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TISCHLER FIBRATIONS OF OPEN, FOLIATED SETS

by J. CANTWELL (*) and L. CONLON (**)

Introduction.

Let M be a smooth, closed n-manifold, $\mathcal{F}$ a foliation of M of codimension one. Unless otherwise specified, we will assume only that $\mathcal{F}$ has C^∞ leaves integral to a C^0 hyperplane field ($\mathcal{F}$ is said to be of class C^{0+}). We will further require that M be orientable and that $\mathcal{F}$ be transversely orientable.

If each leaf of $\mathcal{F}$ is everywhere dense without holonomy, then [10., Theorem 4] implies the existence of a transverse, holonomy invariant, positive measure, finite on compact sets. As in the proof of [10., Theorem 6], it follows that M admits a possibly new C^∞ structure in which the C^∞ structures of the leaves of $\mathcal{F}$ are unchanged and in which $\mathcal{F}$ is defined by a closed, nonsingular 1-form ω. By a theorem of D. Tischler [11], the manifold M, in this new structure, fibers smoothly over S^1 and such fibrations can be found arbitrarily C^∞-close to $\mathcal{F}$. Also, as seems to be well known to experts, these approximating fibrations can be chosen so that the leaves of $\mathcal{F}$ are regular coverings of the fibers in a very natural way, the covering group being a subgroup of co-rank 1 in the group $P(\omega) = \mathrm{Im}\,(\omega : \pi_1(M)\to\mathbf{R})$ of periods of ω.

More generally, suppose that $U \subset M$ is an open, connected, $\mathcal{F}$-saturated subset, each leaf of $\mathcal{F}|U$ being dense in U with trivial holonomy. Such sets are prominent among the fundamental building blocks of C^2 foliations [1], [13]. For instance, such a set U is the

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necessary ambience for any leaf at finite level with an « exotic » nonexponential growth type [1, (3.6) and (3.7)]. Let $\hat{U}$ be the completion of U in the sense of G. Hector [8] and P. Dippolito [5]. This is a manifold with finitely many boundary components [5, Proposition 2] and, generally, it is not compact. The foliation $\mathscr{F}$ induces a C^{0+} foliation $\hat{\mathscr{F}}$ of $\hat{U}$ having each component of $\partial\hat{U}$ as a leaf. The above method of finding a new C^∞ structure generalizes to $\hat{U}$, making $\hat{\mathscr{F}}$ a C^∞ foliation, C^∞-trivial at $\partial\hat{U}$, such that $\hat{\mathscr{F}}|U(=\mathscr{F}|U)$ is defined by a closed, nonsingular 1-form ω on U.

Here we investigate the possibility of smoothly approximating $\hat{\mathscr{F}}$ over precompact regions by a C^∞ foliation $\mathscr{F}^*$ (called a *Tischler foliation*) of $\hat{U}$, C^∞-trivial at $\partial\hat{U}$, such that $\mathscr{F}^*|U$ fibers U over S^1. When that is possible, we further investigate the possibility of choosing these fiberings of U so that the leaves of $\mathscr{F}|U$ are regular coverings of the fibers in a suitably natural way. These questions are of interest, of course, only for $\dim(M) > 2$.

If $\dim(M) = 3$, we find that Tischler foliations always exist (2.1), but we give smooth counterexamples in all dimensions greater than three (4.5). A condition guaranteeing the existence of Tischler foliations in arbitrary dimensions is that the period group $P(\omega)$ be free abelian (2.2). In particular, this gives Tischler foliations if (1) $\hat{U}$ is compact, or (2) each leaf of $\mathscr{F}|U$ has two dense ends, or (3) $\mathscr{F}$ is transversely analytic (cf. (3.10), (3.11), and Remark (2) following (3.11)). This condition on $P(\omega)$ also implies the result about regular coverings (3.8), but even on 3-manifolds, where Tischler foliations always exist, the regular covering property often fails when $P(\omega)$ is not free abelian (3.9).

1. Technical preliminaries.

Fix $M, \mathscr{F}$, and $U \subset M$ as in the introduction. Fix a transverse, smooth, 1-dimensional foliation $\mathscr{L}$. As in [1, (1.6)], obtain the transverse, invariant measure μ for $\mathscr{F}|U$ and the associated C^0 flow Φ : $\mathbf{R} \times M \to M$, nonsingular precisely on U, having as flow lines in U the leaves of $\mathscr{L}|U$, and preserving the foliation $\mathscr{F}$. Let $P(\mu) \subset \mathbf{R}$ be the additive subgroup of periods of μ [1, (1.7)]. That is, $t \in P(\mu)$ if and only if Φ_t carries some (hence every) leaf of $\mathscr{F}|U$ onto itself.

The following is proven by reasoning, familiar-to-specialists, entirely similar to that in [10, Theorem 6].

(1.1) LEMMA. – *There is a possibly new differentiable structure on* $\hat{U}$ *under which*

(1) $\hat{\mathscr{F}}$ *is of class* C^∞ *and is* C^∞*-trivial at* $\partial\hat{U}$;

(2) *The differentiable structure on each leaf of* $\hat{\mathscr{F}}$ *remains unchanged;*

(3) $\mathscr{F}|U$ *is defined by a closed, nonsingular form* $\omega \in A^1(U)$, *and* $P(\mu) = P(\omega)$.

Indeed, a new C^∞ structure is chosen in $\hat{U}$ so that the local leaves of $\hat{\mathscr{L}}$ (the 1-dimensional foliation of $\hat{U}$ induced by $\mathscr{L}$) are the level sets of the first $n-1$ local coordinates, and the flow parameter of Φ provides the n^{th} coordinate. Of course, at the boundary this n^{th} coordinate takes values $\pm\infty$, where we use a smooth structure on $[-\infty,\infty]$ relative to which the group of translations acts smoothly and is C^∞-flat at $\pm\infty$. The coordinate transformations are of the form $x_i = x_i(\bar{x}_1,\ldots,\bar{x}_{n-1})$, $1 \leqslant i \leqslant n-1$, $x_n = \bar{x}_n + c$, c constant, so (1) and (2) follow. The form ω will be well defined on U by the local formulas $\omega = dx_n$. The equality of $P(\mu)$ and $P(\omega)$ is elementary.

We are going to express ω in terms of a carefully chosen basis of $H^1(\hat{U};\mathbf{R})$.

Decomposition of ω. – Recall Dippolito's decomposition [5, Theorem 1] of $\hat{U}$ into a compact, connected manifold K with corners, called the *nucleus*, and noncompact « arms » $\hat{U}_j \cong B_j \times [-1,1]$, $1 \leqslant j \leqslant r$, where B_j is a complete, non-compact, connected, $(n-1)$-*dimensional submanifold of a component of* $\partial\hat{U}$, ∂B_j is compact and connected, and each $\{x\} \times [-1,1]$ is a leaf of $\hat{\mathscr{L}}$. By attaching to K successively larger chunks of the arms, we construct a sequence of nuclei

$$K = K_0 \subset K_1 \subset \cdots \subset K_i \subset \cdots$$

such that $\hat{U} = \bigcup_{i\geqslant 0} K_i$ and each $K_i \subset \operatorname{int}(K_{i+1})$ (interior relative to $\hat{U}$). Remark that the number of arms attached to K_i may become unbounded as $i \to \infty$.

The inclusions $K_i \hookrightarrow \hat{U}$ induce homomorphisms λ_i :

$$H_1(K_i;\mathbf{R}) \longrightarrow H_1(\hat{U};\mathbf{R}),$$

and we set $A_i = \operatorname{Im}(\lambda_i)$, a subspace of $H_1(\hat{U};\mathbf{R})$ of finite dimension $n(i)$. Remark that $H_1(\hat{U};\mathbf{R}) = \bigcup_{i\geqslant 0} A_i$.

Choose *integral* cycles $\sigma_1, \ldots, \sigma_{n(0)}$ in U which represent a basis of A_0, integral cycles $\sigma_{n(0)+1}, \ldots, \sigma_{n(1)}$, $n(1) \geqslant n(0)$, in U which represent a possibly trivial extension of this basis to a basis of A_1, etc. This gives rise to a possibly infinite basis $[\sigma_1]$, $[\sigma_2], \ldots, [\sigma_k], \ldots$ of $H_1(\hat{U};\mathbf{R})$.

