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A CLASS OF LOCALLY CONVEX SPACES WITHOUT \mathcal{C} -WEBS

by Manuel VALDIVIA

The linear spaces we use are defined over the field K of the real or complex numbers. Space means non-zero separated locally convex topological linear space. Given a space E we denote by E' its topological dual. The dimension of E is denoted by $\dim E$. \hat{E} is the completion of E . If E_n is a one-dimensional space, $n = 1, 2, \dots$, we put $\varphi = \bigoplus_{n=1}^{\infty} E_n$.

A space E is unordered Baire-like if given a sequence (A_n) of closed absolutely convex sets of E , with $E = \bigcup_{n=1}^{\infty} A_n$, there is a positive integer n_0 such that A_{n_0} is a neighbourhood of the origin in E .

A space E is suprabarrelled if given an increasing sequence (E_n) of subspaces of E covering E there is a positive integer n_0 such that E_{n_0} is barrelled and dense in E , [7].

A space E is Baire-like if given an increasing sequence (A_n) of closed absolutely convex sets of E covering E there is a positive integer n_0 such that A_{n_0} is a neighbourhood of the origin in E .

If A is a bounded absolutely convex subsets of a space E we denote by E_A the linear hull of A endowed with the topology derived from the gauge of A . A is completing if E_A is a Banach space.

In what follows E and F are spaces and n is any non negative integer. We denote by $E \otimes_{\pi} F$ and $E \otimes_{\epsilon} F$ the tensor product $E \otimes F$ endowed with the projective topology and the topology of the by-equicontinuous convergence respectively. $E \hat{\otimes}_{\epsilon} F$ is the completion of $E \otimes_{\epsilon} F$. We set

$$R_n(E, F) = \{z \in E \otimes F : \text{rank } z \leq n\}.$$

We write $\hat{R}_n(E, F)$ to denote the closure of $R_n(E, F)$ in $E \hat{\otimes}_{\epsilon} F$.

MAIN LEMMA. — $\hat{R}_n(E, F) = R_n(\hat{E}, \hat{F})$.

Proof. — We proceed by complete induction. The result is obvious for $n = 0$. We suppose that the property is true for a positive integer p . Let

$$(1) \quad \left\{ \sum_{j=1}^{p+1} x_j^i \otimes y_j^i : i \in I, \geq \right\}$$

be a net in $R_{p+1}(E, F)$ converging in $E \hat{\otimes}_\varepsilon F$ to an element $z \neq 0$. Since $E' \otimes F'$ separate points in $E \hat{\otimes}_\varepsilon F$, we can select $u \in E'$, $v \in F'$ such that $u \otimes v(z) \neq 0$. Obviously

$$(2) \quad \lim \left\{ \sum_{j=1}^{p+1} u(x_j^i)v(y_j^i) : i \in I, \geq \right\} = u \otimes v(z) \neq 0.$$

Since (1) is a Cauchy net in $E \otimes_\varepsilon F$ given $\delta > 0$ and an equicontinuous subset V of F' there is an $i_0 \in I$ such that

$$\sup \left\{ \left| \sum_{j=1}^{p+1} (u(x_j^h)w(y_j^h) - u(x_j^k)w(y_j^k)) \right| : w \in V \right\} < \delta, \quad \text{for } h, k \geq i_0$$

and therefore, according to (2),

$$(3) \quad \left\{ \sum_{j=1}^{p+1} u(x_j^i)y_j^i : i \in I, \geq \right\}$$

converges in \hat{F} to $y \neq 0$.

From (2) we deduce the existence of a subnet of (1), for which we use the same notation, such that at least one of the nets

$$\{u(x_j^i) : i \in I, \geq\}, \quad j = 1, 2, \dots, p+1,$$

has all its elements different from zero. Without loss of generality we suppose that $u(x_1^i) \neq 0, \forall i \in I$.

