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ON THE FRACTIONAL PARTS OF $x/n$ AND RELATED SEQUENCES. III
by B. SAFFARI and R.-C. VAUGHAN

1. Introduction.

The object of this paper is to investigate the behaviour of $\Phi_{x,y}(x, h)$ (for notation see [2] and [3]) when

$$h(n) = \frac{1}{\log n} \quad (n > 1)$$

and $h(n) = \log n$. In contradistinction to the case $h(n) = 1/n$ it is immediately apparent that the behaviour of $\Phi_{x,y}$ is non-trivial even when $y$ is a large as $e^x$. For simplicity we only investigate the situation when $\mathcal{A}$ is the Toeplitz transformation formed from the simple Riesz means $(R, \lambda_n)$ with $\lambda_n = 1$.

Theorems 1 and 2 deal with the case $h(n) = 1/\log n$, whereas Theorem 3 deals with $h(n) = \log n$. While it is well known ([1], Example 2.4, p. 8) that the sequence $\log n$ is not uniformly distributed modulo 1, Theorem 3 shows that it is uniformly distributed in the present context.

2. Theorems and proofs.

2.1. Let

$$(2.1) \quad \Xi_{x,y}(x) = y^{-1} \sum_{2 \leq n \leq y} c_n(x/\log n).$$

**Theorem 1.** — Suppose that $0 < x < 1$ and $\log y \ll x^2$. Then

$$\Xi_{x,y}(x) = x + O(xy^{-1} (\log x)^{-1}) + O(x^{-1} \log^2 y).$$
Corollary 1.1. — Suppose that $x = o(y \log x)$ and 
$\log y = o\left(\frac{1}{x^2}\right)$ as $x \to \infty$. Then

$$
E_{x,y}(x) \to \alpha \text{ as } x \to \infty.
$$

Proof. — Clearly by (2.1),

$$
yE_{x,y}(x) = S(0) - S(\alpha)
$$

where

$$
S(\beta) = \sum_{m=1}^{\infty} \sum_{\substack{2 \leq n \leq y \atop n \leq e^{\beta/(m+\beta)}}} 1
$$

and $0 \leq \beta < 1$. Let

$$
M_\beta = \left\lfloor \frac{x}{\log y} - \beta \right\rfloor
$$

and

$$
T(\beta) = \sum_{M_\beta < m \leq \frac{x}{\log y} - \beta} \sum_{2 \leq n \leq e^{\beta/(m+\beta)}} 1.
$$

Then, by (2.3),

$$
S(\beta) = T(\beta) + ([y] - 1)M_\beta.
$$

By (2.5),

$$
T(\beta) = \sum_{M_\beta < m \leq \frac{x}{\log y} - \beta} e^{\beta/(m+\beta)} + \sum_{2 \leq n \leq e^{\beta/H}} \frac{x}{\log n} - H e^{\beta/H} + O(H) + O(e^{\beta/H})
$$

where $H$ is a real number at our disposal. Hence, by (2.4),

$$
T(0) - T(\alpha) = \sum_{M_\beta < m \leq H} e^{\beta/m} - \sum_{M_\beta < m \leq H - \alpha} e^{\beta/(m+\alpha)} + O(H) + O(e^{\beta/H})
$$

whenever $H > M_\beta + 1$. Thus

$$
T(0) - T(\alpha) = I(0) - I(\alpha) + O(H) + O(e^{\beta/H}),
$$

where

$$
I(\beta) = \int_{M_\beta}^{H - \beta} ([u] - M_\beta e^{\beta/(u+\beta)}) \frac{x}{(u + \beta)^2}.
$$

Let $b(u)$ denote the first Bernoulli polynomial modulo one,
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b(u) = \{u\} - 1/2. Then, by (2.9),

\begin{equation}
I(\beta) = \int_{M_0+\frac{1}{2}}^H (\nu - M_0 - \beta - 1/2)e^{\pi i \nu x^2} d\nu
- \int_{M_0+\frac{1}{2}}^H b(u - \beta)e^{\pi i \nu x^2} d\nu.
\end{equation}

The argument now divides into two cases according as \(M_0 = M_x\) or \(M_0 = M_x + 1\).

