Ann. Inst. Fourier, Grenoble
28, 3 (1978), 203-208

RADON-NIKODYM PROPERTY FOR VECTOR-VALUED INTEGRABLE FUNCTIONS

by Surjit Singh KHURANA

It is proved in ([6], Theorem 1) that if a Banach space E possesses Radon-Nikodym (R – N) property, then the Banach space $L_p(E, \lambda)$, $1 < p < \infty$, of Bochner p-integrable functions also possesses this property. In this paper we give a new proof of the corresponding result when E is a Frechet space (i.e. a complete metrizable locally convex space [5]).

Let $(Y, \mathcal{B}, \nu)$ be a positive measure which is non-trivial (i.e., there exists a $B \in \mathcal{B}$ such that $0 < \nu(B) < \infty$) with $\mathcal{R} = \{A \in \mathcal{B} : (2(A)) < \infty\}$, E a Frechet space with $\{P_i\}$ an increasing sequence of semi-norms on E generating the topology of E, and $L_p(E, \lambda)$ the equivalence classes of strongly p-power integrable functions $Y \longrightarrow E$, $1 \leqslant p < \infty$. (A function $f : X \longrightarrow E$ is called strongly p-power integrable if there exists a sequence $\{f_n\}$ of $\mathcal{R}$-simple E-valued functions of Y such that (i) $f_n \longrightarrow f$ a.e. $[\nu]$, and (ii) $\int [P_i(f_n - f)]_i^p \, d\nu \longrightarrow 0$, $\forall i$. The increasing sequence of semi-norms

$$N_{i,p}, \; N_{i,p}(f) = \left[\int [P_i(f)]^p \, d\nu \right]^{1/p}$$

makes $L_p(E, \lambda)$ a Frechet space. We use the definition of [4] for a Frechet space to have R – N property.

E = K, we denote $L_p(E, \nu)$ by $L_p(\nu)$ and the corresponding norm by $\|\cdot\|_p$.

THEOREM. – *Suppose* E *is a Frechet space with* R – N *property and* $(Y, \mathcal{B}, \nu)$ *a non-trivial positive measure space. Then* $L_p(E, \nu)$ *has* R – N *property for* $1 < p < \infty$.

Proof. – Using ([3], Theorem 5 (iv)) it is sufficient to prove the R – N property for every separable closed subspace; this means we can assume that E is separable ([3], Theorem 5). Let $(X, \mathcal{U}, \lambda)$ be a finite measure space, $\mu : \mathcal{U} \longrightarrow L_p(E, \nu)$ a measure of finite variation (i.e., $\forall i$, the variation of μ relative to $N_{p,i}$ is finite, [4]), absolutely continuous with respect to λ. Assume first that $\nu(Y) < \infty$ and let $\lambda \times \nu$ be the product of λ and ν on the σ-algebra $\mathcal{U} \times \mathcal{B}$.

For an $A \in \mathcal{U}, B \in \mathcal{B}$, define $\omega(A \times B) = \int_B \mu(A)\, d\nu \in E$ (since $\nu(Y) < \infty$, $P_i(\mu(A)) \in L_p(\nu)$, $\forall i$, implies $P_i(\mu(A)) \in L_1(\nu)$). Take $\{A_i \times B_i\}$ a disjoint sequence in $X \times Y$ $(A_i \in \mathcal{U}, B_i \in \mathcal{B})$ and let $\cup A_i \times B_i = A \times B$ $(A \in \mathcal{U}, B \in \mathcal{B})$. Fix an $f \in E'$. $f \circ \mu : \mathcal{U} \longrightarrow L_p(\nu)$ is of bounded variation and absolutely continuous relative to λ. Since $L_p(\nu)$ has R – N property, there exists a function $\phi : X \times Y \longrightarrow K$ such that

$$f \circ \mu(A) = \int_A \phi(x, y)\, d\lambda\,(x), \ \forall A \in \mathcal{U};$$

it is routine verification that $\phi(x, y) \in L_1(\lambda \times \nu)$. Thus

$$\int_{B_i} f \circ \mu(A_i)\, d\nu = \int_{A_i \times B_i} \phi(x, y)\, d(\lambda \times \nu)$$

