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A NOTE ON REARRANGEMENTS OF FOURIER COEFFICIENTS

by Hugh L. MONTGOMERY

Let $\{\varphi_k\}$ be a sequence of functions on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, with the property that they are uniformly bounded,

$$\|\varphi_k\|_\infty \leqslant \mathrm{M}, \tag{1}$$

and satisfy a Bessels inequality

$$\sum_k \left|\int_0^1 f\varphi_k\right|^2 \leqslant \mathrm{M}^2 \int_0^1 |f|^2. \tag{2}$$

For the sake of simplicity we suppose that M has the same value in (1) and (2); this does not occasion any loss of generality. Suppose that $\sum_k |a_k|^2 < \infty$. Then

$$f(x) = \sum_k a_k\varphi_k(x) \tag{3}$$

is a member of $\mathrm{L}^2(\mathbf{T})$, since the dual of (2) asserts that

$$\int_0^1 \left|\sum_k a_k\varphi_k\right|^2 \leqslant \mathrm{M}^2 \sum_k |a_k|^2. \tag{4}$$

In this note we obtain bounds for $\int_\mathrm{E} |f|^2$ in terms of the measure of the set E and the numbers $|a_k|$. Following Hardy and Littlewood, we let the numbers $a_0^*, a_1^*, \ldots$ be the numbers $|a_k|$, permuted so that $a_n^*\searrow$. Then we set

$$f^*(x) = \sum_{n=0}^{\infty} a_n^* \cos 2\pi nx. \tag{5}$$

THEOREM 1. — *Let* $\{\varphi_k\}$ *be a sequence of functions satisfying* (1) *and* (2), *let* f *and* f^* *be defined by* (3) *and* (5). *Then* *for any measurable set* $E \subseteq \mathbf{T}$, *with measure* $|E| = 2\theta$, *we have*

$$\int_E |f|^2 \leqslant 20M^2 \int_{-\theta}^{\theta} |f^*|^2. \tag{6}$$

If $C \in L^2(\mathbf{T})$, $C(x) \sim \sum_{n=0}^{\infty} C_n \cos 2\pi nx$, and if $C_n \searrow$, then $C = C^*$, so (6) implies that

$$\int_E |C|^2 \leqslant 20 \int_{-\theta}^{\theta} |C|^2,$$

where $E \subseteq \mathbf{T}$, $|E| = 2\theta$. Thus, although it is not necessarily true that $C(x)$ is decreasing on $\left[0, \frac{1}{2}\right)$, in a certain sense it is still the case that C is largest near 0.

Using a simple inequality of. A. Baernstein [2], we shall derive from Theorem 1 the following.

Theorem 2. — *Let* ψ *be a convex increasing function from* $[0, \infty)$ *to* $\mathbf{R}$. *Then, in the above notation,*

$$\int_E \psi(|f|^2) \leqslant \int_{-\theta}^{\theta} \psi(20\, M^2 |f^*|^2).$$

Taking $\psi(t) = t^{q/2}$, we see from the above that

$$\|f\|_q \leqslant 5M \|f^*\|_q \quad (q \geqslant 2). \tag{7}$$

Inequalities of this type have a long history. Hardy and Littlewood [3, 4] proved that

$$\|f\|_q \leqslant c_q \|f^*\|_q \quad (q \geqslant 2) \tag{8}$$

in the case $\varphi_k(x) = e^{2\pi ikx}$, $-\infty < k < +\infty$. Littlewood [6] has shown that c_q is bounded in this case, and F. R. Keogh [5] has shown that $c_p \to 1$ as $q \to \infty$. In the opposite direction, Littlewood [7] showed that $c_q > 1$ except when q is an even integer. Consequently, the constant 20 in Theorems 1 and 2 can not be replaced by 1. R. E. A. C. Paley [9] extended (8) to the case of arbitrary uniformly bounded orthonormal φ_k (see Zygmund [11, XII § 5] for a simple proof). Theorem 2 does not seem to follow from the special case (7), since in general a convex increasing function $\psi(t)$ is not comparable to a sum $\sum_r c_r t^{a_r}$, $c_r \geqslant 0$, $a_r \geqslant 1$.