The case \(M_0 = M_x\). Write \(M\) for the common value. Then, by (2.10),

\begin{equation}
I(0) - I(\alpha) = \int_M^{M+\alpha} \left(\nu - M - \frac{1}{2}\right)e^{\pi i \nu x^2} d\nu + \alpha \int_{M+\alpha}^H e^{\pi i \nu x^2} d\nu
- \int_M^H b(\nu)e^{\pi i \nu x^2} d\nu + \int_{M+\alpha}^H b(\nu - \alpha)e^{\pi i \nu x^2} d\nu.
\end{equation}

The first integral contributes \(\ll e^{x\sqrt{M}}xM^{-2}\), the second is \(\alpha (e^{\pi i (M+\alpha)} - e^{\pi i H})\) and by partial integration the last two are easily seen to contribute \(\ll e^{\pi i (M+M)M^{-2}}\). Hence, by (2.8),

\begin{equation}
(2.11) \quad T(0) - T(\alpha) = \alpha e^{\pi i (M+\alpha)} + O(H)
+ O(e^{\pi i H}) + O(e^{\pi i M}xM^{-2}).
\end{equation}

Recall that \(M = M_0 = \lfloor x/\log y \rfloor\) and \(\log y \ll x^{1/2}\). Thus \(e^{\pi i (M+\alpha)} = \exp(\log y + O(x^{-1} \log^2 y)) = y(1 + O(x^{-1} \log^2 y))\) and \(e^{\pi i M}xM^{-2} = O(yx^{-1} \log^2 y)\). Hence, by (2.2), (2.6) and (2.11)

\begin{equation}
y\Xi_{x,y}(\alpha) = ay + O(H) + O(e^{\pi i H}) + O(yx^{-1} \log^2 y).
\end{equation}

The choice \(H = \frac{x}{\log (x/\log x)}\) now gives the desired conclusion.

The case \(M_0 = M_x + 1\). Write \(M\) for \(M_x\). Then, by (2.10),

\begin{equation}
I(0) - I(\alpha) = (\alpha - 1) \int_{M+1}^H e^{\pi i \nu x^2} d\nu
- \int_{M+\alpha}^{M+1} \left(\nu - M + \alpha - \frac{1}{2}\right)e^{\pi i \nu x^2} d\nu
+ O(e^{\pi i (M+\alpha)}x(M + \alpha)^{-2}).
\end{equation}

Now proceeding as in the previous case we obtain

\begin{equation}
T(0) - T(\alpha) = (\alpha - 1)y + O(H) + O(e^{\pi i H}) + O(yx^{-1} \log^2 y).
\end{equation}
Since \( M_0 = M_x + 1 \), this with (2.6) and (2.2) and the choice \( H = \frac{x}{\log (x/\log x)} \) gives the required result once more.

2.2. One might expect that the theorem holds even when \( y \) is close to \( e^x \), but this is false. In fact the next theorem indicates that Theorem 1 is essentially best possible, at least as for as the upper bound on \( y \) is concerned.

**Theorem 2.** — Suppose that \( 0 < \alpha < 1 \), \( \frac{1}{2} < \theta < 1 \) and \( y = \exp (x^\theta) \). Then \( \limsup_{x \to \infty} \Xi_{x,y}(\alpha) = 1 \) and
\[
\liminf_{x \to \infty} \Xi_{x,y}(\alpha) = 0.
\]

**Proof.** — We begin by following the proof of Theorem 1 as far as (2.7). Suppose that \( 0 < \beta < 1 \),
\[
y = \exp (x^\beta),
\]
and
\[
H = x.
\]
Then, by (2.4),
\[
\frac{x}{(M_\beta + 2)(M_\beta + 3)} \geq x^{-1} (\log y)^2 = x^{2\theta - 1}.
\]
Thus, by (2.13),
\[
\sum_{M_\beta + 1 < m < H - \beta} e^{x(m+\beta)} \leq xe^{x(M_\beta+1+\beta)} \exp (-C_n x^{2\theta-1}).
\]
Hence, by (2.7) and (2.13),
\[
(2.14) \quad T(0) - T(\alpha) = (e^{x(M_\beta+1)} - e^{x(M_\beta+1+\alpha)}) (1 + O(x^{-1})) + O(x).
\]
To obtain the inferior limit, let \( N \) be a large natural number and let
\[
(2.15) \quad x = x_N = (N + \alpha)^{1-\theta}.
\]
Then, by (2.4) and (2.12), \( M_0 = M_x = N \). Hence, by (2.2), (2.6), (2.12), (2.14) and (2.15),
\[
y \Xi_{x,y}(\alpha) = e^{x(N+1)} = o(y)
\]
as \( N \to \infty \).
For the superior limit, take instead

\[(2.16) \quad x = x_N = N^{1-\alpha} \]