Choose closed forms $\omega_1, \omega_2, \ldots$ in $A^1(\hat{U})$ such that $\omega_i(\sigma_j) = \delta_{ij}$. If σ_j does not represent an element of A_i, then $\omega_j|K_i = dh$ for some smooth $h : K_i \to \mathbf{R}$. One smoothly extends h to $\hat{h} : \hat{U} \to \mathbf{R}$ by standard techniques and replaces ω_j by $\omega_j - d\hat{h}$ so as to guarantee that $\omega_j|K_i \equiv 0$. Thus, each point of $\hat{U}$ has a neighborhood on which only finitely many of the forms ω_j are not identically zero.

A further wrinkle is needed in the choice of these forms. Let W be a neighborhood of $\partial\hat{U}$ in $\hat{U}$ such that (see figure 1) :

(*a*) $\hat{U} - K_0 \subset W$;

(*b*) the components of $W \cap K_0$ are disjoint collar neighborhoods of the respective components of $\partial\hat{U} \cap K_0$, fibered by $\hat{\mathscr{L}}|(W \cap K_0)$.

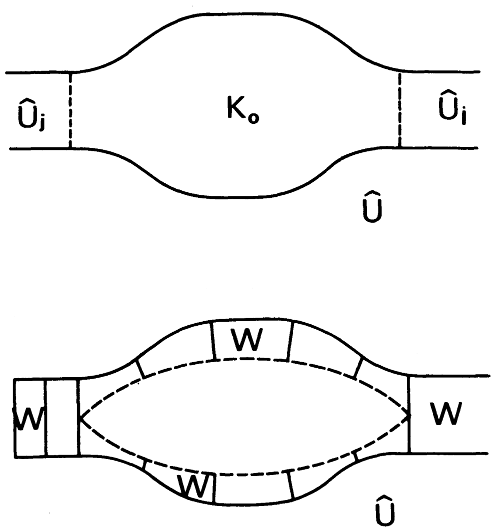

Fig. 1.

Thus, in each component of $W \cap K_0$, we have a canonical choice of projection p into $\partial\hat{U}$ along the leaves of $\hat{\mathscr{L}}$. In each component of $\hat{U} - K_0$, we have two such choices of p.

Fix ω_j. We will find a closed form $\eta \in A^1(\partial\hat{U})$ and a smooth function $h : W \to \mathbf{R}$ such that $\omega_j|W = p^*(\eta) + dh$ unambigously. Damping h

off to zero near the boundary of W in U and extending by 0 defines a smooth function $\hat{h} : \hat{U} \to \mathbf{R}$ such that $\omega_j - d\hat{h}$ vanishes on the tangents to $\hat{\mathscr{L}}$ both near $\partial\hat{U}$ and outside of (say) K_1. We replace ω_j with $\omega_j - d\hat{h}$. We have to take precautions to insure that the local finiteness of $\{\omega_j\}_{j\geqslant 1}$ is not destroyed. Here are more details.

(1) For each component L_k of $\partial\hat{U}$, choose $\eta_k \in A^1(L_k)$ that pulls back via p to the appropriate part of W as a form cohomologous to ω_j.

(2) If L_k and L_q are two components of $\partial\hat{U}$ such that some arm $\hat{U}_i \cong B_i \times [-1,1]$ has $B_i \times \{-1\} \subset L_k$, $B_i \times \{1\} \subset L_q$, the forms η_k and η_q restrict to cohomologous forms on B_i, so similar adjustments as above allow us to assume that these restrictions are equal. This guarantees the non-ambiguity of $p^*(\eta)$.

(3) If $\omega_j | K_i \equiv 0$, we can choose η to vanish on $K_i \cap \partial\hat{U}$ and h to vanish on $W \cap K_i$. This guarantees the local finiteness.

Let $c_j = \omega(\sigma_j)$ and consider the sum $\hat{\omega} = \sum_j c_j\omega_j$. This sum is locally finite and each ω_j is closed, so $\hat{\omega}$ is a closed 1-form on $\hat{U}$. Also, $\hat{\omega}$ vanishes on the tangents to the leaves of $\hat{\mathscr{L}}$ both near $\partial\hat{U}$ and in $\hat{U} - K_1$.

Since $H^1(U;\mathbf{R})$ is the dual vector space to $H_1(U;\mathbf{R})$ and $U \hookrightarrow \hat{U}$ is a homotopy equivalence, the following lemmas are easy consequences of our constructions.

(1.2) Lemma. — *There is a smooth function* $g : U \to \mathbf{R}$ *such that* $\omega = \hat{\omega}|U + dg$. *Near* $\partial\hat{U}$ and in $\hat{U} - K_1$, *the restrictions of* ω *to the leaves of* $\mathscr{L}|U$ agree with those of dg. *In particular,* dg *is nonsingular in those regions and it is unbounded near* $\partial\hat{U}$.

(1.3) Lemma. — *Let* $W_0 \subset \hat{U}$ *be an open, relatively compact set. Fix* $i \geqslant 1$ *such that* $W_0 \subset K_i$. *If numbers* $\tilde{c}_j \in \mathbf{R}$ *are chosen,* $j \geqslant 1$, *so that* $\tilde{c}_1, \ldots, \tilde{c}_{n(i)}$ *are sufficiently near* $c_1, \ldots, c_{n(i)}$ *respectively, then* $\tilde{\omega} = \sum_j \tilde{c}_j(\omega_j|U) + dg$ *is a closed, nonsingular* 1-*form on* U, *defining a foliation* $\tilde{\mathscr{F}}$ *transverse to* $\mathscr{L}|U$, *and* $\tilde{\omega}|(W_0 \cap U)$ *is as* C^∞-*close to* $\omega|(W_0 \cap U)$ *as desired.*

Practically as immediate is the following.

(1.4) LEMMA. — *The foliation* $\tilde{\mathscr{F}}$ *of* (1.3) *can be extended to a* C^∞ *foliation* $\mathscr{F}^*$ *of* $\hat{U}$, C^∞*-trivial at* $\partial\hat{U}$, *by letting each component of* $\partial\hat{U}$ *be a leaf.*

Indeed, the local flows on U produced by $\tilde{\omega}$ and having flow lines along $\mathscr{L}|U$ agree with Φ outside a compact subset of U, hence they can be assembled into a smooth global flow $\tilde{\Phi}$ on U that preserves $\tilde{\mathscr{F}}$. Since $\tilde{\Phi}$ and Φ agree near $\partial\hat{U}$, any coordinate system $x_1, \ldots, x_n$ in a neighborhood of $\partial\hat{U}$, having as $\hat{\mathscr{F}}$-plaques the level sets of x_n, $0 \leqslant x_n \leqslant \infty$ (or $-\infty \leqslant x_n \leqslant 0$), is readily C^∞-transformed to a coordinate system

$$\tilde{x}_1 = x_1, \ldots, \tilde{x}_{n-1} = x_{n-1}, \qquad \tilde{x}_n = x_n + \tau(x_1, \ldots, x_{n-1})$$

having as $\mathscr{F}^*$-plaques the level sets of $\tilde{x}_n$. On overlaps, the coordinate transformations are of the form

$$\tilde{x}_i = \tilde{x}_i(\tilde{y}_1, \ldots, \tilde{y}_{n-1}), \qquad 1 \leqslant i \leqslant n-1, \qquad \tilde{x}_n = \tilde{y}_n + c.$$

Of course, as usual, we stipulate that the level sets of the first $n-1$ coordinates be plaques of $\mathscr{L}$.

Remarks. — (1) the foliation $\mathscr{F}^*$ extends over M to a C^0 foliation, again denoted $\mathscr{F}^*$, such that $\mathscr{F}^*|(M-U) = \mathscr{F}|(M-U)$. One can then show that, in a certain reasonable sense, $\mathscr{F}^*$ is uniformly close to $\mathscr{F}$.

(2) The group $P(\tilde{\omega})$ of periods is equal to the set of numbers $t \in \mathbf{R}$ such that $\tilde{\Phi}_t$ carries each leaf of $\tilde{\mathscr{F}}$ onto itself. It is elementary that the foliation $\tilde{\mathscr{F}}$ fibers U over S^1 if and only if $P(\tilde{\omega})$ is infinite cyclic.

2. Existence of Tischler foliations.

We keep all of the same conventions and notations as in Section 1.