If one were to consider, in place of f^*, a function

$$f^{-}(x) = \sum_{n=0}^{\infty} a_n * \varphi_n(x),$$

then one does not in general expect the inequality

$$\|f\|_q \leqslant c_q \|f^{-}\|_q \quad (q \geqslant 2)$$

to be valid, even when the φ_n are given in some natural order. (See G. A. Bachelis [1], and H. S. Shapiro [10]). However, in the special case of ordinary Dirichlet series, there are good reasons to believe that something positive may be said. For example, we can formulate a

Conjecture. — Let $\varepsilon > 0$, and $2 \leqslant q \leqslant 4$. Then for $T \geqslant 2$, $N > N_0(\varepsilon, q)$, we have

$$\int_{-T}^{T} \left| \sum_{n=1}^{N} a_n n^{-it} \right|^q dt \leqslant (T + N^{q/2}) N^{q/2+\varepsilon},$$

for arbitrary coefficients a_n satisfying $|a_n| \leqslant 1$.

The above is known to be true when $q = 2$, $q = 4$; thus by Hölder's inequality it suffices to consider the case $T = N^{q/2}$. The Conjecture is of special interest in multiplicative number theory, since from it one can deduce (see Montgomery [8, Theorem 12.6]) that the interval $\left(x, x + x^{\frac{1}{2}+\varepsilon}\right)$ contains a prime number, for all $x > x_0(\varepsilon)$.

We now prove Theorem 1. We have only countably many functions φ_k, so without loss of generality we may suppose that $0 \leqslant k < \infty$. Let π be the permutation such that $a_n^* = |a_{\pi(n)}|$. Put $N = [(2\theta)^{-1}]$, and set

$$\mathcal{N} = \{\pi(n) : 0 \leqslant n \leqslant N\}.$$

Thus $\mathcal{N}$ is the set of indices of the $N + 1$ coefficients of largest absolute value. Break the sum (3) into two parts,

$$f = \sum_{n \in \mathcal{N}} + \sum_{n \notin \mathcal{N}} = f_1 + f_2,$$

say. On one hand,

$$\int_E |f_1|^2 \leqslant \|f_1\|_\infty^2 \int_E 1 \leqslant 2\theta \left(M \sum_{n=0}^{N} a_n^* \right)^2,$$

in view of (1). On the other hand, from (4) we see that

$$\int_E |f_2|^2 \leqslant \int_0^1 |f_2|^2 \leqslant M^2 \sum_{n>N} a_n^{*2}.$$

For each x, $|f|^2 \leqslant 2|f_1|^2 + 2|f_2|^2$, so on combining the above we find that

$$(9) \qquad \int_E |f|^2 \leqslant 4\theta \left(M \sum_{n=0}^{N} a_n^*\right)^2 + 2M^2 \sum_{n>N} a_n^{*2}.$$

It now remains to relate the right hand side above to $\int_{-\theta}^{\theta} |f^*|^2$. Let $K(x) = \max(0, 1 - |x|\theta^{-1})$ for $|x| \leqslant \frac{1}{2}$. Then

$$\int_{-\theta}^{\theta} |f^*|^2 \geqslant \int_{-\frac{1}{2}}^{\frac{1}{2}} K|f^*|^2 = \frac{1}{2} \sum_{m,n=0}^{\infty} a_m^* a_n^* (\hat{K}(m+n) + \hat{K}(m-n)).$$

Now $\hat{K}(m) = \theta \left(\frac{\sin \pi m\theta}{\pi m\theta}\right)^2 \geqslant 0$, so

$$(10) \qquad \frac{1}{2} \sum_{m,n=0}^{\infty} a_m^* a_n^* \hat{K}(m-n) \leqslant \int_{-\theta}^{\theta} |f^*|^2.$$

If $|m - n| \leqslant N$ then

$$(11) \quad \hat{K}(m-n) \geqslant \theta \left(\frac{\sin \pi N\theta}{\pi N\theta}\right)^2 \geqslant \theta \left(\frac{\sin \frac{1}{2}\pi}{\frac{1}{2}\pi}\right)^2 = 4\pi^{-2}\theta,$$

since $N \leqslant (2\theta)^{-1}$. But $a_n^* \geqslant 0$, so

$$(12) \quad \theta \left(\sum_{0 \leqslant n \leqslant N} a_n^*\right)^2 \leqslant \frac{1}{4}\pi^2 \sum_{0 \leqslant m,\, n \leqslant N} \hat{K}(m-n) a_m^* a_n^*.$$

If $0 < n - N \leqslant m \leqslant n$ then $a_m^* \geqslant a_n^*$, so from (11) we find that

$$\sum_{n-N \leqslant m \leqslant n} a_m^* \hat{K}(m-n) \geqslant 4\pi^{-2}\theta(N+1) a_n^* \geqslant 2\pi^{-2} a_n^*,$$

since $N + 1 > (2\theta)^{-1}$. Hence

$$(13) \qquad \sum_{n>N} a_n^{*2} \leqslant \frac{1}{2}\pi^2 \sum_{\substack{n>N \\ n-N \leqslant m \leqslant n}} \hat{K}(m-n) a_m^* a_n^*.$$

Combining (9) with (12), (13), we find that

$$\int_E |f|^2 \leqslant \pi^2 \theta M^2 \sum_{m,n=0}^{\infty} a_m^* a_n^* \hat{K}(m-n).$$

But $2\pi^2 < 20$, so by (10) our proof is complete.

