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KA-SING LAU

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THE CLASS OF CONVOLUTION OPERATORS ON THE MARCINKIEWICZ SPACES

by Ka-Sing LAU (*)

1. Introduction.

Throughout the paper, the functions we consider will be complex valued, Borel measurable on R. For $1 \leqslant p < \infty$, we will let

$$\mathcal{M}^p = \left\{ f : \ \|f\|_{\mathcal{M}^p} = \overline{\lim_{T\to\infty}} \left(\frac{1}{2T} \int_{-T}^{T} |f|^p \right)^{1/p} < \infty \right\}$$

and

$$\mathcal{V}^p = \left\{ g : \|g\|_{\mathcal{V}^p} = \overline{\lim_{\epsilon\to 0^+}} \left(\frac{1}{2\epsilon} \int_{-\infty}^{\infty} |g(u+\epsilon) - g(u-\epsilon)|^p \, du \right)^{1/p} < \infty \right\}.$$

The space $\mathcal{M}^p$ is called the *Marcinkiewicz space.* The space $\mathcal{V}^p$ was introduced by Hardy and Littlewood [3] in order to study the fractional derivatives and is called the *integrated Lipschitz class.* By identifying functions whose difference has zero norm, it was proved that both $\mathcal{M}^p$ and $\mathcal{V}^p$ are Banach spaces [4], [8]. These spaces have also been studied in detail in [2], [3], [7], [10], [11], [12]. Let $\mathcal{W}^p$ denote the class of functions f in $\mathcal{M}^p$ such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f|^p$$

exists; then $\mathcal{W}^p$ is a "non-linear" closed subspace of $\mathcal{M}^p$. In [13], Wiener introduced the integrated Fourier transformation $g = W(f)$ of an f in $\mathcal{W}^2$ as

$$g(u) = \frac{1}{2\pi} \left(\int_{-\infty}^{-1} + \int_{1}^{\infty} \right) f(x) \frac{e^{-iux}}{-ix} \, dx + \frac{1}{2\pi} \int_{-1}^{1} f(x) \frac{e^{-iux} - 1}{-ix} \, dx . \qquad (1.1)$$

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We call this transform the *Wiener transformation*. By using a deep Tauberian theorem, he showed that

$$\|f\|_{\mathscr{M}^2} = \|W(f)\|_{\mathscr{V}^2}, \quad f \in \mathscr{W}^2 .$$

Recently, this result has been extended by Lee and the author [8] to include the fact that the Wiener transformation $W : \mathscr{M}^2 \longrightarrow \mathscr{V}^2$ is a surjective isomorphism. Moreover, the exact isomorphic constants have also been obtained. The theorem is an analog of the Plancherel theorem in the classical L^2 case. For $1 < p < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, W also defines a bounded linear operator from $\mathscr{M}^p$ into $\mathscr{V}^{p'}$.

It is the purpose of this paper to study the convolution operators on the Marcinkiewicz space $\mathscr{M}^p$, $1 \leqslant p < \infty$, and on the closed subspace $\mathscr{M}_r^p$ of regular functions f (i.e.,

$$\lim_{T \to \pm\infty} \frac{1}{T} \int_T^{T+a} |f|^p = 0$$

for $a > 0$). Some results related to this subject can be found in [2], [14], [15].

In [2], Bertrandias showed that for each bounded regular Borel measure μ on R, the convolution operator $\Phi_\mu : \mathscr{M}^p \longrightarrow \mathscr{M}^p$ given by $\Phi_\mu(f) = \mu * f$ is well defined and $\|\Phi_\mu\|_{\mathscr{M}^p} \leqslant \|\mu\|$. In § 2, we show that if μ satisfies $\int_R |x| \, d|\mu| < \infty$, then the restriction map $\Phi_\mu : \mathscr{M}_r^p \longrightarrow \mathscr{M}_r^p$ satisfies

$$\lim_{T \to \infty} \frac{1}{2T} \int_R |(\chi_T \Phi_\mu - \Phi_\mu \chi_T) f|^p = 0,$$

where χ_T is the characteristic function of $[-T, T]$. This is used to prove that for any bounded regular Borel measure μ,

$$\|\Phi_\mu\|_{\mathscr{M}_r^p} = \|\Phi_\mu\|_{L^p},$$

where $\|\Phi_\mu\|_{L^p}$ is the norm of the convolution operator Φ_μ on $L^p (= L^p(R))$ (Theorem 2.4).

Let $\mathscr{I}_{\mathscr{M}_r^p}$ ($\mathscr{I}_{L^p}$) denote the norm closure of the family of convolution operators on $\mathscr{M}_r^p$ (L^p, respectively). It follows from the result mentioned above that $\mathscr{I}_{\mathscr{M}_r^p}$ is isometrically isomorphic to $\mathscr{I}_{L^p}$.

However, under the strong operator topologies, the structures of the two spaces are quite different. We prove that in $\mathcal{S}_{\mathcal{M}^p_r}$, the strong operator sequential convergence and the norm convergence coincide (Theorem 2.6).

In § 3, we consider the convolution operator under the Wiener transformation $W : \mathcal{M}^p \longrightarrow \mathcal{V}^{p'}$, $1 < p \leqslant 2$. One of the difficulties in defining the multiplication operators on $\mathcal{V}^p$ is that even for a very "nice" function h, the pointwise multiplication

$$(h \cdot g)(u) = h(u) \cdot g(u), \quad g \in \mathcal{V}^p \tag{1.2}$$

does not give a function in $\mathcal{V}^p$. Let

$$\mathcal{D}^{1/p} = \{h : h(u+\epsilon) - h(u) = o(\epsilon^{1/p}) \text{ uniformly on } u\},$$

it is shown that if $g \in \mathcal{V}^p \cap L^p$ and $h \in \mathcal{D}^{1/p}$, then (1.2) defines a function in $\mathcal{V}^p$. In [8, Theorem 3.3], it was proved that for each $g \in \mathcal{V}^p$, there exists a $g' \in \mathcal{V}^p \cap L^p$ such that $\|g - g'\|_{\mathcal{V}_p} = 0$. Hence, for the above h, $h \cdot g$ can be defined to be the equivalence class in $\mathcal{V}^p$ containing $h \cdot g'$ (defined by (1.2)) where $g' \in \mathcal{V}^p \cap L^p$ and $\|g - g'\|_{\mathcal{V}^p} = 0$. The main result of this section is that for $1 < p \leqslant 2$ and for any bounded regular Borel measure μ such that the Fourier-Stieltjes transformation $\hat{\mu}$ is in $\mathcal{D}^{1/p'}$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, then W yields

$$W(\mu * f) = \hat{\mu} \cdot W(f), \quad f \in \mathcal{M}^p .$$

In particular, if μ satisfies $\int_{R} |x| \, d|\mu| < \infty$, then $\hat{\mu} \in \mathcal{D}^{1/p'}$ and the above equality holds.

