^2+y^2-1)^a((x-1)^2+y^2-1)^b$ for some integers $a, b > 0$ and 3) $g_2/(x^2+y^2-1)^a((x-1)^2+y^2-1)^b$ satisfies the same condition as 1). Let U be the connected component of $\mathbf{R}^2 - S$ corresponding to M_2. By the definition of f and by 1), g_2 takes the same sign on $\mathbf{R}^2 - M_2 - S$. Hence a and b in 2) are even. Therefore g is negative on M_2. This is a contradiction. Hence f is not equivalent to any rational function.

PROPOSITION 8. – *Let f be a C^∞ positive proper function on $\mathbf{R}^n$ with the bounded set* S *of critical points. Let f_1, f_2 be smooth rational functions on $\mathbf{R}^n$ such that $f = f_1$ in a neighborhood of* S *and on $\{|X| \geqslant K\}$ for a number* K, *$f_2^{-1}(0) = S$ and that f_2 satisfies the conditions on g_i in Lemma* 4. *Then f is equivalent to $f_1 + f_2 f_3$ for some polynomial f_3.*

Proof. – We can assume that f_2 is non-negative and moreover proper. The reason is the following. We only have to see $f_2(x) \geqslant |x|^{-2N}$, $|x| \geqslant L$ for some integers N, L, because $(1+|x|^{2N+2})f_2(x)$ is proper and has the same properties as f_2. We identify $\mathbf{R}^n$ as S^n-{a point a} where $S^n = \{x \in \mathbf{R}^{n+1} | |x|=1\}$. Consider the graphs of f_2 and $g = |x|^{-2}$ in a small neighborhood of a. Let P_1, P_2 be the respective closures of them. Then P_1, P_2 are closed semi-algebraic sets satisfying in a neighborhood of $a \times 0$

$$P_1 \cap S^n \times 0 = a \times 0 \quad \text{or} \quad \varnothing, \qquad P_2 \cap a \times \mathbf{R} = a \times 0 .$$

It is well-known [1] that any two closed semi-algebraic sets are regularly situated. Consider P_1 and $S^n \times 0$. Then it follows from the regular situation [1] that

$$|t| \geqslant \operatorname{dist}(x,a)^{N'} \qquad \text{for} \qquad (x,t) \in P_1$$

in a neighborhood of $a \times 0$ where N' is a constant, and here dist means the distance in the metric on S^n. This argument shows also

$$|t|^{N''} \geqslant \operatorname{dist}(x,a) \qquad \text{for} \qquad (x,t) \in P_2$$

for some N''. These inequalities mean that

$$f_2(x) \geqslant g^{N'N''}(x) = |x|^{-2N'N''}$$

in a neighborhood of a.

Put

$$\begin{aligned} F(x,t) &= tf_1(x) + (1-t)f_2(x), \\ B_\varepsilon &= \{(x,t) \in \mathbf{R}^{n+1} | 0 \leqslant t \leqslant 1, < x, \operatorname{grad}_x F(x,t) > \\ &\qquad \leqslant -\varepsilon |x| \, |\operatorname{grad}_x F(x,t)| \} \end{aligned}$$

for $\varepsilon > 0$, where $\operatorname{grad}_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$. Then B_ε is a semi-algebraic set. We want the property that B_ε is bounded. Assume it unbounded. We proved in Sketch of the proof of Theorem 1′ that B is bounded. In the same way we see that there exists an unbounded one-dimensional C^ω smooth semi-algebraic set $B'_\varepsilon \subset B_\varepsilon$ whose germ at a is connected. Here $\mathbf{R}^n = S^n - \{a\}$. Let v be a vector field on B'_ε such that $|v| = 1$. Then we also proved that $(y/|y|, v_y)$ tends to $(b, \pm b)$ for some $b \in \mathbf{R}^n$ as $y \in B'_\varepsilon$ tends to infinity. We can assume that the limit is (b,b). Hence, by the definition of B_ε, we have

$$\langle v_y, \operatorname{grad}_x F(y) \rangle < 0 \qquad \text{for} \qquad y \in B'_\varepsilon$$

if $|y|$ is large enough.

First, consider the case $f_1 = f_2$. Then $\mathrm{grad}_x \mathrm{F} = \mathrm{grad}\, \mathrm{F}$. Hence the above inequality means that the restriction of $\mathrm{F} = f_1$ on B'_ε is monotone decreasing as y tends to infinity. This contradicts the positivity and the properness of f_1. Thus B_ε is shown bounded in the case $f_1 = f_2$. We also proved that B in Proof of Remark 6 is bounded.

For general f_1, f_2, we have to modify them. Let $0 < \delta \leqslant \pi/6$. We saw already that the angle of the vector x and $\mathrm{grad}\, f_2(x)$ is smaller than $\pi/2 + \delta \leqslant 2\pi/3$ for any large $|x|$. At any point x where the angle is smaller than $2\pi/3$, we have

$$\begin{aligned}|\mathrm{grad}\,(1+|x|^{2p})f_2| &= |(1+|x|^{2p})\,\mathrm{grad}\, f_2 + 2p|x|^{2p-2} f_2 x| \\ &\geqslant \max\{3^{1/2}|(1+|x|^{2p})\,\mathrm{grad}\, f_2|/2,\ 3^{1/2}p|x|^{2p-1}|f_2|\}.\end{aligned}$$

Hence, replacing f_2 by $(1+|x|^{2p})f_2$ with large p if necessary, we can assume that $|\mathrm{grad}\, f_2(x)|$ tends to infinity in an arbitrarily large polynomial order of $|x|$ as $x \to \infty$. As $\dfrac{\partial \mathrm{F}}{\partial t} = f_1 - f_2$, if we change f_1 by $f_1 + f_2$, the degree of $\dfrac{\partial \mathrm{F}}{\partial t}$ is independent of p. Hence, then the angle of $\mathrm{grad}\, \mathrm{F}$ and $\mathrm{grad}_x \mathrm{F}$ can be assumed to tends to 0 as $x \to \infty$. Then the unboundedness of B_ε implies it of $\mathrm{B}_{t\varepsilon'}$ which is defined similarly by $\mathrm{grad}\, \mathrm{F}$ in place of $\mathrm{grad}_x \mathrm{F}$ for any $\varepsilon' > \varepsilon$. The boundedness of B_ε for $f_1 = f_2$ implies that $\mathrm{B}_{t\varepsilon'}$ is bounded. Hence B_ε is shown bounded. Assume

$$\mathrm{B}_{1/2} \subset \{|x| < \mathrm{K}\}. \tag{1}$$

Then for any point x in $\mathbf{R}^n$ with $|x| \geqslant \mathrm{K}$, the angle of the vector x and $c_1\, \mathrm{grad}\, f_1(x) + c_2\, \mathrm{grad}\, f_2(x)$ is smaller than $2\pi/3$ if $c_1, c_2 \geqslant 0$ and $c_1 + c_2 > 0$. Particularly the vector fields

$$cx + c_1\, \mathrm{grad}\, f_1(x) + c_2\, \mathrm{grad}\, f_2(x)$$

for $c, c_1, c_2 \geqslant 0$ with $c + c_1 + c_2 > 0$ are non-singular on $\{|x| \geqslant \mathrm{K}\}$ (2). It follows also that $\mathrm{S} \subset \{|x| < \mathrm{K}\}$.

We put $\mathrm{U}_i = \{|x| \leqslant i\mathrm{K}\}$ for $i = 1, 2, 3$. From the assumption, $(f - f_1)/f_2$ is of class C^∞. Let f'_3 be a polynomial on $\mathbf{R}^n$ which is close to $(f - f_1)/f_2$ on U_3 in the C^∞ topology. Put

$$f''_3 = (|x|/2\mathrm{K})^{2p}, \qquad f_3 = f'_3 + f''_3$$

for a large integer p. Compare f and $f_1 + f_2 f_3$. Then the difference is a product of f_2 and a C^∞ function small on U_1. Since f_2 satisfies the conditions on g_i in Lemma 4, we can apply Lemma 5 to $f, f_1 + f_2 f_3$ and $U = U_1$. Hence there exists a C^∞ diffeomorphism τ of $\mathbf{R}^n$ which is close to the identity in the Whitney topology such that

$$f \circ \tau = f_1 + f_2 f_3 \qquad \text{on } U_1 .$$

Consider the vector fields $Y_1 = \operatorname{grad} f \circ \tau$ and $Y_2 = \operatorname{grad}(f_1 + f_2 f_3)$. They are equal on U_1 and Y_1 is non-singular on U_1^c. We want to see that Y_2 is non-singular on U_1^c, that the angle of Y_1 and Y_2 is smaller than π at each point of U_1^c and that $f_1 + f_2 f_3$ is proper. If we can do this, the proposition is proved as follows. Put

$$Y = |Y_1| Y_2 + |Y_2| Y_1 .$$

Then we have

$$Y(f \circ \tau),\ Y(f_1 + f_2 f_3) > 0 \qquad \text{on } U_1^c .$$

Hence $f \circ \tau$ and $f_1 + f_2 f_3$ are monotone on any integral curve of Y in U_1^c. Let x be a point of $\mathbf{R}^n$, $C : (-\infty, \infty) \to \mathbf{R}^n$ be the integral curve of Y passing x such that $C(0) = x$. We want to find a point $\mu(x)$ in the curve such that

$$f \circ \tau \circ \mu(x) = (f_1 + f_2 f_3)(x) .$$

If x is in U_1, we put $\mu(x) = x$. If $x \in U_1^c$, let η_1, η_2 be numbers if exist such that $\eta_1 < 0 < \eta_2$,

$$C(\eta_1), C(\eta_2) \in U_1, \quad \text{and} \quad C((\eta_1, \eta_2)) \subset U_1^c .$$

Then

$$f \circ \tau \circ C(\eta_i) = (f_1 + f_2 f_3) \circ C(\eta_i) \qquad \text{for} \qquad i = 1, 2,$$

and $f \circ \tau \circ C$, $(f_1 + f_2 f_3) \circ C$ are monotone on (η_1, η_2). Hence there exists uniquely η_3 in (η_1, η_2) such that

$$f \circ \tau \circ C(\eta_3) = (f_1 + f_2 f_3) \circ C(0) = (f_1 + f_2 f_3)(x) .$$

We put $\mu(x) = C(\eta_3)$. If there is not such η_2, $f \circ \tau \circ C(\eta)$ and $(f_1 + f_2 f_3) \circ C(\eta)$ tend to infinity as η tends to infinity because of the properness of $f \circ \tau$, $f_1 + f_2 f_3$. Hence we can define $\mu(x)$. The differentiability of μ is clear.

