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THE SPACE $\mathscr{D}(\Omega)$ IS NOT B_r-COMPLETE

by Manuel VALDIVIA

The purpose of this paper is to study certain classes of locally convex spaces which have non-complete separated quotients. Consequently, we obtain some results about B_r-completeness. In particular, we prove that the L. Schwartz space $\mathscr{D}(\Omega)$ is not B_r-complete, where Ω is a non-empty open subset of R^m.

The vector spaces which are used here are defined on the field of the real or complex numbers K. We shorten « Hausdorff locally convex space » to « space ». If $\langle G, H \rangle$ is a dual pair, we denote by $\sigma(G, H)$ and $\mu(G, H)$ the weak topology and the Mackey topology on G, respectively.

Sometimes we symbolize by $L[\mathscr{F}]$ a space L with the topology $\mathscr{F}$. We denote by L' the topological dual of L. If L_n are copies of L, $n = 1, 2, \ldots,$ we denote by L^N the product $\prod_{n=1}^{\infty} L_n$ and by $L^{(N)}$ the direct sum $\bigoplus_{n=1}^{\infty} L_n$. If A is an absolutely convex bounded and closed subset of L, L_A is the linear hull of A with the locally convex topology which has as fundamental system of neighbourhoods of the origin the family $\left\{\frac{1}{n} A : n = 1, 2, \ldots\right\}$. We say that a sequence (x_n) in L is a Cauchy (convergent) sequence in the sense of Mackey, if there exists an absolutely convex closed and bounded subset B of L, such that (x_n) is a Cauchy (convergent) sequence in L_B. Then it is natural to define a locally complete space L as a space in which every Cauchy sequence in the sense of Mackey is convergent in the sense of Mackey in L. If a subset M of L is such that for every point x of L there exists a sequence (x_n) in M which converges to x in the sense of Mackey, M is called locally dense in L. L is called a dual locally complete space if $L'[\sigma(L', L)]$ is locally complete. If L is topologically isomorphic to P write $L \simeq P$.

Let Ω be a non-empty open subset of the m-dimensional euclidean space R^m. Then $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$ are the L. Schwartz spaces with the strong topologies. By ω we denote the product of countably many one-dimensional spaces. Let s be the vector space of all the rapidly decreasing sequences, with the metric topology. The dual of s with the Mackey topology is denoted by s'.

A space T is called B_r-complete if every dense subspace M of $T'[\sigma(T', T)]$ coincides with T' when $M \cap A$ is $\sigma(T', T)$-closed in A for every equicontinuous set A that belongs to T'.

Throughout this paper, E denotes an infinite-dimensional Fréchet space, different from ω, and F a Fréchet space which is not Banach.

In order to prove Theorem 1, we need the following result, which we gave in [8] : a) *Let* G *be an infinite-dimensional space such that in* $G'[\sigma(G', G)]$ *there is an equicontinuous total sequence. Let* H *be a space with a separable absolutely convex weakly compact total subset. If* H *is infinite-dimensional, there is a linear mapping* u *continuous and injective from* G *into* H, *such that* $u(G)$ *is separable, dense in* H *and* $u(G) \neq H$.

In Theorem 1, (E_n) is a sequence of infinite-dimensional spaces such that in E_n there is a weakly compact absolutely convex and separable subset which is total in E_n, $n = 1, 2, \ldots$

THEOREM 1. — *Let* $L = (E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$. *Then there is a quotient of* $L[\sigma(L', L)]$ *which is topologically isomorphic to a locally dense and non-closed subspace of* $\prod_{n=1}^{\infty} E_n[\sigma(E_n, E_n')]$.

Proof. — Since F is a Fréchet space which is not Banach, there exists a closed subspace G of F such that $F/G \simeq \omega$, (see [2], remark at the end of § 31, 4.(1), p. 432). Let f be the canonical mapping from F onto F/G and let J be the identity mapping on E. The tensorial product $J \otimes f$ is a topological homomorphism from $E \widehat{\otimes} F$ onto

$$E \widehat{\otimes} (F/G) \simeq E \widehat{\otimes} \omega,$$

(see [1], Chapter I p. 38). Further, if H_n is an one-dimensional space, then

$$E \widehat{\otimes} \omega = E \widehat{\otimes} \left(\prod_{n=1}^{\infty} H_n\right) \simeq \prod_{n=1}^{\infty} (E \widehat{\otimes} H_n) \simeq E^{N},$$

(see [1], Chapter I p. 46).