First, assume that $\dim(M) = 3$. Fix an open, relatively compact subset $W_0 \subset \hat{U}$ and fix $i > 0$ such that $W_0 \subset K_i$. Consider the decomposition of $\hat{U}$ into the nucleus K_i and arms $\hat{U}_j \cong B_j \times [-1,1]$, $1 \leqslant j \leqslant r$. Thus, each $\partial B_j \cong S^1$, so $K_i \cap \hat{U}_j \cong S^1 \times [-1,1]$. Also, the homomorphism $H_*(K_i \cap \hat{U}_j; \mathbf{Z}) \to H_*(\hat{U}_j; \mathbf{Z})$ identifies with $H_*(\partial B_j; \mathbf{Z}) \to H_*(B_j; \mathbf{Z})$ and this is one-one.

In this situation, the Mayer-Vietoris sequence yields a short exact sequence

$$0 \to \mathbf{Z}^r \to H_1(K_i; \mathbf{Z}) \oplus H_1(\hat{U}_1; \mathbf{Z}) \oplus \cdots \oplus H_1(\hat{U}_r; \mathbf{Z}) \to H_1(\hat{U}; \mathbf{Z}) \to 0.$$

Here, $\mathbf{Z}^r = H_1\left(\bigcup_{j=1}^{r} (K_i \cap \hat{U}_j);\mathbf{Z}\right)$ is generated by the cycles ∂B_j and each $H_1(\hat{U}_j;\mathbf{Z}) = H_1(B_j;\mathbf{Z})$ is free abelian on a basis that contains the cycle ∂B_j. It follows that $H_1(\hat{U};\mathbf{Z}) = A \oplus B$ where A is the (finitely generated) image of $H_1(K_i;\mathbf{Z})$ induced by the inclusion $K_i \hookrightarrow \hat{U}$ and B is free abelian. In the choice of integral cycles $\sigma_1, \sigma_2, \ldots,$ as in Section 1, we can arrange that $\{\sigma_1,\ldots,\sigma_{n(i)}\}$ gives a basis of $A/(\text{torsion})$ and that $\{\sigma_j\}_{j>n(i)}$ gives a basis of B. Thus, the forms ω_j, $j \leqslant n(i)$, annihilate B.

Choose the numbers $\tilde{c}_1, \ldots, \tilde{c}_{n(i)}$ to be rational and as close to $c_1, \ldots, c_{n(i)}$, respectively, as desired. For $j > n(i)$, set $\tilde{c}_j = 0$.

Since $P(\tilde{\omega}) \subset \mathbf{R}$ is generated by $\tilde{\omega}(\sigma_j) = \tilde{c}_j$, $j \geqslant 1$, the above choices force $P(\tilde{\omega})$ to be infinite cyclic. By the final remark in Section 1, we obtain the following.

(2.1) THEOREM. – *If* $\dim(M) = 3$ *and if* $W_0 \subset \hat{U}$ *is open and relatively compact, then there exist Tischler foliations* $\mathcal{F}^*$ *of* $\hat{U}$ *that are arbitrarily* C^∞*-close to* $\hat{\mathcal{F}}$ *on* W_0.

By similar, but slightly more delicate choices of the cycles σ_j and the rational numbers $\tilde{c}_j$, we will prove the following.

(2.2) THEOREM. – *If* $\dim(M) \geqslant 3$, *if* $W_0 \subset \hat{U}$ *is open and relatively compact, and if* $P(\omega)$ *is free abelian, then there exist Tischler foliations* $\mathcal{F}^*$ *of* $\hat{U}$ *that are arbitrarily* C^∞*-close to* $\hat{\mathcal{F}}$ *on* W_0. *Furthermore,* $\mathcal{F}^*$ *can be chosen so that*

$$\operatorname{Ker}(\omega : \pi_1(U) \to \mathbf{R}) \subset \operatorname{Ker}(\tilde{\omega} : \pi_1(U) \to \mathbf{R}).$$

The final assertion in (2.2) will guarantee that the leaves of $\mathcal{F}|U$ are regular coverings of the fibers of $\tilde{\mathcal{F}} = \mathcal{F}^*|U$ in a natural way (3.8). The corresponding assertion is absent from (2.1) due to a wealth of counter-examples (3.9).

Proof of (2.2). – Since $P(\omega)$ is free abelian, the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\omega) \longrightarrow H_1(U;\mathbf{Z}) \xrightarrow{\omega} P(\omega) \longrightarrow 0$$

can be split. Since $H_1(\hat{U};\mathbf{Z}) = H_1(U;\mathbf{Z})$ canonically, we obtain

$$H_1(\hat{U};\mathbf{Z}) = \operatorname{Ker}(\omega) \oplus P$$
$$H_1(\hat{U};\mathbf{R}) = (\operatorname{Ker}(\omega)\otimes\mathbf{R}) \oplus (P\otimes\mathbf{R})$$

such that ω carries P one-one onto $P(\omega)$.

Set $\omega_{(j)} = \omega|(K_j \cap U)$. The inclusions $K_j \subset K_{j+1} \subset \hat{U}$ induce commutative diagrams

$$\begin{tikzcd} \mathrm{Ker}(\omega_{(j)}) \arrow[rr] \arrow[dr, "\gamma_j"'] & & \mathrm{Ker}(\omega_{(j+1)}) \arrow[dl, "\gamma_{j+1}"] \\ & \mathrm{Ker}(\omega) & \end{tikzcd}$$

and $\mathrm{Ker}(\omega) = \bigcup_{j \geqslant 0} \mathrm{Im}(\gamma_j)$. Set $m(j) = \dim(\mathrm{Im}(\gamma_j) \otimes \mathbf{R})$ and choose integral cycles $\rho_1, \ldots, \rho_{m(0)}$ in $K_0 \cap U$ and $\rho_{m(j)+1}, \ldots, \rho_{m(j+1)}$ in $K_{j+1} \cap U$, $j \geqslant 0$, such that the classes $[\rho_1], [\rho_2], \ldots, [\rho_k], \ldots$ define a possibly infinite basis of $\mathrm{Ker}(\omega) \otimes \mathbf{R}$. We can choose the cycles $\sigma_1, \ldots, \sigma_{n(0)}$ (respectively, $\sigma_{n(j)+1}, \ldots, \sigma_{n(j+1)}$) of Section 1 so that $\rho_1, \ldots, \rho_{m(0)}$ (respectively, $\rho_{m(j)+1}, \ldots, \rho_{m(j+1)}$) are among them. Let $\tau_1, \ldots, \tau_{n(0)-m(0)}$ (respectively, $\tau_{n(j)-m(j)+1}, \ldots, \tau_{n(j+1)-m(j+1)}$) be the remaining σ_k's. Finally, let $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ be a possibly infinite basis of the free abelian summand P. One then has a possibly infinite integer matrix $(M_{jk})_{j,k \geqslant 1}$, each row of which has only finitely many nonzero entries, such that, in $H_1(\hat{U};\mathbf{Z})$,

$$[\tau_j] = \sum_{k \geqslant 1} M_{jk} \alpha_k \bmod \mathrm{Ker}(\omega),\ j \geqslant 1.$$

The rows of this matrix are linearly independent over $\mathbf{R}$

Since $\{\sigma_1, \sigma_2, \ldots\} = \{\rho_1, \rho_2, \ldots\} \cup \{\tau_1, \tau_2, \ldots\}$, we can define $p(j)$, $j \geqslant 1$, so that $\sigma_{p(j)} = \tau_j$. If $\sigma_p = \rho_k$, *then* $c_p = 0$ and we set $\tilde{c}_p = 0$. Fix K_i such that $W_0 \subset K_i$ and choose $\tilde{c}_{p(j)}$, $1 \leqslant j < n(i) - m(i)$, rational and as close as desired to $c_{p(j)}$. There exists $r \geqslant n(i) - m(i)$ such that

$$[\tau_j] = \sum_{k=1}^{r} M_{jk} \alpha_k \bmod \mathrm{Ker}(\omega), \qquad 1 \leqslant j \leqslant n(i) - m(i),$$

and there are (not necessarily unique) rational numbers d_k, $1 \leqslant k \leqslant r$, such that

$$\tilde{c}_{p(j)} = \sum_{k=1}^{r} M_{jk}\, d_k, \quad 1 \leqslant j \leqslant n(i) - m(i).$$