In § 4, the results of § 3 are used to prove a Tauberian theorem on $\mathcal{M}^2$. If μ is a bounded regular Borel measure on R such that $\hat{\mu} \in \mathcal{D}^{1/2}$ and $\hat{\mu}(u) \neq 0$ $\forall u \in R$, and if $f \in \mathcal{M}^2$ satisfies

$$\|\mu * f\|_{\mathcal{M}^2} = \lim_{T \to \infty} \left(\frac{1}{2T} \int_{-T}^{T} |\mu * f|^2 \right)^{1/2} = 0,$$

then for any continuous measure $\nu \in M$ such that $\hat{\nu} \in \mathcal{D}^{1/2}$,

$$\|\nu * f\|_{\mathcal{M}^2} = \lim_{T \to \infty} \left(\frac{1}{2T} \int_{-T}^{T} |\nu * f|^2 \right)^{1/2} = 0.$$

This improves a result of Wiener [15, Theorem 29].

2. The Convolution Operators.

Let $\mathscr{M}^p$, $\mathscr{V}^p$ be defined as above. When there is no confusion, we will use the same notation $f \in \mathscr{M}^p$ ($\mathscr{V}^p$) to denote the function f on R as well as the equivalence class of functions in $\mathscr{M}^p$ ($\mathscr{V}^p$, respectively) whose difference from f has zero norm.

Let Φ be a bounded linear operator from a Banach space X into X and let $\|\Phi\|_X$ denote the norm of Φ on X.

PROPOSITION 2.1. – *Let* X *be a closed subspace of* $\mathscr{M}^p$ *such that* $L^p \subseteq X$ *and let* $\Phi : X \longrightarrow X$ *be a linear map. Suppose* Φ *satisfies the following conditions:*

i) *the restriction of* Φ *on* L^p *defines a bounded linear operator* $\Phi : L^p \longrightarrow L^p$,

ii) *for each* $f \in X$, $\displaystyle\lim_{T\to\infty} \frac{1}{2T}\int_R |(\chi_T \Phi - \Phi\chi_T) f|^p = 0$.

Then $\|\Phi\|_X \leqslant \|\Phi\|_{L^p}$.

Proof. – Let $f \in X$. Then

$$\left(\frac{1}{2T}\int_{-T}^{T} |\Phi(f)|^p\right)^{1/p}$$

$$\leqslant \left(\frac{1}{2T}\int_R |\Phi\cdot\chi_T f|^p\right)^{1/p} + \left(\frac{1}{2T}\int_R |(\chi_T\Phi - \Phi\chi_T) f|^p\right)^{1/p}$$

$$\leqslant \|\Phi\|_{L^p}\cdot\left(\frac{1}{2T}\int_{-T}^{T} |f|^p\right)^{1/p} + \left(\frac{1}{2T}\int_R |(\chi_T\Phi - \Phi\chi_T) f|^p\right)^{1/p}.$$

Taking the limit supremum on T yields

$$\|\Phi(f)\|_{\mathscr{M}^p} \leqslant \|\Phi\|_{L^p}\cdot\|f\|_{\mathscr{M}^p}$$

and $\|\Phi\|_X \leqslant \|\Phi\|_{L^p}$. □

Let M be the class of bounded, regular Borel measures on R and let M_1 be the dense subspace of $\mu \in M$ such that

$$\int_R |x|\, d|\mu| < \infty.$$

In [2, p. 19], Bertrandias showed that for each $\mu \in M$, the convolution operator $\Phi_\mu : \mathscr{M}^p \longrightarrow \mathscr{M}^p$ can be defined as the $\mathscr{M}^p$-limit of the functions $\int_{-A}^{B} f(x-y)\,d\mu(y)$ as $A, B \longrightarrow \infty$, $f \in \mathcal{M}^p$. Since $\mathcal{M}^p \subset \mathcal{M}^1$ and

$$\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} |f(x-y)|\, d\,|\mu|\,(y)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2T}\int_{-T}^{T} |f(x-y)|\,dx\,d\,|\mu|\,(y) < \infty,$$

the integral $\int_{-\infty}^{\infty} f(x-y)\,d\mu(y)$ exists for almost all x. We can write the pointwise expression of $\Phi_\mu(f)$ as

$$\Phi_\mu(f)(x) = (\mu * f)(x) = \int_{-\infty}^{\infty} f(x-y)\,d\mu(y).$$

In the following, the convolution operators on the closed subspace $\mathcal{M}_r^p$ of regular functions f (i.e. $\lim\limits_{T\to\pm\infty} \frac{1}{T}\int_T^{T+a} |f|^p = 0$ for $a > 0$) in $\mathcal{M}^p$ will be considered. Note that $f \in \mathcal{M}_r^p$ if and only if $\lim\limits_{T\to\pm\infty} \frac{1}{T}\int_T^{T+1} |f|^p = 0$. Also $\mathcal{W}^p \subset \mathcal{M}^p$. It is easy to show that if $\mu \in M$, $f \in \mathcal{M}_r^p$, then $\mu * f \in \mathcal{M}_r^p$.

