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RATIONAL HOMOTOPY OF SERRE FIBRATIONS

par Jean-Claude THOMAS

1. Preliminaries.

In this paper we adopt the terminology of [8] and [9].

Let A denote the Sullivan functor [16] from topological path connected spaces with base point to commutative graded differential augmented algebras over a field $\mathbf{k}$ of characteristic zero :

$$A : \mathrm{Top} \longrightarrow \mathrm{C.G.D.A.}$$

To each sequence of base point perserving continuous maps, in particular to each Serre fibration,

$$(*)\ F \xrightarrow{j} E \xrightarrow{\Pi} M$$

D. Sullivan [16] associated a commutative diagram (in C.G.D.A.)

$$(D)\qquad \begin{array}{ccccc} (A(M), d_M) & \xrightarrow{A(\pi)} & (A(E), d_E) & \xrightarrow{A(j)} & (A(F), d_F) \\ \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \bar{\phi}} \\ (B, d_B) & \xrightarrow{\iota} & (B \otimes \Lambda X, d) & \xrightarrow{\rho} & (\Lambda X, \bar{d}) \end{array}$$

where :

•) ΛX is the free c.g.a over the graded space $X = \underset{i \geqslant 0}{\oplus} X^i$ and $m^* : H(B, d_B) \longrightarrow H(A(M), d_M) (\cong H(M, \mathbf{k}))$ is an isomorphism.

••) $\iota(b) = b \otimes 1$, $\rho = \epsilon_B \otimes \mathrm{Id}_{\Lambda X}$, where ϵ_B is the augmentation of B.

•••) $\phi^* : H(B \otimes \Lambda X, d) \longrightarrow H(A(E), d_E)$ is an isomorphism.

•v) There exists an homogeneous basis $(e_\alpha)_{\alpha \in K}$ of X indexed by a well ordered set K such that

$$d(1 \otimes e_\alpha) \in B \otimes \Lambda(X_{<\alpha})$$

where we denote by $X_{<\alpha}$ the graded vector space generated by the e_β with $\beta < \alpha$.

The sequence

$$\mathscr{E} : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

is called a K.S-*extension* ([8]), the pair $(\mathscr{E}, \phi)$ a KS-*model* of the sequence $(*)$, $(e_\alpha)_{\alpha \in K}$ a KS-*basis*.

If there exists a K.S basis such that

$$(e_\alpha \in X^i, e_\beta \in X^j, i < j) \Longrightarrow (\alpha < \beta)$$

for all α and β in K and all degrees i and j, the K.S-extension $\mathscr{E}$ (resp. the K.S-model $(\mathscr{E}, \phi)$) is called *minimal*.

When, in the diagram (D), $\bar{\phi}$ induces an isomorphism $\bar{\phi}^*$ between cohomologies, the sequence $(*)$ is called *rational fibration*.

When the base M of $(*)$ is a point, then $((\Lambda X, d), \Phi)$ is a K.S-model of $E = F$ (resp. $((\Lambda X, d), \phi)$) is a minimal K.S model of $E = F$ if $\mathscr{E}$ is minimal).

For all rational fibration $(*)$, with base M, if $(\mathscr{E}, \phi)$ is a minimal K.S model of $(*)$ then $((\Lambda X, \bar{d}), \bar{\phi})$ is minimal K.S model of the fiber F.

Theorem 20.3 of [8] asserts that rational fibrations include Serre fibrations of path connected spaces when one of $H^*(M, \mathbf{k})$ or $H^*(F, \mathbf{k})$ is a graded space of finite type and $\Pi_1(M)$ acts nilpotently in each $H^p(F, \mathbf{k})$.

It can be easily deduced from definitions that if M, E, F are nilpotent spaces with $H(M, \mathbb{Q})$, $H(E, \mathbb{Q})$, $H(F, \mathbb{Q})$ graded spaces of finite type then $(*)$ is a rational fibration if and only the rationalized sequence

$$(**) \quad F_{\mathbb{Q}} \xrightarrow{j_{\mathbb{Q}}} E_{\mathbb{Q}} \xrightarrow{\Pi_{\mathbb{Q}}} M_{\mathbb{Q}}$$

is a rational fibration.

If $((\Lambda X, d), \phi)$ is a K.S minimal model of the topological space M, the graded vector space $\Pi_\psi(M) = \bigoplus_{i \geq 1} \Pi_\psi^i(M)$ of indecomposable elements of ΛX is called the ψ*-homotopy of* M.

Every rational fibration have a long ψ-homotopy sequence,

$$\cdots \longrightarrow \Pi^i_\psi(\mathrm{M}) \xrightarrow{\pi^\#} \Pi^i_\psi(\mathrm{E}) \xrightarrow{j^\#} \Pi_\psi(\mathrm{F}) \xrightarrow{\partial^\#} \Pi^{i+1}_\psi(\mathrm{M}) \longrightarrow \cdots$$

Following [10], if $\dim \Pi^*_\psi(\mathrm{M}) < +\infty$, we call the *Euler homotopy characteristic and the rank* of the space M the integers

$$\chi_\Pi(\mathrm{M}) = \sum_{i=1}^{+\infty} (-1)^i \dim \Pi^i_\psi(\mathrm{M})$$

and

$$rk(\mathrm{M}) = \sum_{i=1}^{+\infty} \dim \Pi^{2i+1}_\psi(\mathrm{M}).$$

If the spaces $\Pi^*_\psi(\mathrm{M})$ and $\mathrm{H}^*(\mathrm{M}, \mathbf{k})$ are finite dimensional, M is called a space of type F ([7]).

2. Main results.

A rational fibration $(*)$ is called *pure* if there exists a K.S-minimal model $(\mathscr{E}, \phi)$ such that

$$d\mathrm{X}^{\mathrm{even}} = 0, \quad d\mathrm{X}^{\mathrm{odd}} \subset \mathrm{B} \otimes \Lambda(\mathrm{X}^{\mathrm{even}}).$$

In this case $(\mathrm{B} \otimes \Lambda(\mathrm{X}^{\mathrm{even}}) \otimes \Lambda(\mathrm{X}^{\mathrm{odd}}), d)$ is a Koszul complex [12] and from [5] when $\mathbf{k} = \mathbf{R}$, and [17] for $\mathbf{k} = \mathbf{Q}$, we have :

THEOREM 1. – *If* G *is a compact connected Lie group and* H *a closed connected subgroup, then every fibre bundle with standard fiber* G/H, *associated to a* G*-principal bundle via the standard action of* G *on* G/H *is a pure fibration.*

In this paper we prove the following results.

