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# ON THE WEAK $L^1$ SPACE AND SINGULAR MEASURES

by Robert KAUFMAN

## Introduction.

The class R of finite, complex measures $\mu$ on $(-\infty, \infty)$ such that $\hat{\mu}(\infty) = 0$, has been intensively investigated (since 1916). For this class $o(1)$ is trivial and for absolutely continuous measures, we have the Riemann-Lebesgue Lemma. We investigate the corresponding $o(1)$ condition for the partial-sum operators

$$S_T(x, \mu) \equiv \int D_T(x - t)\, \mu(dt),$$

$$D_T(t) \equiv (\pi t)^{-1} \sin T t, T > 0.$$

The $o(1)$ condition for $S_T$ depends on the weak $L^1$ norm, defined by

$$\|u\|_1^* \equiv \sup Y\, m\{|u| > Y\};$$

$$\|S_T(\mu)\|_1^* \leqslant C \|\mu\|, \quad 0 < T < +\infty.$$

The weak estimate is an easy consequence of Kolmogorov's estimate for the Hilbert transform [2, Chapter II]. Elementary approximations show that when $\mu = f(x)\,dx$, then $\lim \|S_T(\mu) - f\|_1^* = 0$. When $\mu$ is singular and $\lim \|S_T(\mu) - g\|_1^* = 0$ for a certain measurable $g$, two conclusions can be obtained without great difficulty (see below):

a) $\|S_k(\mu) - S_{k+1}(\mu)\|_1^* \longrightarrow 0$ whence $\hat{\mu}(\infty) = 0$;

b) $S_T(\mu) \longrightarrow 0$ *in measure* as $T \longrightarrow +\infty$

whence $g = 0$ a.e. This leads us to define:

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$W_0$ is the class of measures $\mu$ for which $\|S_T(\mu)\|_1^* \longrightarrow 0$ as $T \longrightarrow +\infty$.

We present an elementary structural property of $W_0$, and then show by example that

(A) There exist $M_0$-sets F carrying no measure $\mu \neq 0$ in $W_0$.

The sets F are defined by a purely metrical property, and they need not be especially small. Their construction is based on an idea from the theory of divergent Fourier series [3I, Chapter VIII].

(B) The set $F_\theta$ of all sums $\sum_0^\infty \pm \theta^m (0 < \theta < 1/2)$ carries a measure $\lambda \neq 0$ in $W_0$, provided $F_\theta$ is an $M_0$-set.

To elucidate example (B) and the next one we recall that $F_\theta$ fails to be an $M_0$-set (or even an M-set) unless $\mu_\theta \in R$, where $\mu_\theta$ is the Bernoulli convolution carried by $F_\theta$ and that $\mu_\theta \in R$ except for certain algebraic numbers $\theta$ [3II, p. 147-156]. Therefore the next example is somewhat unexpected.

(C) When $0 < \theta < 1/2$, then $\mu_\theta \notin W_0$, in fact

$$\|S_T(\mu_\theta)\|_1^* \geqslant c(\theta) > 0$$

for large $T > 0$. We observe in passing that $\mu$ is not known to be singular for $1/2 < \theta < 1$ except when $\mu_\theta \notin R$, e.g., for $\theta^{-1} = (1 + \sqrt{5})/2$.

## 1.

From the weak estimate for $S_T$ it is clear that $W_0$ is norm-closed in the space of all measures. We shall prove that when $\mu \in W_0$ and $\psi \in C^1 \cap L^\infty$, then $\psi\mu \in W_0$; consequently the same is true if only $\psi \in L^1(\mu)$. We need two lemmas; the first was already used implicitly.

LEMMA 1. – *Let* $\mu$ *be a measure such that* $S_k(\mu) - S_{k+1}(\mu) \longrightarrow 0$ *in measure (over finite intervals). Then* $\hat{\mu}(\infty) = 0$, i.e., $\mu \in R$.