(Fubini's theorem) and so $\sum \int_{B_i} f \circ \mu(A_i)\, d\nu = \int_B f \circ \mu(A)\, d\nu$ (unconditional convergence). Since $\langle f, \int_{B_i} \mu(A_i)\, d\nu \rangle = \int_{B_i} f \circ \mu(A_i)\, d\nu$, (simple verification), $\forall f \in E'$, by Pettis-Orlicz theorem,

$$\sum \int_{B_i} \mu(A_i)\, d\nu = \int_B \mu(A)\, d\nu .$$

Also for a finite disjoint collection $\{C_i \times D_i\}$ in $X \times Y$ $(C_i \in \mathcal{U}, D_i \in \mathcal{B})$, $\sum \int_{D_i} f \circ \mu(C_i)\, d\nu = \int_{\cup C_i \times D_i} \phi(x, y)\, d(\lambda \times \nu)$ (previous notation) and so $\left| f \circ \sum \int_{D_i} \mu(C_i)\, d\nu \right| \leq \int |\phi(x, y)|\, d(\lambda \times \nu)$. Combining these results we see that ω can be uniquely extended to a finitely additive set function $\omega : \theta \longrightarrow E$, θ being the algebra generated by $\{A \times B : A \in \mathcal{U}, B \in \mathcal{B}\}$. ω is countably additive, and $\omega(\theta)$ is bounded in E. Since E has R – N property it cannot contain a subspace isomorphic to c_0 ([1]; [3], Theorem 5). From this it easily follows that ω is exhaustive ([2], II; [7], Theorem 4). Thus ω can be uniquely extended to a countably additive measure on the σ-algebra $\mathcal{U} \times \mathcal{B}$ ([2], III). We claim that $\omega \ll \lambda \times \nu$. For an $f \in E'$, $A \in \mathcal{U}, B \in \mathcal{B}$, $f \circ \omega(A \times B) = \int_{A \times B} \phi(x, y)\, d(\lambda \times \nu)$ (previous notations) and so $f \circ \omega(\mathrm{H}) = \int_{\mathrm{H}} \phi \, d(\lambda \times \nu)$, $\forall \mathrm{H} \in \mathcal{U} \times \mathcal{B}$. If $(\lambda \times \nu)(\mathrm{H}) = 0$ we get $f \circ \omega(\mathrm{H}) = 0$ and so $\omega(\mathrm{H}) = 0$.

We now prove that ω is of finite variation. Fix $i \in \mathrm{N}$ and let λ_0 = the finite variation of μ relative to the semi-norm $\mathrm{N}_{i,p}$.

$$\mathrm{H} = \{f \in \mathrm{E}', |f(x)| \leqslant \mathrm{P}_i(x), \forall x \in \mathrm{E}\}$$

is a metrizable compact subset of $(\mathrm{E}', \sigma(\mathrm{E}', \mathrm{E}))$ and so have a countable dense subset $\{f_j\}$. Let $\phi_j \in \mathrm{L}_1(\lambda_0 \times \nu)$ such that

$$f_j \circ \mu(\mathrm{A}) = \int_{\mathrm{A}} \phi_j(x, y) \, d\lambda_0(x)$$

(same reasoning as before).

Let $\varphi_0 = \sup(|\varphi_1|, |\varphi_2|)$ and fix $x \in \mathrm{X}$. Take

$$\mathrm{B}_1 = \{y \in \mathrm{Y} : |\varphi_1(x, y)| = \varphi_0(x, y)\} \text{ and } \mathrm{B}_2 = \mathrm{Y} \backslash \mathrm{B}_1.$$