THEOREM 2. – *For any rational fibration such that the fibre* F *is a space of type* F *with* $\chi_\pi(\mathrm{F}) = 0$ *the following assertions are equivalent :*

i) $(*)$ *is totally non cohomologeous to zero (T.N.C.Z)*

ii) $(*)$ *is a pure fibration.*

Recall that $(*)$ is called T.N.C.Z if $j^* : \mathrm{H}^*(\mathrm{F}, \mathbf{Q}) \longrightarrow \mathrm{H}^*(\mathrm{E}, \mathbf{Q})$ is surjective, which is equivalent [15] when $\mathrm{H}^*(\mathrm{F}, \mathbf{Q})$ and $\mathrm{H}^*(\mathrm{M}, \mathbf{Q})$ are of finite type and $(*)$ is Serre fibration, to :

iii) *The Serre Spectral sequence collapses at the* E_2 *term* $(d_r = 0 \ r \geqslant 2)$.

In particular the hypothesis of theorem 2 are satisfied when F is a homogeneous space G/H with $rk\mathrm{G} = rk\mathrm{H}$, for example if F is a real oriented or complex or quaternionic grassmann manifold, or F = G/T when T is a maximal torus of G or F is a finite product of such spaces. It is proved in [10] that a space M of type F has a χ_π zero iff $\mathrm{H}^{\mathrm{odd}}(\mathrm{M}, \mathbf{Q}) = 0$.

THEOREM 3. – *Every rational fibration such that the fibre* F *is a space of type* F *with* $\chi_\pi(\mathrm{F}) = 0$ *and* $rk(\mathrm{F}) \leqslant 2$ *is a pure fibration.*

This result can be applied when

$$\mathrm{F} = \mathrm{S}^{2n},\ \mathbf{CP}^n,\ \mathbf{HP}^n,\ \mathrm{S}^{2n} \times \mathrm{S}^{2q},\ \mathbf{CP}^q \times \mathrm{HP}^r,\ \mathrm{SP}(2)/\mathrm{U}(2),\ \mathrm{SO}(4)/\mathrm{U}(2),\ \mathrm{U}(2)/\mathrm{U}(1) \times \mathrm{U}(1),\ \mathrm{SO}(5)/\mathrm{SO}(1) \times \mathrm{SO}(3), \ldots$$

It is a particular case of a conjecture of S. Halperin.

Every rational fibration with fibre of type F *and* $\chi_\pi = 0$ *is* T.N.C.Z.

COROLLARY 4. – *If* F *is a path connected topological space of type* F *and* $\chi_\pi = 0$ *and if* G *is a compact connected Lie group operating on* F *then the total space* $\mathrm{F_G}$ *of the fiber bundle*

$$\mathrm{F} \longrightarrow \mathrm{E_G} \underset{\mathrm{G}}{\times} \mathrm{F} \longrightarrow \mathrm{B_G}$$

associated with the operation is intrinsically formal and the Krull dimension of $\mathrm{H_G}(\mathrm{F}, \mathbf{Q}) = \mathrm{H}(\mathrm{F_G}, \mathbf{Q})$ *equals the rank of* G.

COROLLARY 5 (*compare with* [2]). – *There do not exist Serre fibrations* (*) *if one of the following conditions is satisfied:*

i) $\mathrm{H}^{\mathrm{even}}(\mathrm{E}, \mathbf{Q}) = 0$.

ii) E *is a connected Lie group.*

iii) $\mathrm{E} = \mathrm{S}^{2n}$ *except for* $\mathrm{H}^*(\mathrm{F}, \mathbf{Q}) = \mathrm{H}^*(\mathrm{S}^{2n}, \mathbf{Q}))$

and if F *is a non contractile space of type* F *with* $\chi_\pi(\mathrm{F}) = 0$ *and* $rk(\mathrm{F}) \leqslant 2$.

From the Leray-Hirsh theorem we get, that if (*) is T.N.C.Z., then there exists a graded vector space isomorphism

$$f : \mathrm{H}(\mathrm{M} ; \mathbf{Q}) \otimes \mathrm{H}(\mathrm{F}, \mathbf{Q}) \longrightarrow \mathrm{H}(\mathrm{E}, \mathbf{Q})$$

preserving base and fiber cohomology. When f can be chosen to be an algebra isomorphism the fibration $(*)$ is called *cohomologically trivial* (C.T.).

When E, F, M are nilpotent spaces, with rational cohomology algebras of finite type, the rational fibration $(*)$ is called

($\bullet$) homotopically trivial (H.T), or
($\bullet\bullet$) weakly homotopically trivial (W.H.T), or
($\bullet\bullet\bullet$) a σ-fibration ($\sigma \cdot$ F)

if the rational fibration $(**)$

($\bullet$) is trivial or,
($\bullet\bullet$) has a long homotopy exact sequence with a connecting homomorphism $\partial^{\#}$ identically zero

$$(\Pi_\psi(E) = \Pi_\psi(M) \otimes \Pi_\psi(F)), \text{ or}$$

($\bullet\bullet\bullet$) admits a section.

Naturally we have the following diagram

$$\begin{tikzcd}
& (C.T) \arrow[r, Rightarrow] & (T.N.C.Z) \arrow[r, Rightarrow] & (\Pi^* \text{ injective}) \\
(H.T) \arrow[ur, Rightarrow] \arrow[drr, Rightarrow] & & & \\
& & (\sigma \cdot F) \arrow[uur, Rightarrow] \arrow[r, Rightarrow] & (W.H.T)
\end{tikzcd}$$

with all the reversed implications false. We do not know if in the general case (C.T) $\Longrightarrow$ (W.H.T), but we obtain the following results. (For all fibrations $F \longrightarrow E \longrightarrow M$ the spaces are assumed to have cohomology of finite type).

PROPOSITION 6. – a) *Every* T.N.C.Z *rational fibration with fibre* F *such that* $H^*(F, \mathbf{k})$ *is a free commutative graded algebra is* H.T.

b) *Every* C.T *rational fibration with fibre* F *a space of type* F *and* $\chi_\pi = 0$ *is* H.T.

PROPOSITION 7. – a) *Every* σ*-fibration* $(*)$ *such that* M *is* ℓ*-connected and* $\Pi_\psi^i(F) = 0$ *for* $i < r$ *and* $i \geqslant r + \ell$ *is* H.T.

b) *Every rational fibration such that* $\dim H^*(F, \mathbf{k}) < +\infty$ *and* M *is a coformal space* [13], [14] *with spherical cohomology zero in dimension 2p if* $\Pi_\psi^{2p-1}(F) \neq 0$ *is* W.H.T.