Since E is different from ω, in $E'[\sigma(E', E)]$ there is an equicontinuous sequence whose closed linear hull M is infinite dimensional. Let $M^{\perp}$ be the orthogonal subspace of E to M, set $T = E/M^{\perp}$, and let h be the canonical mapping from E^{N} onto $E^{N}/(M^{\perp})^{N} \simeq (E/M^{\perp})^{N} = T^{N}$.

Now we use the result a) and be obtain a linear continuous and injective mapping u_n, from T into E_n, such that $u_n(T)$ is dense in E_n and $u_n(T) \neq E_n$. We define a mapping g from $T^{N} \times \bigoplus_{n=1}^{\infty} E_n$ into $\prod_{n=1}^{\infty} E_n$ such that if

$$x = (x_1, x_2, \ldots, x_n, \ldots) \in T^{N}$$
$$y = (y_1, y_2, \ldots, y_n, \ldots) \in \bigoplus_{n=1}^{\infty} E_n,$$

then

$$g(x, y) = (u_1(x_1) + y_1, u_2(x_2) + y_2, \ldots, u_n(x_n) + y_n, \ldots).$$

The mapping g is a weak homomorphism from $T^{N} \times \bigoplus_{n=1}^{\infty} E_n$ into $\prod_{n=1}^{\infty} E_n$, such that $g\left(T^{N} \times \bigoplus_{n=1}^{\infty} E_n\right)$ is locally dense and non-closed in $\prod_{n=1}^{\infty} E_n$, (see the proof of Theorem 2 in [7]).

If I is the identity mapping of $\bigoplus_{n=1}^{\infty} E_n$, then it suffices to take the quotient of $L[\sigma(L, L')]$ by the kernel of the mapping $g \circ (h \circ (J \otimes f) \times I)$ in order to obtain the conclusion of the theorem. q.e.d.

COROLLARY 1.1. — *Let* (E_n) *be a sequence of separable infinite-dimensional Fréchet spaces. Then there is a quotient of*

$s \times \bigoplus_{n=1}^{\infty} E_n$, *topologically isomorphic to a non-closed dense subspace of* $\prod_{n=1}^{\infty} E_n$.

Proof. — Since E_n is a separable Fréchet space there exists an absolutely convex compact separable subset of E_n, which is total in E_n.

We put
$$T = \{(x, y) \in R^2 : x^2 + y^2 = 1\}.$$

Then $\mathscr{D}(T) \simeq s$ and $\mathscr{D}(T \times T) \simeq s$, (see [1], Chapter II, p. 54). Since $\mathscr{D}(T) \hat{\otimes} \mathscr{D}(T) \simeq \mathscr{D}(T \times T)$, (see [1], Chapter II, p. 84-85), we have that $s \hat{\otimes} s \simeq s$. On the other hand, s is an infinite-dimensional Fréchet space, different from ω, which is not Banach.

Using Theorem 1 we obtain a subspace P of $s \times \bigoplus_{n=1}^{\infty} E_n$ such that $\left(s \times \bigoplus_{n=1}^{\infty} E_n\right)/P$ is weakly isomorphic to a dense non-closed subspace of $\prod_{n=1}^{\infty} E_n$. Since $\left(s \times \bigoplus_{n=1}^{\infty} E_n\right)/P$ is a Mackey space and every subspace of $\prod_{n=1}^{\infty} E_n$ is also a Mackey space, we have that $\left(s \times \bigoplus_{n=1}^{\infty} E_n\right)/P$ is topologically isomorphic to a dense non-closed subspace of $\prod_{n=1}^{\infty} E_n$ q.e.d.

Note 1. — O. G. Smoljanov proves in [6] that there is a closed subspace H in $s \times \omega^{(N)}$ such that $(s \times \omega^{(N)})/H$ is a non closed dense subspace of ω. This result is a consequence of our Corollary 1.1, identifing ω with E_n, $n = 1, 2, \ldots$, and noticing that $\omega^N \simeq \omega$.