LEMMA 2.2. – *Let* $\mu \in M_1$ *and let* $\Phi_\mu : \mathcal{M}_r^p \longrightarrow \mathcal{M}_r^p$ *be the convolution operator. Then* Φ_μ *satisfies*

$$\lim_{T\to\infty} \frac{1}{2T}\int_R |(\chi_T \Phi_\mu - \Phi_\mu \chi_T) f|^p = 0, \quad f \in \mathcal{M}_r^p.$$

Proof. – Let $f \in \mathcal{M}_r^p$ and let $\|\mu\| = 1$. For any $\epsilon > 0$, there exists an $a > 0$ such that

$$\int_{R\setminus[-a,a]} |y|\, d\,|\mu| < \epsilon$$

and a $T_0 > 1$ such that for $|T| > T_0$,

$$\frac{1}{2T}\int_T^{T+a} |f|^p < \epsilon$$

and for $T > T_0$,

$$\frac{1}{2T}\int_{-T}^{T} |f|^p \leqslant \|f\|_{\mathcal{M}^p}^p + \epsilon.$$

Now for $T > T_0$,

$$\int_{-\infty}^{\infty} |(\chi_T \Phi_\mu - \Phi_\mu \chi_T) f|^p$$
$$= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} (\chi_T(x) - \chi_T(x-y)) f(x-y)\, d\mu(y) \right|^p dx$$
$$\leqslant \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\chi_T(x) - \chi_T(x-y)) f(x-y)|^p\, d|\mu|(y)\, dx$$
$$= \iint_E |f(x-y)|^p\, d|\mu|(y)\, dx$$

where $E = E_1 \cup E_2 \cup E_3 \cup E_4$ with

$$E_1 = \{(x,y) : -T \leqslant x \leqslant T,\ x + T \leqslant y\},$$
$$E_2 = \{(x,y) : -T \leqslant x \leqslant T,\ y \leqslant x - T\},$$
$$E_3 = \{(x,y) : T \leqslant x,\ x - T \leqslant y \leqslant x + T\},$$

and

$$E_4 = \{(x,y) : x \leqslant -T,\ x - T \leqslant y \leqslant x + T\}.$$

On the region E_1, we have

$$\iint_{E_1} |f(x-y)|^p\, d|\mu|(y)\, dx$$
$$\leqslant \int_0^a \int_{-T}^{y-T} |f(x-y)|^p\, dx\, d|\mu|(y) + \int_a^\infty \int_{-T}^{T} |f(x-y)|^p\, dx\, d|\mu|(y)$$
$$\leqslant \left(\int_0^a d|\mu|\right)\left(\int_{-T-a}^{-T} |f(z)|^p\, dz\right) + \int_a^\infty \int_{-(T+y)}^{T+y} |f(z)|^p\, dz\, d|\mu|(y)\ .$$

This implies that

$$\frac{1}{2T} \iint_{E_1} |f(x-y)|^p\, d|\mu|(y)\, dx$$
$$\leqslant \epsilon + (\|f\|^p_{\mathscr{M}^p} + \epsilon) \int_a^\infty \frac{T+y}{T}\, d|\mu|(y)$$
$$\leqslant \epsilon + 2(\|f\|^p_{\mathscr{M}^p} + \epsilon)\,\epsilon\,. \tag{2.1}$$

Similarly, we can show that the inequality (2.1) also holds for E_i, $i = 2,3,4$. This completes the proof. □

It follows from Proposition 2.1 and Lemma 2.2 that

$$\|\Phi\|_{\mathscr{M}^p_r} \leqslant \|\Phi\|_{L^p}\,.$$

To obtain the reverse inequality, the following is required.

LEMMA 2.3. – *Let* $\mu \in M_1$ *and let* $f \in L^p$. *For any* $\epsilon > 0$, *there exists an* $\tilde{f} \in \mathcal{M}_r^p$ *such that*

i) $\|\tilde{f}\|^p_{\mathcal{M}^p} \leqslant \|f\|^p_{L^p} + \epsilon$,

ii) $\|\mu * \tilde{f}\|^p_{\mathcal{M}^p} \geqslant \|\mu * f\|^p_{L^p}$.

Proof. – Without loss of generality, we may assume that $\operatorname{supp} f \subseteq [-A, A]$, $\operatorname{supp} \mu \subseteq [-B, B]$ and $A, B > 1$. Let $C = A + B$, then $\operatorname{supp}(\mu * f) \subseteq [-C, C]$.

Let $T_1 = C$ and let $f_1 = f$. Suppose that T_{n-1}, f_{n-1} have been chosen, choose T_n such that

$$T_n > T_{n-1} + 2nC, \quad \frac{T_n}{T_n + 2nC} \geqslant \left(1 - \frac{1}{n}\right)$$

and

$$\frac{1}{T_n - C} \int_0^{T_n} \Big| \sum_{m=1}^{n-1} f_m \Big|^p < \frac{\epsilon}{2} .$$

Let

$$f_n = \frac{T_n}{n} \sum_{k=0}^{n-1} g_k ,$$

where

$$g_k(x) = f(x - T_n - 2kC) .$$

Since each f_n is composed of n disjoint copies of f and all of the f_n's are disjoint, it follows that the sequence $\{\mu * f_n\}$ has the same property. Let

$$\tilde{f} = 2^{1/p} \sum_{n=1}^{\infty} f_n .$$

To see that $\tilde{f} \in \mathcal{M}_r^p$, observe that $\tilde{f}$ is supported by

$$E = \bigcup_{n=1}^{\infty} [T_n - C, T_n + (2n-1)C],$$

and that

$$\overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{-T}^{T} |\tilde{f}|^p = \overline{\lim_{\substack{T \to \infty \\ T \in E}}} \frac{1}{2T} \int_{-T}^{T} |\tilde{f}|^p .$$

If n_0 is such that $\dfrac{T_{n_0}}{T_{n_0} - C} \|f\|^p_{L^p} \leqslant \|f\|^p_{L^p} + \dfrac{\epsilon}{2}$, then for $n > n_0$ and for $T \in [T_n - C, T_n + (2n-1)C]$,

$$\begin{aligned}\frac{1}{2\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}}|\widetilde{f}|^p &\leqslant \frac{1}{\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}}\left|\sum_{m=1}^{n} f_m\right|^p\\ &\leqslant \frac{1}{\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}}|f_n|^p + \frac{\epsilon}{2}\\ &\leqslant \frac{\mathrm{T}_n}{n\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}}\sum_{k=0}^{n-1}|g_k|^p\,dx + \frac{\epsilon}{2}\\ &\leqslant \frac{\mathrm{T}_n}{\mathrm{T}_n - \mathrm{C}}\|f\|_{\mathrm{L}^p}^p + \frac{\epsilon}{2}\\ &\leqslant \|f\|_{\mathrm{L}^p}^p + \epsilon .\end{aligned}$$