We define a C^∞ map $\mu' : \mathbf{R}^n \to \mathbf{R}^n$ in the same way such that

$$f \circ \tau(x) = (f_1 + f_2 f_3) \circ \mu'(x).$$

By the definition of μ, μ', the composition $\mu \circ \mu'$ is the identity. Hence μ is a diffeomorphism. Thus $f \circ \tau$ and $f_1 + f_2 f_3$ are equivalent.

It is trivial that $f_1 + f_2 f_3$ is proper if we take p so large that $2p > \deg f'_3$. We want to see the non-singularity of Y_2 on U_1^c. Consider Y_2 and the functions on $U_3 - U_1$. As $(f - f_1)/f_2$ vanishes there, f'_3 is chosen small. Hence we need only

$$|\mathrm{grad}\,(f_1 + f_2 f''_3)| \geqslant \delta > 0 \quad \text{on} \quad U_3 - U_1$$

for any integer p and with a constant δ. From the property (2) and the equality

$$\mathrm{grad}\,(f_1 + f_2 f''_3) = \mathrm{grad}\, f_1 + (|x|/2K)^{2p}\, \mathrm{grad}\, f_2 + (p/2K^2)(|x|/2K)^{2p-2} f_2 x,$$

it is sufficient to see

$$f_{cc'}(x) = \mathrm{grad}\, f_1 + c\, \mathrm{grad}\, f_2 + c'x \geqslant \delta > 0 \quad \text{on} \quad U_3 - U_1$$

for $c, c' \geqslant 0$. By (2), we have

$$f_{cc'}(x) > 0 \quad \text{on} \quad U_3 - U_1$$

for $c, c' \geqslant 0$. If there were points $(c_i, c'_i, x_i) \in \mathbf{R}^+ \times \mathbf{R}^+ \times (U_3 - U_1)$, $i = 1, 2, \ldots$ such that $f_{c_i c'_i}(x_i) \to 0$ as $i \to \infty$, we had a contradiction as follows. Choosing a subsequence of the points, we can assume that $(1/(c_i + c'_i), c_i/(c_i + c'_i), x_i)$ tends to $(0, c'', x_0)$ as i tends to infinity. Then $f_{c_i c'_i}(x_i) \to 0$ means that

$$c''\, \mathrm{grad}\, f_2 + (1 - c'')x = 0 \quad \text{at } x_0.$$

Since $1 - c'' \geqslant 0$, this contradicts (2).

We consider Y_2 on U_3^c. Take p so large that

$$f'_3 + f''_3 > 0, \ \ 2\,|\mathrm{grad}\, f'_3| < |\mathrm{grad}\, f''_3| \quad \text{on} \quad U_3^c.$$

These inequalities mean $f_3 > 0$ and that the angle of x and $\operatorname{grad} f_3(x)$ is smaller than $\pi/6$ on U_3^c. Hence, by (1), Y_2 is non-singular on U_3^c. These arguments prove also that the angle of Y_1 and Y_2 is smaller than π on U_1^c. Thus the proposition is proved.

Using the same method as the above proof, we prove easily

COROLLARY 8′. — *A non-constant* C^∞ *function* f *on* $\mathbf{R}$ *is equivalent to a rational function if and only if the critical point set is finite, the derivative is nowhere flat, and if* $f(x)$ *tends to* a *as* x *tends to infinity where* a *is a real number or* $\pm\infty$.

Proof of Theorem 2. — Let f be the function stated in the theorem. We assume f to be positive valued. Let $S = \{s_1, \dots, s_k\}$ be the set of critical points. By the assumption, there are rational functions $\varphi_1, \dots, \varphi_k$ on $\mathbf{R}^n$ such that f_{s_i} and φ_{is_i} are equivalent. Here we take orientation preserving local diffeomorphisms of the equivalences. Let $\rho, \rho_i : \mathbf{R}^n \to \mathbf{R}^n$, $i = 1, \dots, k$, be rational mappings defined by

$$\rho(x) = 2\varepsilon x/(1+|x|^2), \qquad \rho_i(x) = \rho(x-s_i)+s_i$$

for constant $\varepsilon > 0$. Then the set of critical points of ρ is $\{|x|=1\}$, and we have

$$\rho(\{|x|=1\}) = \{|x|=\varepsilon\}, \qquad |\rho| \leqslant \varepsilon \quad \text{and} \quad \rho^{-1}(0) = 0.$$

Take ε so small that $\varphi'_i = \varphi_i \circ \rho_i$ for each i is smooth and that the set of critical points of φ'_i is contained in $\{s_i\} \cup \{|x-s_i|=1\}$. Moreover, for small ε, φ'_i does not take the value $\varphi'_i(s_i)$ at any critical point in $\{|x-s_i|=1\}$. The reason is the following. A point x with $|x-s_i| = 1$ is a critical point of φ'_i if and only if x is a critical point of the restriction of φ'_i on $\{|x-s_i|=1\}$. Hence we only need to observe the critical points and the critical values of the restriction of φ_i on $\{|x-s_i|=\varepsilon\}$. By the assumption, $\varphi_i^{-1}(\varphi_i(s_i))$ has an isolated singularity at s_i. It is well-known that for $x \in \varphi_i^{-1}(\varphi_i(s_i))$ near to s_i, the angle of the vector $x - s_i$ and the tangent space of $\varphi_i^{-1}(\varphi_i(s_i))$ at x is smaller than $\pi/2$, and hence the angle of $x - s_i$ and $\operatorname{grad}\varphi_i$ is not 0 nor π. This implies that the restriction of φ_i on $\{|x-s_i|=\varepsilon\}$ is not critical at any point x of $\varphi_i^{-1}(\varphi_i(s_i))$.

It is trivial that for each i φ_{is_i} and φ'_{is_i} are equivalent by an orientation preserving diffeomorphism.

We put

$$\psi_i = (\varphi_i' - \varphi_i'(s_i))^{2n} + \sum_{j=1}^{n} \left(\frac{\partial \varphi_i'}{\partial x_j}\right)^4$$

for each i. Then we have $\psi_i^{-1}(0) = \{s_i\}$, and Lemma 3 implies that φ'_{is_i} and $\varphi'_{is_i} + a_i\psi_{is_i}$ are equivalent for any germ of C^∞ function a_i, because $(\varphi'_{is_i} - \varphi'_i(s_i))^n$ is of the form $\sum_{j=1}^{n} b_j \frac{\partial \varphi'_{is_i}}{\partial x_j}$ for germs of C^∞ functions b_j with $b_j(s_i) = 0$ [7]. Put

$$\psi_i' = \psi_i + \prod_{j \neq i} \psi_j, \quad f_2 = \prod_{i=1}^{k} \psi_i,$$

$$f_1 = \sum_{i=1}^{k} \varphi_i'(1 - \psi_i/\psi_i') + |x|^{2p} f_2$$

for each i and a large integer p. Then we have $f_2^{-1}(0) = S$, f_1 and f_2 are smooth rational functions since $\psi_i' > 0$, and we see the properness of f_1 in the same way as in the proof of Proposition 8. On account of $1 - \psi_i/\psi_i' = \left(\prod_{j \neq i} \psi_j\right)/\psi_i'$, f_{1s_i} for each i is of the form $\varphi'_{is_i} + a_i\psi_{is_i}$. Hence f_{s_i} and f_{1s_i} are equivalent by an orientation preserving local diffeomorphism. Then, by the proof of Theorem 1 we can reduce to the case $f = f_1$ in a neighborhood of S and on $\{|x| \geqslant K\}$ for a number K. Therefore f, f_1 and f_2 satisfy all the conditions in Proposition 8. Thus the theorem is proved.

Problem 9. – Is a C^∞ function on $\mathbf{R}^n$ equivalent to a polynomial if it is proper, the number of critical values is finite and the germ at each point is locally equivalent to a polynomial germ?

Remark 10. – The condition $f = f_1$ on $\{|x| \geqslant K\}$ in Proposition 8 is not necessary. It is sufficient to consider $f_1 + |x|^{2m} f_2$ in place of f_1 for large m, if $n \neq 4,5$.

Remark 11. – Assume that a C^∞ function f on $\mathbf{R}^5$ satisfies the conditions in Theorem 1 or 2 and that $f^{-1}(a)$ is diffeomorphic to S^4 for a large number a. Then f is equivalent to a polynomial, a rational function respectively, because W_i in the proof of Lemma 7 are diffeomorphic to $S^4 \times [0,1]$ (see [2]).

3. C^∞ functions on $\mathbf{R}^3$.

In this section we consider functions on $\mathbf{R}^n$ on the condition $(*)$ that *the absolute value of the differential is larger than a positive constant outside a bounded set.*

By Lemma 7′, any C^∞ proper function with the bounded set of critical points on $\mathbf{R}^n$, $n \neq 4,5$, is equivalent to a function satisfying the condition $(*)$. Hence we ask the next question. Can we replace the properness condition in Theorem 1, 1′, 2 by $(*)$? This is impossible for $n \geqslant 4$. The author was pointed the next example by Y. Matsumoto.

Example. — Let W be the 3-dimensional contractible manifold of J. H. C. Whitehead [12]. Then $W \times \mathbf{R}$ is diffeomorphic to $\mathbf{R}^4$, and W is not diffeomorphic to the interior of any compact manifold with boundary. Hence there exists a C^∞ function f on $\mathbf{R}^4$ without critical point such that $f^{-1}(0)$ is not diffeomorphic to any algebraic set. It is easy to modify f so that $|df| = 1$.

We know examples for $n \geqslant 5$ too. Therefore we assume $n = 3$. The case $n = 2$ will be considered in detail in the next section.

The condition $(*)$ assures a sort of regularity of the function near infinity as the following lemmas. It is not easy to weaken it essentially. Let us consider the case where a C^∞ function f has no critical point. For $n = 1$, f is equivalent to a polynomial if and only if f is surjective. For $n = 2, 3, f$ is equivalent to x_1 if the absolute value of the differential is larger than a positive constant, where (x_1, x_2) or (x_1, x_2, x_3) is an affine coordinate system, because each level is contractible in this case. But $f = (1 + y \sin x) \sin x$ has no critical point and is surjective but not equivalent to any rational function. We remark that a polynomial without critical point is not necessarily surjective. For example, $(1 + x(xy + 1))^2 + x^2$.

