ON SOME ERGODIC PROPERTIES
FOR CONTINUOUS AND AFFINE FUNCTIONS

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1. Introduction.

Let $X$ be a compact Hausdorff space, let $C(X)$ denote the space of continuous real-valued functions on $X$, and let $T$ be a positive linear operator of $C(X)$ into itself. Choquet and Foias [1] have considered convergence properties of the iterates $T^n$ of $T$ and the associated arithmetic means $S_n = n^{-1} \sum_{r=0}^{n-1} T^r$. In particular, they obtained the following two results [1, Théorèmes 13, 1]:

**Theorem 1.1.** — If, for some non-negative function $f$ in $C(X)$, $S_n f$ converges pointwise to a continuous strictly positive function, then the convergence is uniform on $X$.

**Theorem 1.2.** — If, for each $x$ in $X$, $\inf \{(T^n 1)(x): n \geq 1\} < 1$, then $T^n 1$ converges to 0 uniformly on $X$.

Choquet and Foias showed that the condition that the limit in theorem 1.1 is strictly positive cannot be removed [1, Exemple 11]. They then raised the following problem:

**Problem 1.** — Suppose that $S_n 1$ converges pointwise to a continuous limit. Is the convergence necessarily uniform?

If $M(X)$ denotes the set of Radon measures on $X$, identified with $C(X)^*$, and $P(X)$ is the set of probability measures in $M(X)$, then $P(X)$ is weak*-compact and convex, its extreme boundary $\partial_e P(X)$ consists of the measures $\epsilon_x$ concentrated at one point $x$.
of $X$, and there is an isometric order-isomorphism $f \mapsto \hat{f}$ of $C(X)$ onto the space $A(P(X))$ of continuous affine real-valued functions on $P(X)$, given by $f(\mu) = \int f \, d\mu$. This raises a second problem.

**Problem 2.** — Suppose that $K$ is a compact convex subset of a locally convex space, and $T$ is a positive linear operator on $A(K)$ such that for each $x$ in $\partial_e K$, $\inf \{(T^n)_1(x): n \geq 1\} < 1$. Does it necessarily follow that $\|T^n\| \to 0$?

In § 2 we shall show (corollary 2.5) that the answer to problem 1 is affirmative, and in § 3 we shall give an example to show that the answer to problem 2 is negative, although it becomes affirmative if $\partial_e K$ is replaced by its closure $\overline{\partial_e K}$ in $K$.

2. Uniform convergence of arithmetic means.

Let $T$ be a positive linear operator on $C(X)$, and $\sigma$ be a non-negative function in $C(X)$. Let $F_\sigma = \sigma^{-1}(0)$ and $G_\sigma$ be the complement of $F_\sigma$ in $X$. For $x$ in $G_\sigma$ and $n \geq 1$ there is a bounded Radon measure $\mu_{x,\sigma}^n$ on $G_\sigma$ such that

$$\int g \, d\mu_{x,\sigma}^n = \sigma(x)^{-1} T^n (g \cdot \sigma) (x)$$

for all functions $g$ in the space $C^b(G_\sigma)$ of continuous bounded real-valued functions on $G_\sigma$. For a Borel-measurable function $f$ defined $\sigma$-a.e. in $G_\sigma$, put $(T_\sigma^n f) (x) = \int f \, d\mu_{x,\sigma}^n$ if the integral exists.

**Lemma 2.1.** — For $x$ in $G_\sigma$, $n \geq 1$ and any bounded Borel function $f$ on $G_\sigma$. $T_\sigma^n (f \cdot \sigma^{-1}) (x) = \sigma(x)^{-1} T^n (\chi_\sigma \cdot f) (x)$, where $\chi_\sigma$ is the characteristic function of $G_\sigma$, and both sides of the equality exist.

**Proof.** — Suppose that $f$ is continuous and non-negative. Let $(g_\lambda)$ be an increasing net of continuous non-negative functions on $X$ with support in $G_\sigma$ and converging pointwise to $\chi_\sigma$. Then $g_\lambda \cdot f \cdot \sigma^{-1} \in C^b(G_\sigma)$, and

$$\sigma(x) \int g_\lambda \cdot f \cdot \sigma^{-1} \, d\mu_{x,\sigma}^n = T^n (g_\lambda \cdot f) (x) = \int g_\lambda \cdot f \, d\mu_{x,1}^n.$$ 

The right-hand integral increases to the finite integral $\int \chi_\sigma \cdot f \, d\mu_{x,1}^n$, so the result follows immediately in this special case.
The case when $f$ is lower semi-continuous follows by approximating $f$ from below by continuous functions, and the general case from the fact that the bounded Borel functions form the smallest linear space containing the lower semi-continuous functions and closed under bounded monotone sequential limits.

