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HIROHIKO SHIMA

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HOMOGENEOUS HESSIAN MANIFOLDS

by Hirohiko SHIMA

Introduction.

In [8] [9] [10] we introduced the notion of Hessian manifolds and studied the geometry of such manifolds. We first recall the definition of Hessian manifolds(*). Let M be a flat affine manifold, i.e., M admits open charts $(U_\alpha, \{x_\alpha^1, \ldots, x_\alpha^n\})$ such that $M = \cup U_\alpha$ and whose coordinate changes are all affine functions. Such local coordinate systems $\{x_\alpha^1, \ldots, x_\alpha^n\}$ will be called affine local coordinate systems. Throughout this paper the local expressions for geometric concepts on M will be given in terms of affine local coordinate systems.

A Riemannian metric g on M is said to be *Hessian* if for each point $p \in M$ there exists a C^∞-function ϕ defined on a neighbourhood of p such that $g_{ij} = \dfrac{\partial^2 \phi}{\partial x^i \partial x^j}\cdot$ Let D denote the covariant differential with respect to the flat affine structure on M. Using D we may define the exterior differentiation for cotangent bundle valued forms. We know that a Riemannian metric g is Hessian if and only if the cotangent bundle valued 1-form g^0 corresponding to g has an exterior differential zero [8];

$$D_X g^0(Y) - D_Y g^0(X) - g^0([X, Y]) = 0$$

for all vector fields X, Y on M. A flat affine manifold provided

(*) In this paper for the sake of brevity we adopt the term of *Hessian* instead of *locally Hessian* used in [8] [9] [10].

with a Hessian metric is called a *Hessian manifold.* As we see (Proposition 0.1), the tangent bundle over a Hessian manifold admits in a natural way a Kählerian structure. Thus the geometry of Hessian manifolds is related with that of certain Kählerian manifolds.

Let M be a Hessian manifold. A diffeomorphism of M onto itself is called an automorphism of M if it preserves both the flat affine structure and the Hessian metric. The set of all automorphisms of M, denoted by Aut(M), forms a Lie group. A Hessian manifold M is said to be homogeneous if the group Aut(M) acts transitively on M.

For homogeneous Kählerian manifolds Vinberg and Gindikin proposed the following conjecture and settled the related problems [1] [14].

Every homogeneous Kählerian manifolds admits a holomorphic fibering, whose base space is holomorphically isomorphic with a homogeneous bounded domain, and whose fiber is, with the induced Kählerian structure, isomorphic with the direct product of a locally flat homogeneous Kählerian manifold and a simply connected compact homogeneous Kählerian manifold.

In this paper we consider analogous problems for homogeneous Hessian manifolds and obtain the following results.

MAIN THEOREM. – *Let* M *be a connected homogeneous Hessian manifold. Then we have*

1) *The domain of definition* E_x *for the exponential mapping* $\exp_x$ *at* $x \in M$ *given by the flat affine structure is a convex domain. Moreover* E_x *is the universal covering manifold of* M *with affine projection* $\exp_x : E_x \longrightarrow M$.

2) *The universal covering manifold* E_x *of* M *has a decomposition* $E_x = E_x^0 + E_x^+$ *where* E_x^0 *is a uniquely determined vector subspace of the tangent space* $T_x M$ *of* M *at* x *and* E_x^+ *is an affine homogeneous convex domain not containing any full straight line. Thus* E_x *admits a unique fibering with the following properties:*

(i) *The base space is* E_x^+.

(ii) *The projection* $p : E_x \longrightarrow E_x^+$ *is given by the canonical projection from* $E_x = E_x^0 + E_x^+$ *onto* E_x^+.

(iii) *The fiber* $E_x^0 + v$ *through* $v \in E_x$ *is characterized as the set of all points which can be joined with* v *by full straight lines contained in* E_x. *Moreover each fiber is an affine subspace of* $T_x M$ *and is a Euclidean space with respect to the induced metric.*

(iv) *Every automorphism of* E_x *is fiber preserving.*

(v) *The group of automorphisms of* E_x *which preserve every fiber, acts transitively on the fibers.*

COROLLARY 1. – *Let* β *denote the canonical bilinear form on a connected homogeneous Hessian manifold* M; $\beta_{ij} = \dfrac{\partial^2 \log F}{\partial x^i \partial x^j}$ *where* $F = \sqrt{\det [g_{ij}]}$. *Then we have*

(i) β *is positive semi-definite.*

(ii) *The null space of* β *at* $x \in M$ *coincides with* E_x^0. *In particular*

(iii) $\beta = 0$ *if and only if* $E_x = T_x M$ *and it is a Euclidean space with respect to the induced metric.*

(iv) β *is positive definite if and only if* E_x *is an affine homogeneous convex domain not containing any full straight line.*

In [5] Kobayashi considered pseudo-distances c_M^a, c_M, d_M^a and d_M on a flat affine (more generally flat projective) manifold M (see also [11]).

COROLLARY 2. – *Let* M *be a connected homogeneous Hessian manifold and let* d *be one of the pseudo-distances on* E_x *listed above. Then the fiber through a point* $v \in E_x$ *is characterized by the set of all points* $w \in E_x$ *such that* $d(v, w) = 0$. *In particular we have:*

(i) $d = 0$ *if and only if* $E_x = T_x M$ *and it is a Euclidean space with respect to the induced metric.*

(ii) d *is a distance on* E_x *if and only if* E_x *is an affine homogeneous convex domain not containing any full straight line.*

COROLLARY 3. – *Let* M *be a connected homogeneous Hessian manifold. If there is no affine map of* $\mathbf{R}$ *into* M *except for constant* *maps, then the universal covering manifold of* M *is an affine homogeneous convex domain not containing any full straight line.*

COROLLARY 4. – *If a connected Lie subgroup* G *of* Aut(M) *acts transitively on a Hessian manifold* M *and if the isotropy subgroup of* G *at a point in* M *is discrete, then* G *is a solvable Lie group.*

COROLLARY 5. – *If a connected homogeneous Hessian manifold* M *admits a transitive reductive Lie subgroup of* Aut(M), *then the universal covering manifold of* M *is a direct product of a Euclidean space and an affine homogeneous convex self-dual cone not containing any full straight line.*

COROLLARY 6. – *A compact connected homogeneous Hessian manifold is a Euclidean torus.*

At the conclusion of this introduction we show the relation between Hessian manifolds and Kählerian manifolds. Let M be a flat affine manifold and let $\pi : \mathrm{TM} \longrightarrow \mathrm{M}$ be the tangent bundle over M with projection π. Then the space TM admits in a natural way a complex structure induced by the flat affine structure on M. Indeed, for an affine local coordinate system $\{x^1, \ldots, x^n\}$ we put $z^i = y^i + \sqrt{-1}\, y^{n+i}$ where $y^i = x^i \circ \pi$, $y^{n+i} = dx^i$, $i = 1, \ldots, n$. The systems $\{z^1, \ldots, z^n\}$ defined as above give a complex structure on TM (cf. [2]).

Let g be a Riemannian metric on M. If we set

$$g^{\mathrm{T}} = \sum_{i,j=1}^{n} (g_{ij} \circ \pi)\, dz^i d\bar{z}^j ,$$

then g^{T} is a Hermitian metric on TM (the definition of g^{T} is independent of the choice of affine local coordinate systems).

PROPOSITION 0.1. – *A Riemannian metric* g *on* M *is Hessian if and only if the corresponding Hermitian metric* g^{T} *on* TM *is Kählerian.*

Proof. – Since the fundamental 2-form ρ of the Hermitian metric g^{T} is expressed locally as

$$\rho = 2 \sum_{i,j=1}^{n} (g_{ij} \circ \pi)\, dy^i \wedge dy^{n+j} ,$$

we know that $d\rho = 0$ if and only if $\dfrac{\partial g_{ij}}{\partial x^k} = \dfrac{\partial g_{kj}}{\partial x^i}$, which is equivalent to g being Hessian (cf. [8]). q.e.d.

1. Proof of Main Theorem 1) .

In this section we prove the first part of Main Theorem along the same line as Koszul [6] [7]. Let M be a Hessian manifold with Hessian metric g. A C^∞-function ϕ defined on an open set U in M is called a *primitive* of g on U if it satisfies the condition $g_{ij} = \dfrac{\partial^2 \phi}{\partial x^i \partial x^j}$ on a neighbourhood of each point in U.

From now on we always assume that M is a connected homogeneous Hessian manifold.

LEMMA 1.1. – *Let $\{x^1, \ldots, x^n\}$ be an affine local coordinate system in* U. *If ϕ is a primitive of g on* U, *then $\dfrac{\partial \phi}{\partial x^j}$ $(j = 1, \ldots, n)$ are regular rational functions in $x^1, \ldots, x^n$* (*).

Proof. – Let $\mathfrak{g}$ be the Lie algebra of the automorphism group Aut(M). For $X \in \mathfrak{g}$ we denote by X^* the vector field on M induced by $\exp(-tX)$. For fixed $p \in U$ there exist a neighbourhood W of p in U and elements $X_1, \ldots, X_n$ in $\mathfrak{g}$ such that the values of the vector fields $X_1^*, \ldots, X_n^*$ at each point $q \in W$ form a basis of the tangent space of M at q. So we have $\dfrac{\partial}{\partial x^j} = \sum_i \eta_j^i X_i^*$ on W, where each η_j^i is a C^∞-function on W. Since X_i^* is an infinitesimal affine transformation, the components ξ_i^j of $X_i^* = \sum_j \xi_i^j \dfrac{\partial}{\partial x^j}$ are affine functions in $x^1, \ldots, x^n$. Therefore η_j^i are rational functions in $x^1, \ldots, x^n$. Since $X^* = \sum_j \xi^j \dfrac{\partial}{\partial x^j}$ $(X \in \mathfrak{g})$ is an infini-

(*) The author learned this result from Professor Koszul.

tesimal isometry and its components are affine functions, we get

$$\frac{\partial^2 X^*\phi}{\partial x^i \partial x^j} = \sum_p \frac{\partial \xi^p}{\partial x^i} g_{pj} + \sum_p \frac{\partial \xi^p}{\partial x^j} g_{pi} + \sum_p \xi^p \frac{\partial g_{ij}}{\partial x^p} = 0,$$

and so $X^*\phi$ is an affine function in $x^1, \ldots, x^n$. Thus $\frac{\partial \phi}{\partial x^j} = \sum_i \eta_j^i X_i^* \phi$ is a regular rational function in $x^1, \ldots, x^n$ on W, and also on U because p is an arbitrary point in U. q.e.d.

We now need the following lemma due to Koszul [7].

LEMMA 1.2. – *Let* M *be a connected flat affine manifold and let* E_x *be the domain of definition for the exponential mapping* $\exp_x$ *at* $x \in M$ *given by the flat affine structure. Then* $\exp_x$ *is an affine mapping from* E_x *to* M *and its rank is maximum at every point in* E_x *and equal to* $\dim M$. *Moreover if* E_x *is convex it is the universal covering manifold of* M *with covering projection* $\exp_x$.

It follows from this lemma that the induced metric $\tilde{g} = \exp_x^* g$ on E_x is Hessian.

LEMMA 1.3. – *There exists a primitive* ψ *of* $\tilde{g}$ *on* E_x.

Proof. – Let $\{y^1, \ldots, y^n\}$ be an affine coordinate system on $T_x M$. Define a 1-form γ_i on E_x by $\gamma_i = \sum_j \tilde{g}_{ij}\, dy^j$. We have then $d\gamma_i = \sum_{k<j} \left(\frac{\partial \tilde{g}_{ij}}{\partial y^k} - \frac{\partial \tilde{g}_{ik}}{\partial y^j}\right) dy^k \wedge dy^j = 0$. Since E_x is star-shaped with respect to the origin 0, by Poincaré Lemma there exists a C^∞-function h_i on E_x such that $\gamma_i = dh_i$. If we define a 1-form γ on E_x by $\gamma = \sum_i h_i\, dy^i$, we get $d\gamma = \sum_{j<i} \left(\frac{\partial h_i}{\partial y^j} - \frac{\partial h_j}{\partial y^i}\right) dy^j \wedge dy^i = 0.$ Again by Poincaré Lemma there exists a C^∞-function ψ such that $\gamma = d\psi$. Thus we have $\tilde{g}_{ij} = \frac{\partial^2 \psi}{\partial y^i \partial y^j}$. q.e.d.

LEMMA 1.4 (Koszul [6]). – *Let* a *be an element in* $T_x M$ *such that* $ta \in E_x$ *for* $0 \leqslant t < 1$ *and* $a \notin E_x$. Then we have

$$\lim_{t \to 1} \psi(ta) = \infty,$$

where ψ *is a primitive of* $\tilde{g}$ *on* E_x.

Proof. – The length of the curve $\exp_x(ta)$ $(0 \leqslant t < \theta)$ with respect to g is given by

$$1(\theta) = \int_0^\theta g(\dot{\exp}_x(ta)\,,\ \exp_x(ta))^{1/2} dt = \int_0^\theta \left(\frac{dF}{dt}\right)^{1/2} dt\,,$$

where $F(t) = \dfrac{d}{dt}\,\psi(ta)$. Since the Riemannian metric g on M is complete because M is homogeneous, we have

$$\lim_{\theta \to 1} 1(\theta) = \lim_{\theta \to 1} \int_0^\theta \left(\frac{dF}{dt}\right)^{1/2} dt = \infty\,.$$

For each $0 \leqslant t_0 < 1$ there exists a primitive ϕ_{t_0} defined on a neighbourhood of $\exp_x(t_0 a)$ such that $\psi = \phi_{t_0} \circ \exp_x$ and so by Lemma 1.1 and 1.2 $F(t)$ is a regular rational function in t $(0 \leqslant t < 1)$. This together with $\lim_{\theta \to 1} \int_0^\theta \left(\frac{dF}{dt}\right)^{1/2} dt = \infty$ means that $F(t)$ has a pole of order $\geqslant 1$ at $t = 1$. Thus we get

$$\lim_{t \to 1} \psi(ta) = \lim_{\theta \to 1} \int_0^\theta F(t)\,dt + \psi(0) = \infty\,. \qquad \text{q.e.d.}$$

According to Lemma 1.4, Lemma 4.2 in [6] and the fact that E_x is star-shaped with respect to the origin 0, E_x is a convex domain in $T_x M$. Moreover by Lemma 1.2 E_x is the universal covering manifold of M with projection $\exp_x : E_x \longrightarrow M$. Thus Main Theorem 1) is completely proved.