THEOREM 2. — *Let* (E_n) *be a sequence of non-Banach Fréchet spaces. Then* $(E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ *has a quotient which is topologically isomorphic to a non-closed dense subspace of* ω.

Proof. — Since E_n is not a Banach space there is a subspace G_n of $(E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ lying in E_n, such that $E_n/G_n \simeq \omega$.

Let $G = \bigoplus_{n=1}^{\infty} G_n$. Then

$$\left[(E \mathbin{\hat{\otimes}} F) \times \bigoplus_{n=1}^{\infty} E_n\right] \Big/ G \simeq (E \mathbin{\hat{\otimes}} F) \times \bigoplus_{n=1}^{\infty} (E_n/G_n) \simeq (E \mathbin{\hat{\otimes}} F) \times \omega^{(N)}$$

and the conclusion follows straightforward from Theorem 1 applied to $(E \mathbin{\hat{\otimes}} F) \times \omega^{(N)}$. q.e.d.

COROLLARY 1.2. — *There is a quotient of* $s^{(N)}$ *which is topologically isomorphic to a non-closed dense subspace of* ω.

Proof. — Since $s \mathbin{\hat{\otimes}} s \simeq s$, it is possible to write $(s \mathbin{\hat{\otimes}} s) \times s^{(N)} \simeq s^{(N)}$. Now we apply Theorem 2. q.e.d.

LEMMA. — *The space* $\mathscr{D}(\Omega)$ *has a closed subspace topologically isomorphic to* $s^{(N)}$.

Proof. — In Ω let (K_n) be a sequence of compact sets such that $K_n \subset \mathring{K}_{n+1}$, $n = 1, 2, \ldots$, being $\mathring{K}_{n+1}$ the interior set of K_{n+1}, and $\bigcup_{n=1}^{\infty} K_n = \Omega$. Suppose that $K_1 \neq \emptyset$. We put $K_0 = \emptyset$. Let R_n be a closed m-dimensional rectangle contained in $\mathring{K}_n \sim K_{n-1}$, $n = 1, 2, \ldots$ Let E_n and F_n be the subspaces of $\mathscr{D}(\Omega)$ formed by the elements of $\mathscr{D}(\Omega)$ whose supports are contained in R_n and K_n, respectively. We now prove that $E = \bigoplus_{n=1}^{\infty} E_n$ is a subspace of $\mathscr{D}(\Omega)$. Let V_n be an absolutely convex neighbourhood of the origin in E_n. Since E_1 is a subspace of the Fréchet space F_1 there is an absolutely convex neighbourhood U_1 of the origin in F_1 such that $U_1 \cap E_1 = V_1$. Suppose that we have constructed in F_n an absolutely convex neighbourhood of the origin U_n such that

$$U_n \cap [F_{n-1} \oplus E_n] = \Gamma(U_{n-1} \cup V_n), \quad n \geqslant 1, \quad U_0 = \{0\},$$

being $\Gamma(U_{n-1} \cup V_n)$ the absolutely convex hull of $U_{n-1} \cup V_n$. Since $F_n \oplus E_{n+1}$ is closed in F_{n+1} and E_{n+1} and F_n are Fréchet spaces, we have that $F_n \oplus E_{n+1}$ is a subspace of F_{n+1} and therefore we can find an absolutely convex neighbourhood U_{n+1} of the origin in F_{n+1} such that

$$U_{n+1} \cap (F_n \oplus E_{n+1}) = \Gamma(U_n \cup V_{n+1}).$$

Let $U = \bigcup_{n=1}^{\infty} U_n$. It is easy to see that $U \cap \left(\bigoplus_{n=1}^{\infty} E_n\right)$. coincides with the absolutely convex hull $\Gamma\left(\bigcup_{n=1}^{\infty} V_n\right)$ of $\bigcup_{n=1}^{\infty} V_n$ and $U \cap F_n$ coincides with U_n.