LEMMA 12. — *Let f be a C^∞ function on $\mathbf{R}^3$ with $(*)$. Let a be a number which is larger than any critical value. Then the number of connected components of $f^{-1}(a)$ is finite. Each component is diffeomorphic to S^2-(a finite points) and divides $\mathbf{R}^3$ into two connected components one of which contains all other components of $f^{-1}(a)$ and the set of critical points.*

Proof. – Let X be the vector field $\operatorname{grad} f/|\operatorname{grad} f|^2$ on the set of regular points of f. Let φ_t be the local one-parameter group of X. Let A be a connected component of $f^{-1}(a)$. By the assumption, $\varphi_t(x)$ is well-defined for $t \geqslant 0$, $x \in f^{-1}(a)$. Put

$$B_1 = \{\varphi_t(x) | t > 0, x \in A\}, \qquad B_2 = \mathbf{R}^3 - A - B_1.$$

Then B_1 is open in $\mathbf{R}^3$ and closed in $\mathbf{R}^3 - A$. Since $\mathbf{R}^3 - A$ has at most two connected components, B_1 and B_2 are the connected components. Hence we have $f > a$ on B_1, and B_2 contains all the critical points and other connected components of $f^{-1}(a)$. If A is compact, B_2 is bounded, and hence f is proper. From Lemma 7′, Lemma 12 follows in this case. Therefore we assume A to be non-compact in the following.

Let N be an integer such that $\{|x| < N\}$ contains all the critical points. Fix a point x_0 in $f^{-1}(a)$ far from $\{|x| \leqslant N\}$. Consider the integral curve of X passing x_0. Then the curve does not pass any point of $\{|x| \leqslant N\}$. The reason is the following. If there were $t_0 < 0$ such that

$$|\varphi_{t_0}(x_0)| = N, \qquad |\varphi_t(x_0)| > N \qquad \text{for} \qquad t_0 < t \leqslant 0,$$

$f(x_0) - f(\varphi_{t_0}(x_0))$ should be large because of $(*)$. This is impossible. It follows that $\varphi_t(x_0)$ is well-defined for all t and that f takes all the values $\mathbf{R}$ on the curve. Hence, if a connected component A′ of $f^{-1}(a)$ is contained in $\{|x| > K\}$ for large K, $\varphi_t(x)$ is well-defined for $t \in \mathbf{R}, x \in A'$. Then $\{\varphi_t(x) | t \in \mathbf{R}, x \in A'\}$ is open and closed in $\mathbf{R}^3$. This is a contradiction. Therefore there is not such A′. This shows the finiteness of the connected components of $f^{-1}(a)$.

It rests to find a diffeomorphism from A to S^2-[a finite point set]. Assume the existence of Jordan curves C_1, C_2 in A which intersect transversally at one point. Then the pair of $\varphi_\varepsilon(C_1)$ and C_2 or of C_1 and $\varphi_\varepsilon(C_2)$ for small $\varepsilon > 0$ twists, say $\varphi_\varepsilon(C_1)$ and C_2. $\varphi_t(C_1)$ for $t \geqslant 0$ is well-defined and tends to infinity as $t \to \infty$. On the other hand $\varphi_t(C_1)$ for $t > 0$ does not intersect with C_2. This is a contradiction. Hence there are no such Jordan curves. This means that, if there is a compact connected 2-dimensional submanifold of A with boundary, this is diffeomorphic to S^2-finite disjoint open 2-disks. Let K be a large number, M be a compact connected submanifold of A with boundary containing $A \cap \{|x| \leqslant K\}$. Let $\alpha_1, \ldots, \alpha_m$ be the connected components of the boundary of M. Assume that there are a Jordan curve C_1 in $A - M$

and a simple open curve $C_2 : (0,1) \to A - M$ such that they intersect transversally at one point and that the image of C_2 is closed in A. Then $\varphi_t(C_2)$ for all t is well-defined, because K is large. Hence we have a contradiction in the same way as above. This implies that $A - M$ has m connected components and that the closure of each connected component is a manifold with boundary one of α_i and is diffeomorphic to a closed disk or to a closed disk $-$ a point. Therefore A is diffeomorphic to $S^2 -$ a finite point set. Thus the lemma is proved.

LEMMA 13. — *Let f_1, f_2 be C^∞ functions on $\mathbf{R}^3$ with $(*)$, a be a larger number than any critical value of f_1, f_2. Assume that $f_1^{-1}(a)$ is transformed to $f_2^{-1}(a)$ by an orientation preserving C^∞ diffeomorphism of $\mathbf{R}^3$. Then there exists π a C^∞ diffeomorphism of $\mathbf{R}^3$ such that π is the identity on a given bounded set and that $f_1 \circ \pi$ and f_2 are equal outside a bounded set.*

Proof. — Let φ_t be the local one-parameter group of the vector field $\operatorname{grad} f_1/|\operatorname{grad} f_1|^2$. By Lemma 12, $f_1^{-1}(a) \cup \{\infty\}$ has a triangulation. Let K be a subpolyhedron whose complement is bounded in $\mathbf{R}^3$. Choose K so small that $\varphi_t(x)$ is well-defined for all $t \in \mathbf{R}$, $x \in K$. Choose a triangulation of $f_1^{-1}(b) \cup \{\infty\}$ compatible with $\varphi_{b-a}(K)$, where b is a smaller number than any critical value of f_1 and f_2. Since

$$K_1 = \{f_1 \geqslant a\} \cong f_1^{-1}(a) \times [0,\infty),$$
$$K_2 = \{f_1 \leqslant b\} \cong f_1^{-1}(b) \times (-\infty,0].$$
$$K_3 = \{\varphi_t(x) | b-a \leqslant t \leqslant 0, x \in K - \infty\} \cong (K - \infty) \times [0,1],$$

$$\mathbf{R}^3 = K_1 \cup K_2 \cup K_3 \cup (\text{a compact subset } K_4),$$

$\mathbf{R}^3 \cup \{\infty\}$ has a triangulation compatible with $f_1^{-1}(a) \cup \{\infty\}$. This argument shows also that we can assume $f_1 = f_2$ on $K_1 \cup K_3$. We only need to reduce to the case $f_1 = f_2$ on K_2.

Let A be a connected component of $f_1^{-1}(a)$. At first we show that A is imbedded in $\mathbf{R}^3$ in a standard form. By Lemma 12, A is diffeomorphic to S^2-(k points), $k > 0$. Let $M \subset \{x \in \mathbf{R}^3 | |x| = 1\}$ be a connected C^∞ 2-manifold with boundary such that the boundary consists of connected $X'_1, \ldots, X'_k$. Put

$$A' = M \cup \{|x| \geqslant 1, x/|x| \in \partial M\}.$$

We want to see that

$$(\mathbf{R}^3 \cup \{\infty\}, \mathrm{A} \cup \{\infty\}, \{\infty\}) \quad \text{and} \quad (\mathbf{R}^3 \cup \{\infty\}, \mathrm{A}' \cup \{\infty\}, \{\infty\})$$

are p.l. homeomorphic. Let B_1, $\mathrm{B}_2(\mathrm{B}'_1, \mathrm{B}'_2)$ be the connected components of $\mathbf{R}^3 - \mathrm{A}$ $(\mathbf{R}^3 - \mathrm{A}')$ respectively such that

$$\mathrm{B}_1 \cup \mathrm{A} \cong \mathrm{A} \times [a, \infty) \qquad (\mathrm{B}'_1 \cup \mathrm{A}' \cong \mathrm{A}' \times [a, \infty)).$$

Then a p.l. homeomorphism from A to A' can be extended to $\mathrm{B}_1 \cup \mathrm{A} \to \mathrm{B}'_1 \cup \mathrm{A}'$. Let the extension be denoted by τ. We want to extend τ to $\mathrm{B}_2 \cup \mathrm{A} \to \mathrm{B}'_2 \cup \mathrm{A}'$. If $k = 1$, it is well-known that this is possible, because $\mathrm{A} \cup \{\infty\}$ is combinatorial 2-sphere contained in $\mathbf{R}^3 \cup \{\infty\} = \mathrm{S}^3$. We show the existence of an extension inductively on k. Assume $k \geqslant 2$. Choose a triangulation of $\mathbf{R}^3 \cup \{\infty\}$ compatible with $\mathrm{A} \cup \{\infty\}$ and $\{\infty\}$. The star of $\{\infty\}$ in the triangulation of $\mathrm{A} \cup \{\infty\}$ or $\mathrm{A} \cup \mathrm{B}_2 \cup \{\infty\}$ is the cone of k 1-spheres or k 1-disks respectively. Since the imbedding of such cone in $\mathbf{R}^3 \cup \{\infty\}$ is unique, there exist closed neighborhoods U, U' of $\{\infty\}$ in $\mathbf{R}^3 \cup \{\infty\}$ such that

$$(\mathrm{U}, \mathrm{U} \cap (\mathrm{A} \cup \{\infty\}), \{\infty\}) \quad \text{and} \quad (\mathrm{U}', \mathrm{U}' \cap (\mathrm{A}' \cup \{\infty\}), \{\infty\})$$

are p.l. homeomorphic. Put $\mathrm{X}_i = \tau^{-1}(\mathrm{X}'_i), i = 1, \ldots, k$. Then the homotopy class $[\mathrm{X}_1]$ of X_1 in $\mathrm{B}_2 \cup \mathrm{A}$ is zero, because of

$$[\mathrm{X}'_1] = 0 \qquad \text{in} \qquad \tau_1((\mathrm{B}'_2 \cap \mathrm{U}') \cup \mathrm{A}').$$

It follows from Dehn's lemma that there exists $\mathrm{Y}_1 \subset \mathrm{B}_2 \cup \mathrm{A}$ a p.l. 2-disk such that $\mathrm{Y}_1 \cap \mathrm{A} = \partial \mathrm{Y}_1 = \mathrm{X}_1$. Let Y'_1 be the closure of the connected component of $\{|x| = 1\} - \mathrm{M}$ whose boundary is X'_1. Let us extend τ naturally to $\mathrm{Y}_1 \cup \mathrm{A} \to \mathrm{Y}'_1 \cup \mathrm{A}'$. We write the extension as the same τ. Consider $\mathrm{Y}'_1 \cup \{|x| \geqslant 1, x/|x| \in \mathrm{X}'_1\}$ and its inverse image under τ. Then, by the same reason as the case $k = 1$, τ can be extended to one connected component of $\mathrm{B}_2 - \mathrm{Y}_1$. Therefore we have reduced to the case in which A is diffeomorphic to S^2-$(k-1)$ points. Hence, by the induction assumption, we have an extension of τ. Thus A is imbedded in $\mathbf{R}^3$ in a standard form.