2. Normal Hessian algebras.

Let Ω be an affine homogeneous domain in $\mathbf{R}^n$ with an invariant Hessian metric g. In this section we first show that Ω admits a simply transitive triangular subgroup of $\mathrm{Aut}(\Omega)$ and using this we construct a normal Hessian algebra (Definition 2.3). According to Theorem 2.1 the study of affine homogeneous domains with invariant Hessian metric is reduced to that of normal Hessian algebras.

Let $A(n)$ denote the group of all affine transformations of $\mathbf{R}^n$ and $\mathrm{Aff}(\Omega)$ the set of all elements in $A(n)$ leaving Ω invariant. Then it is easy to see that $\mathrm{Aff}(\Omega)$ is a closed subgroup of $A(n)$. Denoting by $I(\Omega)$ the group of all isometries of Ω with respect to the Hessian metric g it follows $\mathrm{Aut}(\Omega) = \mathrm{Aff}(\Omega) \cap \mathrm{I}(\Omega)$. A subgroup of $\mathrm{A}(n)$ is said to be algebraic if it is selected from $\mathrm{A}(n)$ by polynomial equations connecting the coefficients of an affine transformation in an affine coordinate system.

LEMMA 2.1. – *Let* N *be the normalizer of the identity component of* $\mathrm{Aff}(\Omega)$ *in* $\mathrm{A}(n)$. *Then* N *is algebraic and* N, $\mathrm{Aff}(\Omega)$ *have the same identity component.*

For the proof see Vinberg [13].

PROPOSITION 2.1. – *The identity component* $\mathrm{Aut}_0(\Omega)$ *of* $\mathrm{Aut}(\Omega)$ *coincides with that of an algebraic group in* $\mathrm{A}(n)$.

Proof. – Let $\{x^1, \ldots, x^n\}$ be an affine coordinate system on $\mathbf{R}^n$. For $a \in \mathrm{A}(n)$ we denote by $\mathbf{f}(a) = [\mathbf{f}(a)^i_j]$ and $\mathbf{q}(a) = [\mathbf{q}(a)^i]$ the linear part and the translation part of a respectively, where $x^i \circ a = \sum_j \mathbf{f}(a)^i_j x^j + \mathbf{q}(a)^i$. An element $a \in \mathrm{Aff}(\Omega)$ is contained in $\mathrm{I}(\Omega)$ if and only if $\sum_{r,s} \mathbf{f}(a)^r_i \mathbf{f}(a)^s_j g_{rs}(ap) = g_{ij}(p)$ holds for all $p \in \Omega$. Let ϕ be a primitive of g on Ω. Then by Lemma 1.1 the functions $g_{ij} = \dfrac{\partial^2 \phi}{\partial x^i \partial x^j}$ defined on Ω are rational functions in $x^1, \ldots, x^n$. Therefore we may regard g_{ij} as rational functions on $\mathbf{R}^n$ with respect to $x^1, \ldots, x^n$. Put

$$\mathrm{H} = \left\{ a \in \mathrm{A}(n) \,\middle|\, \sum_{r,s} \mathbf{f}(a)^r_i \mathbf{f}(a)^s_j g_{rs}(ax) = g_{ij}(x) \quad \text{for all } x \in \mathbf{R}^n, \; i, j = 1, \ldots, n \right\}.$$

Then H is an algebraic group in $\mathrm{A}(n)$ and $\mathrm{Aut}(\Omega) = \mathrm{Aff}(\Omega) \cap \mathrm{H}$. Therefore by Lemma 2.1 $\mathrm{Aut}_0(\Omega)$ coincides with the identity component of the algebraic group $\mathrm{N} \cap \mathrm{H}$. q.e.d.

PROPOSITION 2.2. – *The isotropy subgroup of* $\mathrm{Aut}_0(\Omega)$ *at a point in* Ω *is a maximal compact subgroup of* $\mathrm{Aut}_0(\Omega)$.

Proof. – Let K be the isotropy subgroup of $\mathrm{Aut}_0(\Omega)$ at $p \in \Omega$. Since $\mathrm{Aff}(\Omega)$ and H are closed in $\mathrm{A}(n)$, $\mathrm{Aut}_0(\Omega)$ is closed in $\mathrm{A}(n)$ and so K is closed in $\mathrm{A}(n)$. Let $\{x^1, \ldots, x^n\}$ be an affine coordinate system such that $x^i(p) = 0$ and $g_{ij}(p) = \delta_{ij}$

where δ_{ij} is Kronecker's delta. Representing affine transformations in terms of $x^1, \ldots, x^n$ it follows $K \subset O(n)$ where $O(n)$ is the orthogonal matrix group. Therefore K is a compact subgroup of $\mathrm{Aut}_0(\Omega)$. Let K′ be a maximal compact subgroup of $\mathrm{Aut}_0(\Omega)$ containing K. Then there exists a fixed point $p' \in \Omega$ for K′ because Ω is a convex domain. Taking $a \in \mathrm{Aut}_0(\Omega)$ such that $ap' = p$ we get $aK'a^{-1} \subset K$. Since $aK'a^{-1}$ is a maximal compact subgroup of $\mathrm{Aut}_0(\Omega)$ we obtain $K = aK'a^{-1}$ and so K is a maximal compact subgroup of $\mathrm{Aut}_0(\Omega)$. q.e.d.

A subgroup T of $A(n)$ is said to be *triangular* if the linear parts of the transformation in T can be written as upper triangular matrices with respect to some affine coordinate system.

By Proposition 2.1 and by a theorem of Vinberg [12] we get a decomposition $\mathrm{Aut}_0(\Omega) = TK$, where T and K are a maximal connected triangular subgroup and a maximal compact subgroup of $\mathrm{Aut}_0(\Omega)$ respectively, and $T \cap K$ consists of the unit element only. Using this together with Proposition 2.2 we have

PROPOSITION 2.3. – *Let Ω be an affine homogeneous domain in $\mathbf{R}^n$ with an invariant Hessian metric. Then Ω admits a simply transitive triangular subgroup of* $\mathrm{Aut}(\Omega)$.

Choose a point $o \in \Omega$ and an affine coordinate system $\{x^1, \ldots, x^n\}$ such that $x^i(o) = 0$ $(i = 1, \ldots, n)$. Let T be a connected triangular subgroup of $\mathrm{Aut}(\Omega)$ acting simply transitively on Ω and $\mathfrak{t}$ the Lie algebra of T. For $X \in \mathfrak{t}$ we denote by X^* the vector field on Ω induced by a one parameter subgroup of $\exp(-tX)$. We have then $X^* = -\sum_i \left(\sum_j f(X)^i_j x^j + q(X)^i\right) \dfrac{\partial}{\partial x^i}$, where $f(X)^i_j$ and $q(X)^i$ are constants determined by X. Let V be the tangent space of Ω at o. Define mappings $q : \mathfrak{t} \longrightarrow V$ and $f : \mathfrak{t} \longrightarrow \mathfrak{gl}(V)$ by

$$q(X) = \sum_i q(X)^i \left(\frac{\partial}{\partial x^i}\right)_0 ,$$

$$f(X)\, q(Y) = \sum_{i,j} f(X)^i_j\, q(X)^j \left(\frac{\partial}{\partial x^i}\right)_0 .$$

Then we have

(1) f is a representation of $\mathfrak{t}$ in V.

(2) q is a linear isomorphism from $\mathfrak{t}$ onto V satisfying

$$q([X, Y]) = f(X)q(Y) - f(Y)q(X) \quad \text{for} \quad X, Y \in \mathfrak{t}.$$

We now define an operation of multiplication in V by the formula

$$x \cdot y = f(q^{-1}(x))y \quad \text{for} \quad x, y \in V. \tag{3}$$

The algebra V with this multiplication is called *the algebra of the affine homogeneous domain* Ω *with respect to the point* $o \in \Omega$ *and the simply transitive connected triangular group* T. *Using the notation*

$$x \cdot y = L_x y = R_y x,$$

$$[x \cdot y \cdot z] = x \cdot (y \cdot z) - (x \cdot y) \cdot z,$$

from (1) (2) we get

$$[L_x, L_y] = L_{x \cdot y - y \cdot x}, \tag{4}$$

$$[x \cdot y \cdot z] = [y \cdot x \cdot z], \tag{5}$$

$$[L_x, R_y] = R_{x \cdot y} - R_y R_x, \tag{6}$$

for $x, y, z \in V$. The conditions (4), (5) and (6) are mutually equivalent.

DEFINITION 2.1 – *An algebra satisfying one of the conditions (4) (5) (6) is said to be left symmetric* (cf. Vinberg [13]).

DEFINITION 2.2. – *A left symmetric algebra is said to be normal if all operators* L_x *have only real eigenvalues* (cf. [13]).

Let $\langle\ ,\ \rangle$ denote the inner product on V given by the Hessian metric. Then we have

$$\langle x \cdot y, z \rangle + \langle y, x \cdot z \rangle = \langle y \cdot x, z \rangle + \langle x, y \cdot z \rangle \tag{7}$$

for all $x, y, z \in V$ (cf. [8]).

DEFINITION 2.3. – *A left symmetric algebra endowed with an inner product satisfying (7) is called a Hessian algebra.*

Summing up the obtained results, we have

PROPOSITION 2.4. – *Let Ω be an affine homogeneous domain with an invariant Hessian metric. Then the algebra of Ω with respect to a point in Ω and a simply transitive connected triangular group is a normal Hessian algebra.*

Conversely we shall prove that a normal Hessian algebra determines an affine homogeneous domain with an invariant Hessian metric.

Let V be a normal Hessian algebra endowed with an inner product $\langle\, ,\rangle$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V with respect to $\langle\, ,\rangle$ and $\{x^1, \ldots, x^n\}$ the affine coordinate system on V given by $v = \sum_i x^i(v)\, e_i$ for all $v \in \mathrm{V}$. We denote by $\mathbf{f}(a) \in \mathrm{GL(V)}$ and $\mathbf{q}(a) \in \mathrm{V}$ the linear part and the translation part of $a \in \mathrm{A}(n)$ respectively; $av = \mathbf{f}(a)v + \mathbf{q}(a)$. For $v \in \mathrm{V}$ we define an infinitesimal affine transformation X_v^* by

$$\mathrm{X}_v^* = -\sum_{i,j} (\mathrm{L}_{v_j}^i x^j + v^i) \frac{\partial}{\partial x^i}, \tag{8}$$

where $\mathrm{L}_{v_j}^i$, v^i are the components of L_v, v with respect to $\{e_1, \ldots, e_n\}$; $\mathrm{L}_v e_j = \sum_i \mathrm{L}_{v_j}^i e_i$, $v = \sum_i v^i e_i$. From (4) it follows

$$[\mathrm{X}_v^*, \mathrm{X}_w^*] = \mathrm{X}_{v.w - w.v}^* \quad \text{for } v, w \in \mathrm{V}, \tag{9}$$

and so $\mathfrak{t}(\mathrm{V}) = \{\mathrm{X}_v^* \mid v \in \mathrm{V}\}$ forms a Lie algebra. Let T(V) denote the connected Lie subgroup of $\mathrm{A}(n)$ generated by $\mathfrak{t}(\mathrm{V})$. We denote by $\Omega(\mathrm{V})$ the open orbit of T(V) through the origin 0; $\Omega(\mathrm{V}) = \mathrm{T(V)}\,0$, which we call *the affine homogeneous domain corresponding to* V.

We first show that T(V) acts simply transitively on $\Omega(\mathrm{V})$. By (8) the isotropy subgroup B of T(V) at 0 is discrete. Suppose $b \in \mathrm{B}$. Since the exponential mapping $\exp : \mathfrak{t}(\mathrm{V}) \longrightarrow \mathrm{T(V)}$ is surjective because T(V) is triangular, there exists $\mathrm{X}_w^* \in \mathfrak{t}(\mathrm{V})$ such that $b = \exp \mathrm{X}_w^*$. If we put $b' = \exp 1/2\, \mathrm{X}_w^*$, then we have $0 = b0 = b'^2 0 = \mathbf{f}(b')\,\mathbf{q}(b') + \mathbf{q}(b')$ and so $\mathbf{f}(b')\,\mathbf{q}(b') = -\,\mathbf{q}(b')$. Since $\mathbf{f}(b') = \exp(-\,1/2\, \mathrm{L}_w)$ and since L_w is triangular, the eigenvalues of $\mathbf{f}(b')$ are all positive. This means $b'0 = \mathbf{q}(b') = 0$ and so $b' = \exp 1/2\, \mathrm{X}_w^* \in \mathrm{B}$. By the same argument we have $\exp 1/2^n\, \mathrm{X}_w^* \in \mathrm{B}$ for all non-negative integer n. Thus $\mathrm{X}_w^* = 0$ because B is discrete. Therefore B consists of the unit element only and T(V) acts simply transitively on $\Omega(\mathrm{V})$.