*Proof.* – $|D_k(t) - D_{k+1}(t)| \leqslant \min(1, |t|^{-1}) \equiv K(t)$, say, and $K \in L^2(-\infty, \infty)$. Thus the functions $|S_k(\mu) - S_{k+1}(\mu)|$ have a common majorant $\int K(x-t)\,|\mu|(dt)$ in $L^2$. The hypothesis on

$S_k - S_{k+1}$ then yields $\| S_k - S_{k+1} \|_2 \longrightarrow 0$. This means that $\int_k^{k+1} (|\hat{\mu}(t)|^2 + |\hat{\mu}(-t)|^2)\, dt \longrightarrow 0$ so $\hat{\mu}(\infty) = 0$, because $\hat{\mu}$ is uniformly continuous.

LEMMA 2. – *Let* $\mu \in R$ *and* $\psi \in C^1 \cap L^\infty$. *Then as* $T \longrightarrow +\infty$ $\| S_T(x, \psi \cdot \mu) - \psi(x) S_T(x, \mu) \|_1^* \longrightarrow 0$.

*Proof.* – Since $\mu$ can be approximated in norm by measures $\mu_n \in R$, each of compact support, we can suppose that $\mu$ itself has compact support, say $|t| \leqslant a$. Now $S_T(\psi \cdot \mu) - \psi S_T(\mu)$ converges to $0$ uniformly on $[-a-1, a+1]$, being equal to

$$\pi^{-1} \int \sin T(t-x) \cdot \varphi(x, t)\, \mu(dt),$$

with $\varphi(x, t) = (t-x)^{-1} [\psi(t) - \psi(x)]$; $\varphi(x, t)$ is jointly continuous. This is sufficient to obtain the uniform convergence claimed.

For $|x| > a + 1$ we write

$$x S_T(x, \mu) = \pi^{-1} \int \sin T(t-x) \cdot \sigma(x, t)\, \mu(dt)$$

with $\sigma(x, t) = x(t-x)^{-1}$; now $|\sigma| \leqslant a + 1$ and

$$\left| \frac{\partial}{\partial t} \sigma(x, t) \right| \leqslant a + 1,$$

for $|t| \leqslant a$. Therefore $x S_T(\mu, x) \longrightarrow 0$ as $T \longrightarrow +0$, uniformly for $|x| \geqslant a + 1$. The same applies to $x S_T(x, \psi \cdot \mu)$, because $\psi \cdot \mu \in R$, and these inequalities show that $\psi S_T(\mu) - S_T(\psi \cdot \mu) \longrightarrow 0$.

## 2. Examples.

I. Let F be a compact set in $(-\infty, \infty)$, $0 < \alpha < 1$, $(\epsilon_j)$ a sequence decreasing to $0$; for each $j$, let $F = \cup F_k^j$, where

$$\operatorname{diam}(F_k^j) \leqslant \epsilon_j,\ d(F_k^j, F_\ell^j) \geqslant \epsilon_j^\alpha,\ k \neq \ell.$$

Then F carries no probability measure $\mu$ in $W_0$ (and hence no signed measure $\mu \neq 0$ in $W_0$).

We define the following property of a number $\beta$ in $[0, 1)$, relative to $\mu$ and the sequence of partitions $F = \cup F_k^j$:

(**) The total $\mu$-measure of the sets $F_k^j$, such that $\mu(F_k^j) > \epsilon_j^\beta$, tends to $0$, as $j \longrightarrow +\infty$.

Plainly $\beta = 0$ has property (**), because $\mu$, being an element of R, can have no discontinuities. We shall prove that if $\beta$ has property (**), and $0 \leqslant \beta < \alpha$, then $\gamma = \beta + (1-\alpha)/2$ has property (**). This leads to a contradiction as soon as $\gamma > \alpha$, since the number of sets $F_k^j \neq \emptyset$ is $0(\epsilon_j^{-\alpha})$.