Let CM denote the cone $\{x \in \mathbf{R}^3 - \{0\} \mid |x| \leqslant 1, x/|x| \in \mathrm{M}\} \cup \{0\}$. Then there exists a p.l. homeomorphism

$$g : (\mathbf{R}^3 \cup \{\infty\}, \mathrm{A} \cup \mathrm{B}_1 \cup \{\infty\}, \{\infty\}) \to (\mathbf{R}^3 \cup \{\infty\}, \mathrm{CM}, \{0\}).$$

Next we show that $f_1^{-1}(a)$ is imbedded in $\mathbf{R}^3$ in a standard form. Let K_1' be a connected component of $K_1 - A - B_1$. The set $\{|x| \leqslant \varepsilon\} - CM, 0 < \varepsilon \leqslant 1$, consists of k connected components. Since $K_1' \cup \{\infty\}$ is contractible, $g(K_1') \cap \{|x| \leqslant \varepsilon\}$ is contained in one connected component of $\{|x| \leqslant \varepsilon\} - CM$, for sufficiently small ε. Hence we can assume that $g(K_1')$ is contained in one connected component of $\{|x| \leqslant 1\} - CM$. Repeating this argument for other connected components of $K_1 - A - B_1$, and imbedding them in $\mathbf{R}^3$ in a standard form, we obtain a p.l. 2-manifold M_1 with boundary contained in $\{x \in \mathbf{R}^3 \,|\, |x| = 1\}$, and a p.l. homeomorphism

$$(\mathbf{R}^3 \cup \{\infty\}, K_1 \cup \{\infty\}, \{\infty\}) \to (\mathbf{R}^3 \cup \{\infty\}, CM_1, \{0\}).$$

By the same reason, there exist a p.l. 2-manifold M_2 with boundary disjoint with M_1 in $\{|x| = 1\}$, and a p.l. homeomorphism

$$h: (\mathbf{R}^3 \cup \{\infty\}, K_1 \cup \{\infty\}, K_2 \cup \{\infty\}, \{\infty\}) \to (\mathbf{R}^3 \cup \{\infty\}, CM_1, CM_2, \{0\}).$$

Put

$$\psi_t(x) = \begin{cases} h \circ \varphi_t \circ h^{-1}(x) & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Then $\psi_t(x)$ is well-defined in a neighborhood of $x = 0$ in $\mathbf{R}^3$ for $b - a \leqslant t \leqslant a - b$ and an isotopy such that

$$\psi_{b-a}(C(\partial M_1) \cap \{|x| \leqslant \varepsilon\}), \quad \psi_{a-b}(C(\partial M_2) \cap \{|x| \leqslant \varepsilon\})$$

for small $\varepsilon > 0$ are neighborhoods of 0 in $C(\partial M_2), C(\partial M_1)$ respectively. Hence, modifying ψ_t, we reduce to the case where $\psi_t(x)$ is defined on $\{|x| \leqslant 1\}$ for $b - a \leqslant t \leqslant a - b$ satisfying

$$\psi_{b-a}(C(\partial M_1)) = C(\partial M_2),$$

$$\bigcup_{b-a<t<0} \psi_t(C(\partial M_1)) = C\{|x| = 1, x \notin M_1 \cup M_2\}.$$

Particularly

$$\psi_{b-a}(\partial M_1) = \partial M_2,$$

$$\bigcup_{b-a<t<0} \psi_t(\partial M_1) = \{|x| = 1, x \notin M_1 \cup M_2\}.$$

Thus we proved that $(\{|x| = 1\}, M_2)$ is uniquely determined by M_1 up to homeomorphisms of $\{|x| = 1\}$ identical on M_1.

The last statement implies the existence of a homeomorphism ρ of $\mathbf{R}^3$ such that

$$\rho = \text{identity on } \mathrm{K}_1 \cup \mathrm{K}_3, \qquad \rho(f_1^{-1}(b)) = f_2^{-1}(b).$$

Hence there exists a diffeomorphism from $f_1^{-1}(b)$ to $f_2^{-1}(b)$ which is the identity on $f_1^{-1}(b) - \mathrm{K}_4$. This diffeomorphism induces naturally a diffeomorphism from $\{f_1 \leqslant b\}$ to $\{f_2 \leqslant b\}$. Therefore there exists a proper C^∞ imbedding ρ' of $\{|x| \geqslant d\}$, for some large number d, into $\mathbf{R}^3$ such that $f_1(x) = f_2 \circ \rho'(x)$ if $|x| \geqslant d$. By a theorem in [8], ρ' is extended to a diffeomorphism ρ'' of $\mathbf{R}^3$. It is easy to modify ρ'' so that it is the identity on a given bounded set. Thus the lemma is proved.

The following corresponds to Theorems 1, 1′, 2.

THEOREM 3. — *A* C^∞ *function* f *on* $\mathbf{R}^3$ *is equivalent to a polynomial if* f *satisfies* $(*)$, *the number of critical points is finite and the Milnor number of the germ at each critical point is finite.*

Proof. — Let $\mathrm{S} = \{s_1, \ldots, s_k\}$ be the set of critical points. We assume $f(s_i) > 0$ for all i. We transform $f^{-1}(0)$ to an algebraic set by a C^∞ diffeomorphism of $\mathbf{R}^3$ as follows. We regard $\mathbf{R}^3$ as S^3-$\{a\}$ where $\mathrm{S}^3 = \{x \in \mathbf{R}^4 \mid |x| = 1\}$. By the argument of standard form of $f^{-1}(x)$ in Proof of Lemma 13, we know that the germ of $f^{-1}(0) \cup \{a\}$ at a is a cone of a finite number of circles. Hence there exists a C^∞ function ξ on S^3 such that $f^{-1}(0)$ can be transformed to $\xi^{-1}(0) \cap \mathbf{R}^3$, a is the unique zero critical point of ξ, and that ξ is analytic near a. Let ξ' be a polynomial approximation of ξ such that $\xi - \xi'$ is p-flat at a for some large p. Then we have a C^r diffeomorphism π of $\mathrm{S}^3, 0 < r < \infty$, such that $\pi(a) = a, \xi = \xi' \circ \pi$ in a neighborhood of $\xi^{-1}(0)$, $\xi'^{-1}(0) = \pi(\xi^{-1}(0))$, and that π is of class C^∞ on $\mathbf{R}^3$. The reason is the following. A fundamental calculation of derivation shows that

$$(\xi - \xi')/\left(\left(\frac{\partial \xi}{\partial x_1}\right)^2 + \left(\frac{\partial \xi}{\partial x_2}\right)^2 + \left(\frac{\partial \xi}{\partial x_3}\right)^2\right)$$

is a small C^r function in a neighborhood of $\xi^{-1}(0)$ and of class C^∞ outside a. We see in the same way that Lemmas 3, 4, 5 hold true in the case C^r too and that the resultant diffeomorphisms are of class C^∞ at any point where the functions are so. Since the restriction of ξ' on $\mathbf{R}^3$ is a rational function ν/ν' for some polynomial ν, ν' on $\mathbf{R}^3$ with $\nu > 0$, we assume from the beginning $f^{-1}(0) = \nu^{-1}(0)$ and that ν has no critical point in $\nu^{-1}(0)$. Let ν be positive on S and ℓ be the same as in the proof of Theorem 1. Let g be a positive polynomial on $\mathbf{R}^3$ such that $g - f/\nu$ and hence $g\nu - f$ are ℓ-flat at each s_i. Then we reduce to the case $f = g\nu$ in a neighborhood of S.

We put

$$h_m(x) = \prod_{i=1}^{k} |x - s_i|^{2m}, \qquad g_m = g + h_m$$

with $2m > \ell$, $2km > \deg g$. Now we will see that $g_m\nu$ satisfies $(*)$ for large m. Put

$$C_{m\varepsilon} = \{x \in \mathbf{R}^3 | f(x) \geqslant 0, <x, \operatorname{grad} g_m\nu(x)> \leqslant -\varepsilon|x||\operatorname{grad} g_m\nu(x)|\}$$

for $\varepsilon > 0$. Then the argument in Proof of Proposition 8 shows that $g_m\nu$ is decreasing as x tends to infinity on any 1-dimensional semi-algebraic subset of $C_{m\varepsilon}$.

Generally speaking, if a polynomial defined on a semi-algebraic set $\subset \mathbf{R}^n$ is not bounded, there exists a one-dimensional semi-algebraic subset the restriction on which of the polynomial is not bounded. We prove this as follows. Imbed algebraically $\mathbf{R}^n, \mathbf{R}$ in S^n, S^1 respectively so that $\mathbf{R}^n = S^n - \{a\}$ and $\mathbf{R} = S^1 - \{b\}$, consider the closure of the graph of the polynomial in $S^n \times S^1$, and then take a connected one-dimensional semi-algebraic subset of the closure containing (a,b). Then (the projection of the subset onto S^n)-$\{a\}$ satisfies the condition.

Hence it follows that $g_m\nu$ is bounded on $C_{m\varepsilon}$. We saw in Sketch of the proof of Theorem 1′ that the set of points x of M such that the angle of x and the tangent space of $f^{-1}(0)$ at x is smaller than given $\varepsilon > 0$ is bounded. Since $g_m\nu$ is regular on $(g_m\nu)^{-1}(0) = f^{-1}(0)$, it follows that the angle of x and $\operatorname{grad} g_m\nu(x)$ tends to $\pi/2$ as x tends to infinity on $f^{-1}(0)$. Hence $C_{m\varepsilon} \cap f^{-1}(0)$ is bounded. Take a sufficiently large integer m', and observe $g_{m'}\nu$ on $C_{m\varepsilon}$. Then we see in the same way as at the beginning of Proof of Proposition 8 that $g_{m'}\nu$ is proper on $C_{m\varepsilon}$. Hence $g_{m'}\nu$ is proper and bounded on $C_{m\varepsilon} \cap C_{m'\varepsilon}$. This means that $C_{m\varepsilon} \cap C_{m'\varepsilon}$ is bounded.