Now we denote by g the T(V)-invariant Riemannian metric on $\Omega(V)$ satisfying $g_{ij}(0) = \delta_{ij}$ (Kronecker's delta). It follows then

$$g_{ij}(a0) = \sum_{p} \mathsf{f}(a^{-1})_i^p \, \mathsf{f}(a^{-1})_j^p \quad \text{for} \quad a \in T(V) \tag{10}$$

where $\mathsf{f}(a)_i^j$ are the components of $\mathsf{f}(a)$ with respect to $\{e_1, \dots, e_n\}$. Denoting by $\exp tX_v^*$ the one parameter group generated by X_v^* we get $\left.\frac{d}{dt}\right|_{t=0} \mathsf{f}(\exp tX_v^*) = -L_v$ and $\left.\frac{d}{dt}\right|_{t=0} \mathsf{q}(\exp tX_v^*) = -v$. Choose an element $a \in T(V)$ and define an isomorphism $v \longrightarrow v'$ of V by $a^{-1} \exp tX_v^* \, a = \exp tX_{v'}^*$. Then we have

$$\begin{aligned} v' &= \mathsf{f}(a)^{-1} L_v \mathsf{q}(a) + \mathsf{f}(a)^{-1} v = L_{v'} \mathsf{f}(a)^{-1} \mathsf{q}(a) + \mathsf{f}(a)^{-1} v, \\ L_{v'} &= \mathsf{f}(a)^{-1} L_v \, \mathsf{f}(a). \end{aligned} \tag{11}$$

Let D denote the natural flat linear connection on $\Omega(V)$ given by $D dx^i = 0$. Put $A_{X^*} = L_{X^*} - D_{X^*}$ where L_{X^*} and D_{X^*} are the Lie differentiation and the covariant differentiation by a vector field X^* respectively. We have

$$(A_{X_u^*} X_v^*)_x = -\sum_i (L_u L_v x + L_u v)^i \left(\frac{\partial}{\partial x^i}\right)_x, \tag{12}$$

for all $x \in \Omega(V)$. Since $A_{X_u^*}$ is a derivation of the algebra of tensor fields and maps every function into zero and since $L_{X^*} g = 0$, it follows

$$(D_{X_u^*} g)(X_v^*, X_w^*) = g(A_{X_u^*} X_v^*, X_w^*) + g(X_v^*, A_{X_u^*} X_w^*). \tag{13}$$

Using (10) (11) (12) we obtain

$$\begin{aligned} &g(a0)((A_{X_u^*} X_v^*)_{a0}, (X_w^*)_{a0}) \\ &\quad = \sum_{i,j,p} \mathsf{f}(a^{-1})_i^p \, \mathsf{f}(a^{-1})_j^p (L_u L_v a0 + L_u v)^i (L_w a0 + w)^j \\ &\quad = \sum_p (\mathsf{f}(a^{-1})(L_u L_v \mathsf{q}(a) + L_u v))^p (\mathsf{f}(a^{-1})(L_w \mathsf{q}(a) + w))^p \\ &\quad = \sum_p (L_{u'} L_{v'} \mathsf{f}(a)^{-1} \mathsf{q}(a) + L_{u'} \mathsf{f}(a)^{-1} v)^p (L_{w'} \mathsf{f}(a)^{-1} \mathsf{q}(a) \\ &\qquad\qquad + \mathsf{f}(a)^{-1} w)^p \\ &\quad = \sum_p (u' \cdot v')^p w'^p \\ &\quad = \langle u' \cdot v', w' \rangle. \end{aligned}$$

This together with (7) (13) implies

$$(D_{X_u^*} g)(X_v^*, X_w^*) = (D_{X_v^*} g)(X_u^*, X_w^*),$$

and so g is a Hessian metric (cf. [8]).

Let Ω be an affine homogeneous domain in $\mathbf{R}^n$ with an invariant Hessian metric and V the normal Hessian algebra of Ω with respect to $0 \in \Omega$ and a simply transitive triangular group. Identifying the tangent space V of Ω at 0 with $\mathbf{R}^n$ the domain $\Omega(V)$ corresponding to V coincides with Ω. Therefore we have

THEOREM 2.1. – *Let* V *be a normal Hessian algebra. Then the domain* $\Omega(V)$ *constructed as above is an affine homogeneous domain with invariant Hessian metric. All affine homogeneous domains with invariant Hessian metric are obtained in this way.*

DEFINITION 2.4 (cf. [3]). – *A normal left symmetric algebra* U *is called a clan if it admits a linear function* ω *satisfying the condition*

(i) $\omega(x \cdot y) = \omega(y \cdot x)$ *for all* $x, y \in U$,

(ii) $\omega(x \cdot x) > 0$ *for all* $x \neq 0 \in U$.

Remark. – Let U be a clan with ω. If we put $\langle x, y \rangle = \omega(x \cdot y)$, then $\langle\, ,\, \rangle$ is an inner product on U satisfying the condition (7) and so U is a normal Hessian algebra.

The following theorem is due to Vinberg [13].

THEOREM 2.2. – *Let* V *be a clan. Then the domain* $\Omega(V)$ *is an affine homogeneous convex domain not containing any full straight line. All affine homogeneous convex domains not containing any full straight line are obtained in this way.*

3. Structure of normal Hessian algebras.

In this section we state a fundamental theorem for normal Hessian algebras. Let V be a normal Hessian algebra.

DEFINITION 3.1. – *Let* W *be a vector subspace of* V.

(a) W *is called a commutative subalgebra of* V *if* $W \cdot W = \{0\}$.

(b) W *is said to be an ideal of* V *if* $W \cdot V \subset W$ *and* $V \cdot W \subset W$.

THEOREM 3.1. – *Let* V *be a normal Hessian algebra. Then* V *is decomposed into the semi-direct sum* $V = I + U$, *where* I *is a commutative ideal of* V *and* U *is a subalgebra with an element* s *satisfying the following properties:*

(i) $s \cdot s = s$,

(ii) *the restriction of* L_s *on* U *is diagonalizable and has eigenvalues* 1, 1/2,

(iii) $R_s = 2L_s - 1$ *on* U,

where 1 *is the identity transformation of* U. (*An element* s *in* U *satisfying the above conditions is called a principal idempotent of* U.)

The proof of this theorem is carried out by induction on the dimension of normal Hessian algebras in an analogous way as Gindikin and Vinberg [1] [14].

For later use we prepare some lemmas.

LEMMA 3.1. – *Let* W *be an ideal of* V. *Then the orthogonal complement* $W^{\perp}$ *of* W *in* V *is a subalgebra.*

Proof. – Let $x, y \in W^{\perp}$ and $a \in W$. We have then

$$\langle a, x \cdot y \rangle = -\langle x \cdot a, y \rangle + \langle a \cdot x, y \rangle + \langle x, a \cdot y \rangle = 0.$$

This implies $x \cdot y \in W^{\perp}$. q.e.d.

LEMMA 3.2. – *Let* u *be a non-zero element in* V *and let* $P = \{p \in V \mid p \cdot u = 0\}$. *Suppose* P *is invariant by* L_u. *Then for* $p \in P$, $x \in V$ *we have*

(i) $L_u(p \cdot x) = (L_u p) \cdot x + p \cdot (L_u x)$,

(ii) $\exp tL_u(p \cdot x) = (\exp tL_u p) \cdot (\exp tL_u x)$,

(iii) $\dfrac{d}{dt} \langle \exp tL_u p, \exp tL_u x \rangle = \langle u, \exp tL_u(p \cdot x) \rangle$.

Proof. – (i) follows from

$$u \cdot (p \cdot x) = (u \cdot p) \cdot x + p \cdot (u \cdot x) - (p \cdot u) \cdot x.$$

(ii) is a consequence of (i). Using (7) in 2 and (ii) we obtain

$$\begin{aligned}
&\frac{d}{dt}\langle \exp t\mathrm{L}_u p\,,\ \exp t\mathrm{L}_u x\rangle \\
&\qquad = \langle \mathrm{L}_u \exp t\mathrm{L}_u p\,,\ \exp t\mathrm{L}_u x\rangle + \langle \exp t\mathrm{L}_u p\,,\ \mathrm{L}_u \exp t\mathrm{L}_u x\rangle \\
&\qquad = \langle (\exp t\mathrm{L}_u p)\cdot u\,,\ \exp t\mathrm{L}_u x\rangle + \langle u\,,\ (\exp t\mathrm{L}_u p)\cdot(\exp t\mathrm{L}_u x)\rangle \\
&\qquad = \langle u\,,\ \exp t\mathrm{L}_u(p\cdot x)\rangle . \qquad\qquad \text{q.e.d.}
\end{aligned}$$

LEMMA 3.3. – *Let* W *be a subspace of* V. *Suppose that an element* $a \neq 0 \in \mathrm{V}$ *satisfies the following conditions:*

(a) $a \cdot a = \epsilon a$, *where* $\epsilon = 0, 1$,

(b) L_a *and* R_a *leave* W *invariant,*

(c) a *is orthogonal to* $\mathrm{W}\cdot\mathrm{W}$.

Then we have:

(i) *If* $\epsilon = 0$, $\mathrm{L}_a = \mathrm{R}_a = 0$ *on* W.

(ii) *If* $\epsilon = 1$, *the restriction of* L_a *on* W *is symmetric and its eigenvalues are* $0, 1/2$. *Moreover* $\mathrm{R}_a = 2\mathrm{L}_a$ *on* W.

Proof. – From (6) in 2, (a) and (b) it follows

$$[\mathrm{L}_a, \mathrm{R}_a] = \epsilon \mathrm{R}_a - \mathrm{R}_a^2 \quad \text{on W}. \tag{1}$$

By (c) we have

$$\langle a\cdot x, y\rangle + \langle x, a\cdot y\rangle = \langle x\cdot a, y\rangle + \langle a, x\cdot y\rangle = \langle x\cdot a, y\rangle$$

for all $x, y \in \mathrm{W}$. This implies

$$\mathrm{L}_a + {}^t\mathrm{L}_a = \mathrm{R}_a \quad \text{on W}. \tag{2}$$

Put $\mathrm{S} = \epsilon\mathrm{R}_a - \mathrm{R}_a^2$. S being commutative with R_a we have $\mathrm{Tr_W\,S}^2 = \mathrm{Tr_W}[\mathrm{L}_a, \mathrm{R}_a]\mathrm{S} = \mathrm{Tr_W}[\mathrm{L}_a\mathrm{S}, \mathrm{R}_a] = 0$. This means $\mathrm{S} = 0$ on W because S is symmetric on W by (2) and so

$$\mathrm{R}_a^2 = \epsilon\mathrm{R}_a \ \text{ on W}, \quad [\mathrm{L}_a, \mathrm{R}_a] = 0 \ \text{ on W}. \tag{3}$$

Suppose $\epsilon = 0$. The facts that R_a is symmetric on W and that $\mathrm{R}_a^2 = 0$ on W imply $\mathrm{R}_a = 0$ on W. Using this and (2), L_a is skew symmetric on W and its eigenvalues are purely imaginary. Therefore we must have $\mathrm{L}_a = 0$ on W. Suppose $\epsilon = 1$. Since $\mathrm{R}_a^2 = \mathrm{R}_a$ on W the eigenvalues of R_a on W are $0, 1$. From (2) it follows $\mathrm{L}_a - {}^t\mathrm{L}_a = 2\mathrm{L}_a - \mathrm{R}_a$ on W. Since $[\mathrm{L}_a, \mathrm{R}_a] = 0$

on W and since the eigenvalues of L_a, R_a on W are real, the eigenvalues of $2L_a - R_a$ on W are real. On the other hand $L_a - {}^tL_a$ is skew symmetric and its eigenvalues are purely imaginary. Therefore we have $L_a - {}^tL_a = 2L_a - R_a = 0$ on W and so ${}^tL_a = L_a$ on W, $R_a = 2L_a$ on W. This means (ii). q.e.d.

The following lemmas 3.4*-3.7* are immediate consequences of Theorem 3.1.

LEMMA 3.4.* – *Let* U_λ *denote the eigenspaces of* L_s *on* U *corresponding to* λ. *Then we have:*

(i) $U = U_1 + U_{1/2}$,

$U_\lambda . U_\mu \subset U_{\mu-\lambda+1}$.

(ii) U *is a clan.*

Proof. – For $x \in U_\lambda$, $y \in U_\mu$ we have

$$s\cdot(x\cdot y) = (s\cdot x)\cdot y + x\cdot(s\cdot y) - (x\cdot s)\cdot y$$
$$= \lambda x\cdot y + \mu x\cdot y - (2\lambda - 1)x\cdot y = (\mu - \lambda + 1)x\cdot y$$

and so $x\cdot y \in U_{\mu-\lambda+1}$. Define a linear function ω on U by

$$\omega(x) = \frac{1}{\lambda}\langle s, x\rangle \quad \text{for} \quad x \in U_\lambda .$$

Let $x \in U_\lambda$, $y \in U_\mu$. Using

$$\langle s\cdot x, y\rangle + \langle x, s\cdot y\rangle = \langle x\cdot s, y\rangle + \langle s, x\cdot y\rangle,$$

$\mu - \lambda + 1 \neq 0$ and $x\cdot y \in U_{\mu-\lambda+1}$ we get

$$\langle x, y\rangle = \frac{1}{\mu - \lambda + 1}\langle s, x\cdot y\rangle = \omega(x\cdot y).$$

Thus we have $\langle x, y\rangle = \omega(x\cdot y)$ for all $x, y \in U$. Therefore U is a clan. q.e.d.

LEMMA 3.5.* – (i) *The restriction of* L_s *on* I *is symmetric and its eigenvalues are* 0, 1/2.

(ii) *Let* I_λ *denote the eigenspace of* L_s *on* I *corresponding to* λ. *Then we have* $I = I_0 + I_{1/2}$,

$$U_\lambda . I_\mu \subset I_{\mu-\lambda+1}, \quad I_\lambda \cdot U_\mu \subset I_{\mu-\lambda} .$$

(iii) $R_s = 2L_s$ *on* I.

Proof. – Since I is a commutative ideal of V and since $s \cdot s = s$, applying Lemma 3.3 it follows that the restriction of L_s on I is symmetric and its eigenvalues are 0, $1/2$ and moreover $R_s = 2L_s$ on I. Let $x \in U_\lambda$, $a \in I_\mu$. By Theorem 3.1 (iii) we obtain

$$\begin{aligned} s \cdot (x \cdot a) &= (s \cdot x) \cdot a + x \cdot (s \cdot a) - (x \cdot s) \cdot a \\ &= \lambda x \cdot a + \mu x \cdot a - (2\lambda - 1) x \cdot a = (\mu - \lambda + 1) x \cdot a \end{aligned}$$

and $x \cdot a \in I_{\mu - \lambda + 1}$. Let $a \in I_\lambda$, $x \in U_\mu$. By (iii) we have

$$\begin{aligned} s \cdot (a \cdot x) &= (s \cdot a) \cdot x + a \cdot (s \cdot x) - (a \cdot s) \cdot x \\ &= \lambda a \cdot x + \mu a \cdot x - 2\lambda a \cdot x = (\mu - \lambda)\, a \cdot x \end{aligned}$$

and so $a \cdot x \in I_{\mu - \lambda}$. q.e.d.