Moreover, for any T such that $\mathrm{T}_n - \mathrm{C} \leqslant \mathrm{T} \leqslant \mathrm{T}_{n+1} - \mathrm{C}$,

$$\frac{1}{2\mathrm{T}}\int_{\mathrm{T}}^{\mathrm{T}+1}|\widetilde{f}| \leqslant \frac{\mathrm{T}_n}{n\mathrm{T}}\|f\|_{\mathrm{L}^p}^p \leqslant \frac{1}{n}\cdot\frac{\mathrm{T}_n}{\mathrm{T}_n - \mathrm{C}}\|f\|_{\mathrm{L}^p}^p .$$

Hence $\widetilde{f} \in \mathcal{M}_r^p$ and satisfies i). To prove ii), we let

$$\mathrm{T} = \mathrm{T}_n + (2n-1)\,\mathrm{C} .$$

Then

$$\begin{aligned}\frac{1}{2\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}}|\mu * \widetilde{f}|^p &= \frac{1}{2\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}}\left|\sum_{m=1}^{n}\mu * f_m\right|^p\\ &\geqslant \frac{1}{2\mathrm{T}}\int_{-\mathrm{T}}^{\mathrm{T}}|\mu * f_n|^p\\ &\geqslant \frac{\mathrm{T}_n}{\mathrm{T}_n + (2n-1)\,\mathrm{C}}\|\mu * f\|_{\mathrm{L}^p}^p .\end{aligned}$$

This implies that

$$\|\mu * \widetilde{f}\|_{\mathcal{M}^p}^p \geqslant \|\mu * f\|_{\mathrm{L}^p}^p . \qquad \square$$

THEOREM 2.4. – *Let* $1 \leqslant p < \infty$ *and let* $\mu \in \mathrm{M}$. *Then the convolution operator* $\Phi_\mu : \mathcal{M}_r^p \longrightarrow \mathcal{M}_r^p$ *satisfies* $\|\Phi_\mu\|_{\mathcal{M}_r^p} = \|\Phi_\mu\|_{\mathrm{L}^p}$.

Proof. – It follows from Proposition 2.1, Lemma 2.2 and Lemma 2.3 that $\|\Phi_\mu\|_{\mathcal{M}_r^p} = \|\Phi_\mu\|_{\mathrm{L}^p}$ for $\mu \in \mathrm{M}_1$. For $\mu \in \mathrm{M}$, there exists a sequence $\{\mu_n\}$ in M_1 which converges to μ. Since

$$\|\Phi_\mu - \Phi_{\mu_n}\|_{\mathcal{M}_r^p} \leqslant \|\Phi_\mu - \Phi_{\mu_n}\|_{\mathcal{M}^p} \leqslant \|\mu - \mu_n\| ,$$

it follows that

$$\|\Phi_\mu\|_{\mathcal{M}_r^p} = \lim_{n\to\infty}\|\Phi_{\mu_n}\|_{\mathcal{M}_r^p} = \lim_{n\to\infty}\|\Phi_{\mu_n}\|_{\mathrm{L}^p} = \|\Phi_\mu\|_{\mathrm{L}^p} . \qquad \square$$

Let $\mathcal{I}_{\mathcal{M}_r^p}$ $(\mathcal{I}_{L^p})$ denote the norm closure of the class of convolution operators on $\mathcal{M}_r^p(L^p$, respectively), Theorem 2. 4 implies that $\mathcal{I}_{\mathcal{M}_r^p}$ and $\mathcal{I}_{L^p}$ are isometrically isomorphic. However, under the strong operator topologies, the two classes of operators are different (Theorem 2.6).

LEMMA 2.5. – *Let* $\{\Phi_{\mu_n}\}$ *be a sequence in* $\mathcal{I}_{\mathcal{M}_r^p}$. *Suppose* $\{\Phi_{\mu_n}\}$ *converges to zero under the strong operator topology. Then* $\{\Phi_{\mu_n}\}$ *converges to zero under the norm topology.*

Proof. – If the lemma were not true, then it follows from Theorem 2.4 and by passing to subsequence, we can assume that there exists a sequence $\{f_n\}$ in L^p and an $a > 0$ such that

$$\|f_n\|_{L^p} = 1 \quad \text{and} \quad \|\mu_n * f_n\|_{L^p}^p > a \quad \forall n \in N .$$

We will construct an $\tilde{f} \in \mathcal{M}_r^p$ such that

$$\|\mu_n * \tilde{f}\|_{\mathcal{M}^p}^p \geqslant a \quad \forall n \in N .$$

This contradicts the hypothesis that $\{\Phi_{\mu_n}\}$ converges to zero under the strong operator topology.

Without loss of generality assume that for each n ,

$$\operatorname{supp} f_n \subseteq [-A_n , A_n] , \quad \operatorname{supp} \mu_n \subseteq [-B_n , B_n] ,$$

and $\{A_n\}$, $\{B_n\}$ are increasing. Let $C_n = A_n + B_n$. In the following, we will define two sequences $\{T_n\}$ and $\{h_n\}$. Let $T_1 = C_1$, $h_1 = f_1$. Given T_{n-1} , h_{n-1} , choose T_n such that

$$T_n > T_{n-1} + 2nC_{n-1} + C_n , \quad \frac{T_n}{T_n + (2n+1)C_n} \geqslant \left(1 - \frac{1}{n}\right)$$

and

$$\frac{1}{T_n} \int_0^{T_n} \left| \sum_{m=1}^{n-1} h_m \right|^p < 1 .$$

Let

$$h_n(x) = \frac{T_n}{n} \sum_{k=1}^{n} f_k(x - T_n - 2(k-1)C_n)$$

and let

$$\tilde{f} = 2^{1/p} \sum_{n=1}^{\infty} h_n ,$$

then the same proof as in Lemma 2.3 shows that $\tilde{f} \in \mathcal{M}_r^p$ and $\|\mu_n * \tilde{f}\|_{\mathcal{M}^p} \geqslant a$. □

The following theorem follows immediately from Lemma 2.5.

THEOREM 2.6. – *Let $\mathcal{J}_{\mathcal{M}_r^p}$ be the closure of the family of convolution operators on $\mathcal{M}_r^p$. Then $\mathcal{J}_{\mathcal{M}_r^p}$ is a Banach algebra such that the strong operator sequential convergence and the norm convergence coincide.*

Note that under the strong operator topology, $\mathcal{J}_{L^p}$ is metrizable on bounded sets, hence Theorem 2.6 does not hold for $\mathcal{J}_{L^p}$.