Since $\mathscr{D}(\Omega)$ is the inductive limit of the sequence (F_n) we have that U is a neighbourhood of the origin in $\mathscr{D}(\Omega)$ and therefore E is a subspace of $\mathscr{D}(\Omega)$. On the other hand, E is complete, hence E is closed in $\mathscr{D}(\Omega)$.

If P and Q are two compact subsets of R^p and R^q, respectively, then

$$\mathscr{D}_P(R^p) \hat{\otimes} \mathscr{D}_Q(R^q) \simeq \mathscr{D}_{P\times Q}(R^{p+q}),$$

(see [1], Chapter II, p. 84). If α_r and β_r are two real numbers such that $\alpha_r < \beta_r$, $r = 1, 2, \ldots, m$, and $I_r = [\alpha_r, \beta_r]$, then $\mathscr{D}_{I_r}(R^1) \simeq s$, (see [3], p. 176). We have

$$\mathscr{D}_{I_1\times I_2}(R^2) \simeq \mathscr{D}_{I_1}(R^1) \hat{\otimes} \mathscr{D}_{I_2}(R^1) \simeq s \hat{\otimes} s \simeq s,$$
$$\mathscr{D}_{I_1\times I_2\times I_3}(R^3) \simeq \mathscr{D}_{I_1\times I_2}(R^2) \hat{\otimes} \mathscr{D}_{I_3}(R^1) \simeq s \hat{\otimes} s \simeq s,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathscr{D}_{I_1\times I_2\times\cdots\times I_m}(R^m) \simeq \mathscr{D}_{I_1\times I_2\times\cdots\times I_{m-1}}(R^{m-1}) \hat{\otimes} \mathscr{D}_{I_m}(R^1) \simeq s \hat{\otimes} s \simeq s$$

and therefore $E_n \simeq s$, hence $\bigoplus_{n=1}^{\infty} E_n \simeq s^{(N)}$. q.e.d.

THEOREM 3. — *Given a space* E, *suppose that* E *has a subspace* G *topologically isomorphic to* $s^{(N)}$. *Then* E *has a separated quotient which is not locally complete.*

Proof. — According to Corollary 1.2, there is in G a closed subspace H so that G/H is topologically isomorphic to a non-closed dense subspace of ω. Since G/H is a closed subspace of E/H and G/H is not locally complete, we have that E/H is not locally complete. q.e.d.

COROLLARY 1.3. — *There is a quotient of* $\mathscr{D}(\Omega)$ *wich is not locally complete.*

Proof. — It is immediate from the Lemma and Theorem 3.
q.e.d.

Note 2. — O. G. Smoljanov prove in [6] that there is a quotient of $\mathscr{D}(R^1)$ which is topologically isomorphic to a non-closed dense subspace of ω. Since $\mathscr{D}(\Omega)$ has a quotient topologically isomorphic to $\mathscr{D}(R^1)$, [10], Corollary 1.3 can be obtained from the result of Smoljanov.

The following result, which will be used afterwards was proved in [9] : *b*) *Let* G *be a Banach space of infinite dimension. If a space* L *is not dual locally complete, there is a linear mapping* g *of* L *into* G, *with closed graph, which is not weakly continuous.*

THEOREM 4. — *In* $\mathscr{D}'(\Omega)$ *there is a closed subspace* G *such that, if* H *is an arbitrary infinite-dimensional Banach space, there is a linear mapping from* G *into* H, *with closed graph, which is not weakly continuous.*

Proof. — Corollary 1.3 gives a closed subspace L of $\mathscr{D}(\Omega)$ such that $\mathscr{D}(\Omega)/L$ is not locally complete. If G is the orthogonal subspace of $\mathscr{D}'(\Omega)$ to L, then G is not dual locally complete. Now it suffices to use result *b*). q.e.d.

We shall use the following result that we proved in [10] : *c*) *Let* G *be a* (LF)-*space. If there is in* G *a closed subspace* H *such that* G/H *is a non complete separable metrizable space, then* $G'[\mu(G', G)]$ *is not* B_r-*complete.* We take G separable in [10], however the proof is valid also if G is non-separable.