LEMMA 3.6*. – *The commutative ideal* I *of* V *is characterized by the set of all points* $x \in V$ *such that* $x \cdot x = 0$.

Proof. – Suppose $x \cdot x = 0$. If $x = a + y$ where $a \in I$ and $y \in U$, we have $0 = x \cdot x = a \cdot y + y \cdot a + y \cdot y$ and so $y \cdot y = 0$. By Lemma 3.4* (ii) there exists a linear function ω on U satisfying the conditions in Definition 2.4. Since $\omega(y \cdot y) = 0$, we have $y = 0$ and $x = a \in I$. q.e.d.

LEMMA 3.7*. – *The subspaces* I_0, $I_{1/2}$ *and* U *are mutually orthogonal with respect to* $\langle\ ,\ \rangle$.

Proof. – By Lemma 3.5* (i) I_0 and $I_{1/2}$ are orthogonal. For $a \in I_\lambda$ we have

$$0 = \langle s \cdot a, s \rangle + \langle a, s \cdot s \rangle - \langle a \cdot s, s \rangle - \langle s, a \cdot s \rangle = (-3\lambda + 1) \langle a, s \rangle$$

and so $\langle a, s \rangle = 0$ because $\lambda = 0, 1/2$. This implies s and I are orthogonal. Applying this, for $a \in I_\lambda$, $x \in U_\mu$ we obtain

$$0 = \langle s \cdot a, x \rangle + \langle a, s \cdot x \rangle - \langle a \cdot s, x \rangle - \langle s, a \cdot x \rangle = (\mu - \lambda) \langle a, x \rangle$$

and

$$\begin{aligned} 0 &= \langle s \cdot x, a \rangle + \langle x, s \cdot a \rangle - \langle x \cdot s, a \rangle - \langle s, x \cdot a \rangle \\ &= (\lambda - \mu + 1) \langle a, x \rangle . \end{aligned}$$

This shows $\langle a, x \rangle = 0$. Therefore I and U are orthogonal. q.e.d.

4. The case $u \cdot u = u$.

Since V is a normal left symmetric algebra, by Lie's Theorem there exists an element $u \neq 0 \in V$ such that $x \cdot u = \kappa(x) u$ for all $x \in V$, where κ is a linear function on V. Multiplying u by non-zero scalar (if necessary) the following two cases are possible ;

$$u \cdot u = u,$$
$$u \cdot u = 0.$$

In this section we consider the case $u \cdot u = u$ and prove the following.

PROPOSITION 4.1. – *Suppose* $u \cdot u = u$. *Then the operator* L_u *is diagonalizable and has eigenvalues* $0, 1/2, 1$. *Denoting by* V_λ *the eigenspace of* L_u *corresponding to* λ *we have:*

(i) $V = V_1 + V_{1/2} + V_0$ (*orthogonal decomposition*).

(ii) $V_1 = \{u\}$.

(iii) $u \cdot p = \frac{1}{2} p, \quad p \cdot u = 0 \quad$ for $\quad p \in V_{1/2}$.

(iv) $u \cdot q = 0, \quad q \cdot u = 0 \quad$ for $\quad q \in V_0$.

(v) $V_0 \cdot V_{1/2} \subset V_{1/2}, \quad V_{1/2} \cdot V_0 \subset V_{1/2},$

$V_0 \cdot V_0 \subset V_0, \quad V_{1/2} \cdot V_{1/2} \subset V_1$.

In particular $V_1 + V_{1/2}$ *is an ideal of* V *with principal idempotent* u *and* V_0 *is a subalgebra.*

Let P denote the kernel of R_u ;

$$P = \{p \in V \mid p \cdot u = 0\}. \tag{1}$$

Then we have

$$L_u P \subset P, \tag{2}$$

$$V = \{u\} + P. \tag{3}$$

Indeed for $p \in P$ we have

$$(u \cdot p) \cdot u = u \cdot (p \cdot u) + (p \cdot u) \cdot u - p \cdot (u . u) = 0,$$

which implies (2). (3) follows from $x - \kappa(x) u \in P$ for all $x \in V$.

LEMMA 4.1. – *The restriction of* L_u *on* P *is diagonalizable and has eigenvalues* $0, 1/2$.

Proof. – By Lemma 3.2 for $p \in \mathrm{P}$ we have

$$\frac{d}{dt}\langle \exp t\mathrm{L}_u p\,,\ \exp t\mathrm{L}_u u\rangle = \langle u\,,\ \exp t\mathrm{L}_u(p\cdot u)\rangle = 0\,,$$

and so

$$\langle \exp t\mathrm{L}_u p\,,\ u\rangle = ae^{-t}\,, \tag{4}$$

where a is a constant determined by p not depending on t. Using this for $x = cu + p \in \mathrm{V}$ $(c \in \mathbf{R},\ p \in \mathrm{P})$ we obtain

$$\begin{aligned}\langle u\,,\ \exp t\mathrm{L}_u x\rangle &= \langle u\,,\ ce^t u + \exp t\mathrm{L}_u p\rangle \\ &= \langle u\,,\ \exp t\mathrm{L}_u p\rangle + c\,\langle u\,,u\rangle e^t = ae^{-t} + be^t\,,\end{aligned} \tag{5}$$

where $a\,,b$ are constants determined by x not depending on t. Applying Lemma 3.2 and (5) we have for $p\,,q \in \mathrm{P}$

$$\frac{d}{dt}\langle \exp t\mathrm{L}_u p\,,\ \exp t\mathrm{L}_u q\rangle = \langle u\,,\ \exp t\mathrm{L}_u(p\cdot q)\rangle = ae^{-t} + be^t\,,$$

and consequently

$$\langle \exp t\mathrm{L}_u p\,,\ \exp t\mathrm{L}_u q\rangle = -\,ae^{-t} + be^t + c\,, \tag{6}$$

where $a\,,b$ and c are constants determined by $p\,,q$ not depending on t. From (6) it follows that L_u is diagonalizable on P. Indeed, if L_u is not diagonalizable on P there exist non-zero elements $p\,,q \in \mathrm{P}$ such that $\mathrm{L}_u p = \lambda p$, $\mathrm{L}_u q = \lambda q + p$. We have then

$$\begin{aligned}\langle \exp t\mathrm{L}_u p\,,\ \exp t\mathrm{L}_u q\rangle &= \langle e^{\lambda t}p\,,\ e^{\lambda t}q + te^{\lambda t}p\rangle \\ &= te^{2\lambda t}\langle p\,,p\rangle + e^{2\lambda t}\langle p\,,q\rangle\,,\end{aligned}$$

which contradicts to (6). Let λ be an eigenvalue of L_u on P and $p \neq 0 \in \mathrm{P}$ an eigenvector corresponding to λ. It follows then

$$\frac{d}{dt}\langle \exp t\mathrm{L}_u p\,,\ \exp t\mathrm{L}_u p\rangle = 2\lambda\,\langle p\,,p\rangle\,e^{2\lambda t}\,.$$

On the other hand (6) implies

$$\frac{d}{dt}\langle \exp t\mathrm{L}_u p\,,\ \exp t\mathrm{L}_u p\rangle = ae^{-t} + be^t\,.$$

Therefore we obtain

$$2\lambda\,\langle p\,,p\rangle\,e^{2\lambda t} = ae^{-t} + be^t\,, \tag{7}$$

consequently $\lambda = 0\,,\ 1/2\,,\ -1/2$. By (4) we get $\langle p\,,u\rangle\,e^{(\lambda+1)t} = a$, so $\langle p\,,u\rangle = 0$ and $a = 0$ because $\lambda + 1 \neq 0$. Thus we have

$$\langle p, u\rangle = 0 \quad \text{for all} \quad p \in \mathrm{P}, \tag{4'}$$

$$\langle u, \exp t\mathrm{L}_u x\rangle = be^t \quad \text{for} \quad x \in \mathrm{V}, \tag{5'}$$

$$2\lambda\langle p, p\rangle e^{2\lambda t} = be^t. \tag{7'}$$

(7') shows $\lambda = 0, 1/2$. q.e.d.

Let P_λ denote the eigenspace of L_u in P corresponding to λ. From Lemma 4.1 and (3) it follows

$$\mathrm{V} = \mathrm{V}_1 + \mathrm{V}_{1/2} + \mathrm{V}_0, \tag{8}$$

where $\mathrm{V}_1 = \{u\}$, $\mathrm{V}_{1/2} = \mathrm{P}_{1/2}$ and $\mathrm{V}_0 = \mathrm{P}_0$.

LEMMA 4.2. – *The decomposition (8) is orthogonal and we have* $\mathrm{P}_\lambda \cdot \mathrm{P}_\mu \subset \mathrm{V}_{\lambda+\mu}$.

Proof. – For $p \in \mathrm{P}_\lambda$ and $q \in \mathrm{P}_\mu$ we have

$$u\cdot(p\cdot q) = (u\cdot p)\cdot q + p\cdot(u\cdot q) - (p\cdot u)\cdot q = (\lambda+\mu)\,p\cdot q.$$

This implies $\mathrm{P}_\lambda \cdot \mathrm{P}_\mu \subset \mathrm{V}_{\lambda+\mu}$. The orthogonality of $\{u\}$ and P follows from (4'). Applying this for $p \in \mathrm{P}_{1/2}$ and $q \in \mathrm{P}_0$ we obtain $1/2\,\langle p, q\rangle = \langle u\cdot p, q\rangle = -\langle p, u\cdot q\rangle + \langle p\cdot u, q\rangle + \langle u, p\cdot q\rangle = 0$ because $p\cdot q \in \mathrm{P}_{1/2}$. Thus $\mathrm{P}_{1/2}$ and P_0 are orthogonal. q.e.d.

The assertion of Proposition 4.1 follows from Lemma 4.2 and (8).

5. The case $u\cdot u = 0$.

The purpose of this section is to prove the following.

PROPOSITION 5.1. – *Suppose* $u\cdot u = 0$. *Then there exists a commutative ideal of* V *containing* u.

LEMMA 5.1. – $\mathrm{L}_u^2 = 0$.

Proof. – Let P denote the kernel of R_u; $\mathrm{P} = \{p \in \mathrm{V} \mid p\cdot u = 0\}$. Then we have

$$\mathrm{L}_u\mathrm{V} \subset \mathrm{P}, \tag{1}$$

because $(u\cdot x)\cdot u = u\cdot(x\cdot u) + (x\cdot u)\cdot u - x\cdot(u\cdot u) = 0$ for all $x \in \mathrm{V}$. For $p \in \mathrm{P}$, $x \in \mathrm{V}$ it follows from (1) and Lemma 3.2

$$\begin{aligned}\frac{d^3}{dt^3}&\langle \exp tL_u p\,,\ \exp tL_u x\rangle \\ &= \frac{d^2}{dt^2}\langle u\,,\ \exp tL_u(p\cdot x)\rangle = \langle u\,,\ u\cdot p'\rangle \\ &= -\langle p', u\cdot u\rangle + \langle p'\cdot u\,, u\rangle + \langle u\,, p'\cdot u\rangle = 0\,,\end{aligned}$$

where $p' = L_u \exp tL_u(p\cdot x)\in P$, and consequently

$$\langle \exp tL_u p\,,\ \exp tL_u x\rangle = at^2 + bt + c\,, \tag{2}$$

where a, b, c are constants independent of t. Let λ be an eigenvalue of L_u on P and $p \neq 0 \in P$ an eigenvector corresponding to λ. By (2) we get $e^{2\lambda t}\langle p, p\rangle = at^2 + bt + c$, and so $\lambda = 0$. This together with (1) implies that the eigenvalues of L_u are equal to 0. Assume $L_u^2 \neq 0$. Then there exist non-zero elements $x, y, z\in V$ such that $u\cdot x = 0$, $u\cdot y = x$, $u\cdot z = y$. From this we have $\exp tL_u y = y + tx$, $\exp tL_u z = z + ty + \frac{t^2}{2}x$. Since $y = u\cdot z\in P$, applying (2) we obtain $\langle y + tx, z + ty + \frac{t^2}{2}x\rangle = at^2 + bt + c$. This is a contradiction because $\langle x, x\rangle \neq 0$. Thus we have $L_u^2 = 0$.

q.e.d.

Using $L_u^2 = 0$ we define a filtration of V. Consider the subspaces of V

$$\begin{aligned} V^{(-1)} &= V\,,\\ V^{(0)} &= \{x \in V \mid L_u x \in \{u\}\}\,,\\ V^{(1)} &= L_u V + \{u\},\\ V^{(2)} &= \{u\}\,.\end{aligned}$$

Then we have

LEMMA 5.2. – *The subspaces* $V^{(i)}$ *form a filtration of the algebra* V ;

(i) $V^{(-1)} \supset V^{(0)} \supset V^{(1)} \supset V^{(2)}$,

(ii) $V^{(i)}\cdot V^{(j)} \subset V^{(i+j)}$.

Moreover we have

(iii) $V^{(1)}\cdot V^{(1)} = \{0\}$.