If $k > r$, set $d_k = 0$ and define rational numbers

$$\tilde{c}_{p(j)} = \sum_{k \geqslant 1} M_{jk}\, d_k, \quad j \geqslant 1.$$

This defines $\tilde{c}_p$ for all $p \geqslant 1$ and the corresponding 1-form

$$\tilde{\omega} = \sum_{p \geqslant 1} \tilde{c}_p(\omega_p | U) + dg$$

as in (1.3). Then

$$\hat{\omega} : H_1(\hat{U}; \mathbf{Z}) \to \mathbf{R}$$

annihilates every $[\rho_j]$, hence $\operatorname{Ker}(\omega) \subset \operatorname{Ker}(\tilde{\omega})$. Furthermore, $\tilde{\omega}[\tau_j] = \tilde{c}_{p(j)}$. There is a unique cohomology class $[\gamma] \in H^1(\hat{U}; \mathbf{R})$ that vanishes on $\operatorname{Ker}(\omega)$ and assigns to each α_k the rational number d_k. By the above, $[\gamma]$ assigns to each $[\tau_j]$ the number $\tilde{c}_{p(j)}$, so $[\gamma] = [\tilde{\omega}]$. Thus, $P(\tilde{\omega}) = P(\gamma)$ and this is generated by the finite set $\{d_1, \ldots, d_r\}$ of rational numbers, so $P(\tilde{\omega})$ is infinite cyclic.

Finally, since $\operatorname{Ker}(\omega) \subset \operatorname{Ker}(\tilde{\omega})$ at the level of homology, the corresponding inclusion holds at the level of homotopy. □

3. The regular covering property.

Let L be a leaf of $\mathscr{F}|U$ and let F be a fiber of $\tilde{\mathscr{F}} = \mathscr{F}^*|U$. Fix a reference point $x_0 \in L$ and choose $t_0 \in \mathbf{R}$ such that $\Phi_{t_0}(x_0) \in F$. Consider

Condition (*). There exists a smooth function $\tau : L \to \mathbf{R}$ such that $\tau(x_0) = t_0$ and $\Phi_{\tau(x)}(x) \in F$, $\forall x \in L$.

If Condition (*) is satisfied, we will define $p : L \to F$ by $p(x) = \Phi_{\tau(x)}(x)$ and prove that this is a regular covering space with covering group $G \subset P(\mu) = P(\omega)$ such that $P(\mu) \cong G \oplus \mathbf{Z}$. Since one easily produces countably generated, additive subgroups $P \subset \mathbf{R}$ that do not admit $\mathbf{Z}$ as a direct summand, and since $P(\mu)$ can be any such subgroup [1, (5.5)], we cannot expect Condition (*) always to be satisfied.

(3.1) LEMMA. – *Condition* (*) *holds if and only if*

$$\operatorname{Ker}(\omega : \pi_1(\hat{U}) \to \mathbf{R}) \subset \operatorname{Ker}(\tilde{\omega} : \pi_1(\hat{U}) \to \mathbf{R}).$$

Furthermore, τ *is uniquely determined by* x_0 *and* t_0.

(3.2) COROLLARY. — *Condition* (*) *holds for one choice of initial conditions* L, F, x_0, t_0 *if and only if it holds for all such choices.*

By the final assertion in (2.2) we also have

(3.3) COROLLARY. — *If* $P(\omega)$ *is free abelian, then Tischler foliations can be chosen, arbitrarily* C^∞*-close to* $\hat{\mathscr{F}}$ *on any preassigned precompact region, such that Condition* (*) *holds.*

Proof of (3.1). — Fix a leaf L of $\mathscr{F}$ and a basepoint $x_0 \in L$. Let σ be a piecewise smooth loop in U based at x_0. In standard fashion, using the transverse flow Φ_t, we deform σ to a loop at x_0 of the form $\sigma_1 + \sigma_2$, where σ_1 is a path in L and σ_2 lies along the flow line through x_0. Thus, $\omega(\sigma) = \int_{\sigma_2} \omega$ and this is zero if and only if σ_2 reduces to the single point x_0. Thus, the image of $i_* : \pi_1(L,x_0) \to \pi_1(\hat{U},x_0)$, where i is the inclusion, is exactly $\mathrm{Ker}(\omega)$. The condition that $\mathrm{Ker}(\omega) \subset \mathrm{Ker}(\tilde{\omega})$ becomes the condition that $\tilde{\omega}(\sigma) = 0$ for every piecewise smooth loop σ lying on L.

If Condition (*) holds, define $p_t : L \to \hat{U}$ by $p_t(x) = \Phi_{t\tau(x)}(x)$, $0 \leqslant t \leqslant 1$. This homotopy can be used to deform any 1-cycle σ on L to a 1-cycle $\tilde{\sigma}$ on F, all within U. Thus, $\tilde{\omega}(\sigma) = \tilde{\omega}(\tilde{\sigma}) = 0$.

Conversely, suppose $\tilde{\omega}(\sigma) = 0$ for each piecewise smooth loop σ on L. Fix t_0 so that $\Phi_{t_0}(x_0) \in F$. Given $x \in L$, choose a piecewise smooth path $\gamma : [0,1] \to L$, $\gamma(0) = x_0$ and $\gamma(1) = x$. We want to project γ smoothly along the leaves of $\mathscr{L}|U$ to $\tilde{\gamma} : [0,1] \to F$, $\tilde{\gamma}(0) = \Phi_{t_0}(x_0)$. More precisely, we want to define a piecewise smooth function $\tau_\gamma : [0,1] \to \mathbf{R}$, $\tau_\gamma(0) = t_0$, such that

$$\Phi_{\tau_\gamma(t)}(\gamma(t)) = \tilde{\gamma}(t) \in F, \qquad 0 \leqslant t \leqslant 1.$$

The mere fact that $\mathscr{L}|U$ is transverse to $\hat{\mathscr{F}} = \mathscr{F}^*|U$ does not guarantee that this is possible, but the additional fact that $\hat{\mathscr{F}}$ fibers U over S^1 makes it a straightforward exercise to prove the existence and uniqueness of τ_γ. If $\rho : [0,1] \to L$ also satisfies $\rho(0) = x_0$ and $\rho(1) = x$, then we claim that $\tau_\rho(1) = \tau_\gamma(1)$. Indeed, let $\lambda : [0,1] \to U$ be the curve (along a leaf of $\mathscr{L}|U$)

$$\lambda(t) = \Phi_{t\tau_\gamma(1)+(1-t)\tau_\rho(1)}(x).$$

Either this curve is constant (i.e., $\tau_\gamma(1) = \tau_\rho(1)$) or it is nonsingular and

$\int_\lambda \tilde{\omega} \neq 0$. The cycle $\tilde{\sigma} = \tilde{\rho} + \lambda + \tilde{\gamma}^{-1}$ is homologous in U to the cycle $\sigma = \rho + \gamma^{-1}$. Since σ is a cycle on L,

$$0 = \tilde{\omega}(\sigma) = \tilde{\omega}(\tilde{\sigma}) = \int_\lambda \tilde{\omega},$$

so λ is constant. Consequently, we can define $\tau(x) = \tau_\gamma(1)$ unambiguously, τ is smooth, and $\Phi_{\tau(x)}(x) = \tilde{\gamma}(1) \in \mathrm{F}$. Also, τ is unique since each τ_γ is unique. □

Assuming that Condition (*) holds, we fix the choices of L, F, and τ and we define $p : \mathrm{L} \to \mathrm{F}$ as above. Our candidate for the covering group $\mathrm{G} \subset \mathrm{P}(\mu)$ is as follows.

Definition. — $\mathrm{G} = \{\tau(x_1) - \tau(x_2) | p(x_1) = p(x_2)\}$.

(3.4) Lemma. — *G is a subgroup of* $\mathrm{P}(\mu)$ *and* $\mathrm{P}(\Phi_t(z)) = p(z)$, $\forall t \in \mathrm{G}$, $\forall z \in \mathrm{L}$.

Proof. — If $p(x_1) = p(x_2)$, then

$$\Phi_{\tau(x_1) - \tau(x_2)}(x_1) = \Phi_{-\tau(x_2)}(p(x_1)) = x_2 .$$

In particular, $\Phi_{\tau(x_1)-\tau(x_2)}(\mathrm{L}) = \mathrm{L}$, proving that $\mathrm{G} \subset \mathrm{P}(\mu)$.