3. The Multipliers.

In this section, we will consider the convolution operator under the Wiener transformation. First, we will define the operators on $\mathcal{V}^p$ of multiplying by scalar functions. We need the following proposition which was proved in [8].

PROPOSITION 3.1. – *Let $1 < p < \infty$. Then for any $g \in \mathcal{V}^p$, there exists a $g' \in \mathcal{V}^p \cap L^p$ such that $\|g - g'\|_{\mathcal{V}^p} = 0$.*

The proposition amounts to saying that by identifying functions whose difference has zero norm, each equivalence class has a representation in L^p.

For each $t \in R$, we use τ_t to denote the translation operator defined by

$$(\tau_t g)(u) = g(t + u)$$

where g is a function on R. For each $g \in \mathcal{V}^p$, we can rewrite the definition of $\|g\|_{\mathcal{V}^p}$ as

$$\|g\|_{\mathcal{V}^p} = \overline{\lim_{\epsilon \to 0^+}} (2\epsilon)^{-1/p} \|\tau_\epsilon g - \tau_{-\epsilon} g\|_{L^p} = \overline{\lim_{\epsilon \to 0^+}} \epsilon^{-1/p} \|\tau_\epsilon g - g\|_{L^p} .$$

Let $\mathcal{D}^{1/p}$ be the class of bounded functions on R such that

$$h(u + \epsilon) - h(u) = o(\epsilon^{1/p})$$

uniformly on u. Let $h \in \mathcal{D}^{1/p}$, let $g \in \mathcal{V}^p \cap L^p$ and let $h \cdot g$ be the pointwise multiplication of h and g. Then

$$\begin{aligned}\epsilon^{-1/p} \|\tau_\epsilon(h \cdot g) - h \cdot g\|_{L^p} &\leqslant \epsilon^{-1/p} \|h \cdot (\tau_\epsilon g - g)\|_{L^p} \\ &\quad + \epsilon^{-1/p} \|(\tau_\epsilon h - h) \cdot \tau_\epsilon g\|_{L^p} . \end{aligned} \tag{3.1}$$

Note that

$$\begin{aligned}
&\lim_{\epsilon\to 0^+} \epsilon^{-1/p} \, \|(\tau_\epsilon h - h)\cdot \tau_\epsilon g\|_{L^p} \\
&\qquad = \|\lim_{\epsilon\to 0^+} \epsilon^{-1/p} (h - \tau_{-\epsilon} h)\cdot g\|_{L^p} \quad \text{(by the dominated convergence theorem)} \\
&\qquad = 0\,. \qquad (3.2)
\end{aligned}$$

Hence, (3.1) and (3.2) imply

$$\|h\cdot g\|_{\mathscr{V}^p} \leqslant \|h\|_\infty \cdot \|g\|_{\mathscr{V}^p}\,.$$

It also follows from the above argument that if g and g' are in $\mathscr{V}^p \cap L^p$, then $h\cdot g = h\cdot g'$ in $\mathscr{V}^p$. We define for $h\in\mathscr{D}^{1/p}$ and for each $g\in\mathscr{V}^p$, the multiplication operator $\Psi_h(g)$ to be the equivalence class in $\mathscr{V}^p$ containing $h\cdot g'$ where $g'\in\mathscr{V}^p\cap L^p$ and $\|g - g'\|_{\mathscr{V}^p} = 0$. We still use $h\cdot g$ to denote $\Psi_h(g)$.

Remark. – For an arbitrary $g\in\mathscr{V}^p$, the pointwise multiplication $h\cdot g$ is not necessary a function in $\mathscr{V}^p$. For example, let $h(u) = e^{iu}$ and let $g(u) = 1$, $u\in R$, then the pointwise multiplication $h\cdot g$ is not in $\mathscr{V}^p$.

PROPOSITION 3.2. – *Let* $1 < p < \infty$ *and let* $h\in\mathscr{D}^{1/p}$. *Then the operator* $\Psi_h : \mathscr{V}^p \longrightarrow \mathscr{V}^p$ *defined above is a bounded linear operator with* $\|\Psi_h\|_{\mathscr{V}^p} \leqslant \|h\|_\infty$. *Moreover,*

$$\|\Psi_h(g)\|_{\mathscr{V}^p} = \overline{\lim_{\epsilon\to 0^+}}\, \epsilon^{-1/p} \, \|h\cdot(\tau_\epsilon g - g)\|_{L^p}\,.$$

Proof. – We need only prove the last formula. The expressions (3.1) and (3.2) imply that

$$\|\Psi_h(g)\|_{\mathscr{V}^p} \leqslant \overline{\lim_{\epsilon\to 0^+}}\, \epsilon^{-1/p}\, \|h\cdot(\tau_\epsilon g - g)\|_{L^p}\,.$$

The reverse inequality is obtained by interchanging the first two terms of (3.1) and applying (3.2) again. □

For each $\mu\in M_1$, it follows that

$$\begin{aligned}
\hat\mu'(u) &= \lim_{\epsilon\to 0^+} \frac{\hat\mu(u+\epsilon) - \hat\mu(u)}{\epsilon} = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_{-\infty}^{\infty} (e^{-i(u+\epsilon)x} - e^{-iux})\, d\mu(x) \\
&= -i\int_{-\infty}^{\infty} e^{-ixu}\cdot x d\mu(x)\,.
\end{aligned}$$

Hence, $\hat\mu(u+\epsilon) - \hat\mu(u) = o(\epsilon^{1/p})$ uniformly in u, i.e. $\hat\mu\in\mathscr{D}^{1/p}$.

COROLLARY 3.3. – *Let* $1 < p < \infty$ *and let* $\mu \in \mathrm{M}$ *such that* $\hat{\mu} \in \mathcal{D}^{1/p}$. *Then the operator* $\Psi_{\hat{\mu}} : \mathcal{V}^p \longrightarrow \mathcal{V}^p$ *is a bounded linear operator with* $\| \Psi_{\hat{\mu}} \|_{\mathcal{V}^p} \leqslant \| \hat{\mu} \|_\infty$. *In particular, if* $\mu \in \mathrm{M}_1$, *then* μ *satisfies the inequality.*

Let W be the Wiener transformation defined by (1.1).