Proof. – (i) follows from $u \cdot u = 0$ and $L_u^2 = 0$. Note that

$$(u \cdot x) \cdot (u \cdot y) = 0 \quad \text{for all} \quad x, y \in V. \tag{3}$$

In fact for $x, y \in V$ we have

$$\begin{aligned} 0 &= u \cdot (u \cdot (x \cdot y)) = u \cdot ((u \cdot x) \cdot y + x \cdot (u \cdot y) - (x \cdot u) \cdot y) \\ &= u \cdot ((u \cdot x) \cdot y) + u \cdot (x \cdot (u \cdot y)) - \kappa(x) u \cdot (u \cdot y) = (u \cdot (u \cdot x)) \cdot y \\ &+ (u \cdot x) \cdot (u \cdot y) - ((u \cdot x) \cdot u) \cdot y + (u \cdot x) \cdot (u \cdot y) + x \cdot (u \cdot (u \cdot y)) \\ &\qquad - (x \cdot u) \cdot (u \cdot y) = 2(u \cdot x) \cdot (u \cdot x) \end{aligned}$$

because $L_u^2 = 0$, $V \cdot u \subset \{u\}$ and $L_u V \subset P$. Let

$$u \cdot x + \lambda u, u \cdot y + \mu u \in V^{(1)} \ (x, y \in V, \lambda, \mu \in \mathbf{R}).$$

Using (1) and (3) we get

$$\begin{aligned} (u \cdot x + \lambda u) \cdot (u \cdot y + \mu u) &= (u \cdot x) \cdot (u \cdot y) + \mu (u \cdot x) \cdot u \\ &\quad + \lambda u \cdot (u \cdot y) + \lambda \mu u \cdot u \\ &= 0. \end{aligned}$$

This implies (iii). Let $x \in V^{(0)}$, $u \cdot y + \mu u \in V^{(1)}$ $(y \in V, \mu \in \mathbf{R})$. We have then $u \cdot x = \nu u$ $(\nu \in \mathbf{R})$ and

$$\begin{aligned} x \cdot (u \cdot y + \mu u) &= x \cdot (u \cdot y) + \mu x \cdot u = (x \cdot u) \cdot y + u \cdot (x \cdot y) \\ &\qquad - (u \cdot x) \cdot y + \mu x \cdot u \\ &= \kappa(x) u \cdot y + u \cdot (x \cdot y) - \nu u \cdot y + \mu \kappa(x) u \in V^{(1)}. \end{aligned}$$

In the same way $(u \cdot y + \mu u) \cdot x \in V^{(1)}$. Therefore we have

$$V^{(0)} . V^{(1)} \subset V^{(1)}, \quad V^{(1)} . V^{(0)} \subset V^{(1)}. \tag{4}$$

Let $u \cdot x + \mu u \in V^{(1)}$ $(x \in V, \mu \in \mathbf{R})$ and $y \in V^{(-1)}$. By (iii) we have

$$\begin{aligned} u \cdot ((u \cdot x + \mu u) \cdot y) &= u \cdot ((u \cdot x) \cdot y) + \mu u \cdot (u \cdot y) = (u \cdot (u \cdot x)) \cdot y \\ &\quad + (u \cdot x) \cdot (u \cdot y) - ((u \cdot x) \cdot u) \cdot y + \mu u \cdot (u \cdot y) = 0 \end{aligned}$$

and
$$\begin{aligned} u \cdot (y \cdot (u \cdot x + \mu u)) &= u \cdot (y \cdot (u \cdot x)) + \mu u \cdot (y \cdot u) \\ &= (u \cdot y) \cdot (u \cdot x) + y \cdot (u \cdot (u \cdot x)) - (y \cdot u) \cdot (u \cdot x) \\ &\qquad + \mu u \cdot (y \cdot u) \\ &= 0. \end{aligned}$$

This implies

$$V^{(1)} . V^{(-1)} \subset V^{(0)}, \quad V^{(-1)} . V^{(1)} \subset V^{(0)}. \tag{5}$$

Let $x, y \in V^{(0)}$. We have then $u \cdot x = \mu u$, $u \cdot y = \nu u$ and so

$$\begin{aligned} u \cdot (x \cdot y) &= (u \cdot x) \cdot y + x \cdot (u \cdot y) - (x \cdot u) \cdot y = \mu \nu u \\ &\qquad + \nu \kappa(x) u - \kappa(x) \nu u = \mu \nu u. \end{aligned}$$

This means
$$V^{(0)} \cdot V^{(0)} \subset V^{(0)} . \tag{6}$$
The other relations $V^{(i)} \cdot V^{(j)} \subset V^{(i+j)}$ are trivial. q.e.d.

If $V^{(0)} = V$, then $V^{(2)} = \{u\}$ is a commutative ideal of V and consequently Proposition 5.1 is proved. From now on we assume $V^{(0)} \neq V$. Since $V^{(0)}$ is a subalgebra of dimension less than dim V, by the inductive hypothesis we have $V^{(0)} = I + U$, where I is a commutative ideal of $V^{(0)}$ and U is a subalgebra with a principal idempotent s.

LEMMA 5.3. – $V^{(1)} \subset I$.

Proof. – According to Lemma 3.6* it follows
$$I = \{x \in V^{(0)} \mid x \cdot x = 0\} .$$
This and $V^{(1)} \cdot V^{(1)} = \{0\}$ imply $V^{(1)} \subset I$. q.e.d.

LEMMA 5.4. – $V \cdot I \subset V^{(0)}$, $I \cdot V \subset V^{(0)}$.

Proof. – Let $x \in V$ and $a \in I$. Since I is commutative and since u, $u \cdot x$, $x \cdot u \in I$ by Lemma 5.2 and 5.3, we have
$$u \cdot (x \cdot a) = (u \cdot x) \cdot a + x \cdot (u \cdot a) - (x \cdot u) \cdot a = 0$$
and
$$u \cdot (a \cdot x) = (u \cdot a) \cdot x + a \cdot (u \cdot x) - (a \cdot u) \cdot x = 0 .$$
This means $x \cdot a$, $a \cdot x \in V^{(0)}$. q.e.d.

If $I = V^{(0)}$, Lemma 5.4 implies that I is a commutative ideal of V containing u and Proposition 5.1 is proved. Henceforth we assume $I \neq V^{(0)}$, i.e., $U \neq \{0\}$.

Let s be a principal idempotent of U. Since $V^{(1)} \subset I$ and since $V^{(1)}$ is invariant by L_s and R_s, by Lemma 3.3 we have:

The restriction of L_s on $V^{(1)}$ is symmetric and its eigenvalues are $0, 1/2$. Therefore denoting by $V^{(1)}_\lambda$ the eigenspace of L_s corresponding to λ we obtain the orthogonal decomposition
$$V^{(1)} = V^{(1)}_0 + V^{(1)}_{1/2} . \tag{7}$$
$$R_s = 2L_s \quad \text{on} \quad V^{(1)} . \tag{8}$$
We set $s \cdot u = \alpha u$. From (8) it follows $u \cdot s = 2 s \cdot u = 2\alpha u$. Thus

$$\begin{aligned} L_s u &= \alpha u\,, \\ R_s u &= 2\alpha u\,, \quad \text{where} \quad \alpha = 0\,, 1/2\,. \end{aligned} \tag{9}$$

Consider the graded algebra $\overline{V}$ associated to the filtered algebra V: $\overline{V} = \overline{V}^{(-1)} + \overline{V}^{(0)} + \overline{V}^{(1)} + \overline{V}^{(2)}$, where $\overline{V}^{(i)} = V^{(i)}/V^{(i+1)}$ $(-1 \leqslant i \leqslant 1)$ and $\overline{V}^{(2)} = V^{(2)}$. For $x \in V^{(i)}$ we denote by $\overline{x}$ the element in $\overline{V}^{(i)}$ corresponding to x and by $L_{\overline{x}}$ (resp. $R_{\overline{x}}$) the left (resp. right) multiplication by $\overline{x}$.

Lemma 5.5. – (i) *The mapping* $L_{\overline{u}} : \overline{V}^{(-1)} \longrightarrow \overline{V}^{(1)}$ *is an isomorphism.*

(ii) $L_{\overline{s}} L_{\overline{u}} = L_{\overline{u}}(L_{\overline{s}} - \alpha)$ *on* $\overline{V}^{(-1)}$. *In particular the restriction of* $L_{\overline{s}}$ *on* $\overline{V}^{(-1)}$ *is diagonalizable and its eigenvalues are* α, $\alpha + 1/2$.

(iii) $R_{\overline{s}} L_{\overline{u}} = L_{\overline{u}} R_{\overline{s}}$ *on* $\overline{V}^{(-1)}$.

Proof. – The mapping $L_{\overline{u}} : \overline{V}^{(-1)} \longrightarrow \overline{V}^{(1)}$ is surjective because $\overline{V}^{(1)} = L_u V + \{u\}/\{u\}$. Suppose $L_{\overline{u}}\overline{x} = 0$ $(x \in V^{(-1)})$. Then it follows $u \cdot x \in \{u\}$, consequently $x \in V^{(0)}$ and $\overline{x} = 0$. Thus (i) is proved. By (9) we have

$$\begin{aligned} L_{\overline{s}} L_{\overline{u}} \overline{x} = \overline{s \cdot (u \cdot x)} &= \overline{(s \cdot u) \cdot x} + \overline{u \cdot (s \cdot x)} - \overline{(u \cdot s) \cdot x} \\ &= \overline{u \cdot (s \cdot x)} - \overline{\alpha u \cdot x} = L_{\overline{u}}(L_{\overline{s}} - \alpha)\overline{x} \end{aligned}$$

for all $x \in V^{(-1)}$, which implies $L_{\overline{s}} L_{\overline{u}} = L_{\overline{u}}(L_{\overline{s}} - \alpha)$ on $\overline{V}^{(-1)}$. Using this together with (7) the restriction of $L_{\overline{s}}$ on $\overline{V}^{(-1)}$ is diagonalizable and has eigenvalues α, $\alpha + 1/2$. This shows (ii). By (9) we obtain

$$\begin{aligned} R_{\overline{s}} L_{\overline{u}} \overline{x} &= \overline{(u \cdot x) \cdot s} = \overline{u \cdot (x \cdot s)} + \overline{(x \cdot u) \cdot s} - \overline{x \cdot (u \cdot s)} = \overline{u \cdot (x \cdot s)} \\ &+ \kappa(x)\,\overline{u \cdot s} - 2\overline{\alpha x \cdot u} = \overline{u \cdot (x \cdot s)} = L_{\overline{u}} R_{\overline{s}} \overline{x} \ \text{for all } x \in V^{(-1)}, \end{aligned}$$

which means (iii). q.e.d.

According to Lemma 3.5*, (7) and Lemma 5.5 the operator $L_{\overline{s}}$ leaves each subspace $\overline{V}^{(i)}$ invariant and is diagonalizable on $\overline{V}^{(i)}$. We denote by $\overline{V}^{(i)}_{\lambda}$ the eigenspace of $L_{\overline{s}}$ in $\overline{V}^{(i)}$ corresponding to $\lambda \in R$.

Lemma 5.6. – *Let* $\overline{a} \in \overline{V}^{(-1)}_{\lambda}$. *Then we have*

(i) $L_{\overline{s}}\overline{a} = \lambda\overline{a}$,

(ii) $R_{\overline{s}}\overline{a} = 2(\lambda - \alpha)\overline{a}$.

Proof. – Using Lemma 5.4 and (8) we obtain

$$L_{\overline{u}} R_{\overline{s}}\overline{a} = R_{\overline{s}} L_{\overline{u}}\overline{a} = \overline{R_s L_u a} = \overline{2L_s L_u a} = 2L_{\overline{s}} L_{\overline{u}}\overline{a} = 2L_{\overline{u}}(L_{\overline{s}} - \alpha)\overline{a} = L_{\overline{u}}(2(\lambda - \alpha)\overline{a}).$$

This implies $R_{\overline{s}}\overline{a} = 2(\lambda - \alpha)\overline{a}$ because $L_{\overline{u}} : \overline{V}^{(-1)} \longrightarrow \overline{V}^{(1)}$ is an isomorphism. q.e.d.

For simplicity we denote by $a' \in V^{(1)}$ the element $u \cdot a$ where $a \in V^{(-1)}$.

LEMMA 5.7. –

(i) *If* $\overline{a} \in V_{\lambda}^{(-1)}$, *then* $\overline{a}' \in \overline{V}_{\lambda-\alpha}^{(1)}$.

(ii) *Let* $\overline{a} \in V_{\lambda}^{(-1)}$, $\overline{b} \in \overline{V}_{\mu}^{(-1)}$. *Then we have*

$$\overline{a}' \cdot \overline{b} \, , \; \overline{a} \cdot \overline{b}' \in \overline{V}^{(0)}_{-\lambda+\mu+\alpha} \, .$$

Proof. – From Lemma 5.5 (ii) it follows

$$L_{\overline{s}}\overline{a}' = L_{\overline{s}} L_{\overline{u}}\overline{a} = L_{\overline{u}}(L_{\overline{s}} - \alpha)\overline{a} = (\lambda - \alpha) L_{\overline{u}}\overline{a} = (\lambda - \alpha)\overline{a}' ,$$

which implies (i). Using (i), (8) and Lemma 5.6 (ii) we obtain

$$\overline{s} \cdot (\overline{a}' \cdot \overline{b}) = (\overline{s} \cdot \overline{a}') \cdot \overline{b} + \overline{a}' \cdot (\overline{s} \cdot \overline{b}) - (\overline{a}' \cdot \overline{s}) \cdot \overline{b} = (\lambda - \alpha)\overline{a}' \cdot \overline{b} + \mu\overline{a}' \cdot \overline{b} - 2(\lambda - \alpha)\overline{a}' \cdot \overline{b} = (-\lambda + \mu + \alpha)\overline{a}' \cdot \overline{b}$$

and

$$\overline{s} \cdot (\overline{a} \cdot \overline{b}') = (\overline{s} \cdot \overline{a}) \cdot \overline{b}' + \overline{a} \cdot (\overline{s} \cdot \overline{b}') - (\overline{a} \cdot \overline{s}) \cdot \overline{b}' = \lambda\overline{a} \cdot \overline{b}' + (\mu - \alpha)\overline{a} \cdot \overline{b}' - 2(\lambda - \alpha)\overline{a} \cdot \overline{b}' = (-\lambda + \mu + \alpha)\overline{a} \cdot \overline{b}' .$$

This shows (ii). q.e.d.