Let $t = \tau(x_1) - \tau(x_2) \in \mathrm{G}$. Define $\bar{\tau} : \mathrm{L} \to \mathbf{R}$ by

$$\bar{\tau}(z) = \tau(\Phi_t(z)) + t .$$

Then $\bar{\tau}(x_1) = \tau(x_2) + t = \tau(x_1)$ and

$$\begin{aligned}\Phi_{\bar{\tau}(z)}(z) &= \Phi_{\tau(\Phi_t(z))}(\Phi_t(z)) \\ &= p(\Phi_t(z)) \in \mathrm{F} .\end{aligned}$$

By the uniqueness assertion in (3.1), $\bar{\tau} \equiv \tau$ and, in particular,

$p(z) = p(\Phi_t(z))$, $\forall t \in \mathrm{G}$, $\forall z \in \mathrm{L}$.

Evidently $0 \in \mathrm{G}$. Also, if $t \in \mathrm{G}$ then $-t \in \mathrm{G}$. Let $p(x_1) = p(x_2)$ and $p(y_1) = p(y_2)$. We must show that

$$(\tau(x_1)-\tau(x_2)) + (\tau(y_1)-\tau(y_2)) \in \mathrm{G} .$$

Let $u = \Phi_{\tau(y_1)-\tau(y_2)}(x_2)$. Then $p(u) = p(x_2)$. As above, for $z \in L$,

$$\tau(\Phi_{\tau(y_1)-\tau(y_2)}(z)) + \tau(y_1) - \tau(y_2) = \tau(z) = \tau(\Phi_{\tau(x_2)-\tau(u)}(z)) + \tau(x_2) - \tau(u).$$

By letting $z = x_2$, we obtain

$$\tau(u) + \tau(y_1) - \tau(y_2) = \tau(u) + \tau(x_2) - \tau(u),$$

hence

$$\tau(y_1) - \tau(y_2) = \tau(x_2) - \tau(u).$$

Consequently,

$$\tau(x_1) - \tau(x_2) + \tau(y_1) - \tau(y_2) = \tau(x_1) - \tau(u)$$

and this is an element of G. □

(3.5) LEMMA. – *For each $y \in F$, the natural action $G \times L \to L$ induces a simply transitive action of G on $p^{-1}(y)$.*

Proof. – Let $t \in G$ and $x \in L$, and suppose that $\Phi_t(x) = x$. Then, as in the proof of (3.4),

$$\tau(x) = \tau(\Phi_t(x)) + t = \tau(x) + t,$$

so $t = 0$. That is, G acts on L without fixed points. If $y_1, y_2 \in p^{-1}(y)$, then $\tau(y_1) - \tau(y_2) \in G$ and $\Phi_{\tau(y_1)-\tau(y_2)}(y_1) = y_2$. □

(3.6) PROPOSITION. – *The map $p : L \to F$ is a regular covering and $G \subset P(\mu)$ is the group of covering transformations.*

Proof. – A finite biregular cover of M relative to $(\mathscr{F},\mathscr{L})$ (cf. [2, Section 1], [5]) defines a (generally infinite) biregular cover $\{W_\alpha\}_{\alpha \in A}$ of $\hat{U}$ relative to $(\hat{\mathscr{F}},\hat{\mathscr{L}})$. Fix a biregular cover $\{V_\beta\}_{\beta \in B}$ of $\hat{U}$ for $(\mathscr{F}^*,\hat{\mathscr{L}})$ such that each $\overline{V_\beta}$ lies in some W_α. Given $y \in F$, $x \in p^{-1}(y) \subset L$, and a plaque P_y^* around y coming from a suitable V_β, there is a neighborhood P_x of x in L carried diffeomorphically by p onto P_y^*. Indeed, by a small deformation of $\overline{V_\beta}$ within a surrounding W_α, holding y fixed, we produce a compact biregular neighborhood for $(\hat{\mathscr{F}},\hat{\mathscr{L}})$ meeting exactly the same local flow lines as $\overline{V_\beta}$. If P is an $\hat{\mathscr{F}}$-plaque of this biregular neighborhood, there is some $t \in \mathbf{R}$ such that $\Phi_t(P)$ has interior P_x as desired.

Let P_x and P^*_y be as above. Let $t \in G$ be such that

$$\Phi_t(P_x) \cap P_x \neq \varnothing .$$

Let $z_1, z_2 \in P_x$ such that $z_1 = \Phi_t(z_2)$. Then

$$p(z_1) = p(\Phi_t(z_2)) = p(z_2), \qquad \text{so} \qquad z_1 = z_2 .$$

By (3.5), $t = 0$. It follows that P^*_y is evenly covered by

$$p^{-1}(P^*_y) = \bigcup_{t \in G} \Phi_t(P_x). \qquad \square$$

(3.7) PROPOSITION. – *If* $G \subset P(\mu) = P(\omega)$ *is the group of covering transformations as above, then* $P(\omega) = G \oplus \mathbf{Z}$.

Proof. – Without loss of generality, we assume there is a basepoint $x_0 \in L \cap F$ such that $p(x_0) = x_0$. Indeed, given arbitrary $x_0 \in L$, we can, if necessary, replace (L, x_0) with $(\Phi_{\tau(x_0)}(L), \Phi_{\tau(x_0)}(x_0))$ and τ with $\tau \circ \Phi_{-\tau(x_0)} - \tau(x_0)$. This leaves the subgroup $G \subset P(\omega)$ unchanged.

Both $\mathscr{F}|U$ and $\hat{\mathscr{F}}$ are transversely complete e-foliations of U (cf. [4]). Thus the leaf inclusions induce monomorphisms of fundamental groups and we obtain exact sequences

$$0 \longrightarrow \pi_1(L, x_0) \longrightarrow \pi_1(\hat{U}, x_0) \xrightarrow{\ \omega\ } P(\omega) \longrightarrow 0$$

$$0 \longrightarrow \pi_1(F, x_0) \longrightarrow \pi_1(\hat{U}, x_0) \xrightarrow{\ \tilde{\omega}\ } P(\tilde{\omega}) \longrightarrow 0 .$$

By the first of these, we identify $P(\omega)$ with $\pi_1(\hat{U}, x_0)/\pi_1(L, x_0)$. Since $p : L \to F$ is a regular covering and $p(x_0) = x_0$, we obtain a commutative diagram of inclusions

$$\begin{tikzcd}
\pi_1(L, x_0) \arrow[r, hook] \arrow[d, hook, "p_*"'] & \pi_1(\hat{U}, x_0) \\
\pi_1(F, x_0) \arrow[ur, hook] &
\end{tikzcd}$$

and

$$G = \pi_1(F, x_0)/\pi_1(L, x_0) \subset \pi_1(\hat{U}, x_0)/\pi_1(L, x_0) = P(\omega).$$

By (3.1), $\tilde{\omega}$ vanishes on $\pi_1(L,x_0)$, so the second of the above sequences yields an exact sequence

$$0 \longrightarrow G \longrightarrow P(\omega) \xrightarrow{\tilde{\omega}} P(\tilde{\omega}) \longrightarrow 0.$$

But $P(\tilde{\omega}) \cong Z$ and this sequence splits. □

Combining (3.3), (3.6), and (3.7), we obtain

(3.8) THEOREM. – *If* $P(\omega)$ *is free abelian, then Tischler foliations* $\mathcal{F}^*$ *can be chosen, arbitrarily* C^∞*-close to* $\hat{\mathcal{F}}$ *on any preassigned precompact region, such that there is a natural regular covering map* $p : L \to F$, L *a leaf of* $\mathcal{F}|U$ *and* F *a fiber of* $\hat{\mathcal{F}} = \mathcal{F}^*|U$, *with covering group* G *a direct summand :* $P(\omega) \cong G \oplus \mathbf{Z}$.

If $P \subset \mathbf{R}$ is a countably generated, additive subgroup, an element $a \in P$, $a \neq 0$, will be called *infinitely divisible* if, for suitable, arbitrarily large integers m, one can find $b_m \in P$ such that $mb_m = a$. The group P contains an infinitely divisible element if and only if P is not free abelian (cf. [7], Theorem 19.1, page 93).