Setting $\sum_{j=1}^{p+1} u(x_j^i)y_j^i = v_1^i, i \in I$, we have that

$$\sum_{j=1}^{p+1} x_j^i \otimes y_j^i = \frac{1}{u(x_1^i)} x_1^i \otimes \left(v_1^i - \sum_{j=2}^{p+1} u(x_j^i)y_j^i \right) + \sum_{j=2}^{p+1} x_j^i \otimes y_j^i,$$

and setting

$$z_1^i = \frac{1}{u(x_1^i)} x_1^i, \quad z_j^i = x_j^i - \frac{u(x_j^i)}{u(x_1^i)} x_1^i, \quad v_j^i = y_j^i, \quad j = 2, 3, \dots, p+1,$$

we have that the net (1) can be expressed as

$$\left\{ \sum_{j=1}^{p+1} z_j^i \otimes v_j^i : i \in I, \geq \right\}.$$

Reasoning in the same way as we have done for the net (3) it follows that $\left\{ \sum_{j=1}^{p+1} v(v_j^i)z_j^i : i \in I, \geq \right\}$ converges in \hat{E} to an element $x \neq 0$. According

to (2) again we suppose that $v(v_1^i) \neq 0, \forall i \in I$, and setting $\sum_{j=1}^{p+1} v(v_j^i)z_j^i = u_1^i$ it follows that

$$\begin{aligned} \sum_{j=1}^{p+1} x_j^i \otimes y_j^i &= \sum_{j=1}^{p+1} z_j^i \otimes v_j^i \\ &= \frac{1}{v(v_1^i)} \left(u_1^i - \sum_{j=2}^{p+1} v(v_j^i)z_j^i \right) \otimes v_1^i + \sum_{j=2}^{p+1} z_j^i \otimes v_j^i \\ &= \frac{1}{v(v_1^i)} u_1^i \otimes v_1^i + \sum_{j=2}^{p+1} z_j^i \otimes \left(v_j^i - \frac{v(v_j^i)}{v(v_1^i)} v_1^i \right). \end{aligned}$$

The net $\left\{ \frac{1}{v(v_1^i)} u_1^i \otimes v_1^i : i \in I, \geq \right\}$ converges to $\frac{1}{u \otimes v(z)} x \otimes y$ in $E \hat{\otimes}_\epsilon F$ and according to the induction hypothesis

$$\left\{ \sum_{j=2}^{p+1} z_j^i \otimes \left(v_j^i - \frac{v(v_j^i)}{v(v_1^i)} v_1^i \right) : i \in I, \geq \right\}$$

converges to an element of $R_p(\hat{E}, \hat{F})$ and therefore

$$\hat{R}_{p+1}(E, F) \subset R_{p+1}(\hat{E}, \hat{F}).$$

Finally, since $R_{p+1}(E, F)$ is obviously dense in $R_{p+1}(\hat{E}, \hat{F})$ it follows that $\hat{R}_{p+1}(E, F) = R_{p+1}(\hat{E}, \hat{F})$. q.e.d.

COROLLARY 1. — *If E and F are complete then $R_n(E, F)$ is complete in $E \hat{\otimes}_\epsilon F$.*

COROLLARY 2. — *$R_n(E, F)$ is closed in $E \hat{\otimes}_\epsilon F$.*

Proof. — $R_n(E, F)$ coincides with $R_n(\hat{E}, \hat{F}) \cap (E \otimes F)$ and according to Main Lemma $R_n(\hat{E}, \hat{F})$ is complete in $E \hat{\otimes}_\epsilon F$, hence $R_n(E, F)$ is closed in $E \hat{\otimes}_\epsilon F$. q.e.d.

THEOREM 1. — $E \otimes_{\varepsilon} F$ has a \mathcal{C} -web if and only if one of the two following conditions is satisfied :

1. E has a \mathcal{C} -web and $\dim F \leq \chi_0$;
2. F has a \mathcal{C} -web and $\dim E \leq \chi_0$.

Proof. — If F is finite dimensional and if E has a \mathcal{C} -web then $E \otimes_{\varepsilon} F$ is isomorphic to $E^{\dim F}$ which has a \mathcal{C} -web. If F has countable infinite dimension and if E has a \mathcal{C} -web then $E \otimes_{\varepsilon} F$ has a \mathcal{C} -web since the topology of $E \otimes_{\varepsilon} F$ is coarser than the topology of $E \otimes_{\varepsilon} \varphi$ and this space has a topology coarser than the topology of $E \oplus E \oplus E \oplus \dots$. We reach the same conclusion changing the roles of E and F .

We suppose now that $\dim E > \chi_0$, $\dim F > \chi_0$ and that $E \otimes_{\varepsilon} F$ has a \mathcal{C} -web $(C_{n_1, n_2, \dots, n_p})$. Let $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ be families of linearly independent vectors in E and F respectively such that $\text{card } I > \chi_0$.

