Annales de l'institut Fourier

Yitzhak Weit

On Schwartz's theorem for the motion group

Annales de l'institut Fourier, tome 30, nº 1 (1980), p. 91-107

<http://www.numdam.org/item?id=AIF_1980__30_1_91_0>

Ann. Inst. Fourier, Grenoble
30, 1 (1980), 91-107

ON SCHWARTZ'S THEOREM FOR THE MOTION GROUP

by Yitzhak WEIT

1. Introduction.

Schwartz's Theorem in the theory of mean periodic functions on the real line states that every closed, translation-invariant subspace of the space of continuous functions on $\mathbf{R}$ is spanned by the polynomial-exponential functions it contains [5]. In particular, every translation-invariant subspace contains an exponential function.

In [2] the two-sided analogue of this result was generalized to $SL_2(\mathbf{R})$. However, since [3] it is known that Schwartz's Theorem fails to hold for $\mathbf{R}^n$, $n > 1$.

Our main goal is to show that the two-sided analogue of Schwartz's Theorem holds for the motion group M(2). That is, every closed, two-sided invariant subspace of C(M(2)) contains an irreducible invariant subspace and every such subspace is spanned by a class of functions which replace the polynomial-exponentials on $\mathbf{R}$.

It seems remarkable that the analogue of Schwartz's Theorem holds for the three dimensional Lie groups $SL_2(\mathbf{R})$ and M(2) while it fails to hold for $\mathbf{R}^2$.

In section 3 we verify the two-sided Schwartz's Theorem for the motion group. In section 4 we consider the problem of one-sided spectral analysis. Finally, in section 5, we study some invariant subspaces of continuous functions on $\mathbf{R}^2$. It turns out that the one-sided Schwartz's Theorem for the motion group is intimately connected with a problem of Pompeiu type [1, 4, 7].

2. Preliminaries and Notation.

Let M(2) denote the Euclidean motion group consisting of the matrices $\begin{pmatrix} e^{i\alpha} & z \\ 0 & 1 \end{pmatrix}$, $\alpha \in \mathbf{R}$, $z \in \mathbf{C}$.

Let C(M(2)) denote the space of all continuous functions on M(2) with the usual topology of uniform convergence on compact sets. Let $\mathcal{E}(\mathbf{R}^n)$ be the space of infinitely differentiable functions on $\mathbf{R}^n$ endowed with the topology of uniform convergence of functions and their derivatives on compacta. Let $\mathcal{E}'(\mathbf{R}^n)$ be the dual of $\mathcal{E}(\mathbf{R}^n)$, the space of Schwartz distributions on $\mathbf{R}^n$ having compact support. The pairing between $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{E}'(\mathbf{R}^n)$ is denoted by $\mathrm{T}(f)$ for $f \in \mathcal{E}(\mathbf{R}^n)$ and $\mathrm{T} \in \mathcal{E}'(\mathbf{R}^n)$, and for such f and T we denote by $\mathrm{T} * f$ the convolution of T and f. For $\mathrm{T} \in \mathcal{E}'(\mathbf{R}^n)$, the Fourier transform of T is defined by $\hat{\mathrm{T}}(z) = \mathrm{T}(e^{iz \cdot x})$ where $z \in \mathbf{C}^n$, $x \in \mathbf{R}^n$ and $z \cdot x = z_1 x_1 + \ldots + z_n x_n$. By Paley-Wiener-Schwartz Theorem, the space $\hat{\mathcal{E}}'(\mathbf{R}^n)$ of Fourier transforms of elements of $\mathcal{E}'(\mathbf{R}^n)$ is identified with the space of entire functions of n complex variables of exponential type which have polynomial growth on the real subspace $\mathbf{R}^n$. The topology of $\hat{\mathcal{E}}'(\mathbf{R}^n)$ is so defined as to make the Fourier transform a topological isomorphism.

Let Π denote the group of all rotations of $\mathbf{R}^2$. We denote by $\mathcal{E}'_{(r)}(\mathbf{R}^2)$ the space of all $\mathrm{T} \in \mathcal{E}'(\mathbf{R}^2)$ which satisfy $\mathrm{T} \circ \tau = \mathrm{T}$ for every $\tau \in \Pi$. Let $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ denote the space of Fourier transforms of elements of $\mathcal{E}'_{(r)}(\mathbf{R}^2)$. We notice that each $f \in \hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ is a function of $z_1^2 + z_2^2$ and that for any even function $g \in \hat{\mathcal{E}}'(\mathbf{R})$ the function $\tilde{g}$ where $\tilde{g}(z_1, z_2) = g(\sqrt{z_1^2 + z_2^2})$ belongs to $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$. Let $\mathcal{E}_0(\mathbf{R}^2)$ denote the space of elements of $\mathcal{E}(\mathbf{R}^2)$ having compact support and $\mathcal{E}_0^{(r)}(\mathbf{R}^2)$ the space of radial functions in $\mathcal{E}_0(\mathbf{R}^2)$.

Let $\mathrm{C}(\mathbf{R}^n)$ denote the space of continuous functions on $\mathbf{R}^n$ with the topology of uniform convergence on compacta and $\mathrm{C}^{(r)}(\mathbf{R}^2)$ the radial functions in $\mathrm{C}(\mathbf{R}^2)$. The dual of $\mathrm{C}(\mathbf{R}^n)$ is the space $\mathrm{M}_0(\mathbf{R}^n) \subset \mathcal{E}'(\mathbf{R}^n)$ of all complex-valued Radon measures having compact support. Let $\mathrm{M}_0^{(r)}(\mathbf{R}^2) = \mathrm{M}_0(\mathbf{R}^2) \cap \mathcal{E}'_{(r)}(\mathbf{R}^2)$.

Finally, for $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$ and $z = x + iy \in \mathbf{C}$ let $(\lambda, z) = \lambda_1 x + \lambda_2 y$.

3. Two-sided spectral synthesis.

The two-sided analogue of Schwartz's Theorem in spectral analysis for the motion group is stated in the following:

THEOREM 1. – *Every closed, two-sided invariant subspace of* C(M(2)) *contains either a character of* M(2) *or a function* $g(e^{i\alpha}, z) = e^{i(\lambda, z)}$ *where* $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$ *and* $\lambda_1^2 + \lambda_2^2 \neq 0$. *The two-sided invariant subspace generated by* $e^{i(\lambda, z)}$ *where* $\lambda = (\lambda_1, \lambda_2)$, $\lambda_1^2 + \lambda_2^2 \neq 0$, *is irreducible (minimal).*

Proof. – For $f \in C(M(2))$, $f \neq 0$, let V_f denote the closed subspace generated by the two-sided translates of f.