Now suppose that $T\sigma \leq \beta \sigma$ for some real number $\beta$. Then $T^{(n)}_\sigma 1 \leq \beta^n$, so $T^{(n)}_\sigma$ maps $C^b(G_\sigma)$ into itself. It follows immediately from the definitions that the following identity is valid for $f$ in $C^b(G_\sigma)$: $T^{(m)}_\sigma (T^{(n)}_\sigma f) (x) = (T^{(m+n)}_\sigma f) (x)$. Elementary integration theory shows that this identity is valid for any Borel function $f$ on $G_\sigma$ in the sense that if either expression exists then so does the other and they are equal. We shall therefore write $T^n_\sigma$ instead of $T^{(n)}_\sigma$. This discussion applies in particular to the case $\sigma = 1$ when it is consistent to write $T$ instead of $T^1_\sigma$.

For $x$ in $G_\sigma$, $0 \leq (T^n_\sigma \sigma) (x) \leq \beta^n \sigma (x) = 0$ so $\mu_{x,1}^n(G_\sigma) = 0$. Thus $T^n(\chi_{G_\sigma} \cdot f) = 0$ on $F_\sigma$. Note that this is consistent with lemma 2.1 which gives

$$T^n_\sigma (T^{(n)}_\sigma f \cdot \sigma^{-1}) = \sigma^{-1} T^n (\chi_{G_\sigma} \cdot T^{(n)}_\sigma f)$$

$$T^{m+n}_\sigma (f \cdot \sigma^{-1}) = \sigma^{-1} T^{m+n} (\chi_{G_\sigma} \cdot f).$$

**Lemma 2.2.** Suppose that $T\sigma \leq \sigma$ and $(T^1_\sigma) (x) < 1$ for all $x$ in $F_\sigma$. Then there is a real number $\alpha$ such that $(T^n_\sigma \chi_{G_\sigma}) (x) \leq \alpha$ for all $n \geq 1$ and $x$ in $G_\sigma$.

**Proof.** By continuity and compactness, there is a neighbourhood $U$ of $F_\sigma$ and real numbers $\beta_1 < 1$ and $\beta_2 \geq \beta_1 (1 - \beta_1) \| \sigma \|^{-1}$ such that

$$(T^1_\sigma) (x) \leq \beta_1 \quad (x \in U)$$

$$(T^1_\sigma) (x) \leq \beta_2 \sigma (x) \quad (x \in K \setminus U).$$

Let $\alpha = (1 - \beta_1)^{-1} \beta_2 \| \sigma \|$. Then $T^1 \leq \alpha$ and $T^1 \leq \beta_1 + \beta_2 \sigma$. In particular, $T^1 \chi_{G_\sigma} \leq T^1 \leq \alpha$. Now suppose that $T^n\chi_{G_\sigma} \leq \alpha$ on $G_\sigma$, and take $x$ in $G_\sigma$. Using lemma 2.1 and the fact that $T^1 \sigma \leq 1$,

$$(T^{n+1}_\sigma \chi_{G_\sigma}) (x) = T^n (T \chi_{G_\sigma}) (x) = \sigma (x) T^n_\sigma (\sigma^{-1} \cdot T \chi_{G_\sigma}) (x)$$

$$\leq \sigma (x) T^n_\sigma (\beta_1 \sigma^{-1} + \beta_2) (x)$$

$$\leq \beta_1 (T^n \chi_{G_\sigma}) (x) + \beta_2 \sigma (x)$$

$$\leq \beta_1 \alpha + \beta_2 \sigma (x)$$

$$\leq \alpha.$$
LEMMA 2.3. — Let $F$ be a Borel subset of $X$, $\chi$ be the characteristic function of the complement of $F$ in $X$, and 
$$\delta = \sup \{(T_1)(x): x \in F\}.$$ 
Then 
$$T^n 1 \leq \delta^n + \sum_{r=1}^{n} \delta^{r-1} T^{n-r} (\chi \cdot T_1).$$