3. Some examples and counter examples.

Example 1. – Even if a rational fibration $(*)$ is pure not every KS minimal model $(*)$ need verify

$$dX^{\mathrm{even}} = 0 \quad \text{and} \quad dX^{\mathrm{odd}} \subset B \otimes \Lambda(X^{\mathrm{even}}).$$

Indeed the minimal K.S-extension

$$\mathcal{E}\colon (\Lambda b_1, 0) \xrightarrow{\iota} (\Lambda b_1 \otimes \Lambda(x_2, x_3, x_4, x_7), d) \xrightarrow{\rho} (\Lambda(x_2, x_3, x_4, x_7), \bar{d})$$

with
$$\begin{aligned} db_1 &= 0 & & \\ dx_3 &= x_2^2 & dx_7 &= x_4^2 + 2b_1 x_3 x_4 \\ dx_2 &= 0 & dx_4 &= b_1 x_2^2 \end{aligned}$$

is a K.S-minimal model of a pure fibration

$$(*) \quad S^2 \times S^4 \xrightarrow{j} E \xrightarrow{\Pi} S^1.$$

Example 2. – As a particular case of pure fibration we get the notion of pure space. Evidently in a pure fibration the fiber is a pure space; the converse however is false. In [10] it is proved that a space of type F with χ_π zero is a pure space, but the conjecture and theorem 2 fail if we replace the hypothesis "F is a space of type F with $\chi_\pi(F) = 0$" by the hypothesis "F is a pure space of type F". Indeed consider the rational fibration

$$F_{\mathbf{Q}} \xrightarrow{j} E \xrightarrow{\Pi} S^3$$

with $F = (S^2 V S^4)_7 \cup_\phi e^7$ where $(S^2 V S^4)_7$ is the 7th Posnikov stage of the space $S^2 V S^4$ and $\phi = [S^4, [S^2, S^2]] - [S^2 [S^2, S^4]]$ defined by its K.S-minimal model

$$\mathcal{E} : (\Lambda b_3, 0) \longrightarrow (\Lambda b_3 \times \Lambda(x_2, x_3, x_4, x_5, x_7), d) \longrightarrow (\Lambda(x_i), \bar{d})$$

$$\begin{aligned} db_3 &= 0 & & & & \\ dx_2 &= 0 & dx_4 &= b_3 x_2 & & \\ dx_3 &= x_2^2, & dx_5 &= x_2 x_4 + b_3 x_3, & dx_7 &= x_4^2 + 2b_3 x_5. \end{aligned}$$

Then $\chi_\pi(F) = -1$ and $H^4(E, \mathbf{k}) = 0$, and (*) is neither a pure fibration nor a T.N.C.Z. fibration.

Example 3. – There exists one (unique up to rational homotopy equivalence) Serre fibration

$$(S^2 \vee S^2)_{\mathbf{Q}} \xrightarrow{j} E \xrightarrow{\pi} S^3$$

which is C.T. but not H.T., as it can be easily seen from the calculations of [11].

Example 4. – The universal fiber bundle

$$S^{2n} \longrightarrow B_{SO(2n)} \longrightarrow B_{SO(2n+1)}$$

is T.N.C.Z. and W.H.T. but not C.T.

Example 5. – Let a vector bundle

$$\eta : \mathbf{R}^{2n+1} \longrightarrow E \longrightarrow M$$

and $p_n(\eta)$ its n^{th} Pontryagin class, and

$$\eta_S : S^{2n} \longrightarrow E_S \longrightarrow M$$

its associated sphere bundle. Suppose that η_S is T.N.C.Z. then $p_n(\eta) = 0$ if and only if η_S is H.T.

Example 6. – If a fibration admits a section then it is a σ fibration. The converse is false indeed, consider the σ-fibration

$$S^4 \times S^3 \xrightarrow{j} E \xrightarrow{\pi} S^5$$

of orthonormal two frames on S^5.

4. Proof of theorem 2.

A K.S-extension $\mathcal{E} : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$ is called pure if there exists a K.S-extension

$$\mathcal{E}' : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X', d') \xrightarrow{\rho'} (\Lambda X', \bar{d}')$$

and an isomorphism of K.S extension $(\mathrm{Id}_B, f, \bar{f})\, \mathcal{E} \simeq \mathcal{E}'$ with $d' X'^{\text{even}} = 0$ and $d' X'^{\text{odd}} \subset B \otimes \Lambda(X'^{\text{even}})$.

In view of proposition 1.11 of [8], theorem 2 follows from the following algebraic version.

THEOREM 2'. – *Let $\mathcal{E}$ be a K.S-minimal extension with connected base* B *and* $\dim H(\Lambda X, d) < \infty$, $\dim X^{odd} = \dim X^{even} < +\infty$ *then the two assertions are equivalent :*

i) ρ^* *is surjective*

ii) E *is pure.*

A) First suppose that $\mathcal{E}$ is pure then $\Lambda(X^{even})$ maps into $H(B \otimes \Lambda X, d)$ and from [7] $H(\Lambda X, \bar{d}) = \Lambda(X^{even})/\bar{d}X^{odd} \cdot \Lambda(X^{even})$ so ρ^* is surjective.

B) The converse is in two steps. First we prove that $\mathcal{E}$ is isomorphic to $\mathcal{E}'$ with $d'X^{even} = 0$ and then we show $\mathcal{E}'$ isomorphic to $\mathcal{E}''$ with $d''X^{even} = 0$ and $d''X^{odd} \subset B \otimes \Lambda X^{even}$.

B1) *First step.* – From [10] we can suppose that $\bar{d}$ satisfies

$$\bar{d}X^{even} = 0 \quad \text{and} \quad \bar{d}X^{odd} \subset \Lambda(X^{even}).$$

Since ρ and ρ^* are surjective for all $x \in X^{even}$ there exists $\Phi_x \in (B \otimes \Lambda X) \cap \ker d$ such that $\rho(\Phi_x) = x$.

Then

$$\Phi_x = x + \Omega_x$$

with $\Omega_x \in B^+ \otimes \Lambda X = \ker \rho$. Let x run through a K.S-minimal basis and define a linear map $g : X \longrightarrow B \otimes \Lambda X$ by

$$g(x) = x \quad \text{if} \quad x \in X^{odd}$$
$$g(x) = x + \Omega_x \quad \text{if} \quad x \in X^{even}$$

g extends uniquely to a B-linear algebra isomorphism. $g : B \otimes \Lambda X \longrightarrow B \otimes \Lambda X$. It can be easily proved than g is an isomorphism.