The subspace V_f contains all the functions g where

$$g(e^{i\alpha}, z) = f(e^{i(\alpha+\beta)}, ue^{i\alpha} + e^{i\theta} z + w) \tag{1}$$

for every $\theta, \beta \in \mathbf{R}$ and $u, w \in \mathbf{C}$. Let $u = \theta = w = 0$ in (1).

Then, for a suitable $m \in \mathbf{Z}$ the function

$$\int_0^{2\pi} f(e^{i(\alpha+\beta)}, z)\, e^{-im\beta} d\beta = e^{im\alpha} \int_0^{2\pi} f(e^{i\beta}, z)\, e^{-im\beta} d\beta$$
$$= e^{im\alpha} g_1(z)$$

is non-zero and belongs to V_f. Let N denote the translation-invariant and rotation-invariant subspace of $C(\mathbf{R}^2)$ generated by g_1.

By (1) the functions $e^{im\alpha} g_1(e^{i\theta} z + w)$ belongs to V_f for every $\theta \in \mathbf{R}$ and $w \in \mathbf{C}$. That is, V_f contains all functions $e^{im\alpha}\tilde{g}(z)$ where $\tilde{g} \in N$. In [1] it was proved that every closed, translation-invariant and rotation-invariant subspace of $C(\mathbf{R}^2)$ is spanned by the polynomial-exponential functions it contains. In particular, the subspace N contains therefore an exponential function $e^{i(\lambda, z)}$, $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$ and the function $h(e^{i\alpha}, z) = e^{im\alpha} e^{i(\lambda, z)}$ belongs to V_f. If $\lambda_1^2 + \lambda_2^2 = 0$ then the subspace N contains the constant functions and V_f contains therefore the character $e^{im\alpha}$. Suppose that $\lambda_1^2 + \lambda_2^2 \neq 0$.

Let $h_1 \in \mathcal{E}_0(\mathbf{R}^2)$ of the form $h_1(w) = h_2(r)\, e^{-i\theta m}$ where $w = re^{i\theta}$, and $h_2 \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$ such that $\hat{h}_1(\lambda_1, \lambda_2) \neq 0$.

Then the function:

$$\int_{\mathbf{R}^2} h(e^{i\alpha}, z - e^{i\alpha}w)\, h_1(w)\, dw = \hat{h}_1(\lambda_1, \lambda_2)\, e^{i(\lambda, z)} \tag{2}$$

(here dw denotes Lebesgue measure on $\mathbf{R}^2$) is non-zero and belongs to V_f. It follows, by (1) and the analyticity of the elements of $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ that V_f contains all functions $e^{i(\mu, z)}$ where $\mu = (\mu_1, \mu_2) \in \mathbf{C}^2$ such that $\mu_1^2 + \mu_2^2 = \lambda_1^2 + \lambda_2^2$. To prove the second part of the theorem, let $g(z) = e^{i(\lambda, z)}$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$, $\lambda_1^2 + \lambda_2^2 \neq 0$. Firstly, we will show that V_g contains no character of $M(2)$.

Suppose that $e^{im\alpha} \in V_g$ for some $m \in \mathbf{Z}$. Let $\mu \in C(M(2))$, $\mu(e^{i\alpha}, z) = e^{-im\alpha}\mu_1(z)$ where $\mu_1 \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$ such that $\hat{\mu}_1(\lambda_1, \lambda_2) = 0$ and $\hat{\mu}_1(0,0) \neq 0$. We have

$$\int_{\mathbf{R}^2} e^{i(\lambda, e^{i\theta}z)}\mu_1(z)\, dz = 0$$

for every $\theta \in \mathbf{R}$. Consequently, we deduce

$$\begin{aligned}\int_{M(2)} & e^{i(\lambda, e^{i\theta}z + we^{i\alpha})}e^{-im\alpha}\mu_1(z)\, d\alpha dz \\ &= \int_0^{2\pi}\left[\int_{\mathbf{R}^2} e^{i(\lambda, e^{i\theta}z)}\mu_1(z)\, dz\right] e^{i[(\lambda, we^{i\alpha}) - m\alpha]}\, d\alpha = 0\end{aligned}$$

for every $\theta \in \mathbf{R}$ and $w \in \mathbf{C}$. Namely, μ annihilates the subspace V_g. On the other hand, we have

$$\int_{M(2)} e^{im\alpha}\mu(e^{i\alpha}, z)\, d\alpha dz = \hat{\mu}_1(0,0) \neq 0, \text{ a contradiction.}$$

Suppose that $e^{i(w,z)} \in V_g$ where $w = (w_1, w_2) \in \mathbf{C}^2$. If $\lambda_1^2 + \lambda_2^2 \neq w_1^2 + w_2^2$ then for $\mu_2 \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$ where $\hat{\mu}_2(\lambda_1, \lambda_2) = 0$ and $\hat{\mu}_2(w_1, w_2) \neq 0$ we have

$$\begin{aligned}\int_0^{2\pi}\int_{\mathbf{R}^2} & e^{i(\lambda, e^{i\theta}z + ve^{i\alpha})}\mu_2(z)\, dz d\alpha \\ &= \int_0^{2\pi}\left[\int_{\mathbf{R}^2} e^{i(e^{i\theta}\lambda, z)}\mu_2(z)\, dz\right] e^{i(\lambda, ve^{i\alpha})}\, d\alpha = 0\end{aligned}$$

for each $\theta \in \mathbf{R}$ and $v \in \mathbf{C}$. However, we have

$$\int_0^{2\pi}\int_{\mathbf{R}^2} e^{i(w,z)}\mu_2(z)\, dz d\alpha \neq 0$$

which proves the irreducibility of V_g. This completes the proof.

Schwartz's Theorem in spectral synthesis is described in the following:

THEOREM 2. – *Every closed, two-sided invariant subspace of* C(M(2)) *is spanned by the functions as*

$$g(e^{i\alpha}, z) = e^{im\alpha}\mathrm{Q}(\mathrm{Re}\, z\,, \mathrm{Im} z)\, e^{i(\lambda, z)}$$

that it contains. ($\lambda \in \mathbf{C}^2$ and Q is polynomial).

Proof. – For $f \in \mathrm{C}(\mathrm{M}(2))$, $f \neq 0$ let V denote the closed subspace generated by the two-sided translates of f. Obviously, f is contained in the closed subspace generated by the functions:

$$e^{im\alpha}\mathrm{P}_m(z) = \int_0^{2\pi} f(e^{i(\alpha+\beta)}, z)\, e^{-im\beta} d\beta = e^{im\alpha} \int_0^{2\pi} f(e^{i\beta}, z)\, e^{-im\beta} d\beta$$

where $m \in \mathbf{Z}$.