(3.9) PROPOSITION. – *If* $\dim(M) = 3$ *and* $P \subset \mathbf{R}$ *is a countably generated, additive subgroup that is not free abelian, then* M *admits a transversely orientable* C^∞*-foliation* $\mathcal{F}$ *with* $U \subset M$ *as usual such that* $P(\omega) = P$ *and such that no choice of Tischler foliation* $\mathcal{F}^*$ *satisfies Condition* (*).

Proof. – Exactly as in [1, (5.5)], construct $\mathcal{F}$ such that $\mathcal{F}|U$ has dense leaves without holonomy and such that $P(\omega) = P$. In choosing the representation $\omega = \Sigma c_j(\omega_j|U) + dg$ of Section 1, it is easy to arrange that c_1 be an infinitely divisible element of $P(\omega)$. In fact, we can arrange that $c_1 = mc_j$, for suitable arbitrarily large integers m and suitable $j > 1$. Furthermore, since $c_1 \neq 0$, we can choose the integral cycle σ_1 (such that $\omega_j(\sigma_1) = \delta_{j1}, j \geqslant 1$) to be a closed transversal to $\mathcal{F}|U$. By performing the standard modification of $\mathcal{F}$ along σ_1, introducing a Reeb component with σ_1 as core transversal, we change U so that $\partial\hat{U}$ has one new component, a torus. The new foliation $\mathcal{F}|U$ has the same properties, including the same period group $P(\omega)$, as before. Perturb σ_1 so that it lies in U near the toral boundary component and is transverse to $\mathcal{F}|U$. Let σ_0 also lie in U near the toral boundary, a perturbed meridian circle relative to the Reeb component and lying on a leaf of $\mathcal{F}|U$. Thus,

$\omega(\sigma_0) = 0$. The new system of basic cycles is either unchanged or it is obtained by adjoining σ_0 to $\{\sigma_1, \sigma_2, \ldots\}$, in which case $c_0 = 0$.

Suppose there is a choice of $\mathcal{F}^*$ so that Condition (*) holds. By (3.1), $\tilde{\omega}(\sigma_0) = 0$. Since $\hat{\mathcal{F}}$ fibers U over S^1, $\mathcal{F}^*$ cannot be a product foliation near the new toral component of $\partial\hat{U}$. Thus, $\tilde{\omega}$ is not exact near this torus and it follows that $\tilde{c}_1 = \tilde{\omega}(\sigma_1) \neq 0$. For suitable, arbitrarily large integers m and $j > 1$, we have $\omega(\sigma_1 - m\sigma_j) = 0$, hence $\tilde{\omega}(\sigma_1 - m\sigma_j) = 0$ by (3.1). That is, in $P(\tilde{\omega})$ there are elements $\tilde{c}_1 = \tilde{\omega}(\sigma_1) \neq 0$ and $\tilde{c}_j = \tilde{\omega}(\sigma_j)$ such that $m\tilde{c}_j = \tilde{c}_1$. This contradicts the fact that $P(\tilde{\omega})$ is infinite cyclic. □

Returning to the positive result (3.8), we describe a fairly general situation in which that result applies.

DEFINITION. – *Let* $U \subset M$ *be as usual. If the nucleus* $K \subset \hat{U}$ *can be chosen so that, in each arm* $\hat{U}_j \cong B_j \times [-1,1]$, $\hat{\mathcal{F}}$ *restricts to the product foliation by leaves* $B_j \times \{t\}$, *then* $\hat{\mathcal{F}}$ *is said to be almost trivial.*

(3.10) PROPOSITION. – *The foliation* $\hat{\mathcal{F}}$ *is almost trivial in each of the following cases :*

(*a*) $\hat{U}$ *is compact;*

(*b*) $\mathcal{F}$ *is of class at least* C^2 *and each leaf of* $\mathcal{F}|U$ *has two dense ends;*

(*c*) $\mathcal{F}$ *is transversely analytic.*

Indeed, case (*a*) is vacuously true and, under the additional hypothesis that $\bar{U} - U$ is a union of proper leaves, case (*b*) was proven in [1, (6.9)] and, under the same hypothesis, case (*c*) was pointed out in that same reference. The additional hypothesis can be avoided by using a result of G. Duminy [6] on the structure of semi-proper, exceptional leaves.

(3.11) THEOREM. – *If* $\hat{\mathcal{F}}$ *is almost trivial, then Tischler foliations* $\mathcal{F}^*$ *can be chosen, arbitrarily* C^∞*-close to* $\hat{\mathcal{F}}$ *on any preassigned, precompact region, such that there is a natural regular covering* $p : L \to F$ with covering group $G \cong \mathbf{Z}^k$, *some integer* $k \geqslant 1$.

Proof. – If σ is an integral 1-cycle contained in an arm $\hat{U}_j$, then $\omega(\sigma) = 0$. Thus, $P(\omega)$ is the finitely generated image of $\omega : H_1(K;\mathbf{Z}) \to \mathbf{R}$ and (3.8) applies. □

Remarks. – (1) In case (*a*) of (3.10), if $\partial\hat{U} = \varnothing$ (i.e., $U = M$), then a famous result of H. Hopf [9], together with (3.11), implies that each leaf of

$\mathscr{F}(=\mathscr{F}|U)$ has the same number of ends as does the covering group $G \cong \mathbf{Z}^k$. This number is two if $k = 1$, and it is one if $k > 1$. The fact that the number of ends is either one or two is also a consequence of [3, Proposition 1], in which it is shown that, generally (whether or not Tischler foliations exist), each leaf of $\mathscr{F}|U$ has either one dense end or two such ends. The proof is similar to Hopf's proof, so one might expect to show that, at least when $G \cong \mathbf{Z}^k$, the number of dense ends is the same as the number of ends of G. This often fails, however, even when $\hat{U}$ is compact. For instance, let $\hat{U} \cong S^1 \times S^1 \times [-1,1]$, the leaves of $\mathscr{F}|U$ being dense planes. These leaves have one dense end, the Tischler fibers are cylinders $S^1 \times \mathbf{R}$, and the covering $p : \mathbf{R}^2 \to S^1 \times \mathbf{R}$ has covering group $G \cong \mathbf{Z}$.

(2) In case (*b*) of (3.10), if we assume only that $\mathscr{F}$ is of class C^{0+}, we can apply the argument in [1, Section 6] to show that $P(\mu) \cong \mathbf{Z} \times \mathbf{Z}$. Thus, (3.8) applies to the case of two dense ends without the smoothness hypothesis. In this case, $G \cong \mathbf{Z}$.

(3) It is natural to ask whether the covering map $p : L \to F$, when it exists, respects the growth types of L and F, at least when $G \cong \mathbf{Z}^k$. That is, if $g_L, g_F : \mathbf{Z}^+ \to \mathbf{R}^+$ are growth functions for L and F respectively, and if $G \cong \mathbf{Z}^k$, do $g_L(m)$ and $m^k g_F(m)$ have the same growth type ? If $\hat{\mathscr{F}}$ is almost trivial, the answer is « yes », as is easily deduced from [1, (2.8) and (6.10)]. In general, however, the answer is « no », as the constructive proof of [1, (5.5)] clearly implies.

4. An example.

Without some condition on $P(\omega)$, Tischler foliations do not generally exist. Here we show how to construct an appropiate example in which dim(M) can be an arbitrary integer greater than three. By (2.1), such examples are impossible when $\dim(M) = 3$. In our example, $P(\omega)$ will be the dyadic rationals $\mathbf{Z}[1/2]$. The method of construction may be of some independent interest.

(A) *Generalized Reeb components.* – Let L be an open, connected manifold of dimension $n - 1$, $n \geqslant 3$. Suppose that there is a decomposition

$$L = A \cup B_1 \cup B_2 \cup \cdots \cup B_k \cup \cdots$$

such that

(1) A is a compact, connected, $(n-1)$-dimensional manifold with ∂A connected;

(2) $B_i \cong B_{i+1}$, $i \geqslant 1$, and B_i is a compact, connected, $(n-1)$-dimensional manifold such that ∂B_i has two components, $\partial_+ B_i$ and $\partial_- B_i$;

(3) $A \cap B_1 = \partial A = \partial_- B_1$ and $A \cap B_i = \varnothing$, $i > 1$;

(4) $B_i \cap B_{i+1} = \partial_+ B_i = \partial_- B_{i+1}$, $i \geqslant 1$, and $B_i \cap B_{i+k} = \varnothing$, $i \geqslant 1$, $k \geqslant 2$;

(5) there is a diffeomorphism γ of L onto itself such that $\gamma(A \cup B_1) = A$ and $\gamma(B_{i+1}) = B_i$, $i \geqslant 1$.