THEOREM 3.4 [8]. – *The Wiener transformation* W *defines a bounded linear operator from* $\mathcal{M}^p$ *into* $\mathcal{V}^{p'}$, $1 < p \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$.

In particular, if $p = 2$, *then* W *is an isomorphism from* $\mathcal{M}^2$ *onto* $\mathcal{V}^2$ *with*

$$\| \mathrm{W} \| = \left(\int_0^\infty h(x)\, dx \right)^{1/2}, \qquad \| \mathrm{W}^{-1} \| = \left(\max_{x \geqslant 0} x \widetilde{h}(x) \right)^{-1/2},$$

where

$$h(x) = \frac{2 \sin^2 x}{\pi x^2} \quad \text{and} \quad \widetilde{h}(x) = \sup_{t \geqslant x} h(x), \quad x \geqslant 0 .$$

LEMMA 3.5. – *Let* $1 < p < \infty$ *and let* $h \in \mathcal{D}^{1/p}$. *Suppose* $g \in \mathcal{V}^p$ *and* $g' \in \mathcal{V}^p \cap \mathrm{L}^p$ *are such that* $\| g - g' \|_{\mathcal{V}^p} = 0$. *Then*

$$\overline{\lim_{\epsilon \to 0^+}}\ \epsilon^{-1/p} \| h \cdot (\tau_\epsilon g - \tau_{-\epsilon} g) - (\tau_\epsilon (h \cdot g') - \tau_{-\epsilon} (h \cdot g')) \|_{\mathrm{L}^p} = 0$$

(where the involved multiplications are pointwise multiplication).

Proof. – Observe that

$$\begin{aligned} &\overline{\lim_{\epsilon \to 0^+}}\ \epsilon^{-1/p} \| h \cdot (\tau_\epsilon g - \tau_{-\epsilon} g) - \tau_\epsilon (h \cdot g') - \tau_{-\epsilon} (h \cdot g')) \|_{\mathrm{L}^p} \\ &\qquad \leqslant \overline{\lim_{\epsilon \to 0^+}}\ \epsilon^{-1/p} \| h \cdot (\tau_\epsilon (g - g') - \tau_{-\epsilon} (g - g')) \|_{\mathrm{L}^p} \\ &\qquad + \overline{\lim_{\epsilon \to 0^+}}\ \epsilon^{-1/p} \| (\tau_\epsilon h - h) \cdot \tau_\epsilon g' \|_{\mathrm{L}^p} \\ &\qquad + \overline{\lim_{\epsilon \to 0^+}}\ \epsilon^{-1/p} \| (\tau_{-\epsilon} h - h) \cdot \tau_{-\epsilon} g' \|_{\mathrm{L}^p} . \end{aligned}$$

The first term is not greater than

$$\| h \|_\infty \ \overline{\lim_{\epsilon \to 0^+}}\ \epsilon^{-1/p} \| \tau_\epsilon (g - g') - \tau_{-\epsilon} (g - g') \|_{\mathrm{L}^p}$$

which is equal to $\| h \|_\infty \cdot \| g - g' \|_{\mathcal{V}^p}$ and by hypothesis, it equals zero. By an argument similar to (3.2), the second and the third term are also zero. This completes the proof of the lemma. □

For an $f \in \mathrm{L}^p$, $1 < p \leqslant 2$, we will use $\hat{f}$ to denote the Fourier transformation of f in $\mathrm{L}^{p'}$. It is well known that for the above f,

$$\left(\int_{\mathrm{R}} |\hat{f}(u)|^{p'} \frac{du}{\sqrt{2\pi}}\right)^{1/p'} \leqslant \left(\int_{\mathrm{R}} |f(x)|^p \frac{dx}{\sqrt{2\pi}}\right)^{1/p} :$$

THEOREM 3.6. – *Let* $1 < p \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$. *Then for any* $f \in \mathscr{M}^p$, $\mu \in \mathrm{M}$ *such that* $\hat{\mu} \in \mathscr{D}^{1/p'}$,

$$\mathrm{W}(\mu * f) = \hat{\mu} \cdot \mathrm{W}f \quad \text{in} \quad \mathscr{V}^{p'} .$$

Proof. – First consider the case that μ has bounded support, say, $\operatorname{supp} \mu \subseteq [-\mathrm{A}, \mathrm{A}]$. Without loss of generality assume that $\|\mu\| = 1$ and let

$$\mathrm{W}(f) = g \quad \text{and} \quad \mathrm{W}(\mu * f) = g_1 .$$

In view of Lemma 3.5, it suffices to show that

$$\overline{\lim_{\epsilon \to 0^+}} \ \epsilon^{-1/p'} \|(\tau_\epsilon g_1 - \tau_{-\epsilon} g_1) - \hat{\mu} \cdot (\tau_\epsilon g - \tau_{-\epsilon} g)\|_{\mathrm{L}^{p'}} = 0 .$$

Since $(\tau_\epsilon g - \tau_{-\epsilon} g)$ is the Fourier transformation of

$$h(x) = \sqrt{\frac{2}{\pi}}\, f(x) \frac{\sin \epsilon x}{x} ,$$

it follows that $(\tau_\epsilon g_1 - \tau_{-\epsilon} g_1)$ is the Fourier transformation of

$$h_1(x) = \sqrt{\frac{2}{\pi}}\, (\mu * f)(x) \frac{\sin \epsilon x}{x} ,$$

and both h_1 and h are in L^p (cf. [8, Theorem 5.5]). Hence

$$\begin{aligned}
&(2\epsilon)^{-1/p'} \|(\tau_\epsilon g_1 - \tau_{-\epsilon} g_1) - \hat{\mu} \cdot (\tau_\epsilon g - \tau_{-\epsilon} g)\|_{\mathrm{L}^{p'}} \\
&= (2\epsilon)^{-1/p'} \|(h_1 - h)\hat{}\,\|_{\mathrm{L}^{p'}} \\
&= (2\epsilon)^{-1/p'} \left(\sqrt{2\pi} \int_{-\infty}^{\infty} |(h_1 - h)\hat{}\,|^{p'} \frac{du}{\sqrt{2\pi}}\right)^{1/p'} \\
&\leqslant (2\epsilon)^{-1/p'} (2\pi)^{1/2p'} \left(\int_{-\infty}^{\infty} |h_1 - h|^p \frac{du}{\sqrt{2\pi}}\right)^{1/p} \\
&= \left(\frac{1}{\pi \epsilon^{p-1}} \int_{-\infty}^{\infty} \left| \int_{-\mathrm{A}}^{\mathrm{A}} f(x-y) \left(\frac{\sin \epsilon x}{x} - \frac{\sin \epsilon (x-y)}{x-y}\right) d\mu(y) \right|^p dx \right)^{1/p}
\end{aligned}$$