By [1], each function $e^{im\alpha}\mathrm{P}_m(z)$ is contained in the closed subspace spanned by the functions $e^{im\alpha}\mathrm{Q}(\mathrm{Re}\, z\,, \mathrm{Im} z)\, e^{i(\lambda, z)}$ where $\mathrm{Q}(\mathrm{Re}\, z\,, \mathrm{Im}\, z)\, e^{i(\lambda,z)}$ is contained in the rotation-invariant and translation-invariant subspace of $\mathrm{C}(\mathbf{R}^2)$ generated by $\mathrm{P}_m(z)$, and hence in the two-sided invariant subspace generated by $\mathrm{P}_m(z)$ which completes the proof of the theorem.

4. One-sided spectral analysis.

One-sided spectral analysis of bounded functions on M(2) was studied in [6].

Notation. – Let Γ_w, $w \in \mathbf{C}$, denote the closed subspace of $\mathrm{C}(\mathbf{R}^2)$ spanned by the functions $e^{i(\lambda_1 x + \lambda_2 y)}$ (of $(x\,, y) \in \mathbf{R}^2$) where $\lambda_1^2 + \lambda_2^2 = w^2$. For the characterization of right-invariant subspaces of C(M(2)) we state the following:

THEOREM 3. – *Every closed, right-invariant subspace of* C(M(2)) *contains a function as*

$$g(e^{i\alpha}, z) = e^{im\alpha} g_1(z)\,,\ m \in \mathbf{Z}\,,\ g_1 \neq 0\,.$$

Moreover, if $g_1 \notin \Gamma_0$, *then the closed right-invariant subspace generated by* g *contains a function as* $h(e^{i\alpha}, z) = g_2(z)$.

For $g_2 \in \Gamma_w$ *and* $g_1 \in \Gamma_0$ *the closed right-invariant subspaces generated by* g_2 *and by* $e^{im\alpha} g_1(z)$ *are irreducible.*

Proof. – Let $f \in \mathrm{V}$, $f \neq 0$, where V is a closed right-invariant subspace of C(M(2)). Then V contains all functions f^* such that $f^*(e^{i\alpha}, z) = f(e^{i(\alpha+\beta)}, z - e^{i\alpha}w)$ where $\beta \in \mathbf{R}$ and $w \in \mathbf{C}$. Hence, for a suitable $m \in \mathbf{Z}$ the function

$$\int_0^{2\pi} f(e^{i(\alpha+\beta)}, z)\, e^{-im\beta} d\beta = e^{im\alpha} \int_0^{2\pi} f(e^{i\beta}, z)\, e^{-im\beta} d\beta = e^{im\alpha} g_1(z)$$

is non-zero and belongs to V. Suppose that $g_1 \notin \Gamma_0$. Then if g_1 is a polynomial (in $\operatorname{Re} z$ and $\mathrm{I}_m z$) which is harmonic on $\mathbf{R}^2$ there exists a function $h \in \mathcal{E}_0(\mathbf{R}^2)$, $h(w) = \mu(r)\, e^{im\theta}$, $\mu \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$, $w = re^{i\theta}$, such that $g_1 * h \neq 0$.

Hence the function

$$e^{im\alpha} \int_{\mathbf{R}^2} g_1(z - e^{i\alpha}w)\, h(w)\, dw = \int_{\mathbf{R}^2} g_1(z - w)\, h(w)\, dw = g_2(z) \quad (3)$$

is non-zero and belongs to V.

Otherwise, the closed rotation-invariant and translation-invariant subspace generated by g_1 contains a function $e^{i(\lambda, z)}$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$, $\lambda_1^2 + \lambda_2^2 \neq 0$ [1]. Let $h_1 \in \mathcal{E}_0(\mathbf{R}^2)$, $h_1(w) = \mu_1(r)\, e^{im\theta}$ where

$$\mu_1 \in \mathcal{E}_0^{(r)}(\mathbf{R}^2),\ w = re^{i\theta},\ \textit{such that}\ \hat{h}_1(\lambda_1, \lambda_2) \neq 0 .$$

There exists $\beta \in \mathbf{R}$ such that $h_{1,\beta} * g_1 \neq 0$, where

$$h_{1,\beta}(w) = h_1(e^{i\beta}w) = e^{im\beta} h_1(w) .$$

Hence, $h_1 * g_1 \neq 0$ and proceeding as in (3) we complete the proof of the first part of the theorem.

Let V_1 be the closed right-invariant subspace generated by $g_2(z)$ where $g_2 \in \Gamma_{w_0}$ for some $w_0 \in \mathbf{C}$, $w_0 \neq 0$. We may show, as in the proof of Theorem 1, that V_1 contains no functions as $e^{im\alpha} g_1(z)$ where $g_1 \in \Gamma_0$. Suppose now that $g_3 \in \mathrm{V}_1$ where $g_3 \in \Gamma_{w_1}$, $w_1 \in \mathbf{C}$. To derive the irreducibility of V_1 we will show that $g_3 = \mathrm{C} g_2$ for some $\mathrm{C} \in \mathbf{C}$. Let $\{\Phi_n\}$ be a sequence in $\mathcal{E}_0(\mathbf{R}^2)$ such that

$$\int_{\mathbf{R}^2} g_2(z - e^{i\alpha}w)\, \Phi_n(w)\, dw \xrightarrow[\mathrm{C(M(2))}]{} g_3(z) .$$

Then we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\int_{\mathbf{R}^2} g_2(z - e^{i\alpha}w)\, \Phi_n(w)\, dw \right] d\alpha \xrightarrow[\mathrm{C(M(2))}]{} g_3(z)$$

and

$$\int_{\mathbf{R}^2} g_2(z - e^{i\alpha}w)\, \Phi_n^*(|w|)dw \xrightarrow[\mathrm{C(M(2))}]{} g_3(z) \qquad (5)$$

where

$$\Phi_n^*(|w|) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{-i\alpha}w)\,d\alpha\,,\ \Phi_n^* \in \mathcal{E}_0^{(r)}\,,\ n = 1\,,2\,,\ldots \qquad (6)$$

But for every n we have

$$\int_{\mathbf{R}^2} g_2(z - w)\, \Phi_n^*(|w|)dw = \hat{\Phi}_n^*(w_0)\, g_2(z)\,.$$

Consequently, $g_3 = Cg_2$, as required. Similarly, we verify the irreducibility of the closed right-invariant subspace generated by $g_1(z)e^{im\alpha}$ where $g_1 \in \Gamma_0$.