Let $\mathcal{E}' : (B, d_B) \longrightarrow (B \otimes \Lambda X, g^{-1} dg) \longrightarrow (\Lambda X, \bar{d})$ so that $(\mathrm{Id}_B, g, \mathrm{Id}_{\Lambda X})$ is an isomorphism of K.S-extensions between $\mathcal{E}$ and $\mathcal{E}'$ and $d'(X^{even}) = g^{-1} dg(X^{even}) = 0$.

B2) *Second step.* – Suppose $\mathcal{E}$ is a K.S-minimal extension such that

$$(H_\ell) = \begin{cases} dX^{even} = 0 \\ dX^{odd} \subset (B \times \Lambda(X^{even})) \otimes (B^{\geq \ell} \otimes (\Lambda^+ X^{odd}) \otimes \Lambda(X^{even})) \end{cases}$$

and let $(\hat{B}_\ell \otimes \Lambda X, \hat{d})$ be the quotient c.g.d.a.

$$(B \otimes \Lambda X, d)/(B^{\geq \ell+1} \otimes \Lambda X, d).$$

LEMMA 1. – *In* $(\hat{B}_\ell \otimes \Lambda X, \hat{d})$ *we obtain*

a) $(\ker \hat{d}) \cap (B^\ell \otimes \Lambda X) = (B^\ell \otimes \Lambda(X^{\text{even}})) + (d(B^\ell \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})))$

b) $(\ker \hat{d}) \cap (B^\ell \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})) \subset \hat{d}(B^\ell \otimes \Lambda^+(X^{\text{even}}) \otimes \Lambda(X^{\text{odd}})))$.

Proof. – a) One inclusion in a) is immediate, the second results from the relation $H_+(\Lambda X, \overline{d}) = 0$ where $H_i(\Lambda X, \overline{d})$ is the homology of the Koszul complex

$$\cdots \longrightarrow (\Lambda(X^{\text{even}}) \otimes \Lambda^{i+1}(X^{\text{odd}})) \xrightarrow{\overline{d}} \Lambda(X^{\text{even}}) \otimes \Lambda^i(X^{\text{odd}}) \xrightarrow{\overline{d}} \Lambda(X^{\text{even}}) \otimes \Lambda^{i-1}(X^{\text{odd}}) \longrightarrow$$

and from the relation

$$\hat{d}\phi_i = (1 \otimes \overline{d})\,\phi_i \quad \text{for} \quad \phi_i \in B^\ell \otimes \Lambda^i(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}).$$

b) is true for the same reason.

Clearly if $\mathscr{E}$ satisfies the hypothesis of theorem 2′ then $\mathscr{E}$ satisfies hypothesis (H_1), and since X is a finite dimensional vector space, theorem 2′ results from

LEMMA 2. – *If* $\mathscr{E}$ *satisfies hypothesis* (H_ℓ) *there exists a minimal* K.S-*extension* $\mathscr{E}'$ *isomorphic to* $\mathscr{E}$ *which satisfies* $(H_{\ell+1})$.

Proof. – 1) *Suppose* $\ell = 2\ell'$, so for each $x \in X^{\text{odd}}$, in $(\hat{B}_\ell \otimes \Lambda X, \hat{d})$ we have

$$\hat{d}x = \Phi_x + \sum_{s \geq 1} \phi_{x,2s}$$

with $\Phi_x \in \hat{B}_\ell \otimes \Lambda(X^{\text{even}})$, $\phi_{x,2s} \in B^\ell \otimes \Lambda^{2s}(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})$.

From relation $\hat{d} \circ \hat{d}x = 0$ we deduce

$$0 = \hat{d}\Phi_x + \hat{d}\Big(\sum_{s \geq 1} \Phi_{2s,x}\Big) = (d_B \otimes id)\,\Phi_x + (id \otimes \overline{d})\Big(\sum_{s \geq 1} \Phi_{2s,x}\Big) \in B^{\text{odd}} \otimes \Lambda X \oplus \hat{B}^\ell \otimes \Lambda X.$$

Hence $$0 = \hat{d}\Phi_x = \hat{d}\Big(\sum_{s \geq 1} \Phi_{2s,x}\Big).$$

By lemma 1, $$\hat{d}x = \Phi_x + \sum_{s \geq 1} \hat{d}\Psi_{x,2s+1}$$

with $$\Psi_{x,2s+1} \in B^\ell \otimes \Lambda^{2s+1}(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}).$$

Thus $d\left(x - \sum_{s \geqslant 1} \Psi_{x,2s+1}\right) = \Phi_x + \Omega_x$ with $\Omega_x \in B^{\geqslant \ell+1} \otimes \Lambda X$. The linear map $g : X \longrightarrow B \otimes \Lambda X$ defined by $g(x) = x$ if $x \in X^{even}$ $g(x_\alpha) = x_\alpha - \sum_{s \geqslant 1} \psi_{\alpha,2s+1}$ if (x_α) is a minimal K.S basis of X^{odd} uniquely extends to a c.g.d.a isomorphism $B \otimes \Lambda X \xrightarrow{\cong} B \otimes \Lambda X$ with $g/B = Id_B$.

Define $\mathscr{E}'$ by $(B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, g^{-1} dg) \xrightarrow{\iota} (\Lambda X, \bar{d})$ then $\mathscr{E}'$ satisfies hypothesis $(H_{\ell+1})$.