THEOREM 5. — *Let* (E_n) *be a sequence of infinite-dimensional separable Fréchet spaces. Then the dual of* $(E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ *with the Mackey topology is not* B_r-*complete.*

Proof. — It is direct consequence of result *c*) and Theorem 1.
q.e.d.

THEOREM 6. — *Let* (E_n) *be a sequence of non-Banach Fréchet spaces. Then the dual of* $(E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ *with the Mackey topology is not* B_r-*complete.*

Proof. — It follows straightforward from result *c*) and Theorem 2. q.e.d.

In Theorem 7 we suppose that (E_n) is a sequence of infinite-dimensional spaces such that there is an absolutely convex weakly compact separable subset of E_n which is total in E_n, $n = 1, 2, \ldots$ We can take an infinite-dimensional closed subspace M of $E'[\sigma(E', E)]$ which contains a total compact separable subset.

THEOREM 7. — *Let* $L = (E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$. *Then there is a closed subspace* G *of* $L'[\sigma(L', L)]$ *and a topological homomorphism* f *from* G *into* M^N, *such that* $f(G)$ *is a non-closed locally dense subspace of* M^N.

Proof. — Let $T = E/M^{\perp}$, being $M^{\perp}$ the orthogonal subspace of E to M. Then it is possible to identify $T'[\sigma(T', T)]$ with M.

If $T_n = T$, $n = 1, 2, \ldots$, we can write T^N in the form $\prod_{n=1}^{\infty} T_n$. Let X be the weak dual of $E \hat{\otimes} F$. In the same way as in the proof of Theorem 1, we see that there is a topological homomorphism u from $E \hat{\otimes} F$ onto $\prod_{n=1}^{\infty} T_n$. Let ${}^t u$ be the mapping from

$$\left(\bigoplus_{n=1}^{\infty} T'_n\right)\left[\sigma\left(\bigoplus_{n=1}^{\infty} T'_n, \prod_{n=1}^{\infty} T_n\right)\right] \quad \text{into} \quad X,$$

which is the transposed of u. Let $Y = {}^t u\left(\bigoplus_{n=1}^{\infty} T'_n\right)$ and $Y_n = {}^t u(T'_n)$. Obviously, ${}^t u$ is an injective topological homomorphism and therefore Y and Y_n, $n = 1, 2, \ldots$, are closed subspaces of X which can be identified with the weak duals of $\prod_{n=1}^{\infty} T_n$ and T_n, respectively.

Taking into account result *a*) being $G = E'_n[\mu(E'_n, E_n)]$ and $H = Y_n$, we can find an injective continuous mapping φ_n from $E'_n[\sigma(E'_n, E_n)]$ into Y_n such that $\varphi_n(E'_n)$ is a non-closed dense subspace of Y_n. We define a mapping f from

$$G = Y \times \prod_{n=1}^{\infty} E'_n[\sigma(E'_n, E_n)] \quad \text{into} \quad \prod_{n=1}^{\infty} Y_n$$

such that if

$$y = \sum_{n=1}^{\infty} y_n \in Y, \qquad y_n \in Y_n, \qquad n = 1, 2, \ldots,$$

with $\sum_{n=1}^{\infty} y_n$ having only a finite number of non-zero terms, and

$$x = (x_1, x_2, \ldots, x_n, \ldots) \in \prod_{n=1}^{\infty} E'_n,$$

then

$$f(y, x) = (y_1 + \varphi_1(x_1), y_2 + \varphi_2(x_2), \ldots, y_n + \varphi_n(x_n), \ldots).$$

The mapping f is a topological homomorphism from G into $\prod_{n=1}^{\infty} Y_n \simeq M^N$ such that its image is a non-closed locally dense subset of $\prod_{n=1}^{\infty} Y_n$ (see the proof of Theorem 1). q.e.d.

THEOREM 8. — *Let* (E_n) *be a sequence of infinite-dimensional separable Fréchet spaces. Let* $L = (E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$. *Then there is a separated quotient of* $L'[\sigma(L', L)]$ *which is not locally complete.*

Proof. — In E_n there is an absolutely convex total separable weakly compact subset, and therefore it is possible to apply Theorem 7.