Remark 1. – We don't know whether Theorem 3 characterizes all the irreducible right invariant subspaces as it is not known whether the exponentials are the only functions of $C(\mathbf{R}^n)$, $n > 1$ which generate irreducible translation-invariant subspaces. Whether every translation-invariant subspace of $C(\mathbf{R}^n)$, $n > 1$ contains an irreducible subspace seems to be an open question.

Remark 2. – In view of Theorem 3 the right-sided analogue of Schwartz's Theorem in spectral analysis of continuous functions may be formulated as the following question; does every closed, right-invariant subspace of C(M(2)) contain either a funtion as $e^{im\alpha}g_1(z)$ where $g_1 \in \Gamma_0$, $g_1 \neq 0$ $m \in \mathbf{Z}$, or $g_2(z)$ where $g_2 \in \Gamma_w$, $g_2 \neq 0$, for some $w \in \mathbf{C}$?

Notation. – Let μ_R, $R \geqslant 0$, denote the normalized Lebesgue measure of the circle $\{z : |z| = R\}$. For $f \in C(\mathbf{R}^2)$ let $N_f^{(r)}$ denote the closed subspace spanned by $\{f * \mu_R : R \geqslant 0\}$ and $\tau(f)$ the closed translation-invariant subspace generated by f.

We deduce an equivalent form of the right-sided analogue of Schwartz's Theorem (as formulated in Remark 2).

It is described in

THEOREM 4. – *The following statements are equivalent:*

(i) *The right-sided analogue of Schwartz's Theorem holds for* M(2).

(ii) *Let* $f \in C(\mathbf{R}^2)$, $f \neq 0$. *Then:* (a) If $\tau(f) \cap \Gamma_0 = \{0\}$ *then there exists* $w \in \mathbf{C}$ *such that* $N_f^{(r)} \cap \Gamma_w \neq \{0\}$. (b) *If* $\tau(f) \cap \Gamma_0 \neq \{0\}$ *then, either* $N_f^{(r)} \cap \Gamma_w \neq \{0\}$ *for some* $w \in \mathbf{C}$, *or, there exist* $m \in \mathbf{Z}$, $g \in \Gamma_0$, $g \neq 0$ *and a sequence* $\psi_n \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$ *such that*

$$f * \phi_n \xrightarrow[C(\mathbf{R}^2)]{} g \tag{7}$$

where $\phi_n(r, \theta) = \psi_n(r) e^{-im\theta}$, $n = 1, 2, \ldots$. *(Here* (r, θ) *are the polar coordinates in* $\mathbf{R}^2$).

Proof. – Suppose that the right-sided analogue of Schwartz's Theorem holds for $M(2)$. Let $f \in C(M(2))$ where $f(e^{i\alpha}, z) = f(z)$. Suppose that $\tau(f) \cap \Gamma_0 = \{0\}$. The closed right-invariant subspace W_f generated by f contains no function as $e^{im\alpha} g(z) \neq 0$ where $g \in \Gamma_0$ and $m \in \mathbf{Z}$. Since, otherwise

$$\int_{\mathbf{R}^2} f(z - e^{i\alpha} w)\, \mu_n(w)\, dw \xrightarrow[C(M(2))]{} e^{im\alpha} g(z)$$

implies for $\alpha = 0$ that: $f * \mu_n \xrightarrow[C(\mathbf{R}^2)]{} g$, a contradiction. Hence, W_f contains a function $g_1(z)$ where $g_1 \in \Gamma_w$, $g_1 \neq 0$. In other words, there exist $\Phi_n \in \mathcal{E}_0(\mathbf{R}^2)$, $n = 1, 2, \ldots$, such that

$$\int_{\mathbf{R}^2} f(z - e^{i\alpha} w)\, \Phi_n(w)\, dw \xrightarrow[C(M(2))]{} g_1(z).$$

Hence, by (5) we have:

$$\int_{\mathbf{R}^2} f(z - w)\, \Phi_n^*(|w|)\, dw \xrightarrow[C(M(2))]{} g_1(z)$$

where Φ_n^* are defined in (6). That is, $g_1 \in N_f^{(r)}$ which yields (ii) (a).

Suppose now that $\tau(f) \cap \Gamma_0 \neq \{0\}$. If $W_f \cap \Gamma_v \neq \{0\}$ for some $v \in \mathbf{C}$ then, as proved above, $N_f^{(r)} \cap \Gamma_v \neq \{0\}$ (here, the functions of Γ_v are looked upon as function on $M(2)$). Otherwise, the subspace W_f must contain a function as $e^{im\alpha} g_2(z)$ where $g_2 \in \Gamma_0$, $g_2 \neq 0$, and $m \in \mathbf{Z}$. Namely, there exists $\phi_n \in \mathcal{E}_0(\mathbf{R}^2)$ such that

$$\int_{\mathbf{R}^2} f(z - e^{i\alpha} w)\, \phi_n(w)\, dw \xrightarrow[C(M(2))]{} e^{im\alpha} g_2(z).$$

Hence we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\int_{\mathbf{R}^2} f(z - \xi)\, \phi_n(e^{-i\alpha} \xi)\, d\xi \right] e^{-im\alpha} d\alpha \longrightarrow g_2(z)$$

which yields

$$\frac{1}{2\pi}\int_{\mathbf{R}^2} f(z-\xi)\,\widetilde{\phi}_n(\xi)\,d\xi \longrightarrow g_2(z)$$

where $\widetilde{\phi}_n(\xi) = \widetilde{\psi}_n(r)\,e^{-im\theta}$, $\widetilde{\psi}_n(r) = \int_0^{2\pi} \phi(e^{-i\eta}r)\,e^{-im\eta}d\eta$, $\xi = re^{i\theta}$, and we have shown that (i) implies (ii).

Suppose now that (ii) holds. By Theorem 3 we have to show that for every $f \in \mathrm{C}(\mathrm{M}(2))$, $f(e^{i\alpha}, z) = f(z)$, $f \neq 0$, the subspace W_f contains either a function $g(z)$, $g \neq 0$, $g \in \Gamma_w$, or, a function $g(e^{i\alpha}, z) = e^{im\alpha}g_1(z)$ where $g_1 \in \Gamma_0$, $g_1 \neq 0$ and $m \in \mathbf{Z}$.