Example. – Let $L = \mathbf{R}^2$, let $A = \{v \in \mathbf{R}^2 : \|v\| \leqslant 2\}$, and let

$$B_i = \{v \in \mathbf{R}^2 : 2^i \leqslant \|v\| \leqslant 2^{i+1}\}, \quad i \geqslant 1.$$

Finally, let $\gamma(v) = v/2$.

Under these circumstances, we have a proper nest of compact sets

$$A \supset \gamma(A) \supset \gamma^2(A) \supset \cdots \supset \gamma^k(A) \supset \cdots.$$

The intersection of these sets is a compact, nonempty, γ-invariant set K and γ is a contraction of L to K. In the above example, $K = \{0\}$. In all cases, γ generates a properly discontinuous action of $\mathbf{Z}$ on $L - K$ and $(L-K)/\mathbf{Z}$ is a closed, connected, $(n-1)$-dimensional manifold T. Indeed, T is obtained from B_i by identifying $\partial_+ B_i$ to $\partial_- B_i$ via γ.

Let $I = [0,1]$ and let $h : I \to I$ be a diffeomorphism (into) such that $h(0) = 0$ and $h(t) < t$, $0 < t \leqslant 1$. Thus, h is a contraction to 0. We also assume that h is C^∞-tangent to the identity at $t = 0$.

Let $\varphi : L \times I \to L \times I$ be the diffeomorphism (into) defined by

$$\varphi(x,t) = (\gamma(x), h(t)).$$

Then φ contracts $L \times I$ to $K \times \{0\}$. Let $X = (L \times I) - (K \times \{0\})$. Then X is an n-manifold with boundary and $\varphi : X \to X$ has no fixed points. Indeed, $\{\varphi^k\}_{k \geqslant 0} = \mathbf{Z}^+$ is a properly discontinuous semigroup of diffeomorphisms of X *into* itself. The boundary component $\partial_0 X = (L \times \{0\}) - (K \times \{0\})$ is invariant under this semigroup. The quotient $Y = X/\mathbf{Z}^+$ is an n-manifold with one boundary component,

$$\partial Y = \partial_0 X/\mathbf{Z}^+ \cong (L-K)/\mathbf{Z} = T.$$

The quotient map $X \to Y$ carries $A \times [h(1),1] \cup B_1 \times [0,1]$ onto Y, hence Y is compact. Finally, the foliation of X by leaves $L \times \{t\}$, $0 < t \leqslant 1$, together with the leaf $\partial_0 X$, is invariant under this semigroup and passes to a C^∞ foliation of Y with $\partial Y \cong T$ as one leaf and all other leaves diffeomorphic to L. The noncompact leaves wind in on ∂Y in a very regular way. Indeed, these leaves each have one end and that end is periodic of period ∂Y, in the sense of [2, (6.1)].

Since h is assumed to be C^∞-tangent to the identity at $t = 0$, it follows that the above foliation is C^∞-trivial at ∂Y. Thus, the double of Y yields a closed, C^∞-foliated n-manifold M having exactly one compact leaf, all other leaves being diffeomorphic to L.

Example. – Applying our construction to $L = \mathbf{R}^2$, $\gamma(v) = v/2$, we obtain the Reeb-foliated solid torus with double the standard Reeb foliation of $S^1 \times S^2$.

We call Y, together with the above foliation, a generalized Reeb component. The doubling construction shows that generalized Reeb components do appear as components in C^∞ foliations of suitable closed n-manifolds M.

(B) *A special example.* – Here, we require that $n \geqslant 4$. Let D denote the closed unit disk in $\mathbf{R}^{n-2}$ and let $R = S^1 \times D = \{(\theta,x)\}$, where θ is well defined mod 2π. Choose a smooth map $i : S^1 \times D \to D$ such that, for each θ, $i_\theta : D \to D$ is an imbedding into $\operatorname{int}(D)$ and $i_\theta(D) \cap i_{\theta+\pi}(D) = \varnothing$. It is here that the condition $n \geqslant 4$ is needed (Borsuk-Ulam). Finally, define

$$\begin{aligned} \psi &: R \longrightarrow R \\ \psi(\theta,x) &= (2\theta, i_\theta(x)). \end{aligned}$$

Thus, ψ imbeds R into $\operatorname{int}(R)$ as indicated in figure 2.

Let s denote the successor function, $s(i) = i + 1$, and consider the sequence of imbeddings

$$R \times \{0\} \xrightarrow{\psi \times s} R \times \{1\} \xrightarrow{\psi \times s} \cdots \longrightarrow R \times \{i\} \longrightarrow \cdots.$$

Let L be the $(n-1)$-manifold obtained by passing to the direct limit of this sequence and consider the natural imbeddings $R \times \{i\} \to L$. Let A be the imbedded $R \times \{0\}$ and define B_i inductively by letting

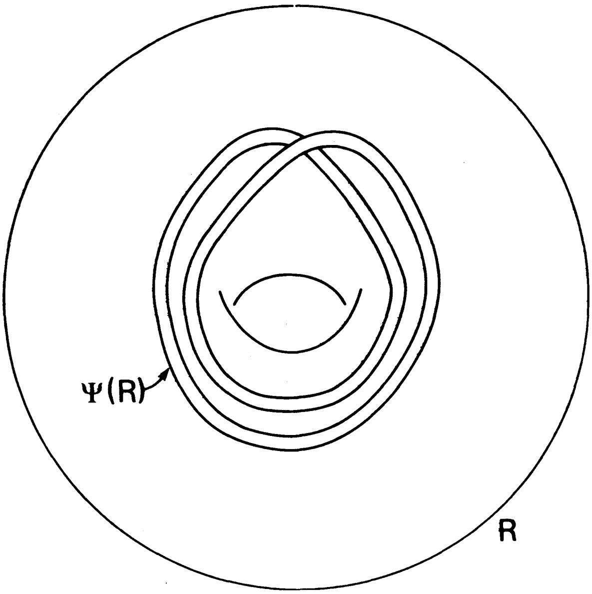

Fig. 2.

$A \cup B_1 \cup \cdots \cup B_i$ be the imbedded $R \times \{i\}$. Finally, define the diffeomorphism $\gamma : L \to L$ via the commutative diagram

$$\begin{tikzcd}
R \times \{0\} \arrow[r, "\psi \times s"] \arrow[d, "\psi \times id"'] & R \times \{1\} \arrow[r, "\psi \times s"] \arrow[dl, "id \times s^{-1}"'] & R \times \{2\} \arrow[r] \arrow[dl, "id \times s^{-1}"'] & \cdots \\
R \times \{0\} \arrow[r, "\psi \times s"] & R \times \{1\} \arrow[r] & \cdots &
\end{tikzcd}$$

It is elementary to check the hypotheses (1) through (5) of (A).

For use in (C), remark that the sequence of fundamental groups

$$\pi_1(R \times \{0\}) \longrightarrow \pi_1(R \times \{1\} \longrightarrow \cdots$$

is exactly

$$Z \xrightarrow{2} Z \xrightarrow{2} \cdots$$

hence $\pi_1(L) = H_1(L) = Z[1/2]$.

(C) *The promised example.* – In the generalized Reeb component of (B), we modify the foliation so that the compact leaf ∂Y remains a leaf, as does the diffeomorphic image in Y of $L \times \{1\}$, but the remainder of the foliation consists of dense leaves without holonomy. Then U will be the diffeomorphic image of $L \times (h(1),1)$ under the quotient map $X \to Y$. Since $\pi_1(U) = \pi_1(L) = \mathbf{Z}[1/2]$, there exists no fibration of U by connected manifolds over S^1. Doubling Y will complete our example.

Let $d\theta \in A^1(R)$ be the closed, nonsingular form pieced together out of the exterior derivatives of the branches of θ. Evidently, $\psi^*(d\theta) = 2\,d\theta$, so we obtain a closed, nonsingular form $\pi \in A^1(L)$ that « restricts » to $2^{-i}\,d\theta$ on $R \times \{i\}$, $i \geqslant 0$. The following is a direct computation.