According to Lemma 3.5* and Lemma 5.3 we get

$$V_{\lambda}^{(0)} \cdot V_{\mu'}^{(1)} \subset V^{(1)} \cap (I_{\lambda} + U_{\lambda}) \cdot I_{\mu'} = V^{(1)} \cap U_{\lambda} \cdot I_{\mu'} \subset V^{(1)} \cap I_{\mu'-\lambda+1} = V^{(1)}_{\mu'-\lambda+1}$$

and

$$V_{\lambda'}^{(1)} \cdot V_{\mu}^{(0)} \subset V^{(1)} \cap I_{\lambda'} \cdot (I_{\mu} + U_{\mu}) = V^{(1)} \cap I_{\lambda'} \cdot U_{\mu} \subset V^{(1)} \cap I_{\mu-\lambda'} = V^{(1)}_{\mu-\lambda'} \, .$$

Thus we have

$$\begin{aligned} V_{\lambda}^{(0)} \cdot V_{\mu'}^{(1)} &\subset V^{(1)}_{-\lambda+\mu'+1} \, , \\ V_{\lambda'}^{(1)} \cdot V_{\mu}^{(0)} &\subset V^{(1)}_{-\lambda'+\mu} \, . \end{aligned} \tag{10}$$

Consider the subspace $W^{(1)}$ of $V^{(1)}$ defined by

$$W^{(1)} = \{a \in V^{(1)} \mid \langle a, u \rangle = 0\}.$$

The subspace $W^{(1)}$ is invariant by L_s. In fact for $a \in W^{(1)}$ using (8), (9) and $V^{(1)} . V^{(1)} = \{0\}$ we have

$$\langle s \cdot a, u \rangle = -\langle a, s \cdot u \rangle + \langle a \cdot s, u \rangle + \langle s, a \cdot u \rangle = -\alpha \langle a, u \rangle + 2\langle s \cdot a, u \rangle = 2\langle s \cdot a, u \rangle$$

and $\langle s \cdot a, u \rangle = 0$, consequently $s \cdot a \in W^{(1)}$. We denote by $W^{(1)}_\lambda$ the eigenspace of L_s in $W^{(1)}$.

LEMMA 5.8. – *Suppose* $\rho' = \nu' - \beta + 1$. *If* $W^{(1)}_{\rho'} . V^{(0)}_\beta \subset \{u\}$, *then* $V^{(0)}_\beta . W^{(1)}_{\nu'} \subset \{u\}$.

Proof. – Let $a_1 \in W^{(1)}_{\nu'}$, $b_1 \in W^{(1)}_{\rho'}$ and $x \in V^{(0)}_\beta$. By (10) we have $x \cdot a_1 \in V^{(1)}_{\rho'}$ and $x \cdot b_1 \in V^{(1)}_{\rho' - \beta + 1}$. Since $b_1 \cdot x \in \{u\}$ and $W^{(1)} . W^{(1)} = \{0\}$, we obtain

$$\langle x \cdot b_1, a_1 \rangle + \langle b_1, x \cdot a_1 \rangle = \langle b_1 \cdot x, a_1 \rangle + \langle x, b_1 \cdot a_1 \rangle = 0.$$

If $\rho' - \beta + 1 \neq \nu'$, the orthogonality of the decomposition $V^{(1)} = V^{(1)}_0 + V^{(1)}_{1/2}$ implies $\langle b_1, x \cdot a_1 \rangle = 0$ and consequently $x \cdot a_1 \in \{u\}$. If $\rho' - \beta + 1 = \nu'$, then $\beta = 1$ and $\rho' = \nu'$. From this it follows $L_x W^{(1)}_{\nu'} \subset V^{(1)}_{\nu'}$. Define the mapping

$$A_x = pr \circ L_x : W^{(1)} \longrightarrow W^{(1)}$$

where pr is the projection from $V^{(1)} = W^{(1)} + \{u\}$ onto $W^{(1)}$. Then we have $\langle A_x b_1, a_1 \rangle + \langle b_1, A_x a_1 \rangle = 0$ for all $a_1, b_1 \in W^{(1)}_{\nu'}$ and so A_x is skew symmetric on $W^{(1)}_{\nu'}$. On the other hand A_x has only real eigenvalues because the eigenvalues of L_x are real. This means $A_x = 0$ on $W^{(1)}_{\nu'}$ and $L_x W^{(1)}_{\nu'} \subset \{u\}$. Thus the proof of this lemma is completed. q.e.d.

LEMMA 5.9. – *Let* $a, b, c \in V^{(-1)}$. *Then the products of* $\overline{a}, \overline{b}'$ *and* $\overline{c}'$ *are equal to* 0 *where* $b' = u \cdot b$ *and* $c' = u \cdot c$.

Proof. – For each $b \in V^{(-1)}$ we denote by b_1 the element in $W^{(1)}$ such that $\overline{b}_1 = \overline{b}'$. Let $\overline{a} \in \overline{V}^{(-1)}_\lambda$, $\overline{b} \in \overline{V}^{(-1)}_\mu$ and $\overline{c} \in \overline{V}^{(-1)}_\nu$. By Lemma 5.7 we see $\overline{b}' = \overline{b}_1 \in \overline{V}^{(1)}_{\mu - \alpha}$, $\overline{c}' = \overline{c}_1 \in \overline{V}^{(-1)}_{\nu - \alpha}$ and $\overline{a} \cdot \overline{b}' \in \overline{V}^{(0)}_{-\lambda + \mu + \alpha}$. We first prove

(i) $(\overline{a} \cdot \overline{b}') \cdot \overline{c}' = 0$.

According to Lemma 5.8, for the proof of (i) it suffices to show

(i)′ $$\bar{d}_1 . (\bar{a} . \bar{b}_1) = 0 \quad \text{for all } \bar{d} \in \bar{V}_\rho^{(-1)},$$

where $\rho = \lambda - \mu + \nu - \alpha + 1$. From $W^{(1)} . W^{(1)} = \{0\}$ it follows

$$\bar{d}_1 . (\bar{a} \cdot \bar{b}_1) = (\bar{d}_1 \cdot \bar{a}) \cdot \bar{b}_1 - (\bar{a} \cdot \bar{d}_1) . \bar{b}_1 .$$

Using Lemma 5.7 and (10) we have

$$\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) \in \bar{V}^{(1)}_{-2\lambda+2\mu-\nu+3\alpha-1},$$
$$(\bar{d}_1 \cdot \bar{a}) \cdot \bar{b}_1 \in \bar{V}^{(1)}_{\nu-3\alpha+2},$$
$$(\bar{a} \cdot \bar{d}_1) \cdot \bar{b}_1 \in \bar{V}^{(1)}_{2\mu-\nu-\alpha} .$$

(A) the case $\alpha = 0$. By Lemma 5.5 we know $\lambda, \mu, \nu = 0, 1/2$. This implies $\nu - 3\alpha + 2 = \nu + 2 = 2, 5/2$. Consequently by (7) we have $(\bar{d}_1 . \bar{a}) \cdot \bar{b}_1 = 0$ and $\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = -(\bar{a} \cdot \bar{d}_1) \cdot \bar{b}_1$. If $\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) \neq 0$, then we obtain $-2\lambda + 2\mu - \nu + 3\alpha - 1 = 2\mu - \nu - \alpha$ and so $\lambda = -\frac{1}{2}$, which is a contradiction. Thus (i)′ holds.

(B) the case $\alpha = \frac{1}{2}$. By Lemma 5.5 we have $\lambda, \mu, \nu = \frac{1}{2}, 1$. Therefore we obtain $\nu - 3\alpha + 2 = \nu + \frac{1}{2} = 1, \frac{3}{2}$, so by (7) $(\bar{d}_1 \cdot \bar{a}) \cdot \bar{b}_1 = 0$ and

(a) $$\bar{d}_1 . (\bar{a} \cdot \bar{b}_1) = -(\bar{a} \cdot \bar{d}_1) \cdot \bar{b}_1 .$$

This shows $\bar{d}_1 \cdot (\bar{a} \cdot \bar{b}_1) = 0$ if $-2\lambda + 2\mu - \nu + 3\alpha - 1 \neq 2\mu - \nu - \alpha$. Thus we may assume $-2\lambda + 2\mu - \nu + 3\alpha - 1 = 2\mu - \nu - \alpha$. Then it follows

(b) $$\alpha = \frac{1}{2}, \quad \lambda = \frac{1}{2}, \quad \rho = -\mu + \nu + 1 .$$

Let $h_1 \in W^{(1)}_{2\mu-\nu-\frac{1}{2}}$. Since $W^{(1)} . W^{(1)} = \{0\}$, we have

(c) $\langle (a \cdot d_1) \cdot b_1 , h_1 \rangle = -\langle b_1 , (a \cdot d_1) \cdot h_1 \rangle + \langle b_1 \cdot (a \cdot d_1), h_1 \rangle .$

Applying Lemma 5.7 and (10) we obtain

$$(\bar{a} \cdot \bar{d}_1) \cdot \bar{h}_1 \in \bar{V}^{(1)}_{3\mu-2\nu-\frac{1}{2}}, \quad \bar{b}_1 . (\bar{a} \cdot \bar{d}_1) \in \bar{V}^{(1)}_{-2\mu+\nu+\frac{3}{2}} .$$

Therefore we have $\langle b_1 , (a \cdot d_1) \cdot h_1 \rangle = 0$ if $\mu - \alpha \neq 3\mu - 2\nu - \frac{1}{2}$, i.e.,

(d) $$\langle b_1 , (a \cdot d_1) \cdot h_1 \rangle = 0 \quad \text{if} \quad \mu \neq \nu .$$

If $-2\mu+\nu+\frac{3}{2}\neq 2\mu-\nu-\frac{1}{2}$, then $\langle b_1\cdot(a\cdot d_1), h_1\rangle=0$. Suppose $-2\mu+\nu+\frac{3}{2}=2\mu-\nu-\frac{1}{2}$. Then we get $\nu=2\mu-1$ and so $\mu=1$, $\nu=1$ or $\mu=\frac{1}{2}$, $\nu=0$. The case $\mu=\frac{1}{2}$, $\nu=0$ is impossible because $\nu=\frac{1}{2}$, 1. Consequently we have

(e) $\langle b_1\cdot(a\cdot d_1), h_1\rangle=0$ except for $\mu=\nu=1$.

(B$'$) The case $\mu\neq\nu$. By (c) (d) (e) we have $(a\cdot d_1)\cdot b_1\in\{u\}$ and so by (a) $\overline{d}_1\cdot(\overline{a}\cdot\overline{b}_1)=-(\overline{a}\cdot\overline{d}_1)\cdot\overline{b}_1=0$.

(B$''$) The case $\mu=\nu=\frac{1}{2}$. It follows then $b_1, h_1, (a\cdot d_1)\cdot b_1$, $(a\cdot d_1)\cdot h_1\in V_0^{(1)}$ and $L_{a.d_1} W_0^{(1)}\subset V_0^{(1)}$. Define the mapping $A_{a\cdot d_1}=pr\circ L_{a\cdot d_1}: W_0^{(1)}\longrightarrow W_0^{(1)}$ where pr is the projection from $V_0^{(1)}=W_0^{(1)}+\{u\}$ onto $W_0^{(1)}$. (c) and (e) imply

$$\langle A_{a\cdot d_1}b, h_1\rangle=-\langle b_1, A_{a\cdot d_1}h_1\rangle$$

and so $A_{a\cdot d_1}$ is skew symmetric. Since the eigenvalues of

$$A_{a\cdot d_1}=pr\circ L_{a\cdot d_1}$$

are all real, we obtain $A_{a\cdot d_1}=0$, $(a\cdot d_1)\cdot b_1\in\{u\}$, and so by (a) $\overline{d}_1\cdot(\overline{a}\cdot\overline{b}_1)=-(\overline{a}\cdot\overline{d}_1)\cdot\overline{b}_1=0$.

Summing up the results mentioned above (A), (B$'$) and (B$''$) we have

$$\text{(f)}\qquad \begin{aligned}&\overline{d}_1.(\overline{a}\cdot\overline{b}_1)=0,\\ &(\overline{a}\cdot\overline{b}_1)\cdot\overline{c}_1=0\end{aligned}$$

except for the case $\alpha=\frac{1}{2}$, $\lambda=\frac{1}{2}$, $\mu=\nu=1$, $\rho=1$.

(B$'''$) The case $\mu=\nu=1$. Then it follows $\alpha=\frac{1}{2}$, $\lambda=\frac{1}{2}$, $\mu=\nu=1$, $\rho=1$. Using $a\cdot b'+a'\cdot b$, $d\cdot b'+d'\cdot b\in V^{(1)}$ and $V^{(1)}\cdot V^{(1)}=\{0\}$, we get

$$\begin{aligned}&\overline{d}_1\cdot(\overline{a}\cdot\overline{b}_1)=\overline{d}'\cdot(\overline{a}\cdot\overline{b}')=-\overline{d}'\cdot(\overline{a}'\cdot\overline{b})=-(\overline{d}'\cdot\overline{a}')\cdot\overline{b}\\ &\quad-\overline{a}'\cdot(\overline{d}'\cdot\overline{b})+(\overline{a}'\cdot\overline{d}')\cdot\overline{b}=-\overline{a}'\cdot(\overline{d}'\cdot\overline{b})=\overline{a}'\cdot(\overline{d}\cdot\overline{b}')=\overline{a}_1\cdot(\overline{d}\cdot\overline{b}_1).\end{aligned}$$

For $h_1\in W_{1/2}^{(1)}$ we obtain

$$\langle a_1 . (d \cdot b_1), h_1 \rangle = - \langle d \cdot b_1, a_1 \cdot h_1 \rangle + \langle (d \cdot b_1) \cdot a_1, h_1 \rangle + \langle a_1, (d \cdot b_1) \cdot h_1 \rangle = \langle (d \cdot b_1) \cdot a_1, h_1 \rangle$$

because $a_1 \cdot h_1 = 0$ and $(\overline{d} \cdot \overline{b}_1) \cdot \overline{h}_1 \in \overline{V}_1^{(1)} = \{0\}$. Since $\overline{a}_1 \cdot (\overline{d} \cdot \overline{b}_1)$, $(\overline{d} \cdot \overline{b}_1) \cdot \overline{a}_1 \in \overline{V}_{1/2}^{(1)}$, we have $a_1 \cdot (d \cdot b_1) - (d \cdot b_1) \cdot a_1 \in \{u\}$ and $\overline{d}_1 \cdot (\overline{a} \cdot \overline{b}_1) = \overline{a}_1 \cdot (\overline{d} \cdot \overline{b}_1) = (\overline{d} \cdot \overline{b}_1) \cdot \overline{a}_1$. (f) implies $(\overline{d} \cdot \overline{b}_1) \cdot \overline{a}_1 = 0$. Thus (i)$'$ holds.