Assuming that $\beta$ has property (**), we form $\lambda = \lambda_j$, by omitting from $F_k$ the intervals $F_k^j$ of $\mu$-measure $> \epsilon_j^\beta$. By Kolmogorov's estimate, $\|S_T(\lambda_j)\|_1^* \longrightarrow 0$, as $j \longrightarrow +\infty$ and $T \longrightarrow +\infty$, independently. Let now $\int^*$ denote an integral over the domain $|x - t| > \epsilon_j^\alpha/2$. Then

$$\int^* |x-t|^{-1}\,\lambda_j(dt) = 0(\epsilon_j^{-\alpha}),\ \text{ if }\ \beta = 0,$$

$$\int^* |x-t|^{-1}\,\lambda_j(dt) = 0(\epsilon_j^{\beta-\alpha})\,(\log \epsilon_j),\ 0 < \beta < \alpha.$$

The first of these is obvious; the second is obtained by packing the subsets $F_k^j$ as close to $x$ as is consistent with the condition $d(F_k, F_\ell) \geqslant \epsilon_j^\alpha$.

For each $k$ such that $\lambda_j(F_k^j) > \epsilon_j^\gamma$, we let $\xi_k$ belong to $F_k^j$ and consider the set defined by

$$(S_k^j):\ \frac{1}{2}\,\lambda(F_k^j)\,\epsilon_j^\sigma < |x - \xi_k| < \lambda(F_k^j)\,\epsilon_j^\sigma\,,$$

$$|\sin \epsilon_j^{-\tau}(x-\xi_k)| > \frac{1}{2}$$

where $\sigma = -\beta + 3\alpha/4 + 1/4$, $\tau = (1+\gamma+\sigma)/2$.

The number $\lambda(F_k^j)\,\epsilon_j^\sigma$ lies between $\epsilon_j^{\beta+\sigma}$ and $\epsilon_j^{\gamma+\sigma}$; we note that $\beta + \sigma > \alpha$, and $\gamma + \sigma = 3/4 + \alpha/4 < 1$. Moreover $\epsilon_j^{-\tau}\epsilon_j = o(1)$, while $\epsilon_j^{-\tau}\lambda(F_k^j)\,\epsilon_j^\sigma \longrightarrow +\infty$.

For each $k$ in question, the Lebesgue measure of $S_k^j$ is asymptotically $c\lambda(F_k^j)\,\epsilon_j^\sigma$, and the different sets are disjoint, because $\lambda(F_k^j)\,\epsilon_j^\sigma = o(\epsilon_j^\alpha)$. We shall prove that $|S_T(\lambda_j)| > c'\epsilon_j^{-\sigma}$ for a certain $c' > 0$, with $T = \epsilon_j^{-\tau} \longrightarrow +\infty$. This will prove that the total $\mu$-measure of the subsets $F_k^j$, such that $\epsilon_j^\gamma < \epsilon_j \leqslant \epsilon_j^\beta$, is $o(1)$.

When $x \in S_k^j$,

$$|S_T(x) - \int_{F_k^j} D_T(x-t)\,\lambda(dt)| < \int^* |x-t|^{-1}\,\lambda(dt),$$

and the error term on the right is $o(\epsilon_j^{-\sigma})$, because $\sigma > \alpha - \beta$.

When $t \in F_k^j$, $t - \xi_k = o(x - \xi_k)$ because $\gamma + \sigma < 1$, and $\sin T(t - x) = \sin T(\xi_k - x) + o(1)$ because $\tau < 1$. This easily leads to the lower bound on $|S_T(x)|$.

Our construction is adapted from Kolmogorov's divergent Fourier series [31, Chapter VIII].

To complete our example, we must present a set F that is also an $M_0$-set. This is known for various $M_0$-sets, but seems to occur explicitly in [1]: there exists a closed set $E \subseteq [0,1]$ and a sequence of integers $N_k \longrightarrow +\infty$ such that

(1) $|N_k x| < N_k^{-1}$ (modulo 1) for $x \in E$, $k \geqslant 1$,

(2) The mapping $y = e^x$ transforms E onto an $M_0$-set.