With the same notation as in Theorem 7, let B be the kernel of f. Then G/B is a closed subspace of $L'[\sigma(L', L)]/B$. Since G/B is not locally complete, we can conclude that $L'[\sigma(L', L)]/B$ is not locally complete. q.e.d.

COROLLARY 1.8. — *Let* H *be a space which contains a subspace* M *topologically isomorphic to* $s^{(N)}$. *Then there is a separated quotient of* $H'[\mu(H', H)]$ *which is not locally complete.*

Proof. — Let $M^{\perp}$ be the subspace of $H'[\mu(H', H)]$ orthogonal to M. Let φ be the canonical mapping from $H'[\mu(H', H)]$ onto $H'[\mu(H', H)]/M^{\perp}$. Since

$$s^{(N)} \simeq s \times s^{(N)} \simeq (s \hat{\otimes} s) \times s^{(N)},$$

we apply Theorem 8 and obtain a separated quotient P of $H'[\mu(H', H)]/M^{\perp}$ which is not locally complete. Let ψ be the canonical mapping from $H'[\mu(H', H)][M^{\perp}$ onto P. If Q is the kernel of $\psi \circ \varphi$ then $H'[\mu(H', H)]/Q$ is topologically isomorphic to P and therefore it is not locally complete. q.e.d.

Note 3. — In [5] D. A. Raikov proves that $\mathscr{D}'(R^1)$ has a separated non-complete quotient. Considering that $\mathscr{D}(\Omega)$ contains a subspace topologically isomorphic to $s^{(N)}$, according to the Lemma, Raikov's result is a particular case of our Corollary 1.8.

THEOREM 9. — *There is a closed subspace* G *of* $\mathscr{D}(\Omega)$ *such that, if* L *is an arbitrary infinite dimensional Banach space, it is possible to find a linear mapping from* G *into* L, *with closed graph, which is not weakly continuous.*

Proof. — Because of Corollary 1.8 and the Lemma, there is a closed subspace H of $\mathscr{D}'(\Omega)$ so that $\mathscr{D}'(\Omega)/H$ is not locally complete. If G is the orthogonal subspace of $\mathscr{D}(\Omega)$ to H, then G is not dual locally complete. It is suffices to apply result *b*). q.e.d.

In order to prove Theorem 10 and Theorem 12 we shall need the following result that we have proved in [10] : *d*) *Let* E *be a separable space. Let* (E_m) *be an increasing sequence of subspaces of* E *so that* $E = \bigcup_{m=1}^{\infty} E_m$. *If there is a bounded set* A *in* E *such that* $A \not\subset E_m$, $m = 1, 2, \ldots$, *there is a dense subspace* F *of* E, $F \neq E$, *so that* $F \cap E_m$ *is finite-dimensional for every* m *positive integer.*

THEOREM 10. — *Let* (E_n) *be a sequence of infinite-dimensional Fréchet spaces. Then* $(E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ *is not* B_r*-complete.*

Proof. — Since every closed subspace of a B_r-complete spaces is B_r-complete, [4], we can suppose that E, F, E_n, $n = 1, 2, \ldots$, are separable spaces.

With the same notation as in the proof of Theorem 7, let B be the kernel of f. Then B is a closed subspace of

$$P = X \times \prod_{n=1}^{\infty} E'_n[\sigma(E'_n, E_n)]$$

which lies in G.

Let (U_n) be a fundamental system of neighbourhoods of the origin in $E \widehat{\otimes} F$ such that $U_n \supset U_{n+1}$, $n = 1, 2, \dots$ Being A_n the polar set of U_n in X, let H_n be the linear hull of A_n with the induced topology of $\sigma(X, E \widehat{\otimes} F)$ and let

$$G_n = H_n \times \prod_{n=1}^{\infty} E'_n[\sigma(E'_n, E_n)].$$

Let z_n be an element in Y_n such that $z_n \notin v_n(E'_n)$. Then the *nth* coordinate of $f(z_n, 0)$ coincides with z_n and the remaining are equal to zero, hence $(f(z_n, 0))$ is a bounded sequence in $\prod_{n=1}^{\infty} Y_n$.

If h is the canonical mapping from P onto P/B, h coincides with f on G and consequently $(h(z_n, 0))$ is a bounded sequence in P/B.