By an easy calculation of gradient, we see that the angles of x and $\operatorname{grad} h_m(x)$ or of x and $\operatorname{grad}(h_{m'} - h_m) = \operatorname{grad}(g_{m'} - g_m)$ tend to 0 as x tends to infinity. Hence it follows that the angle of x and $\operatorname{grad} g_{m'}\nu$ is smaller than it of x and $\operatorname{grad} g_m v$ on $\{f \geqslant 0, |x| \geqslant K\}$ for large K, and that $g_{m'}v$ has no critical point there, here the angle of x and 0 vector is assumed to be π. These imply that $C_{m'\varepsilon} \cap \{|x| \geqslant K\} \subset C_{m\varepsilon}$. Hence $C_{m'\varepsilon}$ is bounded.

Choose K so that

$$C_{m'\frac{1}{2}} \subset \{|x| \leqslant K\}.$$

Then the same calculation of gradient as above shows that

$$|x|^{2k(m''-n')} |\operatorname{grad} g_{m'}v(x)| < c\,|\operatorname{grad} g_{m''}v|$$

on $\{x \in \mathbf{R}^3 \,|\, |x| \geqslant K, f \geqslant 0\}$ for some constant $c > 0$ and any integer m'' larger than m'. Therefore $|\operatorname{grad} g_{m''}v|$ is proper on $\{f \geqslant 0\}$ for sufficiently large m''. We proceed with $\{f \leqslant 0\}$ in the same manner. Thus, from the beginning we can assume $(*)$ of $g_m v$ and the boundednes of $C_{m\varepsilon}$. Then $C_{m\varepsilon'}$, for any $\varepsilon' > 0$, is bounded. If this were unbounded, we should have

$$\frac{\partial g_m v(x(t))}{\partial t} < \text{a negative constant}$$

outside a bounded set, where $t \to x(t), t \in \mathbf{R}_+$, is a semi-algebraic curve in $C_{m''\varepsilon}$ such that $x(t) \to \infty$ as $t \to \infty$ and that t is a canonical parameter of the curve. Hence $g_m v(x(t)) \to -\infty$ as $t \to \infty$. This is a contradiction.

We put

$$f_1 = g_m v, \qquad f_2 = h_m v.$$

As g_m is positive, $\{f_1 \geqslant 0\} = \{f \geqslant 0\}$ and $\{f_1 \leqslant 0\} = \{f \leqslant 0\}$. In Proof of Lemma 13, we saw a standard form of $\{f \leqslant 0\}$. That argument of standard form shows also that if $\{f \leqslant a\}$, for some a, has a standard form, then $(\mathbf{R}^n, \{f \leqslant 0\})$ is diffeomorphic to $(\mathbf{R}^n, \{f \leqslant a\})$. Since f_1 satisfies $(*)$, $\{f_1 \leqslant a\}$ for sufficiently small $a < 0$ has a standard form. Hence we have

$$(\mathbf{R}^3, \{f_1 \leqslant a\}) \cong (\mathbf{R}^3, \{f_1 \leqslant 0\}) = (\mathbf{R}^3, \{f \leqslant 0\}) \cong (\mathbf{R}^3, \{f \leqslant a\}).$$

It follows that $(\mathbf{R}^3, f_1^{-1}(a)) \cong (\mathbf{R}^3, f^{-1}(a))$ for small $a < 0$. Apply Lemma 13 to f and f_1, and we reduce to the case $f = f_1$ in a neighborhood of $S \cup f^{-1}(0)$ and on $\{|x| \geqslant K\}$ for a number K. The rest of the proof goes on just in the same way as it of Proposition 8. We need only the boundedness of

$$B_\varepsilon = \left\{ \begin{array}{l} (x,t) \in \mathbf{R}^{n+1} | 0 \leqslant t \leqslant 1, \ < x, \operatorname{grad}_x F(x,t) > \\ \qquad\qquad \begin{cases} \leqslant -\varepsilon|x|\,|\operatorname{grad}_x F(x,t)| & \text{if } f(x) \geqslant 0 \\ \geqslant \ \ \varepsilon|x|\,|\operatorname{grad}_x F(x,t)| & \text{if } f(x) \leqslant 0 \end{cases} \end{array} \right\}$$

for $\varepsilon > 0$, $F(x,t) = tf_1(x) + (1-t)f_2(x)$. This follows easily from the properties of $\operatorname{grad} g_m v$ and h_m stated above. Thus the theorem is proved.

The non-euclidean case of Theorem 3 is:

THEOREM 3'. — *Let* M *be an affine smooth algebraic variety of dimension* 3, f *be a* C^∞ *function on* M *with the same conditions as in Theorem* 3. *Assume that the boundary of* M *is diffeomorphic to* S^2. *Then* f *is equivalent to a polynomial.*

For the proof, the sketch of the proof of Theorem 1' and the following remark are sufficient. We omit the detail. Lemma 12, 13 are correct for this M, because for large or small a, $f^{-1}(a)$ is contained in M-[a bounded set] which is diffeomorphic to $S^2 \times (0,\infty)$.

THEOREM 4. — *A* C^∞ *function on* $\mathbf{R}^3$ *is equivalent to a rational function, if* f *satisfies* $(*)$, *the number of critical points is finite and the germ at each critical point is equivalent to a rational function germ.*

Proof. — Let $S = \{s_1, \ldots, s_k\}$ be the same as Theorem 3. Assume $f(s_i) > 0$ for all i. By the proof of Theorem 2, we have a proper positive smooth rational function g_1 and a proper non-negative polynomial g_2 such that $f = g_1$ in a neighborhood of S, $g_2^{-1}(0) = S$ and that the germ g_{2s_i} at each point s_i is a linear combination of germs of $\dfrac{\partial f}{\partial x_j}\dfrac{\partial f}{\partial x_{j'}}$ with germs of functions vanishing at s_i as coefficients. We want to find a smooth rational function g_3 on $\mathbf{R}^3$ such that (1) $v = 1 - g_2 g_3$ is regular on $v^{-1}(0)$ and that (2) $v^{-1}(0) = f \circ \tau^{-1}(0)$ for some diffeomorphism τ with $\tau =$ the identity in a neighborhood of S. For this we imbed naturally $\mathbf{R}^3$ in $P^3(\mathbf{R})$, that is, let (t, x_1, x_2, x_3) be homogeneous coordinates of P^3 such that $\mathbf{R}^3 = \{t \neq 0\}$. Let g_2' be the rational function on P^3 whose restriction on $\mathbf{R}^3$ is g_2. Then

$$g_2' = g_2(x_1/t, x_2/t, x_3/t).$$

Put

$$g_2'' = g_2' t^{2\ell}/(t^{2\ell} + x_1^{2\ell} + x_2^{2\ell} + x_3^{2\ell}),$$

where 2ℓ is the degree of g_2. Then g_2'' is smooth on P^3, and its restriction on $\mathbf{R}^3$ or on $P^3 - \mathbf{R}^3$ is equal to $(x_1^{2\ell}+x_2^{2\ell}+x_3^{2\ell})g_2$ or not identical to the zero function respectively. Let a be a point of $P^3 - \mathbf{R}^3$ where g_2'' does not vanish.

Let ξ be a C^∞ function on P^3 such that a C^∞ diffeomorphism of $\mathbf{R}^3$ transforms $f^{-1}(0)$ to $\xi^{-1}(0) \cap \mathbf{R}^3$ fixing S, that a is the unique zero critical point of ξ, that $\xi < 1, > 0$ on $P^3, S \cup (P^3 - \mathbf{R}^3 - \{a\})$ respectively, and that ξ is analytic in a neighborhood of a. Let β be a non-negative C^∞ function on P^3 whose support is a small neighborhood of $g_2''^{-1}(0)$ and which does not vanish on $g_2''^{-1}(0)$. Let ξ' be a polynomial approximation of the well-defined C^∞ function $(1-\xi)/(g_2''+\beta)$ whose difference is p-flat at a for large p. Then $1 - (g_2''+\beta)\xi'$ is an approximation of ξ, and their difference is p-flat at a. Compare $\pm \xi t$ and $\pm(1-(g_2''+\beta)\xi' t)$ in a neighborhood of $\xi^{-1}(0)$. Then, because of $\beta = 0$ there, in the same way as Proof of Theorem 3, we have a C^r diffeomorphism of $P^3, r < \infty$, which is near to the identity, of class C^∞ outside a and transforms $(\xi t)^{-1}(0)$ to $((1-g_2''\xi')t)^{-1}(0)$. Since the C^r diffeomorphism transforms $\{t=0\}$ onto itself, it induces a C^∞ diffeomorphism of $\mathbf{R}^3$. Hence $\nu = 1 - g_2''\xi'$ satisfies the conditions (1), (2) stated at the beginning.

Now we assume $\nu^{-1}(0) = f^{-1}(0)$. We remark $\nu > 0$ on S. By Lemma 4′, the germs of f and $g_1\nu$ at each point s_i are equivalent, and hence we reduce to the case $f = g_1\nu$ in a neighborhood of S. We put

$$f_2 = (1+|x|^{2m})g_2\nu, \qquad f_1 = g_1\nu + f_2$$

for a large integer m. Then, in the same way as the proof of Theorem 3, increasing m, we see $(*)$ of f_2 and the boundedness of $C_{m\varepsilon}$ in the proof and hence that $|\operatorname{grad} f_2(x)|$ tends to infinity of arbitrarily large polynomial order of $|x|$ as x tends to infinity. Hence $(*)$ of f_1 and the boundedness of B_2 in the previous proof can be assumed. The rest of the proof proceeds just in the same way. We omit the detail. Thus the theorem is proved.

We proved already in the proofs above the following $(*)$ case of the Proposition 8.

PROPOSITION 14. — *Let f be a C^∞ function on $\mathbf{R}^3$ with $(*)$. Let S be the set of critical points. Let f_1, f_2 be smooth rational functions on $\mathbf{R}^3$ such that $f = f_1$ in a neighborhood of $S \cup f^{-1}(0)$, $f_2^{-1}(0) = S \cup f^{-1}(0)$, that*

f_2 is regular on $f^{-1}(0)$ and that f_2 satisfies the conditions on g_i in Lemma 4. Then f is equivalent to $f_1 + f_2 f_3$ for some polynomial f_3.