Then $y(E)$ is covered by intervals of length $\leqslant 2eN_k^{-2}$, whose distances are at least $(N_k^{-1} - 2N_k^{-2})$.

In the remaining examples it is occasionally convenient to write $S_T(y)$ in place of $S_T(y, \mu)$, when $\mu = \mu_\theta$.

II. We present example (C) first, because (B) is based on an improvement in one of the inequalities used in (C). For each $n = 0, 1, 2, 3, \ldots$, $F_\theta$ is a union of $2^{n+1}$ sets $E_k$ of diameter $2\theta^{n+1}(1 - \theta)^{-1}$, and mutual distances at least

$$2\theta^{n+1}(1 - 2\theta)\,(1 - \theta)^{-1} \equiv c_1\,\theta^{n+1}\,;\ \mu(E_k) = 2^{-n-1}.$$

The lower bound on the mutual distances gives a Hölder condition on $\mu : \mu(B) \leqslant c_2(\operatorname{diam} B)^\alpha$, where $\alpha = -\log 2/\log\theta < 1$. If $\xi_k$ is the center of $E_k$, we have an identity

$$\int_{E_k} f(t)\,\mu(dt) = 2^{-n-1}\int f(\xi_k + \theta^{n+1}t)\,\mu(dt).$$

For each set $E_k$, we define the set $E_k^\sim$ by the inequality $d(x, E_k) < c_1\theta^{n+1}/3$, so the sets $E_k^\sim$ have distances at least $2c_1\theta^{n+1}/3$. If $x \in E_k^\sim$, then

$$|S_T(x, \mu) - \int_{E_k} D_T(x - t)\,\mu(dt)| < \int_{R - E_k} |x - t|^{-1}\,\mu(dt),$$

and in the last integral, $|x - t| \geqslant 2c_1\theta^{n+1}/3$. Hence, by the Hölder condition, the integral is $\leqslant c_3(\theta^n)^{\alpha-1} = c_3 2^{-n}\theta^{-n}$. The principal term can be evaluated by the identity above, and simplified to the form $2^{-n}\theta^{-n-1}S_{T\theta^{n+1}}(\theta^{-n-1}x - \theta^{-n-1}\xi_k)$.

We observe that

$$\lim \int S_T(x, \mu) f(x)\, dx = \int f(x)\, \mu(dx),$$

for suitable test functions $f$; for example, this is true if $f$ and $\hat{f}$ are integrable. Since $\mu$ is singular, we can find a test function $f$, such that $\|f\|_1 < 1$ and $|\int f(x)\,\mu(dx)| > 2c_3 + 2c_1^{-1}$. Hence $\max |D_T(\mu)| > 2c_3 + 2c_1^{-1}$ for large $T$, say for $T > T_0$.

Let $T > \theta^{-1} T_0$, and let $n \geqslant 0$ be chosen so that $T^* = \theta^{n+1} T$ satisfies the inequalities $T_0 \leqslant T^* \leqslant \theta^{-1} T_0$. Suppose that

$$|D_{T^*}(\theta^{-n-1} x - \theta^{-n-1} \xi_k)| > c_3 + c_1^{-1}.$$

Then $d(\theta^{-n-1} x - \theta^{-n-1} \xi_k, F_\theta) < c_1/3$, since $\pi > 3$, or $d(x, \xi_k + \theta^{n+1} F_\theta) < c_1 \theta^{n+1}/3$, so $x \in E_k^{\sim}$. Hence

$$|D_T(x, \mu)| > c_3 \cdot 2^{-n-1} \theta^{-n-1} - c_3 2^{-n} \theta^{-n} = c_4 2^{-n} \theta^{-n}.$$

But it is easy to see that the set of $x$'s in question has measure at least $c_5 2^n \theta^n$, because $T_0 \leqslant T^* \leqslant \theta^{-1} T_0$, and the functions $D_{T^*}$ have derivatives bounded by $\theta^{-2} T_0^2$. Hence $\|D_T(\mu)\|_1^* \geqslant c_4 c_5$.