Proof. — It is trivial that $T_1 \leq \delta + \chi \cdot T_1$. Suppose the lemma holds for some integer $n$. Then since $T$ is positive, 
$$T^{n+1} 1 \leq \delta^n T_1 + \sum_{r=1}^{n} \delta^{r-1} T^{n+1-r} (\chi \cdot T_1) \leq \delta^{n+1} + \sum_{r=1}^{n+1} \delta^{r-1} T^{n+1-r} (\chi \cdot T_1).$$

THEOREM 2.4. — Let $T$ be a positive linear operator on $C(X)$ and suppose that there is a non-negative continuous function $\sigma$ on $X$ such that $T \sigma \leq \sigma$ and $(T_1)(x) < 1$ whenever $\sigma(x) = 0$. Then 
$$\{T^n 1: n \geq 1\}$$
is uniformly bounded.

Proof. — Take $\alpha$ as in lemma 2.2 and 
$$\delta = \sup \{(T_1)(x): x \in F_\sigma\} < 1.$$ 
By lemma 2.3, for $x$ in $G_\sigma$, 
$$(T^n 1)(x) \leq \delta^n + \alpha \|T_1\| \sum_{r=1}^{n} \delta^{r-1} \leq \delta^n + (1 - \delta)^{-1} \alpha \|T_1\|.$$ 
Also $T^n 1 = T((1 - \chi_\sigma) T^{n-1} 1)$ on $F_\sigma$, so a simple inductive argument shows that $T^n 1 \leq 1$ on $F_\sigma$.

COROLLARY 2.5. — Suppose that $S_n 1$ converges pointwise to a continuous limit $\sigma$. Then the convergence is uniform.

Proof. — It is shown in the proof of [1, Lemme 12] that $T \sigma \leq \sigma$. Hence $\mu_{x,1}^1 (G_\sigma) = 0$ for $x$ in $F_\sigma$, so $T$ induces a positive linear operator $\tilde{T}$ on $C(F_\sigma)$ given by 
$$(\tilde{T}f)(x) = \int_{F_\sigma} f \, d\mu_{x,1}^1.$$ 
Now $\tilde{T}^n 1$ is the restriction of $T^n 1$ to $F_\sigma = \sigma^{-1}(0)$, so 
$$\inf \{\tilde{T}^n 1: n \geq 1\} = 0.$$ 
By theorem 1.2 there is an integer $m$ such that $T^m 1 \leq 1$ on $F_\sigma$. Applying theorem 2.4 to $T^m$, it follows that 
$$\{T^{mn}: n \geq 1\}$$
is uniformly bounded. Hence 
$$\{T^n 1: n \geq 1\}$$
is uniformly bounded. The result now follows from [1, Théorème 10].
3. Affine functions.

We shall now give an example to show that the answer to problem 2 is negative in general, even if K is a simplex.

Example 3.1. — Let N be the linear span in M[0,1] of $\lambda - \epsilon_0$, where $\lambda$ is Lebesgue measure on [0,1], let $\pi: M[0,1] \rightarrow M[0,1]/N$ be the quotient map, and let $K = \pi(P[0,1])$. Then K is a simplex with extreme boundary $\partial_e K = \{ \pi(\epsilon_x): x \in (0,1) \}$, and there is an isometric isomorphism $\Phi$ between $A(K)$ and the space $C_0[0,1]$ of functions $f$ in $C[0,1]$ satisfying $f(0) = \int_0^1 f(x) \, dx$, given by $\Phi^{-1}(f) \circ \pi = \hat{f}$ $(f \in C_0[0,1])$. We shall identify these spaces.

Let $g$ be any continuously differentiable function of [0,1] into itself (in the sense of one-sided derivatives at the end-points) such that

$$g(0) = 0, \quad g'(0) = 1$$
$$g(x) > x, \quad g'(x) \geq 0 \quad (x \in (0,1))$$
$$g(1) = 1, \quad g'(1) = 0. $$

Define the operator $T$ by $(Tf)(x) = g'(x)f(g(x))$. Then $T$ is a positive linear operator of $C_0[0,1]$ into itself.

For any $x$ in $(0,1)$, let $x_0 = x$, $x_r = g(x_{r-1})$. Then $x_r$ increases to the limit 1, so $g'(x_r) \rightarrow 0$. Now

$$(T^n1)(x) = \prod_{r=0}^{n-1} g'(x_r) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$ 

Thus $T$ satisfies all the required properties. However

$$\|T^n\| \geq |(T^n1)(0)| = 1.$$ 

It is noted in [1] that Mokobodzki has shown that problem 2 has an affirmative answer if $\partial_e K$ is closed. This is a special case of the following result, which deals with a general $K$, but assumes a strengthened condition on $T$. The proof is based on one of those given in [1].