X_{A_n} is a Banach space with $Y \cap X_{A_n}$ as a closed subspace. Since

$$\left(\sum_{m=1}^{p} Y_m\right) \cap X_{A_n}$$

is a closed subset of X_{A_n} and

$$\cup \left\{\sum_{m=1}^{p} Y_m : p = 1, 2, \dots\right\} = Y,$$

we can use the Baire theorem and we obtain a positive integer r such that

$$\left(\sum_{m=1}^{r} Y_m\right) \cap X_{A_n} = Y \cap X_{A_n}.$$

Therefore, if q is an integer larger than r

$$Y_q \cap X_{A_n} = \{0\},$$

and thus $z_q \notin H_n$.

Now fix $q > r$ and assume that for some n there exists an element t_n in P such that

$$h(t_n) = h(z_q, 0)$$

with

$$t_n = (y_n, (x_1, x_2, \ldots, x_m, \ldots)),$$
$$y_n \in H_n, \ (x_1, x_2, \ldots, x_m, \ldots) = x \in \prod_{m=1}^{\infty} E'_m.$$

Therefore $h(t_n - z_q, 0)) = 0$ and, since $(z_q, 0) \in G$ and B lies in G, it results that $t_n \in G$. Thus $y_n \in Y$. Hence

$$y_n \in Y \cap H_n = Y \cap X_{A_n} = \left(\sum_{m=1}^{r} Y_m\right) \cap X_{A_n}.$$

We can write

$$y_n = \sum_{p=1}^{r} y_{n_p}, \qquad y_{n_p} \in Y_p$$

and therefore

$$h(t_n) = f(y_n, x) = (y_{n_1} + \varphi_1(x_1), \ y_{n_2} + \varphi_2(x_2), \ldots, \\ y_{n_r} + \varphi_r(x_r), \ \varphi_{r+1}(x_{r+1}), \ \varphi_{r+2}(x_{r+2}), \ldots)$$

and thus the qth component in $h(t_n)$ coincides with $\varphi_q(x_q)$. On the other hand, $h(z_q, 0) = f(z_q, 0)$ has its qth component equal to z_q and since $z_q \notin \varphi_q(E'_q)$ this a contradiction. Consequently $h(z_q, 0) \notin h(G_n)$.

From the last considerations we can assert that the subset of P/B

$$\{h(z_1, 0), h(z_2, 0), \ldots, h(z_n, 0), \ldots\}$$

is bounded and is not included in $h(G_n)$ for any positive integer n. Moreover,

$$P/B = \cup \{h(G_n) : n = 1, 2, \ldots\}.$$

We can apply result $d)$ in order to obtain a dense non closed subspace V in P/B such that $V \cap h(G_n)$ is finite-dimensional, $n = 1, 2, \ldots$ If x_0 is a point lying in the closure of $G_n \cap h^{-1}(V)$ in G_n, then $h(x_0)$ is a point of the closure of $h(G_n) \cap V$ in $h(G_n)$. The space $h(G_n) \cap V$ is finite-dimensional and thus $h(x_0) \in h(G_n) \cap V$. Hence $x_0 \in G_n \cap h^{-1}(V)$ and therefore this space is closed in G_n.

If A is a compact subset of P, let $A^{(1)}$ and $A^{(2)}$ be the projection of A onto X and $\prod_{n=1}^{\infty} E'_n$, respectively. Since (A_n) is a fundamental system of compact subsets in X, there is a positive integer n_0 such that $A^{(1)} \subset A_{n_0}$. Therefore, $A \in G_{n_0}$. Moreover $G_{n_0} \cap h^{-1}(V)$ is closed in G_{n_0}, thus $A \cap h^{-1}(V)$ is compact. Since $h^{-1}(V)$ is a non-closed dense subspace of P it results that $(E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ is not B_r-complete. q.e.d.

THEOREM 11. — *Let* G *be a space which contains a subspace* L *topologically isomorphic to* $s^{(N)}$. *Then* G *is not* B_r*-complete.*

Proof. — Since every closed subspace of a B_r-complete space is B_r-complete, [4], and

$$L \simeq s^{(N)} \simeq s \times s^{(N)} \simeq (s \hat{\otimes} s) \times s^{(N)},$$

it suffices to apply Theorem 10. q.e.d.