Let $f \in \mathrm{C}(\mathbf{R}^2)$, $f \neq 0$ and suppose that $\mathrm{N}_f^{(r)} \cap \Gamma_w \neq \{0\}$ for some $w \in \mathbf{C}$. Then, by definition, there exist $\psi_n \in \mathcal{E}_0^{(r)}(\mathbf{R}^2)$ $n = 1, 2, \ldots,$ and $g \in \Gamma_w$ such that

$$\int_{\mathbf{R}^2} f(z-\xi)\,\psi_n(\xi)\,d\xi \xrightarrow[\mathrm{C}(\mathbf{R}^2)]{} g(z).$$

But we have

$$\int_{\mathbf{R}^2} f(z - e^{i\alpha}\xi)\,\psi_n(\xi)\,d\xi = \int_{\mathbf{R}^2} f(z-\xi)\,\psi_n(\xi)\,d\xi \quad \text{for} \quad n = 1, 2, \ldots,$$

which implies (i).

Finally, suppose that $\tau(f) \cap \Gamma_0 \neq \{0\}$ and that $\mathrm{N}_f^{(r)} \cap \Gamma_w = \{0\}$ for every $w \in \mathbf{C}$. By (ii) (b) we have

$$\begin{aligned}\int_{\mathbf{R}^2} f(z - e^{i\alpha}w)\,\phi_n(w)\,dw &= \int_{\mathbf{R}^2} f(z-\xi)\,\phi_n(e^{-i\alpha}\xi)\,d\xi \\ &= e^{im\alpha}\int_{\mathbf{R}^2} f(z-\xi)\,\phi_n(\xi)\,d\xi\end{aligned}$$

for $n = 1, 2, \ldots,$ which yields, by (7)

$$\int_{\mathbf{R}^2} f(z - e^{i\alpha}\xi)\,\psi_n(\xi)\,d\xi \xrightarrow[\mathrm{C}(\mathrm{M}(2))]{} e^{im\alpha}g(z).$$

This completes the proof.

5. Invariant subspaces of $\mathrm{C}(\mathbf{R}^2)$.

For $f \in \mathrm{C}(\mathbf{R}^2)$ we say that $w \in Sp^{\mathrm{T.R.}}(f)$, $w \in \mathbf{C}$ if the translation-invariant and rotation-invariant subspace generated by f contains an exponential in Γ_w. Actually, the fact announced in [1] that unless $f = 0$ we have $Sp^{\mathrm{T.R.}}(f) \neq \phi$ implies the main results of [1] concerning the Pompeiu problem [4, 7] . By Theorem 4, the one-sided Schwartz's Theorem for the motion group is intimately connected to the following problem:

For $f \in C(\mathbf{R}^2)$ we say that $w \in Sp^{(r)}(f)$, $w \in \mathbf{C}$, $w \neq 0$, if $N_f^{(r)} \cap \Gamma_w \neq \{0\}$, and that $0 \in Sp^{(r)}(f)$ if $N_f^{(r)} \cap \widetilde{\Gamma}_0 \neq \{0\}$, where $\widetilde{\Gamma}_0$ denotes the space of harmonic functions on $\mathbf{R}^2$. Suppose that $f \neq 0$. Does this imply that $Sp^{(r)}(f) \neq \phi$?

Remark 3. – We notice that for $f \in C(\mathbf{R}^2)$ we have $Sp^{(r)}(f) \subseteq Sp^{T.R.}(f)$.

Remark 4. – This question is connected to the following problem of Pompeiu type:

Determine for which family $P \subset M_0(\mathbf{R}^2)$, the only continuous function f on $\mathbf{R}^2$ such that $T(f * \mu_R) = 0$ for all $T \in P$ and $R \geqslant 0$, is the zero function.

Let J_n denote the nth Bessel function of the first kind. By definition, we deduce

$$J_n(r)\,e^{in\theta} = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{ir\cos(\phi-\theta)}\, e^{in\phi}\, d\phi .$$

Hence we have $J_n(wr)\,e^{in\theta} \in \Gamma_w$, $Sp^{(r)}(J_n(wr)\,e^{in\theta}) = \{w\}$ for $w \in \mathbf{C}$, $w \neq 0$ and $N_{I_n}^{(r)}$ is one-dimensional where $I_n(r,\theta) = J_n(wr)\,e^{in\theta}$.

A partial answer to the above question is provided by the following result:

THEOREM 5. – *Let $f \in C(\mathbf{R}^2)$, $f \neq 0$ where*

$$f(r,\theta) = \sum_{m=0}^{N} g_m(r)\,e^{im\theta}, \quad g_m \in C^{(r)}(\mathbf{R}^2) \quad (m = 0,1,\ldots,N).$$

Then $Sp^{(r)}(f) \neq \phi$. If $0 \notin Sp^{(r)}(f)$ there exist λ, $a_m \in \mathbf{C}$ $(m = 0,1,\ldots,N)$, $\lambda \neq 0$, where $\sum_{m=0}^{N} |a_m| > 0$ such that $\sum_{m=0}^{N} a_m J_m(\lambda r)\,e^{im\theta}$ belongs to $N_f^{(r)}$. Moreover, we have

$$Sp^{(r)}(f) = \bigcup_{m=0}^{N} Sp^{(r)}(g_m(r)\,e^{im\theta}).$$

The proof will be accomplished in several lemmas.

LEMMA 6. – Every proper closed ideal in $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ has a common zero in $\mathbf{C}^2$.

Proof. – Let J be a proper closed ideal in $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$ and suppose that the functions in J have no common zeroes. Every $f \in \mathrm{J}$ is a function of $z_1^2 + z_2^2$. That is, there exists an even entire function Q_f of one complex variable such that

$$f(z_1, z_2) = \mathrm{Q}_f(\sqrt{z_1^2 + z_2^2}) \quad \text{and} \quad \mathrm{Q}_f \in \hat{\mathcal{E}}'(\mathbf{R}).$$

Let J^* be the ideal in $\hat{\mathcal{E}}'(\mathbf{R})$ generated by $\{\mathrm{Q}_f : f \in \mathrm{J}\}$.

Obviously, the functions in J^* have no common zeroes. Thus, applying Schwartz's Theorem [5] we deduce that $\mathrm{J}^* = \hat{\mathcal{E}}'(\mathbf{R})$. That is, there exists a sequence $\{\mathrm{P}_n\}$ in J^* converging to 1 in $\hat{\mathcal{E}}'(\mathbf{R})$. Each P_n must be of the form $\sum_{j=1}^{k} \mathrm{T}_j(w)\, \mathrm{S}_j(w)$ where each $\mathrm{T}_j \in \hat{\mathcal{E}}'(\mathbf{R})$ and $\mathrm{S}_j \in \mathrm{J}$. But then the function

$$\sum_{j=1}^{k} \mathrm{T}_j(w)\, \mathrm{S}_j(w) + \sum_{j=1}^{k} \mathrm{T}_j(-w)\, \mathrm{S}_j(-w) = \sum_{j=1}^{k} (\mathrm{T}_j(w) + \mathrm{T}_j(-w))\, \mathrm{S}_j(w)$$

belongs to J since each $\mathrm{T}_j(w) + \mathrm{T}_j(-w)$ belongs to $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$. Hence, $\mathrm{Q}_n(w) = \frac{1}{2}(\mathrm{P}_n(w) + \mathrm{P}_n(-w))$ belongs to J and $\mathrm{Q}_n \longrightarrow 1$ in $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$, a contradiction.