Therefore the proof of (i)$'$ is completed.

Finally we show

(ii) $\overline{c}' \cdot (\overline{a} \cdot \overline{b}') = 0$,

(iii) $(\overline{b}' \cdot \overline{a}) \cdot \overline{c}' = 0$,

(iv) $\overline{c}' \cdot (\overline{b}' \cdot \overline{a}) = 0$.

Using (i) and $V^{(1)} . V^{(1)} = \{0\}$, for $d_1 \in W^{(1)}$ we get

$$\langle c_1 \cdot (a \cdot b_1), d_1 \rangle = - \langle a \cdot b_1, c_1 \cdot d_1 \rangle + \langle (a \cdot b_1) \cdot c_1, d_1 \rangle + \langle c_1, (a \cdot b_1) \cdot d_1 \rangle = 0.$$

This implies (ii). From (i), $b' . a + b \cdot a' \in V^{(1)}$ and $V^{(1)} . V^{(1)} = \{0\}$ we obtain $(\overline{b}' \cdot \overline{a}) \cdot \overline{c}' = - (\overline{b} \cdot \overline{a}') \cdot \overline{c}' = 0$. In the same way (iv) follows from (ii).

According to (i) – (iv) and $V^{(1)} . V^{(1)} = \{0\}$, the proof of this lemma is completed. q.e.d.

LEMMA 5.10. – *Let* $a, b \in V^{(-1)}$. *Then the products of* u, a', b *are equal to* 0 *where* $a' = u \cdot a$.

Proof. – By $V^{(1)} . V^{(1)} = \{0\}$ we obtain

(i) $u \cdot (a' . b) = 0$,

(ii) $u \cdot (b \cdot a') = 0$.

In fact we have $u \cdot (a' . b) = (u \cdot a') \cdot b + a' . (u \cdot b) - (a' . u) \cdot b = 0$ and $u \cdot (b \cdot a') = (u \cdot b) \cdot a' + b \cdot (u \cdot a') - (b \cdot u) \cdot a' = 0$. From (i) it follows

$$\langle (a' . b) \cdot u, u \rangle + \langle u, (a' . b) \cdot u \rangle = \langle u \cdot (a' . b), u \rangle + \langle a' . b, u \cdot u \rangle = 0,$$

so $\langle (a' . b) \cdot u, u \rangle = 0$. This implies

(iii) $(a' . b) \cdot u = 0$.

In the same way by (ii) we get

(iv) $(b \cdot a') \cdot u = 0$.

The other cases easily follow from $V^{(1)} . V^{(1)} = \{0\}$. q.e.d.

From Lemma 5.10 we have

LEMMA 5.10′. – *Let* $a \in V^{(-1)}$ *and* $b^1 \in V^{(1)}$. *Then the products of* a, b^1, u *are equal to* 0.

LEMMA 5.11. – *Let* $a \in V^{(-1)}$ *and* $b^1, c^1 \in V^{(1)}$. *Then the products of* a, b^1, c^1 *are equal to* 0.

Proof. – By Lemma 5.9 we have $(a \cdot b^1) \cdot c^1 \in \{u\}$. Using Lemma 5.10′ and $V^{(1)} \cdot V^{(1)} = \{0\}$ we get

$$\langle u , (a \cdot b^1) \cdot c^1 \rangle = - \langle (a \cdot b^1) \cdot u , c^1 \rangle + \langle u \cdot (a \cdot b^1) , c^1 \rangle + \langle a \cdot b^1 , u \cdot c^1 \rangle = 0 .$$

Thus we have $(a \cdot b^1) \cdot c^1 = 0$. By the same way we obtain

$$c^1 \cdot (a \cdot b^1) = 0 , \quad (b^1 . a) \cdot c^1 = 0 , \quad c^1 \cdot (b^1 \cdot a) = 0 .$$

The other cases follow from $V^{(1)} \cdot V^{(1)} = \{0\}$. q.e.d.

Consider the centralizer Z of $V^{(1)}$ in V ;

$$Z = \{z \in V \mid z \cdot a^1 = a^1 \cdot z = 0 \quad \text{for all} \quad a^1 \in V^{(1)}\} .$$

Then we have

LEMMA 5.12. – Z *is an ideal of* V.

Proof. – Let $z \in Z$, $a \in V$. We have

$$u \cdot (z \cdot a) = (u \cdot z) \cdot a + z \cdot (u \cdot a) - (z \cdot u) \cdot a = 0 ,$$
$$u \cdot (a \cdot z) = (u \cdot a) \cdot z + a \cdot (u \cdot z) - (a \cdot u) \cdot z = 0$$

and so $z \cdot a$, $a \cdot z \in V^{(0)}$. From this $V^{(1)}$ is invariant by $L_{z \cdot a}$, $R_{z \cdot a}$, $L_{a \cdot z}$ and $R_{a \cdot z}$. Using Lemma 5.11 and $V^{(1)} . V^{(1)} = \{0\}$, for $b^1, c^1 \in V^{(1)}$ we get

$$\begin{aligned}\langle L_{z \cdot a} b^1 , c^1 \rangle + \langle b^1 , L_{z \cdot a} c^1 \rangle &= \langle b^1 \cdot (z \cdot a) , c^1 \rangle + \langle z \cdot a , b^1 \cdot c^1 \rangle \\ &= \langle b^1 \cdot (z \cdot a) , c^1 \rangle = \langle (b^1 \cdot z) \cdot a + z \cdot (b^1 \cdot a) - (z \cdot b^1) \cdot a , c^1 \rangle \\ &= \langle z \cdot (b^1 \cdot a) , c^1 \rangle = - \langle z \cdot c^1 , b^1 \cdot a \rangle + \langle c^1 \cdot z , b^1 \cdot a \rangle \\ &\qquad + \langle z , c^1 \cdot (b^1 \cdot a) \rangle \\ &= 0 .\end{aligned}$$

This means that $L_{z \cdot a}$ is skew symmetric. On the other hand the eigenvalues of $L_{z \cdot a}$ are all real. Therefore it must be $L_{z \cdot a} = 0$ on $V^{(1)}$, i.e., $(z \cdot a) \cdot b^1 = 0$ for all $b^1 \in V^{(1)}$. From this it follows

$$\begin{aligned}\langle b^1 \cdot (z \cdot a), c^1 \rangle = &- \langle z \cdot a, b^1 \cdot c^1 \rangle + \langle (z \cdot a) \cdot b^1, c^1 \rangle \\ &+ \langle b^1, (z \cdot a) \cdot c^1 \rangle = 0 \quad \text{for all} \quad b^1, c^1 \in V^{(1)}\end{aligned}$$

and so $b^1 \cdot (z \cdot a) = 0$ for all $b^1 \in V^{(1)}$. Thus we get

(a) $$z \cdot a \in Z.$$

Applying Lemma 5.11 and $V^{(1)} \cdot V^{(1)} = \{0\}$, we obtain

$$\begin{aligned}\langle L_{a \cdot z} b^1, c^1 \rangle + \langle b^1, L_{a \cdot z} c^1 \rangle &= \langle b^1 \cdot (a \cdot z), c^1 \rangle + \langle a \cdot z, b^1 \cdot c^1 \rangle \\ &= \langle b^1 \cdot (a \cdot z), c^1 \rangle = \langle (b^1 \cdot a) \cdot z + a \cdot (b^1 \cdot z) - (a \cdot b^1) \cdot z, c^1 \rangle \\ &= \langle (b^1 \cdot a - a \cdot b^1) \cdot z, c^1 \rangle = - \langle (b^1 \cdot a - a \cdot b^1) \cdot c^1, z \rangle \\ &\quad + \langle c^1 \cdot (b^1 \cdot a - a \cdot b^1), z \rangle + \langle b^1 \cdot a - a \cdot b^1, c^1 \cdot z \rangle = 0\end{aligned}$$

for all $b^1, c^1 \in V^{(1)}$. Consequently $L_{a \cdot z}$ is skew symmetric on $V^{(1)}$. Since the eigenvalues of $L_{a \cdot z}$ are real, we have $L_{a \cdot z} = 0$ on $V^{(1)}$, i.e., $(a \cdot z) \cdot b^1 = 0$ for all $b^1 \in V^{(1)}$. Using this and $V^{(1)} \cdot V^{(1)} = \{0\}$ we get

$$\begin{aligned}\langle b^1 \cdot (a \cdot z), c^1 \rangle &= - \langle a \cdot z, b^1 \cdot c^1 \rangle + \langle (a \cdot z) \cdot b^1, c^1 \rangle + \langle b^1, (a \cdot z) \cdot c^1 \rangle \\ &= 0\end{aligned}$$

for all $b^1, c^1 \in V^{(1)}$ and hence

(b) $$b^1 \cdot (a \cdot z) = 0 \quad \text{for all} \quad b^1 \in V^{(1)}.$$

Therefore we have $a \cdot z \in Z$. (a) and (b) imply that Z is an ideal of V. q.e.d.

Let C denote the center of Z;

$$C = \{c \in Z \mid c \cdot z = z \cdot c = 0 \quad \text{for all} \quad z \in Z\}.$$

Then we have

LEMMA 5.13. – *C is a commutative ideal of V containing u.*

Proof. – From $C \supset V^{(1)}$ it follows $u \in C$. Let $c \in C$, $x \in V$. Since Z is an ideal of V, we have

$$z \cdot (c \cdot x) = (z \cdot c) \cdot x + c \cdot (z \cdot x) - (c \cdot z) \cdot x = 0$$

and $z \cdot (x \cdot c) = (z \cdot x) \cdot c + x \cdot (z \cdot c) - (x \cdot z) \cdot c = 0$ for all $z \in Z$. This implies

(a) $$\mathrm{R}_a = 0 \quad \text{on} \quad \mathrm{Z}$$

where $a = c \cdot x$ or $x \cdot c$. Using this and Lemma 3.2, for $z, z' \in \mathrm{Z}$ we get

(b) $$\frac{d^2}{dt^2} \langle \exp t\mathrm{L}_a z, \exp t\mathrm{L}_a z' \rangle = \frac{d}{dt} \langle a, \exp t\mathrm{L}_a(z \cdot z') \rangle$$
$$= \langle a, \mathrm{L}_a(\exp t\mathrm{L}_a(z \cdot z')) \rangle = - \langle w, a \cdot a \rangle + \langle w \cdot a, a \rangle + \langle a, w \cdot a \rangle = 0,$$

where $w = \exp t\mathrm{L}_a(z \cdot z') \in \mathrm{Z}$. Let λ be an eigenvalue of L_a on Z and z an eigenvector corresponding to λ. Then we have $\frac{d^2}{dt^2} \langle \exp t\mathrm{L}_a z, \exp t\mathrm{L}_a z \rangle = \frac{d^2}{dt^2} \langle z, z \rangle e^{2\lambda t} = (2\lambda)^2 e^{2\lambda t} \langle z, z \rangle$ and by (b) $\lambda = 0$. Thus the eigenvalues of L_a on Z are equal to 0. We show

(c) $$\mathrm{L}_a = 0 \quad \text{on} \quad \mathrm{Z}.$$

Suppose $\mathrm{L}_a \neq 0$ on Z. Then there exist elements $z, w \in \mathrm{V}$ such that $\mathrm{L}_a w = 0$, $w = \mathrm{L}_a z \neq 0$. Since $\exp t\mathrm{L}_a z = z + tw$, we have

$$\frac{d^2}{dt^2} \langle \exp t\mathrm{L}_a z, \exp t\mathrm{L}_a z \rangle = \frac{d^2}{dt^2} \langle z + tw, z + tw \rangle$$
$$= 2 \langle w, w \rangle t + 2 \langle z, w \rangle,$$

which contradicts to (b). Thus (c) holds. (a) and (c) imply $a \in \mathrm{C}$ and consequently $c \cdot x, x \cdot c \in \mathrm{C}$. Therefore C is an ideal of V.

q.e.d.

Proposition 5.1 follows from Lemma 5.13.

6. Proof of Theorem 3.1.

We first consider the case $u \cdot u = u$. By Proposition 4.1 we have the orthogonal decomposition $\mathrm{V} = \{u\} + \mathrm{V}_{1/2} + \mathrm{V}_0$. Since V_0 is a subalgebra, by the inductive assumption we get $\mathrm{V}_0 = \mathrm{I} + \mathrm{U}_0$, where I is a commutative ideal of V_0 and U_0 is a subalgebra with principal idempotent s_0. Put $\mathrm{E} = \{u\} + \mathrm{V}_{1/2}$. Then E is an ideal of V. Let $a \in \mathrm{I}$. Since E is invariant under $\mathrm{L}_a, \mathrm{R}_a$ and is orthogonal to a and since $a \cdot a = 0$, by Lemma 3.3 we obtain $\mathrm{L}_a = \mathrm{R}_a = 0$ on E. From this we know that I is a commutative ideal of V. Put

$$\mathrm{U} = \mathrm{E} + \mathrm{U}_0,$$
$$s = u + s_0,$$

Using Proposition 4.1 (iv), $u \cdot u = u$ and $s_0 \cdot s_0 = s_0$, we have

(i) $s \cdot s = s$.

By Proposition 4.1 (iv) we have $L_u = R_u = 0$ on U_0. Therefore $L_s = L_{s_0}$ is diagonalizable on U_0 and its eigenvalues on U_0 are equal to $1/2$, 1 and moreover $R_s = R_{s_0} = 2L_{s_0} - 1 = 2L_s - 1$ on U_0. Since E is invariant under L_{s_0}, R_{s_0} and is orthogonal to s_0 and since $s_0 \cdot s_0 = s_0$, applying Lemma 3.3 it follows that the restriction of L_{s_0} on E is diagonalizable and its eigenvalues are 0, $1/2$ and that $R_{s_0} = 2L_{s_0}$ on E. Therefore using $L_{s_0} u = 0$, $L_{s_0} V_{1/2} \subset V_{1/2}$ and $L_u = 1/2$ on $V_{1/2}$, $L_s = L_u + L_{s_0}$ is diagonalizable on E and its eigenvalues on E are equal to $1/2$, 1. Since $R_u = 2L_u - 1$ and $R_{s_0} = 2L_{s_0}$ hold on E, we have $R_s = 2L_s - 1$ on E. Thus we obtain

(ii) The restriction of L_s on U is diagonalizable and its eigenvalues on U are equal to $1/2$, 1.