We claim the variation of $\xi = \chi_{\mathrm{B}_1} f_1 \circ \mu + \chi_{\mathrm{B}_2} f_2 \circ \mu$, in $\mathrm{L}_p(\nu)$, does not exceed λ_0. From

$$|\xi(\mathrm{A})|^p = |(\chi_{\mathrm{B}_1} f_1 \circ \mu + \chi_{\mathrm{B}_2} f_2 \circ \mu)(\mathrm{A})|^p$$
$$\leqslant \chi_{\mathrm{B}_1} (\mathrm{P}_i(\mu(\mathrm{A})))^p + \chi_{\mathrm{B}_2} (\mathrm{P}_i(\mu(\mathrm{A})))^p = (\mathrm{P}_i(\mu(\mathrm{A})))^p,$$

we get $\|\xi(\mathrm{A})\|_p \leqslant \mathrm{N}_{i,p}(\mu(\mathrm{A})) \leqslant \lambda_0(\mathrm{A})$ and so the claim is established. Now $\xi(\mathrm{A}) = \int_{\mathrm{A}} (\chi_{\mathrm{B}_1} \varphi_1 + \chi_{\mathrm{B}_2} \varphi_2) \, d\lambda_0$. If $|\xi|$ is the variation of ξ relative to $\mathrm{L}_p(\nu)$, then $|\xi|(\mathrm{A}) = \int_{\mathrm{A}} \|\chi_{\mathrm{B}_1} \varphi_1 + \chi_{\mathrm{B}_2} \varphi_2\|_p \, d\lambda_0$. If $\|\chi_{\mathrm{B}_1} \varphi_1 + \chi_{\mathrm{B}_2} \varphi_2\|_p \geqslant 1 + \eta$ for some $\eta > 0$ on $\mathrm{A} \in \mathcal{U}$, then $\lambda_0(\mathrm{A}) \geqslant |\xi|(\mathrm{A}) \geqslant (1 + \eta)\lambda_0(\mathrm{A})$ which means $\lambda_0(\mathrm{A}) = 0$ and so $\|\chi_{\mathrm{B}_1} \varphi_1 + \chi_{\mathrm{B}_2} \varphi_2\|_p \leqslant 1$ a.e. $[\lambda_0]$. Now

$$\|\chi_{\mathrm{B}_1} \varphi_1 + \chi_{\mathrm{B}_2} \varphi_2\|_1 \leqslant \|\chi_{\mathrm{B}_1} \varphi_1 + \chi_{\mathrm{B}_2} \varphi_2\|_p (\lambda_0(\mathrm{X}))^{1/q} \leqslant (\lambda_0(\mathrm{X}))^{1/q}$$

(Holder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$) means

$$\int \varphi_0(x, y) \, d\nu(y) \leqslant (\lambda_0(\mathrm{X}))^{1/q}, \quad \text{a.e. } [\lambda_0].$$

Since $x \in \mathrm{X}$ was arbitrary we see $\varphi_0(x, y) \in \mathrm{L}_1(\mathrm{X} \times \mathrm{Y}, \lambda_0 \times \nu)$. If $\varphi = \sup(|\varphi_1|, |\varphi_2|, \ldots)$, then proceeding in the same way we prove that $\int \varphi(x, y) \, d\nu(y) \leqslant (\lambda_0(\mathrm{X}))^{1/q}$, a.e. $[\lambda_0]$ and so $\varphi \in \mathrm{L}_1(\mathrm{X} \times \mathrm{Y}, \lambda_0 \times \nu)$ (the set where φ takes values $+\infty$ has measure zero; we put $\varphi \equiv 0$ on that set). Fix $\epsilon > 0$ and let $\{\mathrm{H}_j\}$ be a finite disjoint collection in $\mathcal{U} \times \mathcal{B}$.

$$\sum_j \mathrm{P}_i(\omega(\mathrm{H}_j)) - \epsilon \leqslant \sum_j |f_{k(j)} \circ \omega(\mathrm{H}_j)| \leqslant \sum \int_{\mathrm{H}_j} |\varphi_{k(j)}| \, d(\lambda_0 \times \nu)$$
$$\leqslant \sum \int_{\mathrm{H}_j} \varphi \, d(\lambda_0 \times \nu) \leqslant \int \varphi \, d(\lambda_0 \times \nu)$$

for some finite sequence $\{k(j)\} \subset \mathrm{N}$. This proves ω is of finite variation. Since E has R – N property we get a $g \in \mathrm{L}_1(\mathrm{E}, \lambda \times \nu)$ such that $\int_{\mathrm{B}} \mu(\mathrm{A})\, d\nu = \int_{\mathrm{B}} \int_{\mathrm{A}} g(x,y)\, d\lambda(x)\, d\nu(y)$.