LEMMA 7. – *Let* $f \in \mathrm{C}(\mathbf{R}^2)$ *where* $f(r, \theta) = g(r)\, e^{im\theta}$, $g \in \mathrm{C}^{(r)}(\mathbf{R}^2)$, $g \neq 0$, $m \in \mathbf{Z}$. *Then* $Sp^{(r)}(f) \neq \phi$. *If* $0 \notin Sp^{(r)}(f)$ *there exists* $\lambda \in \mathbf{C}$, $\lambda \neq 0$, *such that* $\mathrm{H} \in \mathrm{N}_f^{(r)}$ *where*

$$\mathrm{H}(r, \theta) = \mathrm{J}_m(\lambda r)\, e^{im\theta}.$$

Proof. – We may assume that $f \in \mathcal{E}(\mathbf{R}^2)$. Let $\mathrm{M}_f^{(r)}$ denote the closed subspace of $\mathcal{E}(\mathbf{R}^2)$ spanned by $\{f * \mu_\mathrm{R} : \mathrm{R} \geqslant 0\}$. For $m \in \mathbf{Z}$ let $\mathcal{E}_m(\mathbf{R}^2)$ denote the closed subspace of functions $s \in \mathcal{E}(\mathbf{R}^2)$ sucht that $s(r, \theta) = h(r)\, e^{im\theta}$. We have $\mathrm{M}_f^{(r)} \subseteq \mathcal{E}_m(\mathbf{R}^2)$.

Let $\mathcal{E}'_m(\mathbf{R}^2) \subset \mathcal{E}'(\mathbf{R}^2)$ denote the dual of $\mathcal{E}_m(\mathbf{R}^2)$.

Let $\mathrm{M}_f^{(r)\perp} = \{\mathrm{T} \in \mathcal{E}'_m(\mathbf{R}^2) : \mathrm{T}(f) = 0,\ f \in \mathrm{M}_f^{(r)}\}$.

Every element of $\hat{\mathcal{E}}'_m(\mathbf{R}^2)$ is of the form $p(r)\,e^{im\theta}$ (as a function on $\mathbf{R}^2$). Let $\mathrm{P} = \{p : \hat{\mathrm{T}}(r\,,\theta) = p(r)\,e^{im\theta}\,,\ \mathrm{T} \in \mathrm{M}_f^{(r)\perp}\}$.

We notice that all functions of P are even or odd depending on m.

Let k be the larger integer such that 0 is a zero of order k for each $p \in \mathrm{P}$. It follows that $\dfrac{p(w)}{w^k}$, $p \in \mathrm{P}$, is an even entire function of w and by complexification of $\dfrac{p(r)}{r^k}$

$$p^*(z_1\,,z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}$$

is an entire function on $\mathbf{C}^2$. The space

$$\mathrm{J}^* = \left\{ p^* : p^*(z_1\,,z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}\,,\ p \in \mathrm{P} \right\}$$

is therefore a closed ideal in $\hat{\mathcal{E}}'_{(r)}(\mathbf{R}^2)$. If $0 \notin \mathrm{S}p^{(r)}(f)\ \mathrm{J}^*$ is a proper ideal.

Hence, by Lemma 6, there exists

$$\lambda^* = (\lambda_1\,,\lambda_2) \in \mathbf{C}^2\ ,\ \lambda_1^2 + \lambda_1^2 = \lambda^2 \neq 0$$

which is a common zero of J^*. Consequently, for each $\mathrm{T} \in \mathrm{M}_f^{(r)\perp}$ we have $\hat{\mathrm{T}}(w) = 0$ where $w = (w_1\,,w_2) \in \mathbf{C}^2\,,\ w_1^2 + w_2^2 = \lambda^2$. It follows that $\mathrm{T}(\mathrm{Q}) = 0$ for $\mathrm{T} \in \mathrm{M}_f^{(r)\perp}$ where

$$\mathrm{Q}(x\,,y) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{i\lambda_1(x\cos\phi + y\sin\phi)}\,e^{im\phi}d\phi\,.$$

But we have

$$\mathrm{Q}(r\,,\theta) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{i\lambda r\cos(\phi-\theta)}\,e^{im\phi}d_\phi = \mathrm{J}_m(wr)\,e^{im\theta}\,.$$

Consequently, $\mathrm{Q} \in \mathrm{M}_f^{(r)} \cap \Gamma_\lambda$ which completes the proof.

Notation. – Let $\mathrm{C}(\mathbf{R}^2\,,\mathbf{C}^\mathrm{N})$ denote the space of all continuous functions on $\mathbf{R}^2$ which take values in $\mathbf{C}^\mathrm{N}$, with the usual topology. Let $\mathrm{M}_0(\mathbf{R}^2\,,\mathbf{C}^\mathrm{N})$ be the dual of $\mathrm{C}(\mathbf{R}^2\,,\mathbf{C}^\mathrm{N})$, the space of vector-valued measures having compact support. For $f \in \mathrm{C}(\mathbf{R}^2\,,\mathbf{C}^\mathrm{N})$, (resp. $\mu \in \mathrm{M}_0(\mathbf{R}^2\,,\mathbf{C}^\mathrm{N})$) let $(f)_n$ (resp. $(\mu)_n$) denote the nth coordinate of f (resp. μ). For $m = (m_1\,,m_2\,,\ldots,m_\mathrm{N}) \in \mathbf{Z}^\mathrm{N}$ let $\mathrm{B}_{(m)}$ denote the closed subspace of $C(\mathbf{R}^2, \mathbf{C}^N)$ which consists of all functions f where

$$(f)_n(r,\theta) = h_n(r)\, e^{im_n\theta} \qquad n = 1, 2, \dots, N.$$

Let $B'_{(m)}$ be the dual of $B_{(m)}$, the space of all $\eta \in M_0(\mathbf{R}^2, \mathbf{C}^N)$ such that $(\eta)_n = \mu_n e^{-im_n\theta}$ where $\mu_n \in M_0^{(r)}(\mathbf{R}^2)$, $n = 1, 2, \dots, N$. We will use the following equality:

$$(J_k(wr')\, e^{ik\theta'}) * (\mu(r')\, e^{im\theta'})\,(r,\theta) = \phi(w)\, J_{k+m}(wr)\, e^{i(k+m)\theta} \tag{8}$$

where $\mu \in M_0^{(r)}(\mathbf{R}^2)$, $w \in \mathbf{C}$, and $\widehat{\mu(r')\, e^{im\theta'}}(r,\theta) = \phi(r)\, e^{im\theta}$. Finally, we notice that $M_0^{(r)}(\mathbf{R}^2)$ acts on $B_{(m)}$ by convolution. Namely, $f \in B_{(m)}$ and $\mu \in M_0^{(r)}(\mathbf{R}^2)$ imply that $f * \mu \in B_{(m)}$.