Since $\bigcup_{n=1}^{\infty} C_n = E \otimes F$ and $\bigcup_{n_{p+1}=1}^{\infty} C_{n_1, n_2, \dots, n_{p+1}} = C_{n_1, n_2, \dots, n_p}$ there is a sequence (n_p) of positive integers and there is a decreasing sequence (J_p) of subsets of I such that $x_i \otimes y_i \in C_{n_1, n_2, \dots, n_p}$, $i \in J_p$, $\text{card } J_p > \chi_0$, $p = 1, 2, \dots$. It follows that we can take linearly independent elements $u_p \in E$, $v_p \in F$, $p = 1, 2, \dots$, such that

$$u_p \otimes v_p \in C_{n_1, n_2, \dots, n_p}.$$

We select a sequence (λ_p) of strictly positive real numbers such that $\{\lambda_p u_p \otimes v_p : p = 1, 2, \dots\}$ is contained in a bounded completing absolutely convex subset A of $E \otimes_{\varepsilon} F$ [3, p. 75]. Since $(E \otimes F)_A$ is a Banach space and since $R_n(E, F)$ is closed in $E \otimes_{\varepsilon} F$ there is a positive integer n_0 such that $R_{n_0}(E, F) \cap (E \otimes F)_A$ has interior point in $(E \otimes F)_A$ and therefore

$$R_{2n_0}(E, F) \cap (E \otimes F)_A = (R_{n_0}(E, F) + R_{n_0}(E, F)) \cap (E \otimes F)_A$$

is a neighbourhood of the origin in $(E \otimes F)_A$ and since $R_{2n_0}(E, F)$ contains any scalar multiple of every vector $z \in R_{2n_0}(E, F)$ it follows that

$$A \subset R_{2n_0}(E, F).$$

But $\sum_{p=1}^{2n_0+1} \frac{1}{2^p} \lambda_p u_p \otimes v_p$ belongs to A and its rank is $2n_0 + 1$. This is a contradiction.

We suppose now that $E \otimes_{\varepsilon} F$ has a \mathcal{C} -web and $\dim F \leq \chi_0$. We

take $y_0 \in F, y_0 \neq 0$. Let T be the mapping from E into $E \otimes_{\varepsilon} F$ such that $T(x) = x \otimes y_0, x \in E$. Then $T(E)$ is isomorphic to E (cf. [3], § 44, 1.(4)). Since $T(E)$ is contained in $R_1(E, F)$ it is easy to prove that $T(E)$ is closed in $E \otimes_{\varepsilon} F$ and thus has a \mathcal{C} -web. Analogously, if $E \otimes_{\varepsilon} F$ has a \mathcal{C} -web and $\dim E \leq \chi_0, F$ has a \mathcal{C} -web. q.e.d.

THEOREM 2. — $E \otimes_{\pi} F$ has a \mathcal{C} -web if and only if one of the two following conditions is satisfied :

1. E has a \mathcal{C} -web and $\dim F \leq \chi_0$;
2. F has a \mathcal{C} -web and $\dim E \leq \chi_0$.

Proof. — On $E \otimes F$ the π -topology is finer than the ε -topology. Hence, if $E \otimes_{\pi} F$ has a \mathcal{C} -web then $E \otimes_{\varepsilon} F$ has a \mathcal{C} -web. Thus we apply the former result to obtain that 1 or 2 is satisfied.

If F is finite dimensional and if E has a \mathcal{C} -web then $E \otimes_{\pi} F = E \otimes_{\varepsilon} F$ has a \mathcal{C} -web. If F has countable infinite dimension and if E has a \mathcal{C} -web then $E \otimes_{\pi} F$ has a \mathcal{C} -web since the topology of $E \otimes_{\pi} F$ is coarser than the topology of $E \otimes_{\pi} \varphi = E \otimes_{\varepsilon} \varphi$. We reach the same conclusion changing the roles of E and F . q.e.d.

THEOREM 3. — If E and F are Suslin spaces then $E \otimes_{\pi} F$ is a Suslin space.