Put $\psi = \mu(\mathrm{A}) - \int_{\mathrm{A}} g(x,y)\, d\lambda(x)$. We get $\int_{\mathrm{B}} \psi\, d\nu = 0$, $\forall \mathrm{B} \in \mathcal{B}$. Fix $i \in \mathrm{N}$ and let $\{f_j\}$ be a countable dense set in the compact metric space

$$\mathrm{H} = \{f \in \mathrm{E}' : |f(x)| \leq \mathrm{P}_i(x),\ \forall x \in \mathrm{E}\} \subset (\mathrm{E}', \sigma(\mathrm{E}', \mathrm{E})).$$

We get $\int_{\mathrm{B}} f_j \circ \psi = 0$, $\forall \mathrm{B} \in \mathcal{B}$ and so $\mathrm{P}_i(\psi) = 0$ a.e. $[\nu]$. Thus $\psi = 0$ a.e. $[\nu]$. Thus $\mu(\mathrm{A}) = \int_{\mathrm{A}} g(x,y)\, d\lambda(x)$. It is easy to verify that $g(\cdot, x) \in \mathrm{L}_1(\mathrm{L}_p(\mathrm{E}, \nu), \lambda)$.

Now we consider the case when $\nu(\mathrm{Y}) = +\infty$. By ([3], Theorem 5) it is enough to prove the result for every closed separable subspace of $\mathrm{L}_p(\mathrm{E}, \nu)$. Let F be a closed separable subspace of $\mathrm{L}_p(\mathrm{E}, \nu)$. It is a simple verification that there exists a $\mathrm{B} \in \mathcal{B}$ with σ-finite ν-measure such that $f = 0$ a.e. $[\nu]$ outside B, $\forall f \in \mathrm{F}$. Thus ([3], Theorem 5) we can assume that ν is σ-finite. Let $\{\mathrm{K}_n\}$ be a $\mathcal{B}$-measurable partition of Y, such that $0 < \nu(\mathrm{K}_n) < \infty$, $\forall n$. Define $\nu_n = \chi_{\mathrm{K}_n} \nu$, $\nu_n : \mathcal{B}_n \longrightarrow [0, \infty)$, $\mathcal{B}_n = \mathcal{B} \cap \mathrm{K}_n$. Given μ as before, we get $\mu_n : \mathcal{U} \longrightarrow \mathrm{L}_p(\mathrm{E}, \nu_n)$, $\mu_n(\mathrm{A}) = \chi_{\mathrm{K}_n} \mu(\mathrm{A}) \in \mathrm{L}_p(\mathrm{E}, \nu_n)$. It is easy to verify that μ_n is of finite variation relative to $\mathrm{L}_p(\mathrm{E}, \nu_n)$ and absolutely continuous relative to λ. Proceeding as before we get $g_n : \mathrm{X} \times \mathrm{K}_n \longrightarrow \mathrm{E}$ such that

$$\mu_n(\mathrm{A}) = \int_{\mathrm{A}} g_n(x,y)\, d\lambda(x) = \int_{\mathrm{A}} \chi_{\mathrm{K}_n} g_n(x,y)\, d\lambda(x).$$

Define $g(x,y) = g_n(x,y)$, $y \in \mathrm{K}_n$, we claim

$$\mu(\mathrm{A}) = \int_{\mathrm{A}} g(x,y)\, d\lambda(x).$$

If $\mu(\mathrm{A}) = f \in \mathrm{L}_p(\mathrm{E}, \nu)$, then

$$\Big(\mathrm{P}_i\big(\mu(\mathrm{A}) - \sum_{j=1}^{k} \mu_j(\mathrm{A})\big)\Big)^p \leq (\mathrm{P}_i(f))^p$$