Problem 15. – In the proofs above, we used the following fact. Let $B = \{x \in \mathbf{R}^3 \mid |x| \leqslant 1\}$. Let $M \subset B$ be a compact C^∞ manifold of codimension 1 with boundary in ∂B. Then $(\operatorname{Int} B, M - \partial M)$ is diffeomorphic to $(\mathbf{R}^3, M')$ where M' is an algebraic set of $\mathbf{R}^3$. We easily see this for dimension 4 too. Is this possible for general dimension?

Problem 16. – Do Theorems 3, 4 remain valid for general dimension n, if we add the condition that $(\mathbf{R}^n, f^{-1}(0))$ is diffeomorphic to $(\mathbf{R}^n$, an algebraic set)?

4. Analytic functions on $\mathbf{R}^2$.

Let us consider the equivalence of a C^∞ function on $\mathbf{R}^2$ to a polynomial. The condition that the germ of the function at each point is equivalent to an analytic function germ is necessary. It is shown in [3], [4] that the function under this condition is globally equivalent to an analytic function. Hence we treat only analytic functions in this section.

The principal ideas of the proofs were used in the proof of II. 2.6 in [3].

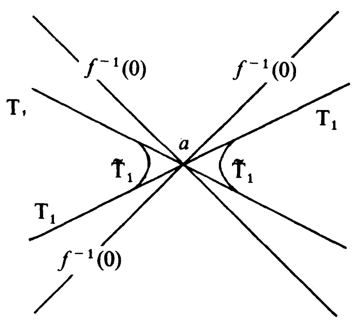
Proof of Theorem 5. – Let f be the function stated in Theorem 5, and S, T, V be the set of critical points, $f^{-1}f(S)$, the set of points x such that $f_x - f(x)$ is not a power of a regular function germ respectively. We remark that S is bounded and that V is finite. We assume f non-proper. The proper case follows more easily. We assume that all the critical values are negative. By the proof of Lemma 12, (or we see easily that) the connected components of $f^{-1}(0)$ are finite k, and for each component E, T and all other components exist in one side of E in $\mathbf{R}^2$. We reduce by the method in the proof of Theorem 3 to the case that $f^{-1}(0)$ is the zero set of a polynomial v regular on $f^{-1}(0)$, and that $f = v$ on $\{|x| \geqslant K\}$ for a large number K. We remark that f is no longer analytic outside a neighborhood of S but it satisfies $(*)$. Let $T_1, T_2, \ldots$ be unions of connected components of T such that for each i, if T_i is bounded then it is connected, that if it is unbounded it consists of all the unbounded components with the same value of f and that $T = T_1 \cup \ldots$. Regard $\mathbf{R}^2$ as S^2-$\{a\}$. Let (r,θ) be a C^∞ polar coordinates of a neighborhood of a in S^2 such that $r = 0$ at a. Put $g = \tan k\theta$, for an integer k. Then in Proof of Lemma 13 we saw the following. For any $c > 0$, there exists a diffeomorphism ρ of $\mathbf{R}^2$ such that

$$f \circ \rho = g \qquad \text{on} \qquad \{0<|r|\leqslant 1\} \cap g^{-1}([-c,c]),$$

and

$$|f \circ \rho| \geqslant c \qquad \text{on} \qquad \{0<|r|\leqslant 1\} \cap g^{-1}(\mathbf{R}-[-c,c])$$

for some k. This k is an invariant. Hence $\{f \leqslant 0\}$ has k connected components in a neighborhood of a, and in each connected component, T_i consists of two curves if T_i is unbounded. Let T_i be modified near a to $\tilde{T}_i$ as the figure so that the two curves become to a curve and that T_i, T_j do not intersect each other and with $f^{-1}(0) \cup \{a\}$. We order $T_1, \ldots$ so that for any $1 \leqslant i < j$, the identity mapping: $\tilde{T}_j \to \mathbf{R}^2 - \tilde{T}_i$ is homotopic to a constant mapping. This is possible, for example, we choose as $\tilde{T}_1$ one of the nearest $\tilde{T}_i$ to $f^{-1}(0)$, as $\tilde{T}_2$ one of the nearest to $\tilde{T}_1 \cup f^{-1}(0)$, and so on.



Using ν, we will find a polynomial whose germ at the zero set is equivalent to the germ of f at T_1. Assume $f(T_1) = -1$. Let $\mathscr{F}$ be the sheaf of germs of C^∞ functions on $\mathbf{R}^2$, $\mathfrak{p}$ be the sheaf of ideals $(f+1)\mathscr{F}$ on T_1 and $\mathscr{F}$ on T_1^c. Then there exist uniquely non-trivial distinct coherent sheaves of ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ and positive integer $\alpha_1, \ldots, \alpha_m$ such that

$$\mathfrak{p} = \prod_{i=1}^{m} \mathfrak{p}_i^{\alpha_i},$$

that if one of $\mathfrak{p}_i^{-1}(0)$ is unbounded then other $\mathfrak{p}_j^{-1}(0)$ are bounded, and that $\{\mathfrak{p}_i\}$ is irreducible in this sense. We remark that the stalk $\mathfrak{p}_{ix}$ at each point x for each i is generated by a regular function germ or by a convergent power series without multiple factor. Let T_{1i} be the zero set of

$\mathfrak{p}_i$ for each i. Let us fix i. Put $U_1 = \mathbf{R}^2 - T_{1i}$ if T_{1i} is a point. If it is not a point, $\mathbf{R}^2 - T_{1i}$ is the disjoint union of two sets U_1 and U_2 such that any point of T_{1i} is adherent to U_1 and U_2 and that $U_1 \supset f^{-1}(0)$, since $\mathbf{R}^2$ is orientable. For any x of $\mathbf{R}^2$, we have a C^∞ function germ ψ at x which generates $\mathfrak{p}_{ix}$ and which is positive on U_1 and negative on U_2. Multiplying ψ by a suitable non-negative C^∞ function with compact support, and summing them, we have a C^∞ function φ_i on $\mathbf{R}^2$ such that

$$\varphi_i^{-1}(0) = T_{1i}, \qquad \varphi_i \mathscr{F} = \mathfrak{p}_i$$

and on $W = \{|x| \geqslant K \text{ or } f(x) \geqslant -\varepsilon\}$

$$\varphi_i = \begin{cases} v^2 + 1 & \text{if } T_{1i} \text{ is compact} \\ v + 1 & \text{if it is not so} \end{cases}$$

for small $\varepsilon > 0$ and large $K > 0$. Here we require that compact $T_{1i} \subset \{|x| \leqslant K\}$.

Now we define a polynomial whose zero set is equivalent to T_{1i} as follows. Assume T_{1i} compact. Then $(\varphi_i - 1)/v^2$ is well-defined and equal 1 on W. Let φ_i' be a polynomial such that $\varphi_i' - (\varphi_i - 1)/v^2$ is small on $\{|x| \leqslant K\}$ and ℓ-flat at $T_{1i} \cap V$ for large ℓ and that φ_i' is positive on W. Then $1 + v^2\varphi_i'$ is a polynomial approximation of φ_i on $\{|x| \leqslant K\}$, and the zero set is contained in $\{|x| \leqslant K\}$.

Consider the case where T_{1j} is not compact. By the proof of Theorem 3, we can assume from the beginning that the set of points x in $\{f \leqslant 0\}$ such that the angle of $-x$ and $\operatorname{grad} v$ is larger than $2\pi/3$ is bounded. Let φ_j' be a proper polynomial on $\mathbf{R}^2$ such that $\varphi_j' - (\varphi_j - 1)/v$ is small on $\{|x| \leqslant K\}$ and ℓ-flat at $T_{ij} \cap V$, that φ_j' is positive on W, and moreover that the angle of x and $\operatorname{grad} \varphi_j'$ is smaller than a small positive number on $\{|x| \geqslant K\}$. Then $1 + v\varphi_j'$ is an approximation of φ_j on $\{|x| \leqslant K\}$. It also follows that $|\operatorname{grad} vg_j'| >$ a positive constant on $\{|x| \geqslant K, f \leqslant 0\}$. Hence there exists a diffeomorphism τ' of $\mathbf{R}^2$ which is near to the identity in the C^0 uniform topology and which transforms $\{v\varphi_j' = -1, |x| \geqslant K\}$ to $\{f = -1, |x| \geqslant K\}$.

Put

$$\varphi = \prod_{i=1}^{m} \varphi_i, \qquad \varphi' = (1 + v\varphi_j') \prod_{i \neq j} (1 + v^2\varphi_1')$$

and

$$\varphi'' = (1+v\varphi_j')^{\alpha_j} \prod_{i \neq j} (1+v^2\varphi_i')^{\alpha_i},$$

where T_{1j} is the non-compact element if exists. Then $\varphi - \varphi'$ is small on $\{|x| \leqslant K\}$ and ℓ-flat at $T_1 \cap V$. Clearly the zero set of φ is T_1, the set of zero critical points is $T_1 \cap V$, and Milnor number of the germ at each zero point is finite [11]. Then we can find C^∞ function h_1, h_2 small on $\{|x| \leqslant K\}$ such that

$$\varphi - \varphi' = h_1\left(\frac{\partial \varphi}{\partial x_1}\right)^2 + h_2\left(\frac{\partial \varphi}{\partial x_2}\right)^2.$$

This fact in the local case is well-known [11]. For the global, it is sufficient to use a partition of unity. Moreover, increasing ℓ, we can assume that h_1, h_2 are ℓ'-flat at $T_1 \cap V$ for large given ℓ'. Hence, applying Lemma 5, and using above τ', we obtain a C^∞ diffeomorphism τ of $\mathbf{R}^2$ that is close to the identity on $\{|x| \leqslant K\}$, equals in a neighborhood of $\{f(x) \geqslant 0\} \cup (S - T_1)$ and satisfies $\varphi \circ \tau = \varphi'$ in a neighborhood of $\tau^{-1}(T_1)$ and $\varphi'^{-1}(0) = \tau^{-1}(T_1)$. Here Proof of Lemma 5 shows that τ-(the identity) is ℓ'-flat at $T_1 \cap V$. Hence it follows that

$$\varphi_j \circ \tau - (1+v\varphi_j'), \qquad \varphi_i \circ \tau - (1+v^2\varphi_j')$$

are ℓ'-flat at $T_1 \cap V$ for above j and any $i \neq j$.