COROLLARY 1.11. — *The space* $\mathscr{D}(\Omega)$ *is not* B_r*-complete.*

Proof. — It suffices to apply the Lemma and Theorem 11. q.e.d.

THEOREM 12. — *Let* (E_n) *be a sequence of infinite-dimensional separable Fréchet spaces. Let* G *be a separable countable inductive limit of Fréchet spaces. If there is a subspace* L *of* G *which is topologically isomorphic to* $(E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$, *then* $G'[\mu(G', G)]$ *is not* B_r*-complete.*

Proof. — Let us suppose that G is the inductive limit of the increasing sequence (G_n) of Fréchet spaces. From Theorem 1 it follows that there is a closed subspace Q of L such that L/Q is topologically isomorphic to a non-closed dense subspace of $\prod_{m=1}^{\infty} E_n$.

If φ is the canonical mapping from G onto G/Q, it leads up to $\varphi(G_n) \not\supset \varphi(L)$, $n = 1, 2, \ldots,$ for if there is a positive integer n_0 such that $\varphi(G_{n_0}) \supset \varphi(L)$, the restriction ψ of φ to $L \cap G_{n_0}$, considering this space as a subspace of G_{n_0}, is continuous and $\psi\,(L \cap G_{n_0}) = \varphi(L)$. Since the subspace $\varphi(L)$ of G/Q is topologically isomorphic to L/Q, which is barrelled, we use the open-mapping theorem, [4], and we obtain that ψ is open and so $\varphi(L)$ is a Fréchet space. But we arrive to a contradiction since $\varphi(L)$ is topologically isomorphic to a non-closed dense subspace of $\prod_{n=1}^{\infty} E_n$.

Consequently, let (n_p) be an increasing sequence of positive integers such that

$$\varphi(G_{n_p}) \cap \varphi(L) \neq \varphi(G_{n_{p+1}}) \cap \varphi(L), \qquad p = 1, 2, \ldots$$

Let us pick an element $x_{p+1} \in \varphi(G_{n_{p+1}}) \cap \varphi(L)$ which is not in $\varphi(G_{n_p})$. Since $\varphi(L)$ is metrizable, there exists a sequence of strictly positive real numbers (α_p) such that the set

$$A = \{\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_p x_p, \ldots\}$$

is bounded. Hence A is a bounded set in G/Q which is not contained in $\varphi(G_{n_p})$, $p = 1, 2, \ldots,$ and therefore, using result *d*), there is a dense subspace F of G/Q, $F \neq G/Q$, such that $F \cap \varphi(G_n)$ is finite-dimensional, $n = 1, 2, \ldots$

Let $Q^{\perp}$ be the orthogonal subspace of Q in $G'[\mu(G', G)]$. If B is an absolutely convex closed subset of G/Q equicontinuous on $Q^{\perp}$ then $(G/Q)_B$ is a Banach space. Since G/Q is a countable inductive limit of Fréchet spaces, we can use Grothendieck's closed graph theorem, [1], applied to the canonical injection from $(G/Q)_B$ into G/Q. Then there is a positive integer n_l such that B is contained in $\varphi(G_{n_l})$, hence

$$B \cap F = B \cap \varphi(G_{n_l}) \cap F$$

and since $\varphi(G_{n_l}) \cap F$ is a finite dimensional space, $B \cap F$ is closed in B. Hence $Q^{\perp}$ is not B_r-complete.

Since each closed subspace of a B_r-complete space is itself B_r-complete, [4], it follows that $G'[\mu(G', G)]$ is not B_r-complete. q.e.d.

Note. — In [10] we proved that the space $\mathscr{D}'(\Omega)$ is not B_r-complete. This result follows also from our Theorem 12, considering that $\mathscr{D}(\Omega)$ is a countable strict inductive limit of separable Fréchet spaces and that it has a subspace L such that $L \simeq s^{(N)} \simeq (s \mathbin{\widehat{\otimes}} s) \times s^{(N)}$.

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