2) *Suppose* $\ell = 2\ell' + 1$. In the same way as in the preceding case we get a K.S minimal extension $\mathscr{E}_1$ and an isomorphism $(Id_B, g_1, Id_{\Lambda X})$ between $\mathscr{E}$ and $\mathscr{E}_1$ such that,

$$\mathscr{E}_1 : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d_1) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

with

$$\begin{cases} d_1(x) = 0 \quad \text{if} \quad x \in X^{even} \\ d_1(x) \in (B^{even} \otimes \Lambda(X^{even})) \oplus (B^\ell \otimes \Lambda^1 X^{odd} \otimes \Lambda X^{even}) \\ \qquad \qquad \oplus (B^{\geqslant \ell+1} \otimes \Lambda X) \quad \text{if} \quad x \in X^{odd}. \end{cases}$$

We put

$$\begin{aligned} B^\ell &= K^\ell \oplus dB^{\ell-1} \quad && \text{if} \quad \ell \geqslant 2 \\ B^1 &= K^1 && \text{if} \quad \ell = 1 . \end{aligned}$$

Using only degree argument, we prove that there exists a minimal K.S-extension $\mathscr{E}_2$ and an isomorphism $(Id_B, g_2, Id_{\Lambda X})$ between $\mathscr{E}_1$ and $\mathscr{E}_2$ such that

$$\mathscr{E}_2 : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d_2) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

with

$$\begin{cases} d_2(x) = 0, \quad \text{if} \quad x \in X^{even} \\ d_2(x) \in (B \otimes \Lambda X^{even}) \oplus (K^\ell \otimes \Lambda^1(X^{odd}) \otimes \Lambda(X^{even})) \\ \qquad \qquad \oplus (B^{\geqslant \ell+1} \otimes \Lambda^+ X^{odd} \otimes \Lambda X^{even}), \quad \text{if} \quad x \in X^{odd} \end{cases}$$

so that in the quotient algebra $(\hat{B}_\ell \otimes \Lambda X, \hat{d}_2)$, we write

$$\hat{d}_2 x_\alpha = \bar{d} x_\alpha + \sum_{r \geqslant 1} \Phi_{\alpha,2r} + \sum_{s=1}^{\alpha-1} \phi_{\alpha,s}\, x_s$$

with (x_α) a K.S-minimal basis of $\mathscr{E}_2$ and $x_\alpha \in X^{odd}$,

$$\Phi_{\alpha,2r} \in B^{2r} \otimes \Lambda(X^{\text{even}}) \qquad \phi_{\alpha,s} \in K^{\ell} \otimes \Lambda(X^{\text{even}}).$$

From the relation $\hat{d} \circ \hat{d} x_\alpha = 0$ and lemma 1 we obtain for each α,

$$d_2(x_\alpha - \theta_\alpha) = \bar{d}\, x_\alpha + \sum_{r \geq 1} \Phi_{\alpha,2n} + \Omega_\alpha$$

with

$$\Omega_\alpha \in B^{\geq \ell+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})$$
$$\theta_\alpha \in B^{\ell} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})$$

and so there exists a minimal K.S-extension $\mathscr{E}'$ and an isomorphism $(\mathrm{Id}_B, g', \mathrm{Id}_{\Lambda X})$ between $\mathscr{E}_2$ and $\mathscr{E}'$ such that

$$\mathscr{E}' : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d') \xrightarrow{\rho} (\Lambda X, \bar{d})$$

and $\mathscr{E}'$ satisfies $H_{\ell+1}$. This ends the proof of lemma 2 .

5. Derivations in Poincaré duality algebras and proof of theorem 3.

Let $(A, d) = (\Lambda(x_1, \ldots, x_n, y_1, \ldots, y_n), d)$ a K.S complex such that the y_i and x_j have respectively even degree $|y_i|$ and odd degree $|x_j|$ and

$$|y_1| \leq |y_2| \leq \cdots \leq |y_n|$$
$$|x_1| \leq |x_2| \leq \cdots \leq |x_n|.$$

Suppose

$$dy_i = 0 \quad i = 1, \ldots, n$$
$$dx_j = f_j \in \Lambda(y_1, \ldots, y_n) \quad j = 1, \ldots, n$$

then (A, d) is a pure K.S complex and from [10] if $\dim H(A,d) < +\infty$ then $H(A, d_A) = \Lambda(y_1, \ldots, y_n)/(f_1, \ldots, f_n)$ is a Poincaré duality algebra of formal dimension

$$N = |f_1| + \ldots + |f_n| - |y_1| - \ldots - |y_n|$$

(i.e.)

i) $H^i(A, d) = 0$ if $i > N$

ii) $H^N(A, d) = \mathbf{k} e$

iii) the bilinear form $\langle\, ,\rangle : H^P(A, d) \times H^{N-p}(A, d) \longrightarrow \mathbf{k}$

defined by $\langle a, b \rangle e = a.b$ is non degenerate.

Since $\dim \mathrm{H}(\mathrm{A}, d) < +\infty$ and $\mathrm{H}^0(\mathrm{A}, d) = \mathbf{k}$, one verifies immediately:

LEMMA 1. – *Any derivation* $\theta \in \mathrm{Der}_{<0}(\mathrm{H}(\mathrm{A}), d)$ *satisfies* $\mathrm{I}_m \theta \cap \mathrm{H}^0(\mathrm{A}, d) = 0$ *and hence maps* $\mathrm{H}^+(\mathrm{A}, d)$ *to itself.*

We put $\overline{y}_i$ the class of y_i in $\mathrm{H}(\mathrm{A}, d)$ and we say that a derivation $\widetilde{\theta}$ of $\mathrm{H}(\mathrm{A}, d_{\mathrm{A}})$ is nilpotent with respect to $(\overline{y}_1, \ldots, \overline{y}_n)$ if $\widetilde{\theta}(y_i)$ is polynomial in $\overline{y}_1, \ldots, \overline{y}_{i-1}$. We denote by $\widetilde{\mathrm{Der}}_{<0}(\mathrm{H}(\mathrm{A}), d)$ the subspace of $\mathrm{Der}_{<0}(\mathrm{H}(\mathrm{A}), d)$ of such derivations.

LEMMA 2. – *Any derivation* $\widetilde{\theta} \in \widetilde{\mathrm{Der}}_{<0}(\mathrm{H}(\mathrm{A}, d))$ *satisfies* $\widetilde{\theta}(\mathrm{H}^{\mathrm{N}}(\mathrm{A}, d)) = 0$.

Proof. – Let m_1 be the largest integer such that $\overline{y}_1^{m_1} \neq 0$ and $\overline{y}_1^{m_1+1} = 0$.

Let m_i be the largest integer such that $(\overline{y}_1^{m_1}, \ldots, \overline{y}_{i-1}^{m_{i-1}})\,\overline{y}_i^{m} \neq 0$ and $(\overline{y}_1^{m_1} \ldots \overline{y}^{m_{i-1}})\,\overline{y}_i^{m_i+1} = 0$, then we obtain an element $\Phi = \overline{y}_1^{m_1}\,\overline{y}_2^{m_2} \ldots \overline{y}_n^{m_n}$ such that for every $a \in \mathrm{H}^+(\mathrm{A}, d)\ a \,.\, \Phi = 0$.