Let x be a point of $T_1 \cap V$. We can assume that φ_i are analytic at x. Then τ showed in Proof of Lemma 5 is automatically analytic at x. Let $\mathcal{O}_x$ be the ring of germs of analytic functions at x. From the next remark and from the unique factorization property of $\mathcal{O}_x$, it follows that

$$(\varphi_j \circ \tau)\mathcal{O}_x = (1+v\varphi_j')\mathcal{O}_x$$
$$(\varphi_i \circ \tau)\mathcal{O}_x = (1+v^2\varphi_i')\mathcal{O}_x.$$

Trivially these equalities hold true for $x \in T_1 - V$ if we replace $\mathcal{O}_x$ by $\mathcal{F}_x$. Remark: Let h_1, h_2 be elements of $\mathcal{O}_x$ such that $h_1 h_2$ has no multiple factor. Then there exists an integer $\ell'' > 0$ such that $h_1(h_2+h_3)$ has no multiple factor for any ℓ''-flat element h_3 of $\mathcal{O}_x$. This is trivial, because an element of $\mathcal{O}_x$ has no multiple factor if and only if it is stable [11].

We have proved $(f \circ \tau + 1)\mathcal{F} = \varphi''\mathcal{F}$. This means that $f \circ \tau + 1$ is divisible by φ'' and the quotient is positive on $\tau^{-1}(T_1)$. Let d be a small positive number. Let χ be a positive polynomial on $\mathbf{R}^2$ whose difference with $((f\circ\tau+1)/\varphi''-d)/\nu^2$ at $T_1\cap V$ is ℓ-flat. Then $f\circ\tau+1-\varphi''(d+\nu^2\chi)$ is ℓ-flat at $T_1\cap V$. By 2.2, 2.3 in [4], there exist germs of C^∞ functions h_1, h_2 at each point x_0 of $T_1\cap V$ such that

$$f\circ\tau+1-\varphi''(d+\nu^2\chi) = h_1\left(\frac{\partial(f\circ\tau+1)}{\partial x_1}\right)^2 + h_2\left(\frac{\partial(f\circ\tau+1)}{\partial x_2}\right)^2 \quad \text{as germs at } x_0.$$

Increasing ℓ, we can assume that h_1, h_2 are ℓ_1-flat at x_0 for large ℓ_1. Assume $x_0=0$. Let ρ be a C^∞ function on $\mathbf{R}^2$ such that

$$0\leqslant\rho\leqslant 1, \qquad \rho(x)=\begin{cases}1 & \text{for} \quad |x|\leqslant 1\\ 0 & \text{for} \quad |x|\geqslant 2.\end{cases}$$

Put $\rho_N(x)=\rho(Nx)$. Then $f\circ\tau+1$ and

$$\varphi''(d+\nu^2\chi)\rho_N+(f\circ\tau+1)(1-\rho_N),$$

for large N, satisfy the conditions in Lemma 4. Hence they are equivalent. Here the diffeomorphism is the identity on $\{|x|\geqslant 1/N\}$. Repeating this argument for other points of $T_1\cap V$, we have a C^∞ function ρ' on $\mathbf{R}^2$ whose support is a neighborhood of $T_1\cap V$, which is equal to 1 in another one, such that $0\leqslant\rho'\leqslant 1$, and that

$$f\circ\tau+1 \quad \text{and} \quad \varphi''(d+\nu^2\chi)\rho' + (f\circ\tau+1)(1-\rho')$$

are equivalent. Compare $\varphi''(d+\nu^2\chi)\rho'+(f\circ\tau+1)(1-\rho')$ with $\varphi''(d+\nu^2\chi)$ in a neighborhood of $\tau^{-1}(T_1)$. Then we see easily that they are equivalent in a neighborhood of $\tau^{-1}(T_1)$. Hence we can reduce to the case

$$f+1=e_1+\nu\mu_1 \quad \text{in a neighborhood of } T_1$$

where e_1 is a non-zero constant and μ_1 is a polynomial satisfying

$$(e_1+\nu\mu_1)^{-1}(0)=T_1.$$

In place of ν, consider

$$\nu_1=\begin{cases}\pm\,\nu(e_1+\nu\mu_1) & \text{if } T_1 \text{ is compact}\\ \nu^2(e_1+\nu\mu_1) & \text{if it is not so.}\end{cases}$$

Then, in the same way as above we obtain a non-zero constant e_2 and a polynomial μ_2 so that we may reduce to

$$f - f(T_2) = e_2 + \nu_1\mu_2$$

in a neighborhood of T_2, and

$$(e_2+\nu_1\mu_2)^{-1}(0) = T_2 .$$

Of course, we have to modify the application of the proof of Theorem 3, because ν_1 has not $(*)$. But, since the essential ideas are the same, we omit this. Here we remark that if the order of $T_1, \ldots$ is wrong, we can not get the last equation and T_2 is only a connected component of the zero set.

Repeating this, we can assume the existence of polynomials $\nu, \lambda_1, \ldots$ on $\mathbf{R}^2$ such that

$$\nu^{-1}(0) = f^{-1}(0), \quad \nu = f \text{ in a neighborhood of } f^{-1}(0)$$
$$\lambda_i^{-1}(f(T_i)) = T_i, \lambda_i = f \text{ in a neighborhood of } T_i$$

for each i and that the zero sets in $\mathbf{C}^2$ of the complexifications of ν and $\lambda_i - f(T_i)$ do not intersect each other. By the last property, there are polynomials $\zeta_0, \zeta_1, \ldots, \eta_0, \ldots$ such that

$$\zeta_0\nu^4 + \eta_0 \prod_i (\lambda_i - f(T_i))^4 = 1$$

$$\zeta_1(\lambda_1 - f(T_1))^4 + \eta_1\nu^4 \prod_{i\neq 1} (\lambda_i - f(T_i))^4 = 1, \ldots .$$

We put

$$\begin{aligned} f'_1 &= \nu(1-\zeta_0\nu^4) + \lambda_1(1-\zeta_1(\lambda_1 - f(T_1))^4) + \ldots \\ f_2 &= \nu \prod_i \left\{(\lambda_i - f(T_i))^4 + \left(\frac{\partial\lambda_i}{\partial x_1}\right)^4 + \left(\frac{\partial\lambda_i}{\partial x_2}\right)^4\right\}(1+|x|^{2r}) \\ f_1 &= f'_1 + f_2 \end{aligned}$$

for a large integer r. Then f is transformed so that $f = f_1$ in a neighborhood of $S \cup f^{-1}(0)$, we have trivially $f_2^{-1}(0) = S \cup f^{-1}(0)$, and f_2 satisfies the conditions on g_i in Lemma 4 because of [7]. Hence the conditions of Proposition 14 are satisfied. Therefore we have the theorem.

We prove in the same way:

THEOREM 5′. — *Let f be an analytic function on $\mathbf{R}^2$. Let $\pi_i, i = 1, \ldots, k, \ldots, m$, be imbeddings of $(a_i, b_i) \times [0,\infty)$ into $\mathbf{R}^2$ for real numbers a_i, b_i such that the complement of the union of the images is bounded, that the images of π_i for $i \leqslant k$ do not intersect each other and that*

$$f \circ \pi_i(x,y) = \begin{cases} \pm\, x^{n_i} + \text{const} & \text{for} \quad i \leqslant k \\ \pm\, y + \text{const} & \text{for} \quad i > k \end{cases}$$

for some integers $n_i \geqslant 1$. Then f is equivalent to a rational function.

Here we do not know if f is equivalent to a polynomial, because the zero sets of the complexifications of similarly defined ν, $\lambda_i - f(T_i)$ may intersect.

In the rest of this section, we consider analytic functions on algebraic varieties. Let $M \subset \mathbf{R}^n$ be an affine smooth algebraic variety of dimension 2.

THEOREM 5″. — *An analytic function f on M with isolated critical points is equivalent to a polynomial, if one of the following conditions is satisfied:*

1) *f is positive, proper;*

2) *$(*)$ or the conditions in Theorem 5′, and that the boundary of M is connected.*

Proof. — If the conditions of Theorem 5′ are satisfied, f is equivalent to a function with $(*)$ because of the isolatedness of the critical points. Hence we assume $(*)$. Let $a_1 > \ldots > a_k$ be all the critical values of f. Put $T_1 = f^{-1}(a_1), \ldots$ and proceed in a similar way to Proof of Theorem 5. The only difference is the definition of $\mathfrak{p}_i$. We change it so that

$$\mathfrak{p} = \prod_{i=1}^{m} \mathfrak{p}_i^{\alpha_i},$$

that $\mathfrak{p}_i^{-1}(0)$ is a point for each $i \geqslant 2$ and that $\{\mathfrak{p}_i\}$ is irreducible in this sense. Then the rest of the proof is the same. We omit the detail.

THEOREM 5‴. — *Assume that M is homeomorphic to one of S^2, $\mathbf{R}^2$ and $P^2(\mathbf{R})$. Then any analytic function on M is equivalent to a polynomial or a rational function under the conditions in Theorem 5 or 5′ respectively.*

Proof. – If $M \cong S^2$ or $\mathbf{R}^2$, we prove this in the same way as Theorem 5. Assume $M \cong P^2(\mathbf{R}) = P^2$. Let T, V be the same ones as in the proof of Theorem 5.

There exists only one connected component of T, say T_1, whose arbitrarily small neighborhood is non-orientable. The reason is the following. Assume that there were two such components T_1, T_2. Then the inverse image of T_1 under a natural covering map $S^2 \to M$ is connected, and its complement in S^2 consists of connected components each of which is diffeomorphic to $\mathbf{R}^2$. Hence the inverse image of T_2 should be contained in a set diffeomorphic to $\mathbf{R}^2$. On the other hand the restriction of the covering map on such set must be injection. Hence we have a contradiction. Assume that there exists not such component T_1. Let $M_1, \ldots, M_k$ be the closures of the connected components of $M - T$. Then if $M_i \cap M_j \neq \varphi$ then one of M_i and M_j is diffeomorphic to $S^1 \times [0,1]$, because the function f is regular on $M - T$. Orient M_1 arbitrarily and M_i compatibly with M_1 if $\partial M_i \cap \partial M_1 \neq \varphi$. If we could continue this operation until every M_i then M should be orientable. Hence there exist a Jordan curve c in M and a homeomorphism $g : M_i \to S^1 \times [0,1]$ for some i such that any small neighborhood of c is non-orientable and that $g(M_i \cap c) = a \times [0,1]$ for some $a \in S^1$. Then the fundamental class $[c] \in H_1(M; \mathbf{Z})$ does not satisfy $2[c] = 0$, because the image of $[c]$ in $H_1(M, \bigcup_{j \neq i} M_j; \mathbf{Z}) = \mathbf{Z}$ is not 0. This contradicts $H_1(M; \mathbf{Z}) = \mathbf{Z}_2$. Let $T_2, \ldots,$ be other connected components of T ordered so that for any $i < j$, the identity mapping: $T_j \to M - T_i$ is homotopic to a constant mapping. We want to transform f so that f is a polynomial in a neighborhood of T_i.