III. The example (B) requires a complicated construction, but relies in essence on small improvements on estimates already used. To estimate $S_T(\mu, x)$ we divide the range of integration into the subsets $\{|x - t| < T^{-1}\}$ and $\{|x - t| > T^{-1}\}$. The second yields an integral $O(T^{1-\alpha})$, by the Hölder condition, and the first yields $T \cdot O(T^{-\alpha}) = O(T^{1-\alpha})$ for the same reason (and the inequality $|D_T| < T$).

We give another estimate on $S_T(x, \mu)$ for large $T$, supposing that $\mu \in R$.

Lemma 3. – *To each* $\epsilon > 0$ *there is a* $T_0$ *such that*

$$|S_T(x, \mu)| < \epsilon d(x, F_\theta)^{-1}$$

*whenever* $T \geqslant T_0$ *and* $d \equiv d(x, F_\theta) \geqslant \epsilon$.

*Proof.* – Let $\delta = d(x, F)$ and observe that

$$\delta S_T(x, \mu) = \pi^{-1} \int \sin T(x - t) \cdot \delta \cdot (x - t)^{-1} \mu(dt).$$

The function $g(t) = \delta \cdot (x - t)^{-1}$ is bounded by 1 on F, and

$|g(t_1) - g(t_2)| \leqslant \delta^{-1} |t_1 - t_2|$ for numbers $t_1, t_2$ in $F_\theta$. Hence the conclusion follows from our assumption that $\mu \in R$ and the Tietze extension theorem.

The inequality of the Lemma can be written in a more useful way. When $t \in F_\theta$, then $|x - t| \leqslant d + 2 \leqslant d(1 + 2\epsilon^{-1})$. Hence $d(x, F_\theta)^{-1} \leqslant (1 + 2\epsilon^{-1}) \int |x - t|^{-1} \mu(dt)$. Suppose now that $x \notin E_k^{\sim}$ so that $d(\theta^{-n-1}x - \theta^{-n-1}\xi_k, F_0) \geqslant c_1 \theta^{n+1}/3$. Using the identity for integrals over $E_k$, we find the following estimate:

If $x \notin E_k^{\sim}$ and $T\theta^{n+1} > T_{00}$, then

$$\left| \int_{E_k} D_T(x - t)\, \mu(dt) \right| < \epsilon \int_{E_k} |x - t|^{-1} \mu(dt).$$

Consequently, when $x \in E_\ell^{\sim}$ and $T\theta^{n+1}$ is sufficiently large (depending on $\epsilon > 0$)

$$|S_T(x, \mu) - 2^{-n-1}\theta^{-n-1} S_{T\theta^{n+1}}(\theta^{-n-1}x - \theta^{-n-1}\xi_\ell)| < \epsilon\, \theta^{n(\alpha-1)}.$$

LEMMA 4. – *To each* $\epsilon > 0$ *there is a* $\delta > 0$ *so that, when* $\theta^{-1} < Y < \delta T^{1-\alpha}$ *then* $Y m\{|S_T(x, \mu)| > Y\} < \epsilon$.

*Proof.* – We choose $n \geqslant 0$ so that $1 < \theta^{n+1} Y^{1/1-\alpha} < \theta^{-1}$; this leads to the inequalities $\theta^{n(\alpha-1)} > Y$, and $T\theta^{n+1} > \delta^{-1}$. For fixed $\ell$, we must estimate the Lebesgue measure of the set defined by

$$|S_{T\theta^{n+1}}(\mu, \theta^{-n-1}x - \theta^{-n-1}\xi_\ell)| > \frac{1}{2} \cdot 2^{n+1}\theta^{n+1} Y.$$

The right hand side exceeds $\frac{1}{2}\theta^{-1}$; when $T\theta^{n+1}$ is large, the measure of the set is at most $\epsilon\theta^{n+1}$; the total for all $\ell$ is at most $\epsilon 2^{n+1}\theta^{n+1} < \epsilon Y^{-1}$. Hence $Y m\{|S_T(x, \mu)| > Y\} < \epsilon$.