$$\leqslant \left(\frac{1}{\pi \epsilon^{p-1}} \int_{-\infty}^{\infty} \int_{-A}^{A} |f(x-y)|^p \left| \frac{\sin \epsilon x}{x} - \frac{\sin \epsilon (x-y)}{x-y} \right|^p d|\mu|(y)\, dx \right)^{1/p}$$

$$\leqslant \left(\frac{1}{\pi \epsilon^{p-1}} \int_{-\infty}^{\infty} \int_{-A}^{A} |f(x-y)|^p \left(\frac{8\epsilon\, |y|}{|x| + |y|} \right)^p d|\mu|(y)\, dx \right)^{1/p}$$

(by [15, p. 157])

$$\leqslant 8\pi^{-1/p} \cdot \epsilon^{1/p} \left(\int_{|y|<1} \int_{-\infty}^{\infty} |f(x-y)|^p \frac{1}{|x|^p + 1}\, dx\, d|\mu|(y) \right.$$

$$\left. + \int_{1<|y|<A} \left(\int_{-\infty}^{\infty} |f(x-y)|^p \frac{1}{|x|^p + 1}\, dx \right) |y|^p\, d|\mu|(y) \right)$$

The fact that $\mathscr{M}^p \subseteq L^p\left(R, \frac{dx}{|x|^p + 1}\right)$ [8, Proposition 2.1] implies that the last two terms of the above inequality are bounded. Hence

$$\overline{\lim_{\epsilon \to 0^+}}\ \epsilon^{-1/p'} \| (\tau_\epsilon g_1 - \tau_{-\epsilon} g_1) - \mu \cdot (\tau_\epsilon g - \tau_{-\epsilon} g) \|_{L^{p'}} = 0 .$$

This completes the proof of the theorem for measures μ with bounded support. Now, for any $\mu \in M_1$, there exists a sequence of $\{\mu_n\}$ with bounded support such that $\|\mu_n - \mu\| \longrightarrow 0$ as $n \longrightarrow \infty$. Corollary 3.3 implies

$$\| \Psi_{\mu_n} - \Psi_\mu \|_{\mathscr{V}^{p'}} \leqslant \| \hat{\mu}_n - \hat{\mu} \|_\infty \leqslant \| \mu_n - \mu \| .$$

Hence

$$W(\mu * f) = \lim_{n \to \infty} W(\mu_n * f) = \lim_{n \to \infty} \hat{\mu}_n \cdot W(f) = \hat{\mu} \cdot W(f) . \qquad \square$$

Let $\mu \in M$ and define the multiplication operator $\Psi_{\hat{\mu}} : \mathscr{V}^p \longrightarrow \mathscr{V}^p$ as the limit of Ψ_{μ_n}, $\mu_n \in \mathscr{D}^{1/p}$.

COROLLARY 3.7. – *Let* $1 < p \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$. *For each* $\mu \in M$, *let* Φ_μ *be the convolution operator of* μ *on* $\mathscr{M}^p$ *and let* $\Psi_{\hat{\mu}}$ *be the multiplication oqerator on* $\mathscr{V}^p$. *Then for any* $f \in \mathscr{M}^p$,

$$W(\Phi_\mu f) = \Psi_{\hat{\mu}} (W(f)) .$$

Let $\mathscr{V}_r^2 = W(\mathscr{M}_r^2)$, then the following result follows from Theorem 2.4, Corollary 3.3, Theorem 3.4 and Theorem 3.6.

COROLLARY 3.8. – *For each* $\mu \in M$, *we have*

$$C^{-1} \| \Phi_\mu \|_{\mathscr{M}_r^2} \leqslant \| \Psi_{\hat{\mu}} \|_{\mathscr{M}_r^2} \leqslant \| \Phi_\mu \|_{\mathscr{M}_r^2} = \| \hat{\mu} \|_\infty$$

where $C = \|W\| \cdot \|W^{-1}\|$.

4. A Tauberian Theorem.

In [15, Theorem 29], Wiener proved a Tauberian theorem on $\mathcal{M}^2$. In this section, by making use of his idea and the results in the previous section, we can simplify his argument and extend the theorem.

LEMMA 4.1. – *Let* $\mu \in \mathrm{M}$ *such that* $\hat{\mu} \in \mathcal{D}^{1/2}$ *and* $\hat{\mu}(u) \neq 0$ *for all* u *in* R. *If* $f \in \mathcal{M}^2$ *is such that* $\|\mu * f\|_{\mathcal{M}^2} = 0$. *Then* $g = \mathrm{W}(f)$ *satisfies*

$$\overline{\lim_{\epsilon \to 0^+}} \frac{1}{\epsilon} \int_{-C}^{C} |g(u+\epsilon) - g(u)|^2 \, du = 0 \qquad \forall C > 0 .$$

Proof. – Since $\hat{\mu}$ is continuous and $\hat{\mu} \neq 0$, there exists a $Q > 0$ such that $|\hat{\mu}(u)| > Q$ for all $u \in [-C, C]$. Hence

$$\begin{aligned}
&\overline{\lim_{\epsilon \to 0^+}} \frac{Q^2}{\epsilon} \int_{-C}^{C} |g(u+\epsilon) - g(u)|^2 \, du \\
&\qquad \leqslant \overline{\lim_{\epsilon \to 0^+}} \frac{1}{\epsilon} \int_{-\infty}^{\infty} |\hat{\mu}(u)|^2 \, |g(u+\epsilon) - g(u)|^2 \, du \\
&\qquad = \|\mathrm{W}(\mu * f)\|_{\mathcal{V}^2}^2 \quad \text{(by Proposition 3.2 and Theorem 3.6)} \\
&\qquad \leqslant \|\mathrm{W}\|^2 \cdot \|\mu * f\|_{\mathcal{M}^2}^2 \\
&\qquad = 0 .
\end{aligned}$$