Necessarily $|\Phi| = \mathrm{N}$ and we may put $e = \overline{y}_1^{m_1} \ldots \overline{y}_n^{m_n}$. Then $\widetilde{\theta}(e) = 0$, since $\widetilde{\theta}$ is nilpotent with respect to $(\overline{y}_1, \ldots, \overline{y}_n)$.

From lemmas 1 and 2 we deduce,

COROLLARY. – *If* $\widetilde{\theta} \in \widetilde{\mathrm{Der}}_{<0}(\mathrm{H}(\mathrm{A}, d))$ *then*

i) $\langle \widetilde{\theta}(a), b \rangle = -\langle a, \widetilde{\theta}(b) \rangle$

ii) $\mathrm{Im}\,\widetilde{\theta} \subset \bigoplus_{i=1}^{\mathrm{N}-1} \mathrm{H}^i$.

LEMMA 3. – *If* $\widetilde{\theta} \in \widetilde{\mathrm{Der}}_{<0}(\mathrm{H}(\mathrm{A}, d))$ *then*

$$(\widetilde{\theta}(\overline{y}_1) = \widetilde{\theta}(\overline{y}_2) = \ldots = \widetilde{\theta}(\overline{y}_{n-1}) = 0) \Longrightarrow (\widetilde{\theta} \equiv 0).$$

Proof. – Suppose that $\widetilde{\theta}(\overline{y}_n) = \Phi' \neq 0$ and let

P_1 be the largest integer such that $\Phi' \overline{y}_1^{\mathrm{P}_1} \neq 0$ and $\Phi' \overline{y}_1^{\mathrm{P}_1+1} = 0$

P_i be the largest integer such that $\Phi' \overline{y}_1^{\mathrm{P}_1} \ldots \overline{y}_i^{\mathrm{P}_i} \ldots \overline{y}_i^{\mathrm{P}_i} \neq 0$ and $\Phi' \overline{y}_1^{\mathrm{P}_1} \ldots \overline{y}_i^{\mathrm{P}_i+1} = 0$.

So we obtain an element $\Psi = \Phi' \overline{y}_1^{P_1} \ldots \overline{y}_n^{P_n}$ such that $\Psi \in H^N(A, d)$ and $\widetilde{\theta}\left(\frac{1}{P_n + 1} \overline{y}_1^{P_1} \ldots \overline{y}_{n-1}^{P_{n-1}}, \overline{y}_n^{P_n+1}\right) = \Psi$ which contradicts part (ii) of the corollary above.

In particular if $n = 2$ since $\hat{\theta}(y_1)$ is always zero,

$$\mathrm{Der}_{<0}(H(A, d)) = 0 .$$

This is what we will need to prove theorem 3.

Proof of Theorem 3.

A) Suppose $\dim \Pi_\psi(F) = 2$ then theorem 3 is equivalent to the following.

THEOREM 3'. – *Let* $\mathscr{E}$ *a* K.S-*minimal extension*

$$(B, d_B) \xrightarrow{\iota} (B \otimes \Lambda(x, y), d) \xrightarrow{\rho} (\Lambda(x, y), \overline{d})$$

such that B *is a connected algebra,* $\dim H((x, y), \overline{d}) < +\infty$ *and* $|x|$ odd, $|y|$ even *then* ρ^* *is surjective.*

Proof. – Since $\dim H(\Lambda(x, y), \overline{d}) < +\infty$, we have $\overline{d}x = \lambda y^m$ with $\lambda \in \mathbf{k} - \{0\}$ and $m \geqslant 2$. Thus

$$dx = \lambda y^m + b_1 y^{m-1} + \ldots + b_m$$

with $|b_i| = i\ |y|$, whence

$$d\left(y + \frac{1}{m\lambda} b_1\right) = 0$$

$$\rho\left(y + \frac{1}{m\lambda} b_1\right) = y$$

and $\rho^* : H(B \otimes \Lambda(x, y)) \longrightarrow \Lambda(y)/(y^m) = H(\Lambda X, \overline{d})$ is surjective.

B) Suppose $\dim \Pi_\psi(F) = 4$ then theorem 3 is equivalent to the following.

THEOREM 3''. – *Let* $\mathscr{E}$ *a* K.S-*minimal extension*

$$(B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, d)$$

such that B *is a connected algebra,* $\dim H(\Lambda X, d) < +\infty$, $\dim X^{\mathrm{odd}} = \dim X^{\mathrm{even}} = 2$ *then* $\mathscr{E}$ *is pure.*

We prove theorem 3″ by induction on ℓ, in the following manner

$$H^1_\ell \Longrightarrow H^2_\ell \Longrightarrow H^3_\ell \Longrightarrow H^1_{\ell+1}$$

where the hypothesis H^i_ℓ are defined by:

$$H^1_\ell = \begin{cases} dx \in B^{\geq \ell} \otimes \Lambda X, & \text{if } x \in X^{\text{even}} \\ dx \in (B \otimes \Lambda(X^{\text{even}})) \oplus (B^{\geq \ell} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{odd}} \end{cases}$$

$$H^2_\ell = \begin{cases} dx \in (B^{\geq \ell} \otimes \Lambda(X^{\text{even}})) \oplus (B^{\geq \ell+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{even}} \\ dx \in (B \otimes \Lambda(X^{\text{even}})) \oplus (B^{\geq \ell} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{odd}} \end{cases}$$

$$H^3_\ell = \begin{cases} dx \in (B^{\geq \ell} \otimes \Lambda(X^{\text{even}})) \oplus (B^{\geq \ell+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}), & \text{if } x \in X^{\text{even}} \\ dx \in (B \otimes \Lambda(X^{\text{even}})) \oplus (B^{\ell} \otimes \Lambda^1(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}) \\ \qquad \oplus B^{\geq \ell+1} \otimes \Lambda X), & \text{if } x \in X^{\text{odd}} \end{cases}$$

To prove $H^1_\ell \Longrightarrow H^2_\ell$ and $H^2_\ell \Longrightarrow H^3_\ell$, we use lemma 1 of IV which again follows from the relation $d \circ d = 0$.

In the case $\ell = 2\ell'$ for degree reasons $H^3_\ell = H^1_{\ell+1}$. When $\ell = 2\ell' + 1$ we prove $H^3_\ell \Longrightarrow H^1_{\ell+1}$.