In view of the inequality $|S_T(\mu, x)| = O(T^{1-\alpha})$, the conclusion of the last lemma holds when $Y > \delta^{-1} T^{1-\alpha}$, $T > 1$, for a certain $\delta > 0$.

In preparation for the next lemma, we recall the identity $(n = 1, 2, 3, \ldots)$

$$\int f(t)\, \mu(dt) \equiv 2^{-n} \sum_{k=1}^{2^n} \int f(\xi_k + \theta^n t)\, \mu(dt).$$

We define $\int f(t)\, \sigma_n(dt) \equiv 2^{-n} \sum_k \int f(\xi_k + \theta^{n+k} t)\, \mu(dt)$. Then

$\sigma_n = g_n \cdot \mu$, where $g_n \geqslant 0$, $g_n$ is continuous on $F_\theta$ and takes the values 0 and $2^k (1 \leqslant k \leqslant 2^n)$. Using the formula for $\sigma_n$ we get an identity

$$S_T(x, \sigma_n) = 2^{-n}\theta^{-n} \sum_k \theta^{-k} S_{T\theta^{n+k}}(\theta^{-n-k}x - \theta^{-n-k}\xi_k).$$

LEMMA 5. — *To each* $\epsilon > 0$, *there is an* $N > 1$ *such that* $\limsup_{T \to +\infty} \| S_T(\sigma_n) \|_1^* < \epsilon$, *if* $n \geqslant N$.

*Proof.* — In calculating $\limsup_{T \to +\infty} \| S_T(\sigma_n) \|_1^*$ we can omit $x$'s outside $(-3, 3)$, because $\sigma_n \in R$. In an obvious notation we write $\sigma_n = \sum_k \sigma_{n,k}$, and observe that, for $T > T_{n,\epsilon}$

$$|S_T(\sigma_n)| < \max_k |S_T(\sigma_{n,k})| + \epsilon/12.$$

When $Y > \epsilon/6$ (the others are trivial, since we suppose that $|x| < 6$),

$$m\{|S_T(\sigma_n)| > 2Y\} \leqslant \sum_k m\{|S_T(\sigma_{n,k})| > Y\}$$
$$= \sum_k \theta^{n+k} m\{|S_{T\theta^{n+k}}(x, \mu)| > 2^n\theta^{n+k}Y\}.$$

Each summand is $O(2^{-n}Y^{-1})$ by Kolmogorov's inequality; if $T\theta^{n+k} > 1$, then the $k$-th term exceeds $\epsilon 2^{-n} Y$ only if

$$\delta(T\theta^{n+k})^{1-\alpha} < Y < \delta^{-1}(T\theta^{n+k})^{1-\alpha},$$

by Lemma 4 and the remark after it, and this inequality occurs for at most $2(1-\alpha)^{-1} \cdot \log\delta/\log\theta$ indices $k = 1, \ldots, 2^n$. (We assume that $Y > \theta^{-1}$, since $S_T(\sigma_n) \longrightarrow 0$ almost everywhere as $T \longrightarrow +\infty$.) This proves our lemma.

A further property of $\sigma_n$, obtained simply by increasing $n$, is the inequality $|\sigma_n(I) - \mu(I)| < \epsilon$ for all intervals $I$.

The next lemma establishes a property of the functional $\| \; \|_1^*$ to simplify the remaining calculations.