LEMMA 8. — *Every closed non-trivial subspace of* $B_{(m)}$, *invariant under* $M_0^{(r)}(\mathbf{R}^2)$ *contains an invariant one-dimensional subspace. Moreover, if* $f \in B_{(m)}$ *such that* $\lambda \in Sp^{(r)}((f)_n)$, $\lambda \neq 0$, *for some* n, $1 \leqslant n \leqslant N$, *then the closed subspace spanned by* $\{f * \mu_R : R \geqslant 0\}$ *contains a function* $g \neq 0$, *such that*

$$(g)_n(r,\theta) = a_n J_{m_n}(\lambda r)\, e^{im_n\theta} \qquad n = 1, 2, \dots, N.$$

Proof. — By induction on N where the case $N = 1$ is provided by Lemma 7. Let $f \in B_{(m)}$ and suppose that $0 \neq \lambda \in Sp^{(r)}((f)_1)$. Let V_f denote the closed subspace of $B_{(m)}$ spanned by $\{f * \mu_R : R \geqslant 0\}$ and $V_f^{\perp} = \{\eta \in B'_{(m)} : \eta(g) = 0,\ g \in V_f\}$. We notice that for $\eta \in V_f^{\perp}$ we have:

$$\sum_{n=1}^{N} (g_n(r)\, e^{im_n\theta}) * (\mu_n e^{-im_n\theta}) = 0 \tag{9}$$

where $(\eta)_n = \mu_n e^{-im_n\theta}$ and $(f)_n = g_n(r)\, e^{im_n\theta}$, $n = 1, 2, \dots, N$.

Thus we may assume that there exists $\eta \in V_f^{\perp}$ such that

$$(J_{m_N}(\lambda r)\, e^{im_N\theta}) * (\mu_N e^{-im_N\theta}) \neq 0. \tag{10}$$

Otherwise, the subspace V_f contains a function g^* such that $(g^*)_n = 0$ for $n = 1, 2, \dots, N-1$, and $(g^*)_N = J_{m_N}(\lambda r)\, e^{im_N\theta}$ which completes the proof. To this end, let $h \in B_{(m')}$ where $(h)_n = (f)_n$ for $n = 1, 2, \dots, N-1$, $m' = (m_1, m_2, \dots, m_{N-1})$ and $B_{(m')} \subset C(\mathbf{R}^2, \mathbf{C}^{N-1})$. By the induction hypothesis the subspace V_h contains a function $h^* \neq 0$ such that

$$(h^*)_n = b_n J_{m_n}(\lambda r)\, e^{im_n\theta} \quad \text{for} \quad n = 1,2,\ldots,N-1\,.$$

That is, there exists a sequence $\{\phi_k\}$, $\phi_k \in M_0^{(r)}(\mathbf{R}^2)$, such that

$$(g_n(r')e^{im_n\theta'} * \phi_k)(r,\theta) \xrightarrow[k\to\infty]{C(\mathbf{R}^2)} b_n J_{m_n}(\lambda r)\, e^{im_n\theta} \tag{11}$$

for $n = 1,2,\ldots,N-1$, where $\sum_{n=1}^{N-1} |b_n| > 0$. Let $\psi_k \in M_0^{(r)}(\mathbf{R}^2)$ where

$$\psi_k = \phi_k * \mu_N e^{-im_N\theta} * \mu_N e^{im_N\theta} \qquad k = 1,2,\ldots,\,.$$

Then by (8), (10) and (11) we obtain:

$$g_n(r)^{im_n\theta} * \psi_k \xrightarrow[k\to\infty]{C(\mathbf{R}^2)} b_n J_{m_n}(\lambda r)\, e^{im_n\theta} * \mu_N e^{-im_N\theta} * \mu_N e^{im_N\theta}$$
$$= b_n C_1 J_{m_n}(\lambda r)\, e^{im_n\theta}$$

for $n = 1,2,\ldots,N-1$ where $C_1 \in \mathbf{C}$, $C_1 \neq 0$.

For $n = N$ we have by (9) and (8):

$$g_N(r)\, e^{im_N\theta} * \psi_k = g_N(r)\, e^{im_N\theta} * \mu_N e^{-im_N\theta} * \phi_k * \mu_N e^{im_N\theta}$$
$$= -\left[\sum_{n=1}^{N-1} g_n(r)\, e^{im_n\theta} * \mu_n e^{-im_n\theta}\right] * \phi_k * \mu_N e^{im_N\theta}\,.$$

Hence we obtain

$$g_n(r)\, e^{im_N\theta} * \psi_k \xrightarrow[k\to\infty]{C(\mathbf{R}^2)} -\left[\sum_{n=1}^{N-1} b_n J_{m_n}(\lambda r)\, e^{im_n\theta} * \mu_n e^{-im_n\theta}\right] * \mu_N e^{im_N\theta}$$
$$= C\, J_0(\lambda r) * \mu_N e^{im_N\theta} = C'\, J_N(\lambda r)\, e^{im_N\theta}\,.$$