(4.1) LEMMA. — *The form* $\eta \in A^1(L)$ *satisfies* $\gamma^*(\eta) = 2\eta$ *and* $P(\eta) = \mathbf{Z}[1/2]$.

Define the contraction $h : I \to I$ so that it imbeds in a flow. More precisely, let $f : I \to \mathbf{R}$ be a smooth map, C^∞-tangent to 0 at $t = 0$, such that $f(t) < 0$, $0 < t \leqslant 1$, let $h_u(t)$ be the local flow on I generated by the vector field $f(t)\dfrac{d}{dt}$ (always defined on all of I for $u \geqslant 0$), and set $h = h_1$. The following is standard.

(4.2) LEMMA. — $h^*(dt/f) = dt/f$ *on* $(0,1]$.

Let $J = [h(1), 1]$ and let $g_0 : J \to \mathbf{R}$ be C^∞ and C^∞-tangent to 0 at the endpoints, $g_0|\mathrm{int}\,(J)$ strictly positive. Let $g_k : h^k(J) \to \mathbf{R}$ be given by

$$g_k(h^k(t)) = 2^{-k}g_0(t), \qquad k \in \mathbf{Z}^+, \qquad t \in J.$$

Finally, define $g : I \to \mathbf{R}$ by

$$\begin{aligned} g|h^k(J) &= g_k \\ g(0) &= 0. \end{aligned}$$

(4.3) LEMMA. — *The function* g *is continuous,* $g|(0,1]$ *is* C^∞ *and* C^∞-*tangent to* 0 *at* $h^k(1)$, $k \geqslant 0$, *and* $h^*(g) = g/2$. *For an appropriate choice of the vector field* $f(t)\dfrac{d}{dt}$, *the function* g *is also* C^∞ *at* $t = 0$ *and* C^∞-*tangent to* 0 *there.*

Proof. — Every assertion is trivial except those concerning the behavior of g at $t = 0$. For each real number $u \geqslant 0$, define $g_u : h_u(J) \to \mathbf{R}$ by $g_u(h_u(t)) = 2^{-u}g_0(t)$. When $u = k \in \mathbf{Z}^+$, this definition agrees with that of

g_k. We want to assure that, for each integer $n \geqslant 1$,

$$\lim_{u \to \infty} g_u^{(n)}(h_u(t)) = 0,$$

uniformly for $t \in \mathrm{J}$.

Inductively, on $\mathrm{J} \times [0,\infty)$ define

$$\begin{aligned} \mathrm{Q}_1(t,u) &= g_0'(t)f(t) \\ \mathrm{Q}_{n+1}(t,u) &= \mathrm{Q}_n'(t,u)f(t) - nf'(h_u(t))\mathrm{Q}_n(t,u) \end{aligned}$$

where Q_n' denotes the derivative with respect to t. Since

$$h_u^*(dt/f) = dt/f, \qquad \forall u \geqslant 0,$$

we have

$$h_u'(t) = f(h_u(t))/f(t), \quad t \in \mathrm{J}.$$

With the aid of this formula, one verifies

$$(*) \qquad g_u^{(n)}(h_u(t)) = \mathrm{Q}_n(t,u)/2^u(f(h_u(t)))^n$$

by induction on $n \geqslant 1$.

If $\mathrm{Q}_n^{(k)}(t,u)$ denotes the k^{th} derivative of Q_n with respect to t, then an elementary induction on n shows that $\mathrm{Q}_n^{(k)}(t,u)$ is uniformly bounded on $\mathrm{J} \times [0,\infty)$ for each fixed integer $k \geqslant 0$. In particular, $\mathrm{Q}_n(t,u)$ is so bounded. Thus, by (*), we must choose f so that $|2^u(f(h_u(t)))^n|$ becomes arbitrarily large, uniformly for $t \in \mathrm{J}$, as $u \to \infty$, for each integer $n \geqslant 1$. This is easily arranged. For example,

$$f(t) = \begin{cases} -\,t^2 e^{-1/t}, & 0 < t \leqslant 1, \\ 0, & t = 0 \end{cases}$$

generates the flow

$$h_u(t) = \begin{cases} (\log(u + e^{1/t}))^{-1}, & 0 < t \leqslant 1 \\ 0, & t = 0 \end{cases}$$

hence

$$|2^u(f(h_u(t)))^n| = 2^u(u + e^{1/t})^{-n}(\log(u + e^{1/t}))^{-2n}. \qquad \square$$

On $\mathrm{L} \times \mathrm{I}$, consider the smooth, nonsingular 1-form $\alpha = fg\eta + dt$.

We also denote by α the restriction of this form to X. Let $U = L \times (h(1),1)$.

(4.4) LEMMA. — *The form* α *is completely integrable and the associated foliation* $\mathscr{H}$ *of* $L \times I$ *is transverse to the intervals* $\{x\} \times I$. *The foliation* $\mathscr{H}|X$ *has the following properties:*

(*a*) $\partial_0 X$ *and* $L \times \{h^k(1)\}$ *are leaves,* $k \in \mathbf{Z}^+$, *and* $\mathscr{H}|X$ is C^∞*-trivial at these leaves;*

(*b*) $\varphi^*(\mathscr{H}|X) = \mathscr{H}|X$;

(*c*) $\mathscr{H}|U$ *is defined by a closed, transversely complete, nonsingular* 1*-form* ω *such that* $P(\omega) = \mathbf{Z}[1/2]$.

Proof. — Since η is closed, $d\alpha = \alpha \wedge (fg)'\eta$, so α is completely integrable. Also, $\alpha(\partial/\partial t) \equiv 1$, so $\mathscr{H}$ is transverse to the interval fibers. Since g is C^∞-tangent to 0 at $t = 0$ and at $t = h^k(1)$, $k \in \mathbf{Z}^+$, (*a*) follows. On $X - \partial_0 X$, $\mathscr{H}$ is also defined by $\alpha/f = g\eta + dt/f$. By (4.1), (4.2), and (4.3), $\varphi^*(\alpha/f) = \alpha/f$. Since $\varphi(\partial_0 X) = \partial_0 X$, (*b*) follows. Finally, $\mathscr{H}|U$ is defined by the closed form $\omega = \eta + dt/fg$. To say that ω is transversely complete means that there is a complete vector field v on U such that $\omega(v) \equiv 1$ (equivalently, $\mathscr{H}|U$ is a transversely complete *e*-foliation in the sense of [4]). The vector field $v = fg\, \partial/\partial t$ satisfies this. For any piecewise smooth 1-cycle σ in U, $\int_\sigma \eta = \int_\sigma \omega$. Thus, $P(\omega) = P(\eta)$ and (*c*) follows from (4.1). □

By part (*b*) of (4.4), $\mathscr{H}|X$ passes to a C^∞ foliation $\mathscr{F}$ of Y. The quotient map imbeds U as an open, $\mathscr{F}$-saturated subset of Y and $\mathscr{F}|U = \mathscr{H}|U$. By parts (*a*) and (*c*) of (4.4), $\hat{\mathscr{F}}$ has all of the properties that we have been assuming in this paper. Also, α has contact of infinite order with dt along $\partial_0 X$, so $\mathscr{F}$ is C^∞-trivial at ∂Y and we can pass to the double M of Y, with the doubled foliation also being denoted by $\mathscr{F}$. As earlier remarked, U does not fiber over S^1 with connected fibers, so we have proven the following.

(4.5) THEOREM. — *For each integer* $n \geqslant 4$, *there exists a closed, orientable n-manifold* M *with a transversely orientable,* C^∞ *foliation* $\mathscr{F}$ *of codimension one and an open, connected,* $\mathscr{F}$*-saturated set* U *of locally dense leaves without holonomy, such that* $\hat{U}$ *admits no associated Tischler foliation.*

Remarks. — (1) One can show that the leaves of $\mathscr{F}|U$ are diffeomorphic to $\mathbf{R}^{n-1}$ and have exponential growth.

(2) Although the product foliation of $\hat{U} \cong L \times [h(1), 1]$ does fiber U over $(h(1), 1) \cong \mathbf{R}$, a simple foliated surgery along a closed transversal to $\mathscr{F} | U$ will alter the example so that the new manifold $\hat{U}$ admits no foliation, tangent to $\partial \hat{U}$, that fibers U over a 1-manifold.

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