We note that once (9) is established, the remainder of the proof can be effected in several ways. In proving (8), Hardy and Littlewood [3] established that

$$\int_0^1 |f^*|^q \approx_q \sum_{n=0}^{\infty} a_n^{*q}(n+1)^{q-2}.$$

One can modify their proof of this (see also Keogh [5]) to show that

$$\sum_{n>\theta^{-1}} n^{-2} \Big(\sum_{0\leqslant m\leqslant n} a_m^* \Big)^2 < c \int_{-\theta}^{\theta} |f^*|^2.$$

Theorem 1 follows easily from the above and (9), apart from the values of constants.

To prove Theorem 2 we require the following result of A. Baernstein [2].

Lemma. — *For* $f \in L^1(\mathbf{T})$, $0 \leqslant \theta \leqslant \frac{1}{2}$, *let* $f^+(\theta) = \sup_E \int_E |f|$, *where the supremum is taken over all measurable sets* $E \subseteq 0, 1)$ *such that* $|E| = 2\theta$. *For two functions* $r, s \in L^1(\mathbf{T})$, *the following are equivalent:*

(*a*) *For all* $\theta \in \left[0, \frac{1}{2}\right)$, $r^+(\theta) \leqslant s^+(\theta)$;

(*b*) *For any* $\psi(t)$, *convex and increasing on* $[0, \infty)$, *we have*

$$\int_0^1 \psi(|r|) \leqslant \int_0^1 \psi(|s|).$$

In the language of this lemma, we find from Theorem 1 that $(|f|^2)^+(\theta) \leqslant (20M|f^*|^2)^+(\theta)$. Hence

$$\|\psi(|f|^2)\|_1 \leqslant \|\psi(20M|f^*|^2)\|_1.$$

However, with a little more care we obtain the full strength of Theorem 2. Let $E \subseteq [0, 1)$ be a set with $|E| = 2\theta$. Put $r = |f|^2\chi_E$, $s = 20M^2|f^*|^2\chi_{(-\theta,\theta)}$. Then by Theorem 1, $r^+ \leqslant s^+$, so by the Lemma above, $\|\psi(r)\|_1 \leqslant \|\psi(s)\|_1$. If $\psi(0)=0$, then this asserts that

$$\int_E \psi(|f|^2) \leqslant \int_{-\theta}^{\theta} \psi(20M^2|f^*|^2).$$

To obtain this for general ψ we have only to add a constant to both sides of the inequality. This completes the proof of Theorem 2.

BIBLIOGRAPHY

[1] G. A. Bachelis, On the upper and lower majorant properties of $L^p(G)$, *Quart. J. Math.* (Oxford), (2), 24 (1973), 119-128.

[2] A. Baernstein, II, Integral means, univalent functions and circular symmetrizations, *Acta Math.*, 133 (1974), 139-169.

[3] G. H. Hardy and J. E. Littelwood, Notes on the theory of series (XIII): Some new properties of Fourier constants, *J. London Math. Soc.*, 6 (1931), 3-9.

[4] G. H. Hardy and J. E. Littelwood, A new proof of a theorem on rearrangements, *J. London math. Soc.*, 23 (1949), 163-168.

[5] F. R. Keogh, Some inequalities of Littlewood and a problem on rearrangements, *J. London Math. Soc.*, 36 (1961), 362-376.

[6] J. E. Littlewood, On a theorem of Paley, *J. London Math. Soc.*, 29 (1954), 387-395.

[7] J. E. Littlewood, On inequalities between f and f^*, *J. London Math. Soc.*, 35 (1960), 352-365.

[8] H. L. Montgomery, Topics in multiplicative number theory, *Lecture Notes in Mathematics*, Springer-Verlag, Vol. 227, (1971), 187 pp.

[9] R. E. A. C. Paley, Some theorems on orthogonal functions, *Studia Math.*, 3 (1931), 226-238.

[10] H. S. Shapiro, Majorant problems for Fourier coefficients, to appear.

[11] A. Zygmund, Trigonometric series, Second Edition, Cambridge University Press, 1968.

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