LEMMA 6. — *Let* $a_i = \| f_i \|_1^*$ $1 \leqslant i \leqslant N$. *Then*

$$\| \Sigma f_i \|_1^* \leqslant (\Sigma a_i^{1/2})^2.$$

*Proof.* — Let $0 \leqslant t_i \leqslant 1$, and $\Sigma t_i = 1$. Then

$$m\{|\Sigma f_i| \geqslant Y\} \leqslant \Sigma m\{|f_i| \geqslant t_i Y\} \leqslant \Sigma t_i^{-1} Y^{-1} a_i.$$

The minimum of the sum is $Y^{-1}(\Sigma\, a_i^{1/2})^2$. With a little more effort, we can obtain the bound $c(1-p)^{-1}(\Sigma\, a_i^p)^{1/p}$, $0<p<1$.

We are now in a position to construct the measure $\lambda$. We shall find probability measures $\lambda_k = f_k\,\mu$, with $f_k \geqslant 0$, $\int f_k d\mu = 1$, such that $\|S_T(\lambda_k)\|_1^* < k^{-1}$ for $T > T_k > T_{k-1} \ldots$ and $|\hat{\lambda}_k(u)| < k^{-2}$ for $u > T_k$. Lemma 5 provides $\lambda_1$; let us suppose that $\lambda_k$ and $T_k$ are known. We find $\sigma_k$ so that $|\sigma_k(I) - \lambda_k(I)| < k^{-1}(1+T_k)^{-2}$ and $\|S_T(\sigma_k)\|_1^* < k^{-4}/25$, and $|\hat{\sigma}_k(u)| < k^{-1}$, for $u > T^0_{k+1} > T_k$. (The construction of $f_{k+1}\mu$ from $f_k\mu$ follows Lemma 5). We now set $\lambda_{k+1} = (1-k^{-1/2})\,\lambda_k + k^{-1/2}\sigma_k$; by Lemma 6, we have for $T > T^0_{k+1}$

$$\|S_T(\lambda_{k+1})\|_1^{*1/2} \leqslant (1-k^{-1/2})^{1/2}k^{-1/2} + k^{-2}/5 .$$

When $k=1$, the last bound is $1/5$, while $(k+1)^{-1} = \frac{1}{2}$. For $k \geqslant 2$, we need the inequality

$$(1-k^{-1/2})^{1/2}k^{-1/2} + k^{-2}/5 < (k+1)^{-1/2},$$

which can be verified with the aid of calculus. Clearly, we have $|\hat{\lambda}_{k+1}(u)| < (k+1)^{-2}$ for $T > T^{00}_{k+1}$; we take $T_{k+1} = T^0_{k+1} + T^{00}_{k+1}$.

By the construction, and integration by parts,

$$|\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| \leqslant k^{-3/2}(1+T_k)^{-2}\,|u|;$$

consequently $|\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| \leqslant k^{-3/2}$ unless $|u| > 1 + T_k$. However, if $|u| > T_{k+1} > T_k$, then $|\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| < 2k^{-2}$. Since $|\hat{\lambda}_k - \hat{\lambda}_{k+1}| \leqslant 2k^{-1/2}$, we have a limit $\varphi(u)$, with

$$|\varphi - \hat{\lambda}_k| = O(k^{-1/2}).$$

Hence $\varphi = \hat{\lambda}$, with $\lambda$ carried by $F_\theta$ and $\lambda \in R$.

In verifying that $\lim \|S_T(\lambda)\|_1^* = 0$ we can calculate the weak norms over $(-3,3)$. Suppose that $T_{k-1} \leqslant T \leqslant T_k$; then

$$|S_T(\lambda_k) - S_T(\lambda)| = O(k^{-1/2}).$$

Since $T \geqslant T_{k-1}$, $\|S_T(\lambda_{k-1})\|_1^* < (k-1)^{-1}$; and finally

$$\|S_T(\lambda_k) - S_T(\lambda_{k-1})\|_1^* = O(k^{-1/2}).$$

Hence $\|S_T(\lambda)\|_1^* = O(k^{-1/2})$ over $(-3,3)$.

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