First, we can assume that

$$\begin{cases} dx \in (K^\ell \otimes \Lambda(X^{\text{even}}) \oplus (B^{\geq \ell+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{even}} \\ dx \in (B \otimes \Lambda X^{\text{even}}) \oplus (K^\ell \otimes \Lambda^1(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})) \\ \qquad \oplus (B^{\geq \ell+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{odd}} \end{cases}$$

with

$$B^\ell = K^\ell \oplus dB^{\ell-1} \quad \text{if } \ell > 1$$

$$B^1 = K^1 \quad \text{if } \ell = 1.$$

In the quotient algebra $(\hat{B} \otimes \Lambda X, \hat{d})$ (4, B_2), we have

$$\begin{cases} dy_i = \psi_i & i = 1, 2 \\ dx_j = \overline{d}x_j + \sum_r \Phi_{j,2r} + \sum_{s=1}^{j-1} \phi_{j,s} x_s & j = 1, 2 \end{cases}$$

for a K.S-minimal basis (y_i, x_j) of X with $|y_i|$ even and $|x_j|$ odd, with

$$\psi_1 \in K^\ell, \quad \psi_2 \in K^\ell \otimes \Lambda(y_1)$$
$$\Phi_{j,2r} \in B^{2r} \otimes \Lambda(y_1, y_2)$$
$$\phi_{j,s} \in K^\ell \otimes \Lambda(y_1, y_2).$$

And from the relation $\hat{d} \circ \hat{d} = 0$ we obtain

$$\hat{d}(\bar{d}x_j) = \sum_{s=1}^{j-1} \phi_{js}\, dx_s .$$

Let (b_ϵ) a base of K^ℓ and put for each $\Phi \in \Lambda(y_1, y_2)$

$$\hat{d}(\Phi) = \sum_\epsilon b_\epsilon \otimes \theta^\epsilon(\Phi).$$

This defines a degree $1\text{-}\ell$ derivation θ^ϵ on $\Lambda(y_1, y_2)$ which respects the ideal $(\bar{d}x_1, \bar{d}x_2)$. So θ^ϵ induces a derivation $\widetilde{\theta}^\epsilon$ on $\Lambda(y_1, y_2)/(\bar{d}x_1, \bar{d}x_2) = H(\Lambda X, \bar{d})$ which is nilpotent with respect to $(\bar{y}_1, \bar{y}_2)$. From our results on such derivations, $\widetilde{\theta}^\epsilon \equiv 0$ and necessarily

$$\theta^\epsilon(y_1) = 0 \qquad \theta^\epsilon(y_2) = \bar{d}\Phi^\epsilon \qquad \Phi^\epsilon \in \Lambda X^{\text{even}} \otimes \Lambda^1 X^{\text{odd}}$$

thus,

$$\begin{cases} \hat{d}_3\left(y_2 + \sum_\epsilon b_\epsilon \otimes \Phi^\epsilon\right) = 0 \\ \hat{d}_3(y_1) = 0 . \end{cases}$$

A standard argument now ends the proof.

6. Proof of the corollaries 4 and 5.

A) COROLLARY 4. – Since $H^{\text{odd}}(F, \mathbf{k}) = H^{\text{odd}}(B_G, \mathbf{k}) = 0$, the Serre spectral sequence collapses at the E_2 term so that the fibration

$$(*) \qquad F \longrightarrow E_G \underset{G}{\times} F \longrightarrow B_G$$

is T.N.C.Z. By [1], $H(B_G, \mathbf{Q}) = \Lambda Z$, $Z = Z^{\text{even}}$ and so $(\Lambda Z, 0)$ is the minimal model for B_G. From theorem 2 there exists a K.S.-minimal model of $(*)$

$$\mathscr{E} : (\Lambda Z, 0) \xrightarrow{\iota} (\Lambda Z \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

with

$$dX^{\text{even}} = 0$$
$$dX^{\text{odd}} \subset \Lambda Z \otimes \Lambda X^{\text{even}}.$$

So we have the Koszul complex,

$$\longrightarrow \Lambda(Z \oplus X^{\text{even}}) \otimes \Lambda^{i+1}(X^{\text{odd}}) \xrightarrow{d} \Lambda(Z \oplus X^{\text{even}}) \otimes \Lambda^{i}X^{\text{odd}} \xrightarrow{d} \Lambda(Z \oplus X^{\text{even}}) \otimes \Lambda^{i-1}X^{\text{odd}} \longrightarrow$$

and we easily verify that $H_+(\Lambda(Z \oplus X), d) = 0$. Thus if x_i is a homogeneous basis of X^{odd} and if we put $dx_i = g_i$ then

$$H(\Lambda(Z \oplus X), d) = H_0(\Lambda(Z \oplus X), d) = \Lambda(Z \oplus X^{\text{even}})/(g_1, \dots, g_n)$$

where $(g_1, \dots, g_n)$ is a regular sequence of $\Lambda(Z \oplus X^{\text{even}})$. This proves directly from commutative algebra that $H(F_G, \mathbf{k})$ is a Cohen Macaulay ring of Krull dimension $\dim Z$ equal to the rank of G and minimalizing $(\Lambda(Z \oplus X), d)$ we obtain the brigaded model of $H(F_G, \mathbf{k})$ in the sense of [11]. This is two stage, and so F_G is intrinsically formal (i.e. F_G is formal and there is no space $M \underset{\mathbf{Q}}{\not\sim} F_G$ such that $H(F_G, \mathbf{k}) = H(M, \mathbf{k})$).

B) Corollary 5. – i) Since $H^{\text{even}}(F, \mathbf{k})$ and $H^{\text{even}}(E, \mathbf{k}) = 0$ the condition j^* surjective is impossible.

ii) From the long exact sequence of ψ-homotopy we deduce that in a pure fibration we have

$$rk(\Pi_{2n}(F)) \leqslant rk(\Pi_{2n}(E))$$

which is impossible if F non contractible and E a Lie group.

iii) A fibration satisfying the hypothesis is pure by Theorem 3 and hence has a K.S minimal model of the form

$$(B, d_B) \longrightarrow (B \otimes \Lambda X, d) \longrightarrow (\Lambda X, \bar{d})$$

with

$$dX^{\text{even}} = 0 \qquad \dim X^{\text{even}} = \dim X^{\text{odd}}$$
$$dX^{\text{odd}} = B \otimes \Lambda X^{\text{even}}.$$

Necessarily $\dim X^{\text{even}} = 1$ and if we choice $x \in X^{\text{odd}} - \{0\}$ $dx = y^p + b_1 y^{p-1} + \dots + b_p$ with $p \geqslant 2$, $y \in X^{\text{even}} - \{0\}$. Since j^* is surjective $p = 2$ then $F_{\mathbf{Q}} \sim S^{2n}$.