Then, by (2.4) and (2.12), \( M_x = M_0 - 1 = N - 1 \), so that, by (2.2), (2.6), (2.12), (2.14) and (2.16),

\[
y^{N, \gamma}(x) \sim e^{x(N+\gamma)} + y \sim y \]

as \( N \to \infty \).

2.3. The latter part of the paper is devoted to \( h(n) = \log n \). It is well known that the sequence \( \log n \) is not uniformly distributed modulo 1, and in view of this the next theorem is perhaps rather surprising. However, one can take the view that \( x \) being permitted to go to infinity, however slowly by comparison with \( y \), crushes any unruly behaviour of the logarithmic function.

Let

\[(2.17) \quad \Omega_{x, \gamma}(x) = y^{-1} \sum_{n \leq y} c_x(x \log n). \]

**Theorem 3.** — Suppose that \( 0 < \alpha < 1, x \geq 2 \) and \( y \geq 2 \). Then

\[
\Omega_{x, \gamma}(x) = \alpha + O(x^{-1} \log x + x^{1/3}y^{-2/3}(\log xy)^{2/3}).
\]

**Corollary 3.1.** — Suppose that \( x^{1/2} \log x = o(y) \) as \( x \to \infty \). Then

\[
\Omega_{x, \gamma}(x) \to \alpha \quad \text{as} \quad x \to \infty.
\]

**Proof.** — Let

\[(2.18) \quad M = [y^{2/3}x^{-1/3}(\log xy)^{-2/3}] + 1. \]

Then, by Theorem 1 of [2] and (2.17),

\[(2.19) \quad \Omega_{x, \gamma}(x) = \alpha \leq y^{-1} + M^{-1} + y^{-1} \sum_{k=1}^{M} k^{-1} \sum_{n \leq y} e(kx \log n). \]

Let

\[(2.20) \quad Y = [y] + \frac{1}{2} \]

and

\[(2.21) \quad T = 4\pi kx. \]
Then, by Lemma 3.12 of Titchmarsh [4],

$$
\sum_{n \leq y} e(kx \log n) = \frac{1}{2\pi i} \int_{1+\frac{1}{\log y}-iT}^{1+\frac{1}{\log y}+iT} \frac{\zeta(s - 2\pi ikx)}{s} \frac{Y^s}{s} \, ds
+ O\left((\frac{Y}{T} + 1) \log xy\right)
$$

where $\zeta$ is the Riemann zeta function. By moving the path of integration to the line $\sigma = 1/\log y$, one obtains

$$
\sum_{n \leq y} e(kx \log n) = \frac{y^{1+2\pi ikx}}{1 + 2\pi ikx} + \frac{1}{2\pi i} \int_{\frac{1}{\log y}-iT}^{\frac{1}{\log y}+iT} \frac{\zeta(s - 2\pi ikx)}{s} \frac{Y^s}{s} \, ds
+ O((kx)^{1/2} + y \log kx)T^{-1}).
$$

Hence, by (2.21),

$$
\sum_{n \leq y} e(kx \log n) \ll (kx)^{1/2} \int_0^t \frac{dt}{t + \frac{1}{\log y}} + (kx)^{-1/2} + \frac{y \log kx}{kx}
$$

$$
\ll (kx)^{1/2} (\log \log y + \log kx) + y (\log kx)(kx)^{-1}.
$$

Thus

$$
\sum_{k=1}^M k^{-1} \left| \sum_{n \leq y} e(kx \log n) \right| \ll (Mx)^{1/2} (\log \log y + \log Mx) + yx^{-1} \log x.
$$

Therefore, by (2.18) and (2.19), we have the theorem.

**BIBLIOGRAPHY**