Proof. — Let $\psi : E \times F \rightarrow E \otimes_{\pi} F$ be the canonical bilinear mapping which is continuous and thus $\psi(E, F) = R_1(E, F)$ is a Suslin topological subspace of $E \otimes_{\pi} F$. On the other hand the mapping

$$\theta : R_1(E, F) \times R_1(E, F) \times \dots \times R_1(E, F) \rightarrow E \otimes_{\pi} F$$

so that

$$\theta(x_1 \otimes y_1, x_2 \otimes y_2, \dots, x_n \otimes y_n) = \sum_{j=1}^n x_j \otimes y_j$$

is π -continuous and therefore

$$\theta(R_1(E, F) \times R_1(E, F) \times \dots \times R_1(E, F)) = R_n(E, F)$$

is a Suslin topological subspace of $E \otimes_{\pi} F$. Finally, $E \otimes_{\pi} F$ is a Suslin space since coincides with $\bigcup_{n=1}^{\infty} R_n(E, F)$, (cf. [2], § 6, N° 2, Prop. 8).

q.e.d.

COROLLARY 1.3. — *If E and F are Suslin spaces then $E \otimes_{\epsilon} F$ is a Suslin space.*

Note 1. — If we consider two Suslin spaces E and F of dimension larger than χ_0 we can apply the former results to obtain that $E \otimes_{\pi} F$ is a Suslin space without a \mathcal{C} -web. Thus the closed graph theory of De Wilde [4] does not contain the closed graph theorem of L. Schwartz, [7] and [8].

Bourbaki [1, p. 43] proves that if E and F are metrizable and F is barrelled then every separately equicontinuous set of bilinear forms on $E \times F$ is equicontinuous. Since every barrelled metrizable space is Baire-like [1], Theorem 4 generalizes this result.

THEOREM 4. — *If E is metrizable and F is a Baire-like space then every separately equicontinuous set \mathcal{B} of bilinear forms on $E \times F$ is equicontinuous.*

Proof. — Let (U_n) be a decreasing basis of closed absolutely convex neighbourhoods of the origin in E . We set

$$V_n = \{y \in F : |B(x,y)| \leq 1, x \in U_n, B \in \mathcal{B}\}, \quad n = 1, 2, \dots,$$

which is a closed absolutely convex subset of F and the sequence (V_n) is increasing. On the other hand if $z \in F$ there is a positive integer p such that $|B(x,z)| \leq 1, x \in U_p, B \in \mathcal{B}$, according to the separate equicontinuity of \mathcal{B} . Thus $z \in V_p$ and since F is Baire-like there is a positive integer q such that V_q is a neighbourhood of the origin in F . Therefore $|B(x,y)| \leq 1, (x,y) \in U_q \times V_q, B \in \mathcal{B}$. q.e.d.

COROLLARY 1.4. — *If E is metrizable and if F is a barrelled space whose completion is Baire, then every set separately equicontinuous of bilinear forms on $E \times F$ is equicontinuous.*

Proof. — It is an immediate consequence of the former theorem and from the fact that every barrelled space whose completion is Baire is a Baire-like, [6]. q.e.d.

Note 2. — It is immediate that our Theorem 4 can be easily generalized to bilinear mapping with range in a third locally convex space. Then this theorem contains another result due to Bourbaki [3, Ex. 1, Chap. III, § 4, p. 44] for locally convex spaces.

PROPOSITION 1. — *If E is a barrelled metrizable space and if F is a Baire-like space then $E \otimes_{\pi} F$ is Baire-like.*

Proof. — Let (W_n) be an increasing sequence of closed absolutely convex subsets of $E \otimes_{\pi} F$ covering $E \otimes F$ and let (U_n) be a decreasing basis of closed absolutely convex neighbourhoods of the origin in E . We set

$$V_n = \{y \in F : x \otimes y \in W_n, x \in U_n\}, \quad n = 1, 2, \dots,$$

which is a closed absolutely convex subsets of F and (V_n) is an increasing sequence.

$$Z_n = \{x \in E : x \otimes z \in W_n\}, \quad n = 1, 2, \dots,$$

which is a closed absolutely convex subset of E and (Z_n) is an increasing sequence covering E . Then there is a positive integer n_0 such that Z_{n_0} is a neighbourhood of the origin in E . We can find a positive integer $p \geq n_0$ such that $U_p \subset Z_{n_0}$ and then $z \in V_p$ and so (V_n) covers F . Since F is a Baire-like there is a positive integer q such that V_q is a neighbourhood of the origin in E and thus W_q is a neighbourhood of the origin in $E \otimes_{\pi} F$.
q.e.d.