Theorem 3.2. — Let $K$ be a compact convex set, let $\overline{\partial_e K}$ be the closure of its extreme boundary, and let $T$ be a positive linear operator on $A(K)$. If, for each $x$ in $\overline{\partial_e K}$, inf $\{(T^n1)(x): n \geq 1\} < 1$, then $\|T^n1\| \rightarrow 0$. 


Proof. — For a bounded real-valued function \( g \) on \( K \), and \( x \) in \( K \), put \( (\widetilde{T}g)(x) = \inf \{(Ta)(x) : a \in A(K), a \geq g \text{ on } \partial_e K\} \). Then \( \widetilde{T}(\lambda g) = \lambda \widetilde{T}g \), \( \widetilde{T}g_1 \leq \widetilde{T}g_2 \) if \( g_1 \leq g_2 \) on \( \partial_e K \), and \( \widetilde{T}a = Ta \) for \( a \) in \( A(K) \).

By compactness of \( \partial_e K \), there is an integer \( r \) and constant \( \alpha > 0 \) such that if \( g_0(x) = \min \{((T + \alpha)^n)x) : 1 \leq n \leq r\} \), then \( g_0 \leq 1 \) on \( \partial_e K \). Then \( (\widetilde{T} + \alpha)g_0 \leq (T + \alpha)1 \) on \( \partial_e K \). Also \( g_0 \leq (T + \alpha)^n1 \), so \( (\widetilde{T} + \alpha)g_0 \leq (T + \alpha)^{n+1}1 \) (1 \( \leq n \leq r\)). Hence, on \( \partial_e K \), \( (\widetilde{T} + \alpha)g_0 \leq g_0 \), so \( \widetilde{T}g_0 \leq (1 - \alpha)g_0 \).

Now \( g_0 \geq \alpha' \), so \( T^n1 \leq \alpha^{-r}(1 - \alpha)^ng_0 \) on \( \partial_e K \). The result now follows.

Similarly one may modify the proof of Théorème 2 of [1] to show that if, under the conditions of theorem 3.2,
\[ \sup \{(T^n1)(x) : n \geq 1\} > 1 \]
for each \( x \) in \( \partial_e K \), then \( \|T^n\| \to \infty \).

Example 3.3. — Let \( \mathcal{H} \) be a complex Hilbert space, and \( x \) be an operator on \( \mathcal{H} \) such that \( x - \alpha \) is compact for some scalar \( \alpha \) with \( |\alpha| < 1 \). Suppose that for each unit vector \( \xi \) in \( \mathcal{H} \), \( \|x^n\| < 1 \) for some \( n \) (possibly dependent on \( \xi \)). If \( x \) is self-adjoint, the spectral theorem may be used to deduce that \( \|x\| < 1 \). However it is easily verified for example that any non-self-adjoint operator \( x \) of rank 1 and norm 1 also satisfies \( \|x^2\| < 1 \).

Let \( A \) be the C*-algebra spanned by the identity and the compact operators on \( \mathcal{H} \), and let \( K \) be its state space. It is well-known that the evaluation map is an isometric order-isomorphism of the self-adjoint part \( A^s \) of \( A \) onto \( A(K) \), and that \( \partial_e K \) consists of the vector states \( \omega_\xi \) (\( \xi \in \mathcal{H}, \|\xi\| = 1 \)) given by \( \omega_\xi(a) = \langle a\xi, \xi \rangle \) together with the unique state \( \phi_0 \) annihilating the compacts [2, Corollaire 4.1.4]. Using the weak compactness of the unit ball of \( \mathcal{H} \) it is easy to see that \( \partial_e K \) consists of states of the form \( \beta\omega_\xi + (1 - \beta)\phi_0 \) (\( \beta \in [0,1] \)).

If \( x \) satisfies the above conditions, and \( T \) is defined by \( Ta = x^*ax \) then \( T \) is a positive linear operator on \( A^s \), and
\[ (\beta\omega_\xi + (1 - \beta)\phi_0)(T^n1) = \beta \|x^n\|^2 + (1 - \beta)|\alpha|^2n < 1 \]
for some \( n \). Theorem 3.2 now shows that \( \|T^n1\| \to 0 \), so \( \|x^n\| \to 0 \).
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BIBLIOGRAPHY


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