and so by dominated convergence theorem $\sum_{i=1}^{k} \mu_i(\mathrm{A})$ converges to $\mu(\mathrm{A})$ in $\mathrm{L}_p(\mathrm{E}, \nu)$. Let $|\mu|$ and $\left|\sum_{j=1}^{k} \mu_j\right|$ be the variations of μ and $\sum_{j=1}^{k} \mu_j$ relative to $\mathrm{N}_{i,p}$. Then

$$|\mu|(\mathrm{A}) \geqslant \left|\sum_{j=1}^{\infty} \mu_j\right|(\mathrm{A}) = \int_{\mathrm{A}} \mathrm{N}_{i,p}\left(\sum \chi_{\mathrm{K}_j} g_j\right) d\lambda .$$

By monotone convergence theorem $\mathrm{N}_{i,p}(g) < \infty$, a.e. $[\lambda]$ and $\int \mathrm{N}_{i,p}(g)\, d\lambda < \infty$. On the set where $\mathrm{N}_{i,p}(g) = +\infty$ we change its value and the value of each of g_n to 0 and so $g(\cdot, x) \in \mathrm{L}_p(\mathrm{E}, \nu)$, $\forall x \in \mathrm{X}$. Now it is easy to verify that $h_n = \sum_{j=1}^{n} \chi_{\mathrm{K}_j} g_j$ converges to g in $\mathrm{L}_p(\mathrm{E}, \nu)$, a.e. $[\lambda]$ and $\mathrm{N}_{i,p}(g - h_n)$, as a function of x, is decreasing as n increases. By monotone convergence theorem, $\int \mathrm{N}_{i,p}(g - h_n)\, d\lambda \longrightarrow 0$. Thus

$$\mathrm{N}_{i,p}\left(\int_{\mathrm{A}} g\, d\lambda - \int_{\mathrm{A}} h_n\, d\lambda\right) \leqslant \int \mathrm{N}_{i,p}(g - h_n)\, d\lambda \longrightarrow 0$$

and so $\int_{\mathrm{A}} h_n\, d\lambda \longrightarrow \int_{\mathrm{A}} g\, d\lambda$ in $\mathrm{L}_p(\mathrm{E}, \lambda)$. But

$$\int_{\mathrm{A}} h_n\, d\lambda = \sum_{j=1}^{n} \mu_j(\mathrm{A}) \longrightarrow \mu(\mathrm{A})$$

and so $\mu(\mathrm{A}) = \int_{\mathrm{A}} g\, d\lambda$. The result now follows easily.

BIBLIOGRAPHY

[1] J. DIESTEL, J.J. UHL, Jr., The Radon-Nikodym property for Banach space valued measures, *Rocky Mountain J. Math.*, 6 (1976), 1-46.

[2] L. DREWNOWSKI, Topological rings of sets, continuous set functions, integration I, II, III, *Bull. Acad. Polon. Sci.*, Ser. Math. Astron. Phys., 20 (1972), 269-276, 277-286, 439-445.

[3] E. SAAB, Dentabilité, points extrémaux et propriété de Radon-Nikodym, *Bull. Soc. Math.*, 99 (1975), 129-134.

[4] E. SAAB, Dentabilité, points extrémaux et propriété de Radon-Nikodym, *C.R. Acad. Sci.*, Paris, 280 (1975), 575-577.

[5] H.H. SCHAEFER, Topological vector spaces, Macmillan, New York (1971).

[6] K. SUNDARESAN, The Radon-Nikodym theorem for Lebesgue-Bochner function spaces, *J. Func. Anal.*, 24 (1977), 276-279.

[7] Ju. B. TUMARKIN, On locally convex spaces with basis, *Doklady Acad. Sci. USSR,* 195 (1970), 1278-1281, English Translation: *Soviet Math.,* 11 (1970), 1672-1675.

[8] P. TURPIN, Convexité dans les espaces vectoriels topologiques généraux, *Disser. Math.,* 131 (1976).

Manuscrit reçu le 21 septembre 1977
Proposé par G. Choquet.

Surjit Singh KHURANA,
Department of Mathematics
University of Iowa
Iowa City, Iowa 52242 (USA).