(iii) $R_s = 2L_s - 1$ on U.

(i) (ii) (iii) imply that s is a principal idempotent of U. Thus in the case $u \cdot u = u$ the proof of Theorem 3.2 is completed.

Next we consider the case $u \cdot u = 0$. By Proposition 5.1 there exists a commutative ideal C of V containing u. Let V' be the orthogonal complement of C in V. By Lemma 3.1 V' is a subalgebra. From the inductive assumption we get $V' = I' + U$, where I' is a commutative ideal of V' and U is a subalgebra with principal idempotent s. Let $a' \in I'$. Since C is invariant under $L_{a'}$, $R_{a'}$ and $C \cdot C = \{0\}$ and since $a' \cdot a' = 0$, by Lemma 3.3 we obtain $L_{a'} = R_{a'} = 0$ on C. This shows that $I = C + I'$ is a commutative ideal of V. Thus the decomposition $V = I + U$ has the desired properties.

Therefore the proof of Theorem 3.1 is completed. q.e.d.

7. Proof of Main Theorem 2) and Corollaries.

Let V be the tangent space of M at x. In view of Main Theorem 1), Proposition 2.4 and Theorem 2.1 V admits a structure of normal Hessian algebra and $E_x = T(V)0$.

Proof of Main Theorem 2). – According to Theorem 3.1 the normal Hessian algebra V is decomposed in $V = I + U$, where I is a commutative ideal of V and U is a clan. Denote by $T(I)$ the commutative normal subgroup of $T(V)$ generated by $\{X_a^* \mid a \in I\}$ and $T(U)$ the subgroup of $T(V)$ generated by $\{X_w^* \mid w \in U\}$. Then we get $T(V) = T(I)\,T(U)$. Let E_x^+ denote the orbit of $T(U)$ through the origin 0; $E_x^+ = T(U)\,0$. For $a \in I$, $v^+ \in E_x^+$ we have $\exp X_a^* v^+ = v^+ + \sum_{k=0}^{\infty} \dfrac{L_a^{k+1}}{(k+1)!}(L_a v^+ + a) = a + a \cdot v^+ + v^+$ because I is a commutative ideal of V. Thus $T(I)v^+ \subset I + v^+$. Suppose $v^+ = h0$ where $h \in T(U)$. Since

$$T(I)\,v^+ = T(I)\,h0 = hh^{-1}T(I)\,h0 = hT(I)\,0 = hI$$

and since h is an affine transformation of V, we obtain $T(I)\,v^+ = I + v^+$.

Therefore, putting $E_x^0 = I$ we get

$$E_x = T(V)\,0 = T(I)\,T(U)0 = T(I)\,E_x^+ = E_x^0 + E_x^+ .$$

Let $p : E_x \longrightarrow E_x^+$ denote the projection from $E_x = E_x^0 + E_x^+$ onto E_x^+. Then E_x admits a fibering with projection p. Since U is a clan, applying Theorem 2.2 (Vinberg's result) the base space E_x^+ is an affine homogeneous convex domain not containing any full straight line. The fiber $p^{-1}(v^+) = T(I)\,v^+ = E_x^0 + v^+$ over $v^+ \in E_x^+$ is an affine subspace of V and a Euclidean space with respect to the induced metric because $T(I)$ is commutative. It is clear that the fiber $E_x^0 + v$ through $v \in E_x$ is characterized as the set of all points which can be joined with v by full straight lines contained in E_x. This implies that our fibering of E_x is unique and that every affine transformation of E_x is fiber preserving. q.e.d.

Proof of Corollary 1. – If we put $\alpha_x(v) = \mathrm{Tr}\, L_v$ for $v \in V$, the value β_x of the canonical bilinear form β at x has an expression (cf. [8]) $\beta_x(v, w) = \alpha_x(v \cdot w)$ for $v, w \in V$. By Theorem 3.1 V is decomposed in $V = I + U$, where I is a commutative ideal of V and U is a clan. I being a commutative ideal of V we get

$$\begin{aligned} &\alpha_x(a) = 0, \\ &\beta_x(a, v) = 0, \quad \text{for} \quad a \in I,\ v \in V. \end{aligned} \tag{1}$$

Because $\langle v \cdot a, b \rangle + \langle a, v \cdot b \rangle = \langle a \cdot v, b \rangle + \langle v, a \cdot b \rangle = \langle a \cdot v, b \rangle$ for $a, b \in I$ and $v \in V$, we have

$$L_v + {}^tL_v = R_v \quad \text{on} \quad I. \tag{2}$$

Since U is a clan, it follows

$$\mathrm{Tr}_{\mathrm{U}}\, \mathrm{L}_{v \cdot v} > 0 \quad \text{for} \quad v \neq 0 \in \mathrm{U}. \tag{3}$$

Using $\mathrm{R}_{v \cdot v} = \mathrm{R}_v \mathrm{R}_v + [\mathrm{L}_v , \mathrm{R}_v]$ and (2) we obtain

$$\mathrm{Tr}_{\mathrm{I}}\, \mathrm{L}_{v \cdot v} = \frac{1}{2}\, \mathrm{Tr}_{\mathrm{I}}\, \mathrm{R}_{v \cdot v} = \frac{1}{2}\, \mathrm{Tr}_{\mathrm{I}}\, \mathrm{R}_v{}^t\mathrm{R}_v \geqslant 0 .$$

From this and (3) we get $\beta_{x_0}(v , v) = \mathrm{Tr}_{\mathrm{I}}\, \mathrm{L}_{v \cdot v} + \mathrm{Tr}_{\mathrm{U}}\, \mathrm{L}_{v \cdot v} > 0$ for all $v \neq 0 \in \mathrm{U}$. This together with (1) implies that β_x is positive semi-definite and that the null space of β_x coincides with $\mathrm{E}_x^0 = \mathrm{I}$.
q.e.d.

Proof of Corollary 2. – Let $v \in \mathrm{E}_x$. Since the fiber $\mathrm{E}_x^0 + v$ through v is an affine subspace of V, it follows $d(v , w) = 0$ for all $w \in \mathrm{E}_x^0 + v$ (cf. [5]). Conversely, suppose $d(v , w) = 0$. Then we get $0 \leqslant c^a_{\mathrm{E}_x^+}\, (p(v), p(w)) \leqslant c^a_{\mathrm{E}_x}\, (v , w) \leqslant d(v , w) = 0$ because the projection $p : \mathrm{E}_x \longrightarrow \mathrm{E}_x^+$ is an affine mapping. By a result of Vey [11] $c^a_{\mathrm{E}_x^+}$ is a distance on E_x^+. This implies $p(v) = p(w)$. Therefore we get $\mathrm{E}_x^0 + v = \{w \in \mathrm{E}_x \mid d(v , w) = 0\}$. q.e.d.

Proof of Corollary 3. – Our assertion follows from the facts that the covering projection $\exp_x : \mathrm{E}_x = \mathrm{E}_x^0 + \mathrm{E}_x^+ \longrightarrow \mathrm{M}$ is an affine mapping and that E_x^0 is an affine subspace of V. q.e.d.

Let G be a connected Lie subgroup of Aut(M) which acts transitively on M and B the isotropy subgroup of G at a point x in M ; $\mathrm{M} = \mathrm{G}/\mathrm{B}$. We denote by $\widetilde{\mathrm{G}}$ the universal covering group of G and by $\pi : \widetilde{\mathrm{G}} \longrightarrow \mathrm{G}$ the covering projection. Then $\widetilde{\mathrm{M}} = \widetilde{\mathrm{G}}/\widetilde{\mathrm{B}}$ is the universal covering manifold of $\mathrm{M} = \mathrm{G}/\mathrm{B}$, where $\widetilde{\mathrm{B}}$ is the identity component of $\pi^{-1}(\mathrm{B})$. Let $\widetilde{\mathrm{N}}$ be the normal subgroup of $\widetilde{\mathrm{G}}$ consisting of all elements in $\widetilde{\mathrm{G}}$ which induce the identity transformation of $\widetilde{\mathrm{M}}$. We put $\mathrm{G}^* = \widetilde{\mathrm{G}}/\widetilde{\mathrm{N}}$, $\mathrm{B}^* = \widetilde{\mathrm{B}}/\widetilde{\mathrm{N}}$. According to Main Theorem 1) it follows that $\widetilde{\mathrm{M}} = \mathrm{G}^*/\mathrm{B}^*$ is a convex domain in $\mathbf{R}^n$ and that G^* is a subgroup of the affine transformation group $\mathrm{A}(n)$ of $\mathbf{R}^n$.

Proof of Corollary 4. – Assume G is not solvable. Since G^* is not solvable, there exists a connected semi-simple Lie subgroup S^* of G^*. Let K^* be a maximal compact subgroup of S^*. Since $\widetilde{\mathrm{M}}$ is a convex domain in $\mathbf{R}^n$, K^* has a fixed point $\widetilde{y}$ in $\widetilde{\mathrm{M}}$.

Therefore we have

$$\dim G^* = \dim \widetilde{M} = \dim G^* \widetilde{y} \leqslant \dim G^* - \dim K^* < \dim G^* ,$$

which is a contradiction. Thus G must be a solvable Lie group. q.e.d.

Proof of Corollary 5. – Let G be a transitive reductive Lie subgroup of Aut(M) and let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g}$ is decomposed into the direct sum $\mathfrak{g} = \mathfrak{c} + \mathfrak{s}$ where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $\mathfrak{s}$ the semi-simple part of $\mathfrak{g}$. Denoting by C^* and S^* the connected Lie subgroup of G^* corresponding to $\mathfrak{c}$ and $\mathfrak{s}$ respectively, we have $G^* = C^*S^*$. Since S^* is a connected semi-simple Lie subgroup of $A(n)$, S^* is closed in $A(n)$ (cf. [15]). Let $\overline{C}^*$ denote the closure of C^* in $A(n)$. Then the subgroup $\overline{C}^* S^*$ is closed in $A(n)$ (cf. [3]) and so coincides with the closure $\overline{G}^*$ of G^* in $A(n)$. It is easy to see that every element in $\overline{G}^*$ preserves the domain $\widetilde{M}$ and leaves invariant the Hessian metric on $\widetilde{M}$. Denoting by K_c^* and K_s^* maximal compact subgroups of $\overline{C}^*$ and S^* respectively, the group $K^* = K_c^* K_s^*$ is a maximal compact subgroup of $\overline{G}^* = \overline{C}^* S^*$ because the center of S^* is finite. Since $\widetilde{M}$ is a convex domain in $\mathbf{R}^n$, K^* has a fixed point $\widetilde{o}$ in $\widetilde{M}$. We may assume that $\widetilde{o}$ is the origin in $\mathbf{R}^n$. The isotropy subgroup $K^{*\prime}$ of $\overline{G}^*$ at $\widetilde{o}$ is contained in an orthogonal group and is closed in $\overline{G}^*$. Thus $K^{*\prime}$ is a compact subgroup of $\overline{G}^*$ containing K^* and so $K^{*\prime} = K^*$. Since $\overline{G}^*$ acts effectively on $\widetilde{M}$, $\overline{K}_c^*$ is reduced to the identity and so $K^{*\prime} = K^* = K_s^*$. We denote by $\overline{\mathfrak{g}}^*, \overline{\mathfrak{c}}^*, \mathfrak{s}^*$ and $\mathfrak{k}_s^*$ the Lie algebras of $\overline{G}^*, \overline{C}^*, S^*$ and K_s^* respectively, and by $\mathfrak{p}_s^*$ the orthogonal complement of $\mathfrak{k}_s^*$ in $\mathfrak{s}^*$ with respect to the Killing form of $\mathfrak{s}^*$. Putting $\mathfrak{k}^* = \mathfrak{k}_s^*$ and $\mathfrak{p}^* = \overline{\mathfrak{c}}^* + \mathfrak{p}_s^*$, we have

$$\overline{\mathfrak{g}}^* = \mathfrak{k}^* + \mathfrak{p}^* , \quad [\mathfrak{k}^* , \mathfrak{k}^*] \subset \mathfrak{k}^* , \quad [\mathfrak{k}^* , \mathfrak{p}^*] \subset \mathfrak{p}^* , \quad [\mathfrak{p}^* , \mathfrak{p}^*] \subset \mathfrak{k}^* .$$

From this using the same argument as in [9] it follows that $\widetilde{M} = G^*/K^*$ is the direct product of a Euclidean space and an affine homogeneous convex self-dual cone not containing any full straight line. q.e.d.

Proof of Corollary 6. – Since M is compact, the automorphism group $G = \mathrm{Aut}(M)$ is compact. Therefore the Lie algebra $\mathfrak{g}$ of G is decomposed into the direct sum $\mathfrak{g} = \mathfrak{c} + \mathfrak{s}$ where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $\mathfrak{s}$ is a compact semi-simple subalgebra of $\mathfrak{g}$. Denote by

C^* and S^* the connected Lie subgroups of G^* corresponding to $\mathfrak{c}$ and $\mathfrak{s}$ respectively. Then $G^* = C^*S^*$, and S^* is compact by a theorem of Weyl. Since $\tilde{M} = G^*/B^*$ is a convex domain, S^* has a fixed point in M. Therefore $B^* \supset S^*$ and so S^* is a normal subgroup of G^* contained in B^*. From this and the effectiveness, S^* is reduced to the identity. Thus we have $\mathfrak{g} = \mathfrak{c}$. Consequently G is commutative and M is a Euclidean torus. q.e.d.

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Hirohiko SHIMA,
Department of Mathematics
Yamaguchi University
Yamaguchi 753 (Japan).