□

LEMMA 4.2. – *Let* ν *be a continuous measure in* M *such that* $\hat{\nu} \in \mathcal{D}^{1/2}$. *Let* $f \in \mathcal{M}^2$ *and let* $g = \mathrm{W}(f)$. *Then*

$$\lim_{C \to \infty} \overline{\lim_{\epsilon \to 0^+}} \frac{1}{\epsilon} \left(\int_{-\infty}^{-C} + \int_{C}^{\infty} \right) |\hat{\nu}(u)|^2 \, |g(u+\epsilon) - g(u)|^2 = 0 .$$

Proof. – We will estimate the following limit :

$$\lim_{\eta \to 0^+} \overline{\lim_{\epsilon \to 0^+}} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|^2 \, |\hat{\nu}(u)|^2 \, |g(u+\epsilon) - g(u)|^2 .$$

Since ν is a continuous measure, $\lim_{|u| \to \infty} \hat{\nu}(u) = 0$. Also note that

$$\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|$$

is bounded, and for any $A > 0$,

$$\lim_{\eta \to 0^+} \left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right| = 0 \text{ uniformly for } u \in [-A, A].$$

For $\epsilon_0 > 0$, there exists A_0 such that for $A \geq A_0$, $|\hat{\nu}(u)| \leq \frac{\epsilon_0}{K_1}$ where $K_1 (> 1)$ is the bound of $\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|$. There exists η_0 such that for $0 < \eta < \eta_0$

$$\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right| < \frac{\epsilon_0}{K_2}, \quad u \in [-A_0, A_0],$$

where K_2 (> 1) is a bound of $\hat{\nu}$ in $[-A_0, A_0]$. Hence, for $0 < \eta < \eta_0$,

$$\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right| \cdot |\hat{\mu}(u)| < \epsilon_0, \quad u \in \mathrm{R},$$

and

$$\overline{\lim_{\epsilon \to 0^+}} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|^2 |\hat{\nu}(u)|^2 \, |g(u + \epsilon) - g(u)|^2 \leq \epsilon_0 \, \|g\|_{\mathscr{V}^2}.$$

This implies

$$\lim_{\eta \to 0^+} \overline{\lim_{\epsilon \to 0^+}} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|^2 |\hat{\nu}(u)|^2 \, |g(u + \epsilon) - g(u)|^2 = 0 .$$

Since $\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right| > \frac{1}{2}$ for any $u\eta > 4$, we have

$$\lim_{\eta \to 0^+} \overline{\lim_{\epsilon \to 0^+}} \frac{1}{\epsilon} \left(\int_{-\infty}^{-4/\eta} + \int_{4/\eta}^{\infty} \right) |\hat{\nu}(u)|^2 \, |g(u + \epsilon) - g(u)|^2 = 0 . \ \square$$

THEOREM 4.3. – *Let* $\mu \in \mathrm{M}$ *such that* $\hat{\mu} \in \mathscr{D}^{1/2}$ *and* $\hat{\mu}(u) \neq 0$ *for all* u *in* R. *Suppose* $f \in \mathscr{M}^2$ *satisfies*

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\mu * f|^2 = 0 .$$

Then for any continuous measure $\nu \in \mathrm{M}$ *such that* $\hat{\nu} \in \mathscr{D}^{1/2}$,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\nu * f|^2 = 0 .$$

Proof. – Lemma 4.1 implies that for any $C > 0$,

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{-C}^{C} |\hat{\nu}(u)|^2 \, |g(u + \epsilon) - g(u)|^2 = 0 .$$

Also by Lemma 4.2,

$$\lim_{C \to \infty} \overline{\lim_{\epsilon \to 0^+}} \frac{1}{\epsilon} \left(\int_{-\infty}^{-C} + \int_{C}^{\infty} \right) |\hat{\nu}(u)|^2 \, |g(u + \epsilon) - g(u)|^2 = 0 .$$

This implies that $\| \hat{\nu} \cdot g \|_{\mathscr{V}^2} = 0$. By Theorem 3.4 and Theorem 3.6, $\| \nu * f \|_{\mathscr{V}^2} = 0$. □

5. Some Remarks.

In Section 2, we proved that the convolution operator $\Phi_\mu : \mathscr{M}_r^p \longrightarrow \mathscr{M}_r^p$ satisfies $\| \Phi_\mu \|_{\mathscr{M}_r^p} = \| \Phi_\mu \|_{L^p}$, we do not know whether or not $\Phi_\mu : \mathscr{M}^p \longrightarrow \mathscr{M}^p$ will satisfy the same equality.

An operator $\Phi : L^p \longrightarrow L^p$ is called a *multiplier* if $\Phi \tau_t = \tau_t \Phi$ for $t \in R$. The relationship of multipliers and the equation $\Phi(f)^{\hat{}} = h \cdot f^{\hat{}}$ for some bounded function h on R is generally well known. Also, the class of multipliers on L^p equals the strong-operator closure of the class of convolution operators. However, nothing is known for the multipliers on $\mathscr{M}^p$. It would be nice to have complete characterizations of the multiplier on $\mathscr{M}^p$, especially on $\mathscr{M}^2$.

In Section 4, we can only prove the Tauberian theorem on $\mathscr{M}^2$ (Theorem 4.3). For $1 < p < 2$, the Wiener transformation is well defined. All the proofs in Section 4 will go through except the last step in Theorem 4.3. It depends on the following statement which has to be justified:

For $1 < p < 2$, *the Wiener transformation* $W : \mathscr{M}^p \longrightarrow \mathscr{V}^{p'}$ *is one to one.*

Note that the statement is true for the Fourier transformation from L^p to $L^{p'}$, $1 \leqslant p < 2$.

In our Tauberian Theorem, we have to assume that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\mu * f|^2 = 0 .$$

We do not know whether the conclusion holds if we let $f \in \mathscr{W}^2$ and replace the zero by a positive number. Also, we do not know whether the condition on μ and ν in Theorem 4.3 can be relaxed.

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Ka-Sing LAU,

University of Pittsburgh
Faculty of Arts and Sciences
Department of Mathematics & Statistics
Pittsburgh, Pa. 15260 (USA).