PROPOSITION 2. — *If E and F are unordered Baire-like spaces and if E is metrizable then $E \otimes_{\pi} F$ is unordered Baire-like.*

Proof. — Let (W_n) be a sequence of closed absolutely convex subsets of $E \otimes_{\pi} F$ covering $E \otimes F$ and let (U_n) be a decreasing basis of closed absolutely convex neighbourhood of the origin in E . We set

$$V_{n,m} = \{y \in F : x \otimes y \in W_n, \forall x \in U_m\}, \quad , m = 1, 2, \dots$$

Obviously $V_{n,m}$ is a closed absolutely convex subset of F . We shall see that $\bigcup_{n,m=1}^{\infty} V_{n,m} = F$.

Indeed, let $z \in F$ and set

$$Z_n = \{x \in E : x \otimes z \in W_n\}, \quad n = 1, 2, \dots,$$

which is a closed absolutely convex subset of E such that $\bigcup_{n=1}^{\infty} Z_n = E$. There is a positive integer p such that Z_p is a neighbourhood of the origin in E . We find a positive integer q such that $U_q \subset Z_p$. Then

$z \in V_{p,q}$. We can find positive integers r, s such that $V_{r,s}$ is a neighbourhood of the origin in F . If $\psi: E \times F \rightarrow E \otimes F$ is the canonical bilinear mapping then $\psi(U_s \times V_{r,s}) \subset W_r$ and thus W_r is a neighbourhood of the origin in $E \otimes_\pi F$. q.e.d.

PROPOSITION 3. — *If E is a metrizable suprabarrelled space and if F is an unordered Baire-like space, then $E \otimes_\pi F$ is suprabarrelled.*

Proof. — Let (G_n) be an increasing sequence of subspaces of $E \otimes_\pi F$ covering $E \otimes F$ and let us suppose G_n is not barrelled, $n = 1, 2, \dots$. We can find in each G_n a barrel T_n which is not a neighbourhood of the origin in G_n . Let W_n be the closure of T_n in $E \otimes_\pi F$, $n = 1, 2, \dots$. We use $V_{n,m}$ and Z_n with the same meaning as before. We set

$$S_n = \{x \in E : x \otimes z \in G_n\}, \quad n = 1, 2, \dots$$

(S_n) is an increasing sequence of subspace of E covering E and therefore there is a positive integer p such that S_p is barrelled and dense in E . If $x \in S_p$ there is a real number $h > 0$ such that $h(x \otimes z) \in T_p \subset W_p$ and thus $hz \in Z_p$, i.e., Z_p is absorbing and thus a barrel in E . It follows that $Z_p \cap S_p$ is a barrel in S_p and therefore a neighbourhood of 0. But S_p is dense in E , so $Z_p = \overline{Z_p \cap S_p}$ is a neighbourhood of the origin in E . Reasoning as we did in the proposition above, we obtain a positive number r such that W_r is a neighbourhood of the origin in $E \otimes_\pi F$, which is a contradiction. According to Proposition 1 there is a positive integer n_0 such that G_n is dense in $E \otimes_\pi F$, $n \geq n_0$. q.e.d.

PROPOSITION 4. — *$E \otimes_\pi F$ is a Baire space if and only if one of the following two conditions is satisfied :*

1. $\dim E < \infty$ and $F^{\dim E}$ is a Baire Space;
2. $\dim F < \infty$ and $E^{\dim F}$ is a Baire space.

Proof. — If the dimension of E is finite then $E \otimes_\pi F$ is isomorphic to $F^{\dim E}$. Analogously if F is finite dimensional then $E \otimes_\pi F$ is isomorphic to $E^{\dim F}$. Suppose that E and F are infinite dimensional and $E \otimes_\pi F$ is a Baire space. We can find a positive integer n_0 such that $R_{n_0}(E, F)$ is of second category in $E \otimes_\pi F$. According to Corollary 2.1, $R_{n_0}(E, F)$ is closed in $E \otimes_\varepsilon F$ and thus it is closed in $E \otimes_\pi F$ hence $R_{2n_0}(E, F)$ is a neighbourhood of the origin in $E \otimes_\pi F$ which is a contradiction since $R_{2n_0}(E, F)$ is not absorbing in $E \otimes F$.

The following Theorem is an immediate consequence of the preceding results, having in mind that every infinite dimensional Fréchet spaces has dimension larger or equal than 2^{\aleph_0} , [6].

THEOREM 5. — *If E and F are infinite dimensional Fréchet spaces the following is true for $E \otimes_{\pi} F$:*

1. *is unordered Baire-like;*
 2. *is not a Baire space;*
 3. *has not a \mathcal{C} -web.*
- If E and F are separable then*
4. *$E \otimes_{\pi} F$ is a Suslin space.*

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