Assume $f(T_1) = 0$. Let $\mathfrak{p}$ be the coherent sheaf of ideals $f\mathscr{F}$ on T_1 and $\mathscr{F}$ on T_1^c. Let

$$\mathfrak{p} = \prod_{i=1}^{m} \mathfrak{p}_i^{\alpha_i}$$

be the unique factorization. Then, for each i $\mathfrak{p}_i$ is locally generated by a germ of a regular function except at $T_{1i} \cap V$, where $T_{1i} = \mathfrak{p}_i^{-1}(0)$. We assume that a small neighborhood of T_{1i} is orientable for $i \leqslant m'$ and non-orientable for $m' < i \leqslant m$. For $i \leqslant m'$, we have a C^∞ function φ_i on M such that $\varphi_i \mathscr{F} = \mathfrak{p}_i$. Let ψ_i be a polynomial approximation of φ_i such that $\psi_i - \varphi_i$ is ℓ-flat at $T_{1i} \cap V$ for a large ℓ.

Fix $i > m'$. By Lemma 2 in [6] there exists C_i a smooth algebraic subset of M of dimension 1 such that $C_i \cap V = \emptyset$, $[C_i] = [T_{1i}]$ in $H_1(M; \mathbf{Z}_2)$ and that C_i is transversal to T_{1i}. Hence $C_i \cup T_{1i}$ is the boundary of a subset M_i' of M. We can assume moreover that M_i' contains $V - T_{1i}$. Let φ_i be a C^∞ function on M such that $\varphi_i^{-1}(0) = C_i \cup T_{1i}$, that $\varphi_i \mathscr{F} = \mathfrak{p}_i$ outside C_i and that φ_i is regular on $C_i - T_{1i}$, of Morse type at $C_1 \cap T_{1i}$ and positive on M_i'. Let $g_1, \ldots, g_k$ be polynomials on M, $h_1, \ldots, h_k$ be C^∞ functions such that

$$g_{j|C_i} = 0, \qquad j = 1, \ldots, k, \sum_{j=1}^{k} h_j g_j = \varphi_i .$$

Let $h_j', j = 1, \ldots, k$ be polynomial approximations of h_j such that $h_j - h_j'$ and hence $\varphi_i - \sum_{j=1}^{k} h_j' g_j$ are ℓ-flat at $T_{1i} \cap (V \cup C_i)$. Put $\psi_i = \sum_{j=1}^{k} h_j' g_j$. We remark that the closure $\overline{\psi_i^{-1}(0) - C_i}$ is algebraic (see the proof of Lemma 2 in [6]). Because of Lemma 5, there exists a C^∞ diffeomorphism τ close to the identity such that

$$\prod_{i=1}^{m} \psi_i = \prod_{i=1}^{m} \varphi_i \circ \pi$$

in a neighborhood of $\pi^{-1}(T_1)$. Here the idea of the modification of T_{1i} to the algebraic set $\pi^{-1}(T_{1i}) = \overline{\psi_i^{-1}(0) - C_i}$ is due to [10]. In the same way as Proof of Theorem 5 we see that

$$\psi_i \mathscr{F} = \varphi_i \circ \pi \mathscr{F} .$$

Consider the complexifications $\tilde{\psi}_i$ of each ψ_i and $\tilde{C}_i$ of C_i. Then the algebraic set $\overline{\tilde{\psi}_i^{-1}(0) - \tilde{C}_i}$ is defined by real polynomials $P_{i1}, \ldots, P_{ii'}$. It follows that

$$\sum_{j=1}^{i'} P_{ij} \mathscr{F} = \mathfrak{p}_i \circ \pi .$$

Hence, for any point x_0 of M, there exists a polynomial P such that $P\mathscr{F}_{x_0} = \prod_{i=1}^{m} (\mathfrak{p}_i \circ \pi)_{x_0}^{\alpha_i}$ since $\mathfrak{p}_i \circ \pi)_{x_0}$ are principal ideals. Let Q be a positive polynomial on M such that $Q(x_0) = 1$ and that Q is small outside a small neighborhood of x_0. Consider $\pm QP$, here the sign is decided so that $\pm$ QP and $f \circ \pi$ take the same sign at each point near to x_0. Take the sum f' of such polynomials for finite points x_0. Then, by the property that any connected component of T_1^c is homeomorphic to $\mathbf{R}^2$, we have

$$f'\mathscr{F} = \prod_{i=1}^{m} (p_i \circ \pi)^{\alpha_i} = p \circ \pi .$$

Approximating by a polynomial the C^∞ function $f \circ \pi / f'$ in a neighborhood of $\pi^{-1}(T_1)$, we reduce in the same way as the above proof to the case that f is equal to a polynomial in a neighborhood of T_1 and that the zero set of the polynomial is T_1.

For $T_j, j > 1$, the modification of f in a neighborhood of T_j proceeds in the same manner as Proof of Theorem 5, because T_j is contained in a set homeomorphic to $\mathbf{R}^2$. Thus the theorem is proved.

THEOREM 6. — *Assume* M *compact and irreducible. Then any analytic function on* M *is equivalent to a rational function if and only if* M *is connected, and all the elements of* $H_1(M; \mathbf{Z}_2)$ *are realizable by algebraic subsets of* M.

Proof. — « Only if » : The connectedness is trivial. Let $C_1, \ldots, C_m$ be Jordan analytic curves in M which are realizations of all the elements of $H_1(M; \mathbf{Z}_2)$. Let $f_1, \ldots, f_m$ be analytic functions on M such that for each $i f_i^{-1}(0) = C_i$, and f_{ix} is $2i$-power of a regular function germ for any x of C_i. We put $f = f_1 \ldots f_m$. Then, by the assumption there exists a diffeomorphism π of M such that $f \circ \pi$ is a polynomial. Hence $\pi^{-1}\left(\bigcup_{i=1}^{m} C_i\right)$ is algebraic. Moreover, considering the zeros of the second partial derivatives of $f \circ \pi$, we see that $\pi^{-1}\left(\bigcup_{i=2}^{m} C_j\right)$ and hence $\pi^{-1}(C_1) \cup \{\text{finite points}\}$ are algebraic. Repeating this, we have realizations by the algebraic sets $\pi^{-1}(C_i) \cup \{\text{finite points}\}, i = 1, \ldots, m$, of all the elements of $H_1(M; \mathbf{Z}_2)$.

« if » : Let $a_1 > \ldots > a_k$ be all the critical values of an analytic function f. Put $T_1 = f^{-1}(a_1), \ldots$. Using the same method as in Proof of Theorem 5‴, we reduce to the case $f = p_i$ in a neighborhood of $T_i, i = 1, \ldots, k$, where p_i is a polynomial such that $p_i^{-1}(a_i) = T_i$.

Then, in the same way as Proof of Theorem 2, we prove the equivalence of f to a rational function. We omit the detail.

In the non-compact case, we prove the followings in the same manner.

THEOREM 6′. — *Assume* M *non-compact, irreducible. Then any positive proper analytic function on* M *with finite critical values is equivalent to a rational function if and only if all elements of* $H_1(M;\mathbf{Z}_2)$ *are realizable by compact algebraic subsets of* M, *and if there is no compact connected component of* M.

Let M′ be a desingularization of the algebraic closure of M in $P^n(\mathbf{R})$.

THEOREM 6″. — *Assume that* M *is non-compact, irreducible, and connected and that the boundary of* M *is connected. Any analytic function on* M *with* $(*)$ *or under the conditions in Theorem* 5′ *is equivalent to a rational function if and only if all elements of* $H_1(M';\mathbf{Z}_2)$ *are realizable by algebraic subsets of* M′.

Remark 17. — The author does not know whether elements of $H_1(M;\mathbf{Z}_2)$ are realizable by algebraic subsets of connected M. By Lemma 2 in [6], if M is non-orientable, at least one non-zero element of $H_1(M;\mathbf{Z}_2)$ is realizable.

Problem 18. — Is any rational function on compact M equivalent to a polynomial ?

5. Equivalence to Nash functions.

The results obtained until now hold true in the problem of equivalence to Nash functions. We use the terminologies Nash manifold and Nash functions in the sense of [4]. We define the boundary of a Nash manifold in the same way as algebraic varieties. Let M be a Nash manifold of dimension $n \neq 4, 5$, f be a C^∞ function on M.

THEOREM 7. — *Assume that* f *is proper, that the number of critical points is finite, that the boundary of* M *is simply connected for* $n \geqslant 6$, *that any connected component of the boundary is not diffeomorphic to* $P^2(\mathbf{R}^2)$ *for* $n = 3$, *and that the germ at each critical point is equivalent to a Nash function germ. Then* f *is equivalent to a Nash function.*

THEOREM 8. — *Assume* $n = 3$, $(*)$ *in* § 3 *and the conditions above on the germs and on the set of critical point. Assume that the boundary of* M *is diffeomorphic to a disjoint union* $S^2 \cup \ldots \cup S^2$. *Then* f *is equivalent to a Nash function.*

THEOREM 9. — *If* $n = 2$, *and if* f *is analytic and satisfies one of the conditions in Theorem* 5 *or* 5′, *then* f *is equivalent to a Nash function.*

These were shown in [3], [4] for M compact. To prove the non-compact case, we need only the following two remarks (see [5]). Any element of $H^1(M; \mathbf{Z}_2)$ is realizable by a smooth semi-algebraic subset of M. M can be Nash imbedded in a Euclidean space so that the closure of the image is a compact topological manifold with boundary and that the boundary of the topological manifold is of class C^∞ with corner. We omit the details.

Remark 19. — In the theorems on equivalence to rational functions or to Nash functions, the properness condition of a function f can be changed to ones that $f^{-1}(a) = \varnothing$ for some $a \in \mathbf{R}$ and that $f(x)$ tends to a as x tends to infinity.

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