Similarly, we may prove that if $0 \in Sp^{(r)}((f)_n)$ for some n, $1 \leqslant n \leqslant N$, then V_f contains a function $g \neq 0$ such that:

$$(g)_n(r,\theta) = a_n r^{m_n} e^{im_n\theta} \qquad n = 1,2,\ldots,N\,.$$

Proof of Theorem 5. – Let $h \in B_{(m)}$, $B_{(m)} \subset C(\mathbf{R}^2, \mathbf{C}^{N+1})$ where $m = (0,1,\ldots,N)$ and $(h)_n(r,\theta) = g_{n-1}(r)\, e^{i(n-1)\theta}$, $n = 1,2,\ldots,N+1$, and suppose that $\lambda \in Sp^{(r)}((h)_{k_0})$, $\lambda \neq 0$, for some k_0, $1 \leqslant k_0 \leqslant N+1$. Then by Lemma 8, there exists a sequence $\{\phi_k\}$, $\phi_k \in M_0^{(r)}(\mathbf{R}^2)$ $k = 1,2,\ldots,$ such that

$$(g_{n-1}(r')\, e^{i(n-1)\theta'} * \phi_k)(r,\theta) \xrightarrow[k\to\infty]{C(\mathbf{R}^2)} a_{n-1} J_{n-1}(\lambda r)\, e^{i(n-1)\theta}$$

for $n = 1, 2, \ldots, N+1$ where $\sum_{n=0}^{N} |a_n| > 0$. Hence, we have

$$\left[\left(\sum_{n=0}^{N} g_n(r')e^{in\theta'}\right) * \phi_k\right](r,\theta) \xrightarrow[k\to\infty]{C(\mathbf{R}^2)} \sum_{n=0}^{N+1} a_n J_n(\lambda r)e^{in\theta}.$$

If $0 \in Sp^{(r)}((h)_{k_0})$ then, similarly, $N_f^{(r)}$ contains $g \in \widetilde{\Gamma}_0$, $g \neq 0$, where $g(r,\theta) = \sum_{n=0}^{N} b_n r^n e^{in\theta}$. Finally, we may easily prove that $Sp^{(r)}(f) \subseteq \bigcup_{m=0}^{N} Sp^{(r)}(g_m e^{im\theta})$ and the result follows.

COROLLARY 6. – *Let* $f \in C(\mathbf{R}^2)$, $f \neq 0$ *where*

$$f(r,\theta) = \sum_{m=0}^{N} g_m(r)e^{im\theta}, \quad g_m \in C^{(r)}(\mathbf{R}^2) \quad (m = 0, 1, \ldots, N).$$

Then the translation-invariant closed subspace $\tau(f)$ *generated by* f *contains an exponential function.*

Proof. – If $0 \in N_f^{(r)}$ then $\tau(f)$ contains a polynomial and hence $1 \in \tau(f)$. Otherwise, by Theorem 5, $g \in \tau(f)$, $g \neq 0$ where:

$$g(r,\theta) = \sum_{m=0}^{N} a_m J_m(\lambda r)e^{im\theta}$$

for some λ, $a_m \in \mathbf{C}$, $\lambda \neq 0$, $(m = 0, 1, \ldots, N)$.

The subspace $\tau(f)$ contains therefore all the functions h where

$$\begin{aligned} h(x,y) &= (g * \mu)(x,y) \\ &= C\sum_{m=0}^{N} a_m \int_{\mathbf{R}^2}\left[\int_0^{2\pi} e^{i\lambda[(x-\alpha)\cos\phi+(y-\beta)\sin\phi]}e^{im\phi}\right]d\mu(\alpha,\beta) \\ &= C\sum_{m=0}^{N} a_m \int_0^{2\pi} \hat{\mu}(\lambda\cos\phi, \lambda\sin\phi)e^{i\lambda(x\cos\phi+y\sin\phi)}e^{im\phi}d\phi \end{aligned}$$

for every $\mu \in M_0(\mathbf{R}^2)$ where $C \in \mathbf{C}$, $C \neq 0$.

Thus $\tau(f)$ contains all the functions u where

$$u(x,y) = \sum_{m=0}^{N} a_m \int_0^{2\pi} s(\phi)e^{i\lambda(x\cos\phi+y\sin\phi)}e^{im\phi}d\phi$$

for every $s \in C[0,2\pi]$, $s(0) = s(2\pi)$. For a sequence $\{s_n\}$ converging to the Dirac mass δ_{ϕ_0} concentrated in ϕ_0 where $\sum_{m=0}^{N} a_m e^{im\phi_0} \neq 0$, we obtain, by passing to the limit, that $v \in \tau(f)$ where

$$v(x, y) = \Big(\sum_{m=0}^{N} a_m e^{im\phi_0}\Big) e^{i(x\lambda \cos \phi_0 + y\lambda \sin \phi_0)}$$

which completes the proof.

Remark 5. – To this end we may introduce the following proof to the fact that every translation-invariant and rotation-invariant closed subspace of $C(\mathbf{R}^2)$ contains an exponential function [1]. Let R_f denote the closed translation-invariant and rotation invariant subspace generated by $f \neq 0$. Then, for a suitable $m \in \mathbf{Z}$ the function g where

$$g(r, \theta) = \int_0^{2\pi} f(r, \theta + \beta)\, e^{-im\beta} d\beta = e^{im\theta} \int_0^{2\pi} f(r, \beta)\, e^{-im\beta} d\beta$$
$$= e^{im\theta} f_1(r)$$

is non-zero and belongs to R_f. Let $\mu \in M_0^{(r)}(\mathbf{R}^2)$ where $\mu(f_1) \neq 0$. Hence the function $g_1 = g * (\mu e^{-im\theta})$ is non-zero and belongs to $R_f \cap C^{(r)}(\mathbf{R}^2)$.

By Lemma 6, or by Lemma 7 for $m = 0$, there exists $\lambda \in \mathbf{C}$ such that $J_0(\lambda r) \in R_f$. Arguing as in the proof of Corollary 6, we deduce that R_f contains the exponentials $e^{i(x\lambda \cos \phi + y\lambda \sin \phi)}$ for every $\phi \in \mathbf{R}$ and the result follows.

BIBLIOGRAPHY

[1] L. BROWN, B.M. SCHREIBER, and B.A. TAYLOR, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier,* Grenoble, 23 (1973), 125-154.

[2] L. EHRENPREIS, and F. MAUTNER, Some properties of the Fourier transform on semi-simple Lie groups II., *Trans. Amer. Math. Soc.,* 84 (1957), 1-55.

[3] D.I. GUREVICH, Counterexamples to a problem of L. Schwartz, *Funct. Anal. Appl.*, 197 (1975), 116-120.

[4] D. POMPEIU, Sur une propriété intégrale des fonctions de deux variables réelles, *Bull. Sci. Acad. Royale Belgique* (5), 15 (1929), 265-269.

[5] L. SCHWARTZ, Théorie générale des fonctions moyenne-périodiques, *Ann. of Math.*, 48 (1947), 857-928.

[6] Y. WEIT, On the one-sided Wiener's theorem for the motion group, to appear in *Ann. of Math.*

[7] L. ZALCMAN. Analyticity and the Pompeiu problem, *Arch. Rational Mech. Anal.*, 47 (1972), 237-254.

Manuscrit reçu le 7 septembre 1979.

Yitzhak WEIT,
University of Haïfa
Department of Mathematics
Mount Carmel
Haïfa (Israël).