7. Proof of propositions 6 and 7.

PROPOSITION 6. – *The two following lemmas are easily proved and the first is well known.*

LEMMA 1. – *A Serre fibration* (*) *is* T.N.C.Z. (*resp.* CT) *if and only if there exists a graded vector space homomorphism (resp. a graded algebra homomorphism)*

$$\tau : H^*(F, \mathbf{k}) \longrightarrow H^*(E, \mathbf{k})$$

such that

$$j^* \tau = \mathrm{Id}_{H^*(F,\mathbf{k})} .$$

LEMMA 2. – *A rational fibration* (*) *is* H.T. *if and only if there exists a* K.S-*minimal model* $(\mathscr{E}, \phi)$ *and a graded differential algebra homomorphism*

$$\sigma : (\Lambda X, \bar{d}) \longrightarrow (A(M) \otimes \Lambda(X), d)$$

such that

$$\rho \circ \sigma = \mathrm{Id}_{\Lambda X} .$$

Remarks. – i) These two lemmas prove in particular that the notions of T.N.C.Z, C.T or H.T fibration are invariant by pull back.

ii) Every T.N.C.Z. Serre fibration is a rational fibration, when base or fibre has finite type.

Proof of a). – Since $H(F, \mathbf{k}) = \Lambda X$, the fibration (*) admits a K.S-minimal model

$$\mathscr{E}: (A(M), d_M) \xrightarrow{\iota} (A(M) \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, 0)$$

with ρ^* surjective. Choose a homogeneous basis of X, $(x_\alpha)_\alpha$ and for each α, an element $c_\alpha \in (A(M) \otimes \Lambda X) \cap \ker d$ such that $\rho^*([c_\alpha]) = x_\alpha$ so that σ in lemma 2 is defined by $\sigma(x_\lambda) = c_\alpha$.

Proof of b). – By Theorem 2 there is a K.S minimal model $\mathscr{E}$ of (*) :

$$\mathscr{E}: (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

with $\dim X^{\mathrm{odd}} = \dim X^{\mathrm{even}}$, $dX^{\mathrm{even}} = 0$, $dX^{\mathrm{odd}} \subset B \otimes \Lambda X^{\mathrm{even}}$. From [10], we have $H(\Lambda X, \bar{d}) = \Lambda X^{\mathrm{even}} / \bar{d}(X^{\mathrm{odd}}) \cdot (\Lambda(X^{\mathrm{even}}))$. Let τ be as in lemma 1; then for each $y \in X^{\mathrm{even}}$, there exists $\alpha_y \in (B \otimes \Lambda X) \cap \ker d$ such that

$$\tau([y]) = [\alpha_y] .$$

One verifies that

$$\rho(\alpha_y) = y + \bar{d}\beta_y^+ \quad \text{with} \quad \beta_y^+ \in \Lambda X^{\text{even}} \otimes \Lambda^{\geq 1} X^{\text{odd}}.$$

Hence

$$\alpha_y = y + d\beta_y^+ + \Omega_y \quad \text{with} \quad \Omega_y \in B^+ \otimes \Lambda X.$$

Put

$$\sigma(y) = \alpha_y - d\beta_y^+$$

then

$$\rho \circ \sigma = \mathrm{Id}\,|_{\Lambda(X^{\text{even}})} \quad \text{and} \quad \sigma^* = \tau .$$

On the other hand, from the formulas

$$\tau[\bar{d}x] = [\sigma(\bar{d}x)] = 0 \quad \text{and} \quad \rho(\sigma\,\bar{d}x) = \bar{d}x\,,\ x \in X^{\text{odd}}\,,$$

we deduce

$$\sigma(\bar{d}x) = \bar{d}x + \Omega_x^+ = d\beta_x$$

with

$$\Omega_x^+ \in B^+ \otimes \Lambda X \quad \text{and} \quad \beta_x \in B \otimes \Lambda X .$$

Thus

$$\sigma(\bar{d}x) = dx + d\hat{\Omega}_x^+ .$$

with $\hat{\Omega}_x^+ \in B^+ \otimes \Lambda X$ so we put

$$\sigma(x) = x + \hat{\Omega}_x^+ .$$

This defines σ as required in lemma 2.

PROPOSITION 7. – *The next lemma is straightforward.*

LEMMA 3. – *A rational fibration* $(*)$ *is a* σ*-fibration (resp.* W.H.T.*) if and only if there exists a* K.S*-minimal model*

$$\mathcal{E} : (B\,, d_B) \xrightarrow{\iota} (B \otimes \Lambda X\,, d) \xrightarrow{\rho} (\Lambda X\,, \bar{d})$$

with B *a connected algebra (resp. with* $B = \Lambda Z$ *the minimal model of* M*) such that :*

$$\forall\, x \in X,\ dx - \bar{d}x \in B^+ \otimes \Lambda^+(X)$$

(resp., $\forall\, x \in X\,,\ dx - \bar{d}x \in (\Lambda^+ Z \cdot \Lambda^+ Z) \oplus (\Lambda^+ Z \otimes \Lambda^+ X))$.

Proof of a). – This results directly from lemma 3.

Proof of b). – Let $(\Lambda Z\,, d_B)$ be a K.S-minimal model of M and $\mathcal{E} : (\Lambda Z\,, d_B) \xrightarrow{\iota} (\Lambda Z \otimes \Lambda X\,, d) \xrightarrow{\rho} (\Lambda X\,, \bar{d})$ a K.S-minimal model of $(*)$. Since M is coformal $d_B Z \subset \Lambda^2 Z$ and since $\dim H(F) < +\infty$, from [6] we deduce that $\partial^{\#}(X^{even}) = 0$.

Suppose that there exists $x \in X^{odd}$ such that $\partial^{\#} x = b \neq 0$ then

$$dx = \bar{d}x + b + \Phi + \Omega$$

with

$$b \in \Lambda^1 Z, \quad \Phi \in \Lambda^1 Z \otimes \Lambda^+ X, \quad \Omega \in \Lambda^{\geq 2} Z \otimes \Lambda X.$$

We can suppose $x = e_{\alpha_0}$ where α_0 is the smallest index in a K.S-minimal basis such that $\partial^{\#} e_\alpha \neq 0$. A simple calculation from $d^2 x = 0$ and the fact that $db \in \Lambda^2 Z$ gives $db = 0$. Hence $[b]$ lives in the spherical cohomology of M and from our hypothesis, b is coboundary which is impossible. This proves $\partial^{\#} = 0$.

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